

Two logarithms II: explicit lower bounds

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A parameter theorem becomes useful only after its integers and inequalities have been checked. We now make that check for arbitrary two-term logarithmic forms, including multiplicative dependence, roots of unity, nonprincipal branches and collisions in the additive grid. The resulting constant is deliberately generous: it permits every numerical estimate to be visible.

We use the preceding lesson's determinant parameter theorem and the written internal *Heights of algebraic numbers*, Theorem 2.8, for the one-place Liouville inequality. The height throughout this lesson is the absolute logarithmic Weil height. The general estimate is followed by a refined determinant argument, complete proofs of the constants 25.2 and 35.1, a rational specialization and both unit-circle estimates. We keep the centered exponents and the full perturbation Taylor expansion to obtain the sharper constants. A derivative matrix then leads to Gouillon's arithmetic and analytic comparison. We prove its polynomial bases, Taylor orders and general geometric zero estimate, and verify the exact cardinality conditions that give full row rank. Freely accessible works of [Laurent 2008], [Gouillon 2003] and [Gouillon 2006] support the full determinant proofs; [Bugeaud 2008] supplies additional rational context.

1. An explicit bound without multiplicative independence

Theorem 7.1. Let $\alpha_1, \alpha_2 \neq 0$ be algebraic, choose any logarithms ℓ_j of them, and put

$$D = [\mathbb{Q}(\alpha_1, \alpha_2) : \mathbb{Q}].$$

Let $a_j = \log A_j$ satisfy

$$a_j \geq \max\{h(\alpha_j), |\ell_j|/D, 1/D\}.$$

For nonzero integers c_1, c_2 , assume $\Gamma = c_1\ell_1 + c_2\ell_2 \neq 0$. Define

$$B' = \frac{|c_1|}{Da_2} + \frac{|c_2|}{Da_1}, \quad B = \max\{10, \log B'\}.$$

Then

$$\log |\Gamma| \geq -21600D^4a_1a_2B^2. \tag{7.1}$$

No independence hypothesis is imposed on the bases. Enlarging a_j changes both the height factor and B' ; keep both changes when comparing bounds.

We prove the theorem in Sections 2–4. The normalizations

$$x = Da_1 \geq 1, \quad y = Da_2 \geq 1, \quad t = BD \geq 10, \quad Q = t^2xy = D^4a_1a_2B^2 \tag{7.2}$$

will make the rounding estimates transparent.

2. Bounding the factorial normalization

We need the following estimate only for very large K .

Lemma 7.2. If $K \geq 25$, then

$$\left(\prod_{j=1}^{K-1} j! \right)^{2/(K(K-1))} \geq \frac{K-1}{5}.$$

Consequently, the parameter b of the preceding lesson satisfies

$$b \leq \frac{5(Rb_2 + Sb_1)}{K-1}. \quad (7.3)$$

Proof. Since $\log z$ and $z \log z$ are increasing on $[1, \infty)$,

$$\log(j!) \geq \int_1^j \log z \, dz \geq j \log j - j,$$

and, writing $m = K - 1$,

$$\sum_{j=1}^m j \log j \geq \int_1^m z \log z \, dz \geq \frac{m^2}{2} \log m - \frac{m^2}{4}.$$

Therefore the logarithm of the left side is at least

$$\log m - \left(1 + \frac{m}{2(m+1)} + \frac{\log m}{m+1} \right).$$

The expression in parentheses decreases for $m \geq 24$: the derivative of its last two terms is

$$\frac{3/2 + 1/m - \log m}{(m+1)^2} < 0.$$

At $m = 24$, it is less than $1 + 12/25 + 3.18/25 = 1.6072 < \log 5$. Exponentiating proves the first assertion. In the definition of b , replace its numerator $(R-1)b_2 + (S-1)b_1$ by the larger $Rb_2 + Sb_1$, and apply that assertion. $\square$

The elementary numerical bounds used here, and $\log 3 \in (1.098, 1.099)$ used below, can all be certified by the rational series for logarithms in the preceding lesson. This avoids rounded floating-point inequalities at the proof's strict boundary.

3. The noncolliding parameter choice

Use the inversion and sign reductions proved in the preceding lesson. In the cancellation case write

$$\Lambda = b_2 \ell_2 - b_1 \ell_1, \quad b_1, b_2 > 0, \quad |\alpha_1|, |\alpha_2| \geq 1, \quad b_1 |\ell_1| \leq b_2 |\ell_2|.$$

The a_j , B' and Q are preserved, up to permuting indices. Suppose first that at least one base is not a root of unity. Choose

$$\begin{aligned} K &= \lfloor 4900txy \rfloor, & L &= \lfloor 4t \rfloor, \\ R_1 &= \lceil 4ty \rceil, & S_1 &= \lceil 4tx \rceil, \\ R_2 &= \lceil 140ty \rceil, & S_2 &= \lceil 140tx \rceil, \end{aligned} \quad (7.4)$$

and take the analytic radius $\rho = 3$. Thus $K \geq 49000$, and every other integer is at least two.

Since $R_1, S_1 \geq L$, the multiplicative grid contains at least L distinct values, by Exercise 1 in the preceding lesson. If the second rectangle's additive map

$$(r, s) \mapsto rb_2 + sb_1, \quad 0 \leq r < R_2, \quad 0 \leq s < S_2$$

is injective, then its cardinality satisfies

$$R_2 S_2 \geq 19600 t^2 xy \geq KL > (K-1)L. \quad (7.5)$$

The equality allowed in the middle is harmless because the last inequality is strict.

Here are bounds incorporating every rounding error:

$$\begin{aligned} K &\geq 4899.9txy, & K-1 &\geq 4899.8txy, & L &\leq 4t, & L-1 &\geq 3.8t, \\ R = R_1 + R_2 - 1 &\leq 144.1ty, & S = S_1 + S_2 - 1 &\leq 144.1tx, & N = KL &\leq 19600Q. \end{aligned} \quad (7.6)$$

For example $R \leq 144ty + 1$, and $ty \geq 10$. Lemma 7.2 gives

$$b \leq \frac{5(144.1)t(yb_2 + xb_1)}{4899.8txy} < B' = \frac{b_2}{x} + \frac{b_1}{y}.$$

Therefore $\log b \leq B$. The previous theorem also proves $b \geq L$, so its logarithm is positive.

The determinant inequality is implied by

$$K(L-1) \log 3 > 2(D+1) \log N + (D+1)KB + 8L(Rx + Sy). \quad (7.7)$$

Indeed $Dh(\alpha_j) \leq Da_j$ and $|\ell_j| \leq Da_j$, so the analytic and height contribution with radius three is at most the last term.

Divide (7.7) by Kt . Its left side is at least $3.8 \log 3 > 4.1724$. The middle contribution is at most

$$(D+1)B/t = 1 + 1/D \leq 2,$$

and the last is at most

$$\frac{8(4)(288.2)}{4899.9} < 1.883.$$

For the first contribution, note

$$\log N \leq \log 19600 + 2 \log t + \log(xy) \leq 2txy.$$

For completeness, $\log 19600 < 10$, $2 \log t \leq t$ for $t \geq 10$, and $\log(xy) \leq xy - 1$. The inequality $10 + t + xy - 1 \leq 2txy$ follows by first setting $xy = 1$, then observing that the right side increases faster as xy increases. Since $D/t = 1/B \leq 1/10$,

$$\frac{2(D+1) \log N}{Kt} \leq \frac{8D}{4899.9t} < 0.001.$$

The entire right side is less than 3.884, proving the strict inequality.

Theorem 6.3 now gives $|\Lambda_*| > 3^{-N}$, in its notation. Since

$$N \log 3 < 19600(1.099)Q < 21550Q,$$

this is almost (7.1). To remove the prefactor, put $M = \max\{LR, LS\}$. If $|\Lambda| \geq 1$, there is nothing to prove. If $1/M \leq |\Lambda| < 1$, then $\log |\Lambda| \geq -\log M$. Otherwise $|\Lambda|M < 1$; both exponentials in the prefactor are at most e , because $b_1, b_2 \geq 1$. Thus

$$\log |\Lambda| > -N \log 3 - 1 - \log M \geq -N \log 3 - 1 - \log L - \log R - \log S. \quad (7.8)$$

The same lower bound follows in the intermediate case.

Using $R \leq 145ty$, $S \leq 145tx$, $L \leq 4t$, we have

$$1 + \log L + \log R + \log S < 13 + 3 \log t + \log(xy) \leq 13 + 3t + xy \leq t^2 xy = Q.$$

For the final inequality, the difference increases with $xy \geq 1$, and at $xy = 1$ it is $t^2 - 3t - 14 > 0$ for $t \geq 10$. Equation (7.8) therefore gives the stronger bound $\log |\Lambda| > -21551Q$.

4. Collisions, periods and the remaining reductions

If the additive rectangle in Section 3 has a collision, subtract the two pairs. There are integers r, s , both nonzero, such that

$$|r| < R_2 < R, \quad |s| < S_2 < S, \quad rb_2 + sb_1 = 0.$$

The strict inequalities $R_2 < R, S_2 < S$ follow from $R_1, S_1 \geq 2$. Put $H = r\ell_1 + s\ell_2$. Then

$$H = -\frac{r}{b_1}\Lambda \neq 0, \quad |\Lambda| = \frac{b_1}{|r|}|H|. \quad (7.9)$$

If $e^H = 1$, the nonzero period has $|H| \geq 2\pi$. Hence $\log |\Lambda| \geq -\log R$, which is much stronger than (7.1).

If $e^H \neq 1$, height arithmetic and one-place Liouville give

$$|e^H - 1| \geq \exp\{-D \log 2 - |r|Dh(\alpha_1) - |s|Dh(\alpha_2)\} \geq \exp\{-D \log 2 - Rx - Sy\}.$$

For $|H| \leq 1$, $|e^H - 1| \leq e|H|$; for $|H| > 1$, the desired lower estimate on $|H|$ is automatic. By (7.9),

$$\log |\Lambda| \geq -\log R - D \log 2 - Rx - Sy - 1.$$

The right side exceeds $-44Q$: use $Rx + Sy \leq 288.2txy$, $\log R \leq R \leq 144.1txy$, $D \log 2 \leq 0.1txy$, $1 \leq 0.1txy$, and $t \geq 10$. Thus a collision is an easier case. We have not assumed multiplicative independence or incorrectly applied Liouville to the zero number $e^H - 1$.

If both bases are roots of unity, then $h(e^\Gamma) = 0$, regardless of the integer coefficients. When $e^\Gamma \neq 1$, Liouville gives $|e^\Gamma - 1| \geq 2^{-D}$; applying $|e^z - 1| \leq e|z|$ for $|z| \leq 1$ yields $\log |\Gamma| \geq -D \log 2 - 1$. When $e^\Gamma = 1$, the nonzero form is a period of magnitude at least 2π . Both cases imply (7.1).

Finally, the same-sign case after inversions has the lower bound proved in the preceding lesson:

$$\log |\Gamma| \geq -\log 4 - D^2(2h(\alpha_j) + \log 2)$$

for a base of modulus greater than one. This exceeds $-Q$: $Q \geq 100D^2$, $Q \geq 100D^3a_j$, and $h(\alpha_j) \leq a_j$. If both moduli are one, inverting either base permits the cancellation setting. These cases exhaust the original form and complete the proof of Theorem 7.1. $\square$

5. Centering the exponents

The smaller constants use more information about the rectangle than the coarse bounds of Theorem 6.3. Here is the precise estimate, together with a proof.

Lemma (centered exponent sums). Let $N = KL \leq RS$, let (r_j, s_j) , $1 \leq j \leq N$, be distinct points in $\{0, \dots, R-1\} \times \{0, \dots, S-1\}$, and let λ_i be the list in which each number

$$-\frac{L-1}{2}, -\frac{L-1}{2} + 1, \dots, \frac{L-1}{2}$$

occurs K times. Set

$$g = \frac{1}{4} - \frac{N}{12RS}, \quad G_1 = \frac{gLRN}{2}, \quad G_2 = \frac{gLSN}{2}.$$

For every permutation τ ,

$$\left| \sum_i \lambda_i r_{\tau(i)} \right| \leq G_1, \quad \left| \sum_i \lambda_i s_{\tau(i)} \right| \leq G_2. \quad (7.13)$$

Proof. We prove the first bound by replacing the discrete points with intervals whose means are the same. Associate to a paired value $(\lambda_i, r_{\tau(i)})$ a measure of mass one on

$$[\lambda_i - \frac{1}{2}, \lambda_i + \frac{1}{2}] \times [r_{\tau(i)}, r_{\tau(i)} + 1],$$

uniform and independent in its two coordinates. Call these coordinates u, v . Summing the measures gives a marginal density K for u on $[-L/2, L/2]$; the marginal density for v on $[0, R]$ is at most S , since at most S selected grid points have any fixed first coordinate. Moreover,

$$\int uv = \sum_i \lambda_i (r_{\tau(i)} + \frac{1}{2}) = \sum_i \lambda_i r_{\tau(i)},$$

because $\sum_i \lambda_i = 0$.

A measure of mass m on $[0, R]$ with density at most S has first moment between $m^2/(2S)$ and $Rm - m^2/(2S)$. To see the lower bound, let $F(v)$ be its mass in $[0, v]$, so $F(v) \leq \min\{Sv, m\}$; integration of $m - F(v)$ over $[0, R]$ gives the first moment and the stated bound. Reflecting v about $R/2$ gives the upper bound.

For $0 \leq z \leq L/2$, each of the sets $u > z$ and $u < -z$ has mass $m(z) = K(L/2 - z)$. Their v -marginals still have density at most S . By expressing u as its positive and negative level-set integrals,

$$\int uv = \int_0^{L/2} \left(\int_{u>z} v - \int_{u<-z} v \right) dz \leq \int_0^{L/2} \left(Rm(z) - \frac{m(z)^2}{S} \right) dz.$$

The last integral equals

$$\frac{KRL^2}{8} - \frac{K^2L^3}{24S} = \frac{LRN}{8} - \frac{LN^2}{24S} = G_1.$$

Replacing every λ_i by its negative gives the lower bound. Interchanging R, S and the two grid coordinates proves the other assertion. $\square$

This proves the centered estimate used in Laurent–Mignotte–Nesterenko and Laurent’s sharper determinant argument. The continuous measure is an exact device for counting the discrete rectangle, rather than a limiting approximation.

6. A sharper arithmetic determinant bound

We first need a useful form of Hadamard’s inequality.

Lemma (largest permutation product). For an N -square complex matrix A ,

$$|\det A| \leq N^{N/2} \max_{\tau} \prod_i |A_{i, \tau(i)}|. \quad (7.14)$$

Proof. The assertion is immediate when $\det A = 0$. Otherwise choose a nonzero permutation product of largest absolute value and permute columns so that it is the diagonal. Put $c_{ij} = \log |A_{ij}| - \log |A_{ii}|$ for nonzero entries. Regard it as the weight of a directed edge from j to i . Every directed cycle has total weight at most zero: a positive cycle would replace the corresponding diagonal entries by a larger permutation product.

Add a starting vertex with edges of weight zero to all vertices. Let u_i be the largest weight of a path from it to i . This maximum exists: deleting cycles never decreases a path’s weight, so a maximizing path can be chosen without repeated vertices. Then $c_{ij} \leq u_i - u_j$, whence

$$|A_{ij}| \leq |A_{ii}| e^{u_i - u_j}.$$

Divide row i by $|A_{ii}| e^{u_i}$ and column j by e^{-u_j} . The resulting entries have modulus at most one. Hadamard bounds its determinant by $N^{N/2}$, while the total scaling factor is exactly the largest permutation product. This proves (7.14). $\square$

Return to the nonzero minor of the binomial matrix in Section 6 of the preceding lesson. Write its row indices as (k_i, τ_i) , and its distinct column points as (r_j, s_j) . Let

$$d = [\mathbb{Q}(\alpha_1, \alpha_2) : \mathbb{Q}], \quad \chi = [\mathbb{R}(\alpha_1, \alpha_2) : \mathbb{R}] \in \{1, 2\}, \quad D_* = \frac{d}{\chi}.$$

Here χ is the local degree of the selected real or complex embedding; thus D_* differs from the $D = d$ in Theorem 7.1. Suppose $|\alpha_1|, |\alpha_2| \geq 1$ at this embedding. Set

$$\begin{aligned} M_1 &= \frac{L-1}{2} \sum_j r_j, & M_2 &= \frac{L-1}{2} \sum_j s_j, \\ b &= \frac{(R-1)b_2 + (S-1)b_1}{2} \left(\prod_{k=1}^{K-1} k! \right)^{-2/(K(K-1))}, & T &= \frac{(K-1)N}{2}. \end{aligned} \tag{7.15}$$

The factor $1/2$ in this b is intentional; it was absent from the coarser theorem.

Lemma (arithmetic bound). The minor satisfies

$$\begin{aligned} \log |\Delta| \geq & -\frac{D_* - 1}{2} N \log N - (D_* - 1)T \log b \\ & + (M_1 + G_1) \log |\alpha_1| + (M_2 + G_2) \log |\alpha_2| \\ & - 2D_* \{G_1 h(\alpha_1) + G_2 h(\alpha_2)\}. \end{aligned} \tag{7.16}$$

Proof. At a nonarchimedean place v , the binomial entries are integers. Expanding the determinant, using the ultrametric inequality and (7.13) for the exponents $\tau_i - (L-1)/2$, gives

$$\log |\Delta|_v \leq M_1 \log |\alpha_1|_v + M_2 \log |\alpha_2|_v + G_1 \log |\alpha_1|_v + G_2 \log |\alpha_2|_v.$$

For an archimedean embedding, shift the binomial variable to its midpoint and replace the binomial basis by its leading monomials divided by $k!$. Within each fixed τ , these are triangular row operations of determinant one. The centered variable has absolute value at most $((R-1)b_2 + (S-1)b_1)/2$. Each permutation product in the transformed determinant is therefore bounded by

$$b^T |\alpha_1|^{M_1} |\alpha_2|^{M_2} \exp\{G_1 |\log |\alpha_1|| + G_2 |\log |\alpha_2||\}.$$

Apply (7.14). Thus the archimedean bound has the same form as the finite-place bound, with the additional constant $\log C = \frac{N}{2} \log N + T \log b$.

Now apply the product formula to $\Delta \neq 0$, with the usual local degrees n_v . The terms $M_j \log |\alpha_j|_v$ sum to zero, and

$$\sum_v n_v \log |\alpha_j|_v = 2d h(\alpha_j).$$

Exclude the selected embedding from the upper bounds on all other places. Its local weight is χ ; the total archimedean weight remaining is $d - \chi$. After division by χ , the constant is $(D_* - 1) \log C$. At the selected embedding the excluded absolute-log term is $G_j \log |\alpha_j|$, because the moduli are at least one. Reinsert the centering term $M_j \log |\alpha_j|$. This gives precisely (7.16). No assertion that the midpoint exponents themselves are integral is needed; only the original binomial determinant is used at finite places. $\square$

The proof uses exactly the height normalization and product formula in the written internal height lesson. In particular the selected complex place has weight two; treating it as weight one would lose the sharper degree D_* .

7. Keeping the full Taylor expansion

Laurent's refinement uses every Taylor order in the perturbation. We begin with an alternative combinatorial proof of the Taylor-order estimate used in the preceding lesson.

Lemma (Taylor orders). Let $1/3 \leq \mu \leq 1$, set $\sigma = (1 + 2\mu - \mu^2)/2$, and let positive integers $\nu_1, \dots, \nu_q$ sum to N . Then

$$\sum_{j=1}^q \binom{\nu_j}{2} + \mu N \sum_{j=1}^q (j-1)\nu_j \geq \frac{\sigma N^2 - N}{2}. \quad (7.17)$$

Proof. On the simplex $\sum x_j = 1$, $x_j \geq 0$, minimize $F(x) = \frac{1}{2} \sum x_j^2 + \mu \sum (j-1)x_j$. Swapping two coordinates shows that a minimizing vector is decreasing. Transferring mass between coordinate one and any positive coordinate j gives the stationarity condition $x_j = x_1 - (j-1)\mu$. If m coordinates are positive, their sum yields

$$x_j = \frac{1}{m} + \left(\frac{m+1}{2} - j \right) \mu, \quad 1 \leq j \leq m.$$

Positivity of the last one requires $\mu < 2/(m(m-1))$; hence $m \leq 2$. For one positive coordinate the minimum is $1/2 \geq \sigma/2$. For two it is attained at $((1+\mu)/2, (1-\mu)/2)$, with value $\sigma/2$. Substitute $x_j = \nu_j/N$ and subtract $N/2$ to obtain (7.17). $\square$

Lemma (analytic bound). Use the same minor and quantities as in Section 6, and suppose $b_1|\ell_1| \leq b_2|\ell_2|$. For $\rho > 1$, if

$$|\Lambda'| \leq \rho^{-\mu N}, \quad \Lambda' = \Lambda \max \left\{ \frac{LS}{2b_2} e^{LS|\Lambda|/(2b_2)}, \frac{LR}{2b_1} e^{LR|\Lambda|/(2b_1)} \right\},$$

then

$$|\Delta| \leq \rho^{-(\sigma N^2 - N)/2} N \{e^N + (e-1)^N\} N! (\rho b)^T \times |\alpha_1|^{M_1} |\alpha_2|^{M_2} e^{\rho(G_1|\ell_1| + G_2|\ell_2|)}. \quad (7.18)$$

Proof. Put $\beta = b_1/b_2$, $\eta = ((R-1) + \beta(S-1))/2$, $z_j = r_j + \beta s_j - \eta$ and $\lambda_i = \tau_i - (L-1)/2$. The row operations of Section 6, together with $\sum_i \lambda_i = 0$, give

$$\Delta = \alpha_1^{M_1} \alpha_2^{M_2} \det \left(\varphi_i(z_j) e^{\lambda_i s_j \Lambda / b_2} \right), \quad \varphi_i(z) = \frac{b_2^{k_i}}{k_i!} z^{k_i} e^{\lambda_i \ell_1 z}.$$

Fractional powers in this identity are defined with the chosen logarithms; their absolute values are unambiguous.

Expand each row's exponential in its full power series. If $\mathbf{n} = (n_1, \dots, n_N)$ is a tuple of nonnegative integers, its term is

$$\Delta_{\mathbf{n}} = \det \left(\varphi_i(z_j) \frac{(\lambda_i s_j \Lambda / b_2)^{n_i}}{n_i!} \right).$$

The series is absolutely convergent: the determinant is a finite sum over permutations, each of whose products is bounded by a product of exponential series.

Let $m_1 < \dots < m_q$ be the distinct row orders n_i , occurring with multiplicities $\nu_1, \dots, \nu_q$. The function obtained by replacing $\varphi_i(z_j)$ by $\varphi_i(x z_j)$ has zero order at least $\sum_j \binom{\nu_j}{2}$. Indeed two rows having both the same n_i and the same Taylor exponent of φ_i are proportional; their common column factor is $z_j^h s_j^{n_i}$. Each group of ν_j equal row orders therefore requires distinct nonnegative Taylor exponents.

For any permutation, (7.13) bounds the product of exponential factors by

$$\exp\{|x|\ell_1|(G_1 + \beta G_2)\} \leq \exp\{|x|(G_1|\ell_1| + G_2|\ell_2|)\}.$$

The midpoint bound for z_j gives the polynomial factor $(|x|b)^T$. Writing $u = LS|\Lambda|/(2b_2)$, Schwarz's estimate thus yields

$$|\Delta_{\mathbf{n}}| \leq \Omega \rho^{-\sum_j \binom{\nu_j}{2}} \frac{u^{\sum_i n_i}}{\prod_i n_i!}, \quad \Omega = N!(\rho b)^T e^{\rho(G_1|\ell_1|+G_2|\ell_2|)}. \quad (7.19)$$

Here is the complete summation. First take tuples with $m_1 = 0$. Write $m_j = (j-1) + r_j$ for $j \geq 2$, with $0 \leq r_2 \leq \dots \leq r_q$. Since $m_j! \geq (j-1)!r_j!$, summing the residual orders and dropping their ordering condition bounds the residual factor by

$$\prod_{j=2}^q \sum_{r \geq 0} \frac{u^{r\nu_j}}{(r!)^{\nu_j}} \leq e^{u \sum_{j=2}^q \nu_j} \leq e^{u \sum_{j=1}^q (j-1)\nu_j}.$$

The first inequality follows by expanding $(\sum_{r \geq 0} u^r/r!)^{\nu_j}$, which includes all displayed diagonal terms. Thus the remaining factor is at most

$$\frac{(ue^u)^{\sum_j (j-1)\nu_j}}{\prod_j ((j-1)!)^{\nu_j}}.$$

There are $N!/\prod \nu_j!$ assignments of the row groups. By $ue^u \leq |\Lambda'| \leq \rho^{-\mu N}$ and (7.17), their total is bounded by

$$\Omega \rho^{-(\sigma N^2 - N)/2} \sum_{q=1}^N \sum_{\substack{\nu_j \geq 1 \\ \sum \nu_j = N}} \frac{N!}{\prod \nu_j!} \prod_{j=1}^q \frac{1}{((j-1)!)^{\nu_j}} \leq \Omega \rho^{-(\sigma N^2 - N)/2} N e^N.$$

The last step is the multinomial theorem with zero multiplicities allowed.

For tuples with $m_1 \geq 1$, write $m_j = j + r_j$ instead. The same reasoning uses $\prod (j!)^{\nu_j}$ and $\sum j\nu_j$; its latter exponent is larger than the one in (7.17). The multinomial sum is now at most $N(e-1)^N$, since $\sum_{j \geq 1} 1/j! = e-1$. Add the two bounds and reinsert $|\alpha_1|^{M_1}|\alpha_2|^{M_2}$. This proves (7.18). If $\Lambda = 0$, only the tuple of all zero orders survives, and the same bound follows directly, with $0^0 = 1$. $\square$

8. The refined parameter theorem

Theorem 7.3 (Laurent's refined determinant estimate). In the cancellation setting, use $D_* = d/\chi$ from Section 6. Let $K \geq 2$ and $L, R_1, R_2, S_1, S_2 \geq 1$ be integers satisfying

$$\#\{\alpha_1^r \alpha_2^s : 0 \leq r < R_1, 0 \leq s < S_1\} \geq L, \quad \#\{rb_2 + sb_1 : 0 \leq r < R_2, 0 \leq s < S_2\} > (K-1)L.$$

Set $R = R_1 + R_2 - 1$, $S = S_1 + S_2 - 1$, $N = KL$, and define g, b by Sections 5–6. Choose $\rho > 1$, $1/3 \leq \mu \leq 1$, and put $\sigma = (1 + 2\mu - \mu^2)/2$. Let positive real a_1, a_2 satisfy

$$a_j \geq \rho|\ell_j| - \log |\alpha_j| + 2D_*h(\alpha_j).$$

If

$$K(\sigma L - 1) \log \rho - (D_* + 1) \log N - D_*(K-1) \log b - gL(Ra_1 + Sa_2) > \epsilon(N), \quad (7.20)$$

where

$$\epsilon(N) = \frac{2}{N} \log \{N!N^{-N+1}(e^N + (e-1)^N)\},$$

then $|\Lambda'| > \rho^{-\mu KL}$, with Λ' from Section 7.

Proof. The zero lemma proved in the preceding lesson works with $K \geq 2$, $L \geq 1$ and positive side lengths; it produces a nonzero N -column minor. In particular $N \leq RS$, so the centered-sum lemma applies. Permute the bases if necessary to order $b_1|\ell_1| \leq b_2|\ell_2|$; all the hypotheses and the conclusion are symmetric.

Suppose the conclusion were false. Apply the arithmetic lower bound (7.16) and analytic upper bound (7.18). Their $M_j \log |\alpha_j|$ terms cancel. Dividing their comparison by $N/2$, using $T = (K - 1)N/2$ and $2G_1/N = gLR$, $2G_2/N = gLS$, gives

$$\begin{aligned} K(\sigma L - 1) \log \rho &\leq (D_* + 1) \log N + D_*(K - 1) \log b + \epsilon(N) \\ &\quad + gLR\{\rho|\ell_1| - \log |\alpha_1| + 2D_*h(\alpha_1)\} \\ &\quad + gLS\{\rho|\ell_2| - \log |\alpha_2| + 2D_*h(\alpha_2)\}. \end{aligned}$$

This contradicts (7.20). The conclusion follows. $\square$

This is Theorem 1 of [Laurent 2008]. Its rank, centered-exponent, arithmetic, analytic and summation arguments have all been supplied here or in the preceding lesson. In particular no unproved arithmetic estimate from Laurent–Mignotte–Nesterenko is being used. The parameter μ measures the relative perturbation size. Retaining its full Taylor expansion changes the zero-order coefficient from one to σ , while allowing the perturbation threshold to use μ .

9. Factorials and the optimized rectangle

To specialize Theorem 7.3 sharply, we need more than the factor five in Lemma 7.2. We establish the required factorial estimate directly.

Lemma (Stirling bounds). For every integer $n \geq 1$,

$$\sqrt{2\pi n}(n/e)^n < n! < \sqrt{2\pi n}(n/e)^n e^{1/(12n)}.$$

Proof. Put $a_n = \log(n!) - (n + \frac{1}{2}) \log n + n$. With $z = (2n + 1)^{-1}$, the convergent logarithm series gives

$$a_n - a_{n+1} = (n + \frac{1}{2}) \log(1 + 1/n) - 1 = \sum_{j \geq 1} \frac{z^{2j}}{2j + 1}.$$

This is positive and less than $z^2/(3(1 - z^2)) = 1/(12n(n + 1))$. Thus a_n has a finite limit c , and $0 < a_n - c < 1/(12n)$.

We determine c without assuming Stirling's formula. Let $I_m = \int_0^{\pi/2} \sin^m v \, dv$. Integration by parts gives $I_m = (m - 1)I_{m-2}/m$, with $I_0 = \pi/2$, $I_1 = 1$. Hence

$$I_{2n} = \frac{\pi}{2} \frac{\binom{2n}{n}}{4^n}, \quad I_{2n+1} = \frac{4^n}{(2n + 1)\binom{2n}{n}}.$$

Since $I_{2n+1} \leq I_{2n} \leq I_{2n-1}$ and $I_{2n-1}/I_{2n+1} = (2n + 1)/(2n)$, the ratio I_{2n}/I_{2n+1} tends to one. Their product is $\pi/(2(2n + 1))$; it follows that $\binom{2n}{n} \sqrt{n}/4^n \rightarrow 1/\sqrt{\pi}$. On the other hand, expressing this same ratio with $a_{2n} - 2a_n$ gives the limit $\sqrt{2}e^{-c}$. Thus $c = \frac{1}{2} \log(2\pi)$. The bounds on $a_n - c$ prove the result. $\square$

Lemma (factorial product). For $K \geq 2$,

$$\begin{aligned} \log \left(\prod_{j=1}^{K-1} j! \right)^{2/(K(K-1))} &\geq \log(K - 1) - \frac{3}{2} \\ &\quad + \frac{\log(2\pi(K - 1)/\sqrt{e})}{K - 1} - \frac{\log K}{6K(K - 1)}. \end{aligned} \tag{7.21}$$

Proof. We first bound $\sum_{j=1}^m j \log j$ from above. For $f(v) = v \log v$, integration by parts on each unit interval gives the trapezoidal error

$$\frac{f(a) + f(a + 1)}{2} - \int_a^{a+1} f(v) \, dv = \frac{1}{2} \int_0^1 u(1 - u) f''(a + u) \, du.$$

Because $f''(v) = 1/v$ is convex, the right side is at most $\frac{1}{12} \int_a^{a+1} f''(v) dv$. Here is a justification of this weighted bound: symmetrize $f''(a+u)$ about $u = 1/2$; the resulting function decreases on $[0, 1/2]$, while $u(1-u) - 1/6$ increases, has one sign change there, and has integral zero. Subtracting the function's value at that sign change makes the product nonpositive on both subintervals. Reflection proves the bound on $[0, 1]$.

Summing the errors yields

$$\sum_{j=1}^m j \log j \leq \frac{m^2 + m}{2} \log m - \frac{m^2}{4} + \frac{1}{4} + \frac{\log m}{12}.$$

Use the identity

$$\sum_{j=1}^m \log(j!) = (m+1) \log(m!) - \sum_{j=1}^m j \log j,$$

and the Stirling lower bound. With $K = m+1$, division by $K(K-1)/2$ gives (7.21), even with $\log(K-1)$ in place of $\log K$ in the last numerator. The displayed weaker version follows. $\square$

We also record the optimized geometric contribution.

Lemma (rectangle contribution). Let $K \geq 2$, $L \geq 1$, $a_1, a_2 > 0$, and suppose

$$R \leq 1 + (1 + \sqrt{K-1})\sqrt{La_2/a_1}, \quad S \leq 1 + (1 + \sqrt{K-1})\sqrt{La_1/a_2}.$$

With $N = KL$, $g = 1/4 - N/(12RS)$, one has

$$\begin{aligned} gL(Ra_1 + Sa_2) &\leq \frac{1}{3}L^{3/2}\sqrt{(K-1)a_1a_2} + \frac{2}{3}L^{3/2}\sqrt{a_1a_2} + \frac{1}{3}L(a_1 + a_2) \\ &\quad - \frac{L^{3/2}\sqrt{a_1a_2}}{6(1 + \sqrt{K-1})}. \end{aligned} \tag{7.22}$$

Proof. The left side equals

$$\frac{L}{4}(Ra_1 + Sa_2) - \frac{KL^2}{12}(a_1/S + a_2/R),$$

which increases in each of $R, S > 0$. Replace them by their upper bounds. Put $q = K-1$, $c = 1 + \sqrt{q}$, $r = \sqrt{La_2/a_1}$, $s = \sqrt{La_1/a_2}$, $u = r + s$, and $V = \sqrt{La_1a_2}$. Then $rs = L$, $a_1 + a_2 = Vu/L$, and the proposed inequality reduces, after multiplying by positive factors, to

$$(2q/c - u/L)(1 + cu + c^2L) \leq K(u + 2cL).$$

The left side minus the right side is

$$2q/c - 2cL - 2cu - u/L - cu^2/L \leq 0.$$

Indeed $q \leq c^2$, $L \geq 1$, $u > 0$. This proves (7.22). $\square$

These estimates supply, within the course, the factorial and rectangle inequalities used in the numerical specialization of Laurent's refined theorem. Their hypotheses allow all the rounded integers used next.

10. A numerical specialization with complete parameter checks

We now specialize Theorem 7.3. The following form of Laurent's Theorem 2 keeps the quantities needed to verify a numerical constant.

Theorem 7.4. Suppose that α_1, α_2 are multiplicatively independent, with moduli at least one and chosen logarithms ℓ_j . Use $D_* = d/\chi$ as above. Choose $\rho > 1$, $1/3 \leq \mu \leq 1$, and define

$$\sigma = \frac{1 + 2\mu - \mu^2}{2}, \quad \lambda = \sigma \log \rho, \quad H = \frac{h_0}{\lambda} + \frac{1}{\sigma}, \quad \omega = 2 \left(1 + \sqrt{1 + \frac{1}{4H^2}} \right), \quad \theta = \sqrt{1 + \frac{1}{4H^2}} + \frac{1}{2H}.$$

Let $a_j \geq \max\{1, \rho|\ell_j| - \log |\alpha_j| + 2D_*h(\alpha_j)\}$, with $a_1a_2 \geq \lambda^2$, and assume

$$h_0 \geq \max \left\{ D_* \left(\log \left(\frac{b_1}{a_2} + \frac{b_2}{a_1} \right) + \log \lambda + 1.75 \right) + 0.06, \lambda, \frac{D_* \log 2}{2} \right\}.$$

Define

$$C = \frac{\mu}{\lambda^3 \sigma} \left(\frac{\omega}{6} + \frac{1}{2} \sqrt{\frac{\omega^2}{9} + \frac{8\lambda\omega^{5/4}\theta^{1/4}}{3\sqrt{a_1a_2}\sqrt{H}} + \frac{4}{3} \left(\frac{1}{a_1} + \frac{1}{a_2} \right) \frac{\lambda\omega}{H}} \right)^2, \quad C' = \sqrt{\frac{C\sigma\omega\theta}{\lambda^3\mu}}.$$

Then, with $U_0 = h_0 + \lambda/\sigma$,

$$\log |\Lambda| \geq -CU_0^2 a_1 a_2 - \sqrt{\omega\theta} U_0 - \log(C'U_0^2 a_1 a_2). \quad (7.23)$$

The symbol h_0 is a positive real parameter, distinct from the height function.

Proof. Choose the unique integer L in

$$H + \sqrt{H^2 + \frac{1}{4}} - \frac{1}{2} < L \leq H + \sqrt{H^2 + \frac{1}{4}} + \frac{1}{2}.$$

Put

$$U = \lambda(L - H), \quad V = L/3, \quad W = \frac{1}{3} \left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{2\sqrt{L}}{\sqrt{a_1a_2}} \right), \quad k = \left(\frac{V}{2U} + \frac{1}{2} \sqrt{\frac{V^2}{U^2} + \frac{4W}{U}} \right)^2.$$

Thus $kU - \sqrt{k}V - W = 0$. Finally choose

$$\begin{aligned} K &= 1 + \lfloor kLa_1a_2 \rfloor, \\ R_1 &= 1 + \lfloor \sqrt{La_2/a_1} \rfloor, \quad S_1 = 1 + \lfloor \sqrt{La_1/a_2} \rfloor, \\ R_2 &= 1 + \lfloor \sqrt{(K-1)La_2/a_1} \rfloor, \quad S_2 = 1 + \lfloor \sqrt{(K-1)La_1/a_2} \rfloor. \end{aligned} \quad (7.24)$$

We verify the needed size estimates first. Since $H \geq 2$,

$$\frac{3 + \sqrt{17}}{4} \leq \sqrt{\omega/\theta} < L/H \leq \sqrt{\omega\theta} \leq \frac{5 + \sqrt{17}}{4}.$$

For any half integer $\gamma \neq 2$,

$$\omega^{\gamma/2} \theta^{-|\gamma/2-1|} H^{\gamma-1} \leq \frac{L^\gamma}{L-H} \leq \omega^{\gamma/2} \theta^{|\gamma/2-1|} H^{\gamma-1}, \quad (7.25)$$

and

$$4H \leq \frac{L^2}{L-H} \leq \omega H.$$

To check (7.25), differentiate $x^\gamma/(x-H)$: its derivative has the sign of $\gamma(x-H) - x$. On the chosen interval it is negative for $\gamma \leq 3/2$, because $x/H \leq (5 + \sqrt{17})/4 < 3$; it is positive for $\gamma \geq 5/2$, because $x/H > (3 + \sqrt{17})/4 > 5/3$. Substituting the two interval endpoints gives (7.25). For $\gamma = 2$, the minimum is at $2H$, and the two endpoints give the same upper value ωH .

In particular $L \geq 4$. Also $K \geq 8$. Here is an explicit lower estimate that checks the latter assertion, using $a_1 a_2 \geq \lambda^2$ and $a_1 + a_2 \geq 2\sqrt{a_1 a_2}$ in the definition of k :

$$\sqrt{kLa_1a_2} \geq \frac{\omega^{3/4}\sqrt{H}}{6\theta^{1/4}} + \frac{1}{2}\sqrt{\frac{\omega^{3/2}H}{9\sqrt{\theta}} + \frac{8\omega^{3/4}\sqrt{H}}{3\theta^{1/4}} + \frac{8\sqrt{\omega}}{3\sqrt{\theta}}}.$$

The right side increases with H : express its terms through the increasing functions $H\omega = 2H + \sqrt{4H^2 + 1}$ and $\sqrt{\omega/\theta} = 1 + \sqrt{1 + 1/(4H^2)} - 1/(2H)$. At $H = 2$ it exceeds $2.66 > \sqrt{7}$, so $1 + [kLa_1a_2] \geq 8$. Consequently $N = KL \geq 32$.

Write $b'' = b_1/a_2 + b_2/a_1$, $g_0 = \gcd(b_1, b_2)$. First suppose

$$b'' > 2\mu\lambda\sigma^{-1}kL^2g_0. \quad (7.26)$$

The additive rectangle is injective. Indeed, since

$$\sqrt{k}L \geq \frac{L^2}{3\lambda(L-H)} \geq \frac{4H}{3\lambda}, \quad \frac{\mu}{\sigma} \geq \frac{3}{7},$$

we have $\mu\lambda\sigma^{-1}\sqrt{k}L \geq 4H/7 > 1$. Condition (7.26) forces either

$$b_1/g_0 > \mu\lambda\sigma^{-1}\sqrt{k}L\sqrt{(K-1)La_2/a_1},$$

or the corresponding bound on b_2/g_0 with the ratio reversed. Thus $b_1/g_0 > R_2 - 1$ or $b_2/g_0 > S_2 - 1$. An additive collision would force these coprime integers to divide the two nonzero coordinate differences, contradicting that bound. Therefore the additive cardinality is $R_2S_2 > (K-1)L$. Multiplicative independence gives multiplicative cardinality $R_1S_1 > L$.

For the factorial parameter, (7.21) and (7.24) give

$$b \leq \frac{(1 + \sqrt{K-1})\sqrt{K}}{2(K-1)\sqrt{k}} b'' \exp \left\{ \frac{3}{2} - \frac{\log(2\pi(K-1)/\sqrt{e})}{K-1} + \frac{\log K}{6K(K-1)} \right\}.$$

Using $\sqrt{k} \geq (3 + \sqrt{17})/(12\lambda)$, this implies

$$\log b \leq \log(\lambda b'') - \frac{\log(2\pi K/\sqrt{e})}{K-1} + f(K),$$

where

$$f(x) = \log \frac{(1 + \sqrt{x-1})\sqrt{x}}{x-1} + \frac{\log x}{6x(x-1)} + \frac{3}{2} + \log \frac{6}{3 + \sqrt{17}} + \frac{\log(x/(x-1))}{x-1}.$$

Every nonconstant term decreases for $x > 1$. For the first term set $q = \sqrt{x-1}$; its derivative with respect to q is $-\{q^2 + q + 2\}/\{q(q+1)(q^2+1)\} < 0$. The other two decrease directly, using $\log x > (x-1)/x$. Since $f(8) < 1.75$, the hypothesis on h_0 yields

$$\log b \leq \frac{h_0 - 0.06}{D_*} - \frac{\log(2\pi K/\sqrt{e})}{K-1}. \quad (7.27)$$

We now check the strict analytic inequality of Theorem 7.3. Insert (7.27) and (7.22) into its left side. The result is at least $\Phi + \Theta$, where

$$\begin{aligned} \Phi &= K(\lambda L - h_0 - \lambda/\sigma) - \frac{L^{3/2}\sqrt{(K-1)a_1a_2}}{3} - \frac{2L^{3/2}\sqrt{a_1a_2}}{3} - \frac{L(a_1 + a_2)}{3}, \\ \Theta &= 0.06(K-1) + h_0 + \frac{L^{3/2}\sqrt{a_1a_2}}{6(1 + \sqrt{K-1})} + D_* \log(2\pi K/\sqrt{e}) - (D_* + 1) \log(KL). \end{aligned}$$

Because $K \geq kLa_1a_2$, $K - 1 \leq kLa_1a_2$, and $\lambda L - h_0 - \lambda/\sigma = U > 0$,

$$\Phi \geq La_1a_2(kU - \sqrt{k}V - W) = 0.$$

Condition (7.26) gives

$$\lambda b'' \geq \frac{32}{21}H^2,$$

by the lower bound on $\sqrt{k}L$. In particular $h_0 > 3.6$, and $L^{3/2}\sqrt{a_1a_2} \geq 2L\lambda > 12$.

Use the lower bound on h_0 in Θ . It gives $\Theta \geq (D_* - 1)\Theta_0 + \Theta_1$, with

$$\Theta_0 = \log(\lambda b'') + 1.75 - \log L + \log(2\pi/\sqrt{e}) > 3,$$

and

$$\Theta_1 \geq 0.06K - \log K + c_0 + \frac{2}{1 + \sqrt{K-1}}, \quad c_0 = -2\log \frac{5 + \sqrt{17}}{4} + \log \frac{2\pi}{\sqrt{e}} + \log \frac{32}{21} + 1.75.$$

The last function is strictly convex for $K > 1$. Its derivative is negative at 19 and positive at 20, so its minimum lies in that interval. Its value at 19 exceeds 0.437, and its derivative there exceeds -0.002 ; the tangent-line bound gives a minimum exceeding $0.435 > 0.4$.

On the other hand, the proved Stirling upper bound gives, for $N \geq 32$,

$$\epsilon(N) \leq \frac{2}{N} \left\{ \frac{3}{2} \log N + \frac{1}{2} \log(2\pi) + \frac{1}{12N} + \log \left(1 + \left(\frac{e-1}{e} \right)^N \right) \right\} < 0.4.$$

Each term divided by N decreases for $N \geq 32$, and substitution at 32 verifies the strict inequality. Hence $\Phi + \Theta > \epsilon(N)$, as required. Theorem 7.3 applies.

To express its conclusion in the original parameters, the definition of k gives

$$\sqrt{k}L = \frac{L^2}{6U} + \frac{1}{2} \sqrt{\frac{L^4}{9U^2} + \frac{8L^{5/2}}{3U\sqrt{a_1a_2}} + \frac{4}{3} \left(\frac{1}{a_1} + \frac{1}{a_2} \right) \frac{L^2}{U}}.$$

Apply (7.25) and its $\gamma = 2$ counterpart, with $U = \lambda(L - H)$. Substitution in this displayed expression proves

$$\mu\lambda\sigma^{-1}kL^2a_1a_2 \leq CU_0^2a_1a_2.$$

Since $KL \leq L + kL^2a_1a_2$ and $\mu\lambda\sigma^{-1}L \leq \lambda L \leq \sqrt{\omega\theta}U_0$, we obtain

$$\log |\Lambda'| \geq -CU_0^2a_1a_2 - \sqrt{\omega\theta}U_0. \quad (7.28)$$

We remove its prefactor explicitly. Applying (7.25) with exponents 3, 4, 9/2 to the expression for $\sqrt{k}L^2$ proves

$$\sqrt{k}L^2 \leq C'U_0^2, \quad C'U_0^2a_1a_2 > e^2, \quad C'/C < 4.$$

For the middle inequality, the lower bound

$$\sqrt{k}L^2 > \frac{\omega^{3/2}\theta^{-1/2}H^2}{3\lambda}$$

and $a_1a_2 \geq \max\{1, \lambda^2\}$ give $C'U_0^2a_1a_2 > 4(3 + \sqrt{17})/3 > e^2$. For the last, ignore the positive supplementary terms in the radical defining C , obtaining $C'/C \leq 3(\sigma/\mu)\sqrt{\theta/\omega} < 4$. The same lower estimate gives

$$L \leq \frac{3\theta}{2\omega} \sqrt{k}L^2a_1a_2.$$

Using $a_j \geq 1$, $K \geq 8$ and (7.24), it follows that

$$\max\{LR, LS\} \leq \left\{ \frac{3}{2} \left(\frac{4}{3 + \sqrt{17}} \right)^2 + 1 + \frac{1}{\sqrt{7}} \right\} \sqrt{k} L^2 a_1 a_2 < 1.86 C' U_0^2 a_1 a_2.$$

Write $Q_0 = C U_0^2 a_1 a_2$, $A_0 = C' U_0^2 a_1 a_2$. We have $Q_0 \geq (16/21) H^2 > 3$ and $\sqrt{\omega\theta} U_0 \geq D_* \log 2$. Unless (7.23) is already true, we may assume $\log |\Lambda| \leq -Q_0 - D_* \log 2 - 2 \leq -Q_0 - 2.6$. Then

$$\frac{|\Lambda|}{2} \max\{LR, LS\} \leq 3.72 Q_0 e^{-Q_0 - 2.6} \leq 12 e^{-5.6}.$$

Consequently each exponential in the prefactor is at most $e^{12e^{-5.6}}$, and

$$|\Lambda'| \leq 0.53 |\Lambda| \max\{LR, LS\} < |\Lambda| A_0.$$

Combine this with (7.28) to prove (7.23) when (7.26) holds.

It remains to cover the complementary case. Put $b_j^* = b_j/g_0$. Multiplicative independence ensures $\gamma = \alpha_2^{b_2^*} \alpha_1^{-b_1^*} \neq 1$. The weighted product formula, applied to $\gamma - 1$ and excluding the chosen place, gives

$$\log \frac{|\gamma - 1|}{\max\{1, |\gamma|\}} \geq -D_* h(\gamma) - (D_* - 1) \log 2.$$

For any logarithm z of γ , $|e^z - 1| \leq |z| \max\{1, |e^z|\}$, by integrating e^{tz} for $0 \leq t \leq 1$. Thus

$$\log |\Lambda| \geq -D_* \{b_1^* h(\alpha_1) + b_2^* h(\alpha_2) + \log 2\} \geq -\frac{1}{2} \left(\frac{b_1^*}{a_2} + \frac{b_2^*}{a_1} \right) a_1 a_2 - D_* \log 2.$$

The complementary inequality to (7.26) bounds this below by $-Q_0 - \sqrt{\omega\theta} U_0$. Since $A_0 > e^2$, it also implies (7.23). This completes the proof. $\square$

Corollary 7.5 (Laurent's constant 25.2). Let α_1, α_2 be multiplicatively independent nonzero algebraic numbers, with any chosen logarithms. Set $D_* = d/\chi$, and choose

$$a_j^0 = \log A_j \geq \max\{h(\alpha_j), |\ell_j|/D_*, 1/D_*\}.$$

For integers c_1, c_2 , not both zero, put

$$B' = \frac{|c_1|}{D_* a_2^0} + \frac{|c_2|}{D_* a_1^0}, \quad M = \max\{\log B' + 0.21, 20/D_*, 1\}.$$

Then

$$\log |c_1 \ell_1 + c_2 \ell_2| \geq -25.2 D_*^4 a_1^0 a_2^0 M^2. \quad (7.29)$$

Proof. In the cancellation setting apply Theorem 7.4 with

$$\rho = 6.3, \quad \mu = 0.56, \quad h_0 = D_* M, \quad a_j = (\rho + 2) D_* a_j^0.$$

The height inequalities are immediate; $a_j \geq 8.3$ and $a_1 a_2 \geq 8.3^2 > \lambda^2$. Also $h_0 \geq 20$, $h_0 \geq D_*$, so the last two cutoffs hold. For the coefficient cutoff, $b'' = B'/8.3$, and

$$-\log 8.3 + \log \lambda + 1.81 < 0.21.$$

Since $D_* \geq 1$, this verifies the full cutoff, including its additive 0.06.

For fixed μ, ρ , write (7.23) as $\log |\Lambda| \geq -C'' h_0^2 a_1 a_2$, where

$$C'' = \left(1 + \frac{\lambda}{h_0 \sigma} \right)^2 \left\{ C + \frac{\sqrt{\omega\theta}}{U_0 a_1 a_2} + \frac{\log(C' U_0^2 a_1 a_2)}{U_0^2 a_1 a_2} \right\}. \quad (7.30)$$

This coefficient decreases in h_0, a_1, a_2 . To check the only less immediate term, put $A_0 = C'U_0^2a_1a_2 > e^2$. The explicit formula for C' shows that A_0 increases in all three variables: its terms are positive products of $H\omega$, $H\theta$, a_1, a_2 , all with positive powers. Meanwhile C' decreases. Since $\log v/v$ is positive and decreases for $v > e$, the product $C' \log A_0/A_0 = \log A_0/(U_0^2a_1a_2)$ decreases. Every other factor in (7.30) decreases directly.

It therefore suffices to substitute $h_0 = 20$, $a_1 = a_2 = 8.3$. The exact interval certificate for these expressions gives

$$25.1203 < C''(8.3)^2 < 25.1205 < 25.2.$$

Substitution in (7.30) proves (7.29).

We finish the reductions, rather than assuming the cancellation notation exhausts all signs. In the same-sign case after inversions, Section 1 of the preceding lesson bounds the form below by $\exp\{-\log 4 - d^2(2h(\alpha_j) + \log 2)\}$. Since $d \leq 2D_*$, $a_j^0 \geq 1/D_*$, $M \geq \max\{20/D_*, 1\}$, and $D_*M \geq 20$, this is stronger than (7.29). For example the height term is at most $8D_*^2a_j^0$, while the magnitude on the right of (7.29) is at least $25.2D_*^3a_j^0M^2$; the ratio is at most $8/(25.2D_*M^2) \leq 8/(25.2 \cdot 20)$. The constant terms are at most $6D_*^2$, whose ratio to the target magnitude is at most $6/(25.2M^2) \leq 6/25.2$. The sum of these two ratios is less than one. If both moduli are one, invert either base to enter the cancellation case.

If exactly one coefficient is zero, the selected-place logarithm bound in the proof of Theorem 7.4 gives $\log |\ell_j| \geq -D_*(h(\alpha_j) + \log 2)$. Multiplicative independence excludes $\alpha_j = 1$. The nonzero integer coefficient cannot decrease its absolute value, and this lower bound is stronger than (7.29) by the same cutoffs. Thus all stated cases are covered. $\square$

Laurent's numerical families

The cutoff $20/D_*$ in Corollary 7.5 balances the leading constant against the size of the coefficients. Other cutoffs give useful bounds for different ranges. The following complete family is due to [Laurent 2008].

Corollary (complex and positive-real families). Under the hypotheses of Corollary 7.5, retain a_j^0, B' , and let m be any even integer from 10 through 30. Put

$$M_1(m) = \max\{\log B' + 0.21, m/D_*, 1\}.$$

For the constant $C_1(m)$ in the table below,

$$\log |c_1\ell_1 + c_2\ell_2| \geq -C_1(m)D_*^4a_1^0a_2^0M_1(m)^2.$$

If both bases exceed one and the chosen logarithms are positive real, use $D_* = [\mathbb{Q}(\alpha_1, \alpha_2) : \mathbb{Q}]$ and put

$$M_2(m) = \max\{\log B' + 0.38, m/D_*, 1\}.$$

Then the stronger real bound is

$$\log |c_1\ell_1 + c_2\ell_2| \geq -C_2(m)D_*^4a_1^0a_2^0M_2(m)^2.$$

Both assertions permit arbitrary integer coefficients that are not both zero. The bases remain multiplicatively independent. The parameter pairs in the table make the numerical verification reproducible.

m	C_1	μ_1	ρ_1	C_2	μ_2	ρ_2
10	32.3	0.54	5.9	25.2	0.52	5.0
12	29.9	0.54	6.0	23.4	0.53	5.1
14	28.2	0.55	6.1	22.1	0.54	5.2

m	C_1	μ_1	ρ_1	C_2	μ_2	ρ_2
16	26.9	0.56	6.2	21.1	0.55	5.2
18	26.0	0.56	6.3	20.3	0.55	5.3
20	25.2	0.56	6.3	19.7	0.56	5.3
22	24.5	0.57	6.4	19.2	0.56	5.4
24	24.0	0.57	6.4	18.8	0.56	5.4
26	23.5	0.57	6.4	18.4	0.57	5.4
28	23.1	0.57	6.5	18.1	0.57	5.4
30	22.8	0.58	6.5	17.9	0.57	5.5

Proof. First take two nonzero coefficients in the cancellation setting. For the complex family choose $\rho = \rho_1, \mu = \mu_1, h_0 = D_*M_1(m)$, and $a_j = (\rho + 2)D_*a_j^0$. The height and product conditions of Theorem 7.4 follow exactly as in Corollary 7.5. For every row of the table, the rational interval certificate verifies

$$-\log(\rho + 2) + \log \lambda + 1.81 < 0.21, \quad \lambda < m, \quad \lambda < \rho + 2.$$

Since $D_* \geq 1$, the first inequality includes the additive 0.06 in the coefficient cutoff. The other two, together with $M_1(m) \geq 1$, verify all remaining hypotheses.

The monotonicity of C'' proved in Corollary 7.5 holds for every fixed pair (ρ, μ) . Therefore

$$C''(h_0, a_1, a_2)(\rho + 2)^2 \leq C''(m, \rho + 2, \rho + 2)(\rho + 2)^2 < C_1(m).$$

The last strict inequality is certified for every row using outward rational intervals. Substitution into (7.30) proves the complex bound.

For the real family choose $\rho = \rho_2, \mu = \mu_2, h_0 = D_*M_2(m)$, and $a_j = (\rho + 1)D_*a_j^0$. The improvement of one in this scale comes from

$$\rho|\ell_j| - \log|\alpha_j| + 2D_*h(\alpha_j) = (\rho - 1)\ell_j + 2D_*h(\alpha_j) \leq (\rho + 1)D_*a_j^0.$$

The same certificate proves

$$-\log(\rho + 1) + \log \lambda + 1.81 < 0.38, \quad \lambda < m, \quad \lambda < \rho + 1,$$

and $C''(m, \rho + 1, \rho + 1)(\rho + 1)^2 < C_2(m)$. Thus all hypotheses and the conclusion follow as before.

For arbitrary signs and a single nonzero coefficient, use the reductions in Corollary 7.5. Their estimates remain stronger throughout both families. Indeed $D_*M_i(m) \geq m$, $M_i(m) \geq 1$, and the sum of the two coarse ratios used there is at most

$$\frac{8}{C_i(m)m} + \frac{6}{C_i(m)} < 0.39 < 1$$

for every entry. Multiplicative independence supplies nonvanishing. This completes all coefficient cases. $\square$

For example, the positive-real bound at $m = 20$ has constant 19.7, while the complex bound at $m = 30$ has constant 22.8. One may evaluate all eleven expressions and take their largest lower bound: decreasing the leading constant also increases the cutoff, so the last row need not give the best bound for a given form.

11. A sharper bound when the real bases are close to one

For positive real bases, a large admissible radius reduces the main coefficient by a factor $(\log E)^3$. The height condition must accompany this gain: a tiny real logarithm by itself does not imply a small Weil height.

Theorem 7.6 (Laurent–Mignotte–Nesterenko). Let α_1, α_2 be multiplicatively independent positive real algebraic numbers. Write $D = [\mathbb{Q}(\alpha_1, \alpha_2) : \mathbb{Q}]$, use real logarithms, and choose

$$a_j^0 = \log A_j \geq \max\{h(\alpha_j), |\log \alpha_j|/D, 1/D\}.$$

For integers c_1, c_2 , not both zero, set

$$B' = \frac{|c_1|}{Da_2^0} + \frac{|c_2|}{Da_1^0}.$$

Suppose

$$2 \leq E \leq 1 + \min_j \frac{Da_j^0}{|\log \alpha_j|}, \quad E \leq \min_j A_j^{3D/2},$$

and put

$$v = \log E, \quad M_E = \max\{\log B' + \log v + 0.47, 10v/D, 1/2\}.$$

Then

$$\log |c_1 \log \alpha_1 + c_2 \log \alpha_2| \geq -\frac{35.1D^4 a_1^0 a_2^0 M_E^2}{v^3}. \quad (7.31)$$

A rational specialization appears in the freely readable author version [Bugeaud 2008]. We prove the full algebraic estimate from the parameter theorem of [Laurent 2008], using the complete determinant argument above and a slightly smaller intermediate coefficient.

Proof. Invert bases below one and change their coefficient signs. Heights and absolute logarithms are preserved. The bases are now greater than one. In the cancellation case, apply Theorem 7.4 with

$$\rho = E, \quad \mu = 0.56, \quad \sigma = 0.9032, \quad \lambda = \sigma v, \quad h_0 = DM_E, \quad a_j = \frac{7}{2}Da_j^0.$$

Indeed,

$$(\rho - 1)|\log \alpha_j| + 2Dh(\alpha_j) \leq 3Da_j^0 < a_j.$$

The cutoffs give $Da_j^0 \geq \max\{1, 2v/3\}$, so $a_j \geq 1$ and $a_1 a_2 \geq \lambda^2$. Also $h_0 \geq 10v$, $h_0 \geq D/2$, which verifies the radius and $D \log 2/2$ cutoffs. Since $b'' = 2B'/7$ and

$$\log \sigma - \log(7/2) + 1.81 < 0.47,$$

the coefficient cutoff, including its additive 0.06, follows from $D \geq 1$.

We bound the coefficient C'' from (7.30). Its established monotonicity allows us to replace h_0 by $10v$ and both a_j by $7v/3$. These values are admissible for the coefficient formulas because $v \geq \log 2$. At these values let

$$H = \frac{11}{\sigma}, \quad \omega = 2 \left(1 + \sqrt{1 + \frac{1}{4H^2}} \right), \quad \theta = \sqrt{1 + \frac{1}{4H^2}} + \frac{1}{2H},$$

and define the constants

$$c = \frac{\mu}{\sigma^4} \left\{ \frac{\omega}{6} + \frac{1}{2} \sqrt{\frac{\omega^2}{9} + \frac{8\sigma(3/7)\omega^{5/4}\theta^{1/4}}{3\sqrt{H}} + \frac{8}{3}(3/7)\frac{\sigma\omega}{H}} \right\}^2,$$

$$c' = \sqrt{\frac{c\omega\theta}{\sigma^2\mu}}, \quad c_A = \frac{5929}{9}c'.$$

Then $C = c/v^3$, $C' = c'/v^3$, $U_0 = 11v$, and $C'U_0^2 a_1 a_2 = c_A v$. Consequently,

$$\left(\frac{7}{2}\right)^2 v^3 C'' \leq \frac{5929}{400} \left\{ c + \frac{9\sqrt{\omega\theta}}{539} + \frac{9\log(c_A v)}{5929v} \right\}. \quad (7.32)$$

The exact interval certificate verifies $c_A \log 2 > e$. The function $\log(c_A v)/v$ therefore decreases for $v \geq \log 2$. Substitution of $v = \log 2$, with outward rational rounding, bounds the right side of (7.32) by $34.41890783 < 35.1$. Since

$$h_0^2 a_1 a_2 = (7/2)^2 D^4 M_E^2 a_1^0 a_2^0,$$

Theorem 7.4 proves (7.31) in the cancellation case.

For completeness, a single nonzero coefficient or two coefficients of the same sign also satisfy the result. The logarithm is then at least one of the positive real logarithms in absolute value. The selected-place product formula used in Theorem 7.4 gives $\log |\log \alpha_j| \geq -D(h(\alpha_j) + \log 2)$. Writing $s_j = Da_j^0$, the target magnitude $T = 35.1h_0^2 s_1 s_2 / v^3$ satisfies

$$T \geq 2340s_j, \quad T \geq 78D.$$

The first inequality uses $h_0 \geq 10v$ and $s_{3-j} \geq 2v/3$; the second uses $h_0^2 \geq (10v)(D/2)$ and $s_1 s_2 \geq 4v^2/9$. Hence $(s_j + D \log 2)/T \leq 1/2340 + \log 2/78 < 1$, proving the remaining cases. $\square$

An absolute cutoff, including dependent bases

An absolute term in the coefficient cutoff permits a simple bound valid for dependent bases as well. The next deduction uses the complete proof of Theorem 7.6 and the height inequality; its cutoff is part of its statement.

Corollary (an absolute coefficient cutoff). Let $\alpha_1, \alpha_2 > 1$ be real algebraic numbers and set $D = [\mathbb{Q}(\alpha_1, \alpha_2) : \mathbb{Q}]$. Choose

$$a_j \geq \max\{h(\alpha_j), \log \alpha_j / D, 1/D\}.$$

Let c_1, c_2 be integers such that $\Gamma = c_1 \log \alpha_1 + c_2 \log \alpha_2 \neq 0$, and define

$$\begin{aligned} B' &= \frac{|c_1|}{Da_2} + \frac{|c_2|}{Da_1}, \quad v = \log E, \\ 2 &\leq E \leq 1 + \min_j \frac{Da_j}{\log \alpha_j}, \\ E &\leq \min_j \exp(3Da_j/2), \\ W &= \max\{3/D, \log B'/v, 1\}. \end{aligned}$$

Then, without a multiplicative-independence hypothesis,

$$\log |\Gamma| \geq -\frac{390D^4 a_1 a_2}{v} W^2. \quad (7.76)$$

The same statement follows with the larger constant 78500. The cutoff 1 in W is retained in both statements.

Proof. First suppose that the bases are multiplicatively independent. Put $U = vW = \max\{3v/D, \log B', v\}$. Since $\log v \leq v - 1$, we have

$$\begin{aligned} \log B' + \log v + 0.47 &\leq 2U - 0.53 < 2U, \\ 10v/D &\leq (10/3)U, \\ 1/2 &\leq (10/3)U. \end{aligned}$$

The last inequality follows from $U \geq v \geq \log 2$. Thus $M_E \leq (10/3)U$ in Theorem 7.6, and $35.1(10/3)^2 = 390$ proves (7.76).

Now suppose that the bases are dependent. Their real logarithms have a positive rational ratio. There are relatively prime positive integers r, s with $\alpha_1^r = \alpha_2^s$. Choose integers p, q with $sp + rq = 1$, and put $\beta = \alpha_1^p \alpha_2^q$. Taking real logarithms gives

$$\beta > 1, \quad \beta^s = \alpha_1, \quad \beta^r = \alpha_2, \\ \Gamma = (c_1 s + c_2 r) \log \beta.$$

The integer in the last expression is nonzero, so $|\Gamma| \geq \log \beta$. Also β belongs to the same number field and

$$h(\beta) = h(\alpha_1)/s = h(\alpha_2)/r \leq a_j \quad (j = 1, 2).$$

Apply the one-place height inequality to $\gamma = 1 - \beta^{-1} \neq 0$. Its degree is at most D , and $h(\gamma) \leq h(\beta) + \log 2$. Consequently

$$\log \beta \geq 1 - \beta^{-1} \geq \exp\{-D(h(\beta) + \log 2)\},$$

and $\log |\Gamma| \geq -Da_j - D \log 2$.

Write $s_j = Da_j$ and let $T = 390D^4 a_1 a_2 W^2 / v$. The height and radius cutoffs give $s_j \geq 1$ and $s_j \geq 2v/3$. Using $W \geq 1$, we obtain

$$T \geq \frac{390D^2 s_1 s_2}{v} \geq 260D^2 s_j \geq 260D.$$

Therefore

$$\frac{Da_j + D \log 2}{T} \leq \frac{1}{260D^2} + \frac{\log 2}{260D} < 1.$$

The elementary dependent-base bound is stronger than (7.76). This also covers a zero coefficient, provided $\Gamma \neq 0$. $\square$

The cutoff certificate checks the scalar margins and actual dependent forms in fields generated by roots of $X^D - 2$. The proof above supplies the uniform argument; finite numerical checks are supplementary.

The exact radius for dependent bases

For dependent positive bases, the exact radius also records the size of the logarithm of their primitive generator. This gives an estimate without an absolute coefficient cutoff in the following cases.

Proposition (dependent bases and the exact radius). Let $\alpha_1, \alpha_2 > 1$ be multiplicatively dependent, let $D = [\mathbb{Q}(\alpha_1, \alpha_2) : \mathbb{Q}]$, and suppose

$$a_j \geq \max\{h(\alpha_j), \log \alpha_j / D, 1/D\}.$$

Choose nonzero integers c_1, c_2 with $\Gamma = c_1 \log \alpha_1 + c_2 \log \alpha_2 \neq 0$. In this proposition the radius is the specified value

$$E = 1 + \min_j \frac{Da_j}{\log \alpha_j}, \quad E \leq \min_j \exp(3Da_j/2), \quad v = \log E. \quad (7.77)$$

Write $\alpha_1 = \beta^{m_1}$, $\alpha_2 = \beta^{m_2}$, where $\beta > 1$ and m_1, m_2 are relatively prime positive integers, as constructed in the proof of (7.76). Set

$$S_j = Da_j, \quad H = \min\{S_1/m_1, S_2/m_2\}, \quad k = c_1 m_1 + c_2 m_2.$$

Then the following is an exact identity:

$$|\Gamma| = \frac{|k|H}{E-1}, \quad \log |\Gamma| \geq \log H - \log(E-1) > \log H - v. \quad (7.78)$$

In particular, put

$$B' = \frac{|c_1|}{Da_2} + \frac{|c_2|}{Da_1}, \quad W_0 = \max\{3/D, \log B'/v\}.$$

The bound

$$\log |\Gamma| \geq -\frac{78500D^4a_1a_2}{v}W_0^2 \quad (7.79)$$

holds if either of the following conditions is satisfied:

1. At least one of α_1, α_2 is not an algebraic unit.
2. $\max\{m_1, m_2\} \leq E^{313999}$.

More generally, $H \geq E^{-313999}$ is sufficient.

Proof. The primitive generator belongs to $K = \mathbb{Q}(\alpha_1, \alpha_2)$ by its Bezout construction. Conversely both bases belong to $\mathbb{Q}(\beta)$, so $K = \mathbb{Q}(\beta)$ and $\deg \beta = D$. Writing $\ell = \log \beta > 0$ in (7.77) gives

$$E-1 = \min_j \frac{S_j}{m_j \ell} = \frac{H}{\ell}.$$

Also $\Gamma = k\ell$, with k a nonzero integer. These two equalities prove (7.78); no estimate for a nearly cancelling pair of real numbers is involved.

Let T denote the positive quantity on the right of (7.79). The height and radius assumptions imply $S_j \geq \max\{1, 2v/3\}$, while $W_0 \geq 3/D$. Therefore

$$T \geq \frac{706500S_1S_2}{v} \geq 314000v. \quad (7.80)$$

Thus $H \geq E^{-313999}$ and (7.78) imply $\log |\Gamma| > -314000v \geq -T$.

For condition 1, β is not an algebraic unit: a positive integral power of an algebraic unit is again an algebraic unit. Let $a \prod_{i=1}^D (X - \beta_i)$ be its primitive irreducible integer polynomial, with $a > 0$. By the height formula, Theorem 2.4,

$$\exp(Dh(\beta)) = a \prod_{i=1}^D \max\{1, |\beta_i|\}. \quad (7.81)$$

This product is at least both the leading coefficient a and the absolute value of the constant coefficient. If both of those integers were one, the polynomial and its reciprocal would be monic integer polynomials for β and β^{-1} . That would make β an algebraic unit. Consequently the product in (7.81) is at least two, so $Dh(\beta) \geq \log 2$. Height homogeneity now gives

$$H \geq Dh(\beta) \geq \log 2.$$

Since $\log 2 > 1/2 > e^{-1}$, equation (7.78) yields $\log |\Gamma| > -v - 1$. On the other hand $v \geq \log 2 > 1/2$, so $T \geq 314000v > v + 1$. This proves (7.79).

For condition 2, let $M = \max\{m_1, m_2\}$. The inequalities $S_j \geq 1$ imply $H \geq 1/M$, and hence

$$\log |\Gamma| > -v - \log M \geq -314000v \geq -T.$$

This proves the second assertion, including the equality case in the condition on M . $\square$

The identity (7.78) depends on the equality specifying E in (7.77). An arbitrarily smaller admissible radius does not give that identity. For a dependent pair outside the two sufficient conditions, (7.78) remains an exact formula and (7.76) remains applicable with its stated cutoff.

Example (radical fields in every degree). Let $D \geq 1$, let $\beta = 2^{1/D}$, and choose an integer

$$m \geq \max\{3, \log(D+1)/\log 2\}.$$

Take $\alpha_1 = \beta^m$, $\alpha_2 = \beta^{m+1}$, $c_1 = -1$, $c_2 = 1$, and $a_1 = m \log 2/D$, $a_2 = (m+1) \log 2/D$. The polynomial $X^D - 2$ is irreducible: a nontrivial monic integer factorization would reduce modulo two to factors of X^D , both with zero constant coefficient, forcing the constant coefficient of the product to be divisible by four. The two bases generate the same field as their ratio β . All conjugates of β have modulus $2^{1/D}$, so $Dh(\beta) = \log 2$, and the chosen heights are exact. Here

$$E = D + 1, \quad H = \log 2, \quad \Gamma = \frac{\log 2}{D}, \quad W_0 = \frac{3}{D}. \quad (7.82)$$

Indeed $m \log 2 \geq \log(D+1)$ verifies the radius cap, and $B' \leq 7/(12 \log 2) < 1$ since $m \geq 3$. Thus (7.79) holds in arbitrarily large degree with no absolute term in W_0 .

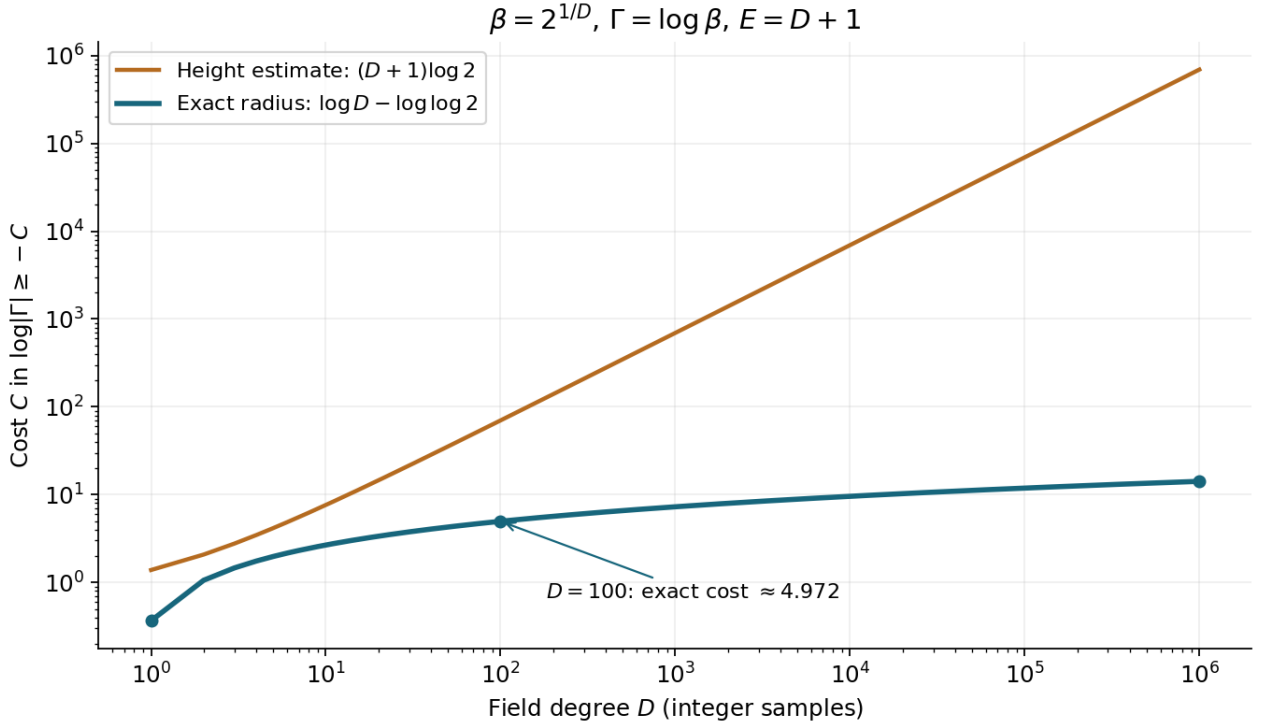


Figure.

For the example (7.82), the exact value of $-\log |\Gamma|$ is $\log D - \log \log 2$. The radius identity recovers this value. The direct one-place height argument gives the larger cost $(D+1) \log 2$; both are valid bounds. Integer degrees are sampled, and both axes use logarithmic scales. The proof is (7.78)–(7.82), rather than an extrapolation of the plot. Radius notation follows the free author exposition Bugeaud 2008. The figure source and rational-interval checks are included.

12. A rational number close to one

Corollary 7.7. Let x_1, x_2, y_1, y_2 be positive integers with

$$x_1 \geq 2, \quad x_1 \neq y_1, \quad y_2 < x_2 \leq \frac{6}{5}y_2.$$

Let $b \geq 1$ be an integer, and suppose $(x_1/y_1)^b \neq x_2/y_2$. Define $\eta > 0$ by

$$\frac{x_2}{y_2} = 1 + x_2^{-\eta}.$$

Then $0 < \eta < 1$, $\eta \log x_2 > 1$, and

$$\log \left| (y_1/x_1)^b (x_2/y_2) - 1 \right| \geq -\frac{35.2}{\eta} \log x_1 \left(\max \left\{ 1 + \frac{\log b}{\eta \log x_2}, 10 \right\} \right)^2. \quad (7.33)$$

Proof. Put $\delta = |(y_1/x_1)^b(x_2/y_2) - 1|$ and $E = x_2^\eta = y_2/(x_2 - y_2)$. The hypotheses give $E \geq 5$, $x_2 \geq 6$, and $E < x_2$. Thus $0 < \eta < 1$ and $\eta \log x_2 = \log E \geq \log 5 > 1$.

Several cases have elementary stronger bounds. If $y_1 > x_1$, then $\delta \geq 1/x_1$. If $y_1 < x_1 \leq 5$, then $\delta \geq 1 - 24/25 = 1/25$. If the two ratios above one are multiplicatively dependent, their prime-exponent vectors lie on the same rational line. Dividing the first vector by the greatest common divisor of its coordinates gives a primitive integer vector; the second is an integer multiple because a primitive vector has an integer linear combination of its coordinates equal to one. Thus

$$x_1/y_1 = (c/d)^u, \quad x_2/y_2 = (c/d)^w$$

for coprime integers $c > d \geq 1$ and positive integers u, w . The exponent $ub - w$ is nonzero, and

$$\delta = |(d/c)^{ub-w} - 1| \geq 1/c \geq 1/x_1.$$

These elementary lower bounds imply (7.33), whose target magnitude exceeds $3520 \log x_1$.

We may now assume $x_1 > y_1$, $x_1 \geq 6$, and multiplicative independence. If $x_1 < E/2$, then

$$(y_1/x_1)^b(x_2/y_2) \leq (1 - 1/x_1)(1 + 1/E) < 1 - \frac{1}{2x_1},$$

again proving the result. If $\delta \geq x_1^{-10}$, the assertion is also immediate. Therefore suppose

$$5 \leq E \leq 2x_1, \quad 0 < \delta < x_1^{-10}.$$

Apply Theorem 7.6 to $\alpha_1 = x_1/y_1$, $\alpha_2 = x_2/y_2$, with $D = 1$, $a_j^0 = \log x_j$. Their reduced numerator heights and real logarithms are at most $\log x_j$, and these cutoffs exceed one. Moreover,

$$E \leq x_1^{3/2}, \quad E < x_2^{3/2}, \quad \log \alpha_2 = \log(1 + 1/E) \leq 1/E.$$

The local logarithm comparison gives

$$b \log \alpha_1 \leq \log \alpha_2 + 2\delta \leq \frac{1 + 4x_1^{-9}}{E} < \frac{1.001}{E}.$$

Since $\log x_1 \geq \log 6 > 1.001$, both inequalities $E \leq 1 + \log x_j / \log \alpha_j$ hold. All radius hypotheses are therefore verified.

Set $v = \log E = \eta \log x_2$ and $B' = b/\log x_2 + 1/\log x_1$. Because $b \geq 1$, $v \leq \log x_2$, $E \leq 2x_1$,

$$v \left(\frac{1}{\log x_2} + \frac{1}{\log x_1} \right) \leq 2 + \frac{\log 2}{\log 6} < 2.5.$$

As $2.5e^{0.47} < 5 \leq E$, it follows that $\log B' + \log v + 0.47 \leq \log b + v$. The other cutoffs give

$$\frac{M_E}{v} \leq \max\{1 + \log b/v, 10\} =: T_b.$$

Theorem 7.6 now yields

$$\log |\log \alpha_2 - b \log \alpha_1| \geq -\frac{35.1}{\eta} (\log x_1) T_b^2.$$

Finally $|\log(1 + u)| \leq 2|u|$ for real $|u| < 1/2$, so the same lower bound for $\log \delta$ loses at most $\log 2$. The extra $0.1(\log x_1) T_b^2 / \eta \geq 10 \log 6 > \log 2$ absorbs that loss, proving (7.33). $\square$

13. The unit circle and nonzero logarithmic periods

Theorem 7.8. Let α have modulus one and not be a root of unity. Write $D = [\mathbb{Q}(\alpha) : \mathbb{Q}]$, and choose any logarithm ℓ . Let

$$a = \log A \geq \max\{h(\alpha), |\ell|/D, 1/D\}.$$

For nonzero integers b_1, b_2 , put

$$B' = \frac{|b_1|}{Da} + \frac{|b_2|}{\pi}.$$

Then

$$\log |b_1 i\pi - b_2 \ell| \geq -68000 D^3 a (\max\{10, \log B'\})^2. \quad (7.10)$$

Proof. Apply Theorem 7.1 with bases $-1, \alpha$, logarithms $i\pi, \ell$, and $a_1 = \pi/D$, $a_2 = a$. The field degree is still D , and $h(-1) = 0$. The form cannot vanish: otherwise exponentiating twice gives $\alpha^{2b_2} = 1$. The bound supplied by (7.1) has constant $21600\pi < 68000$. $\square$

The constant 68000 estimate is proved above. The absence of a multiplicative independence assumption in Theorem 7.1 is essential: -1 is itself torsion.

For powers of a unit-circle number, use the principal logarithm $\ell = i\theta$, $-\pi < \theta \leq \pi$. Given an integer $b > 0$, choose an integer q nearest to $b\theta/(2\pi)$ and put $z = i(b\theta - 2\pi q)$. Then

$$|z| \leq \pi, \quad e^z = \alpha^b, \quad |\alpha^b - 1| = 2|\sin(|z|/2)| \geq \frac{2}{\pi}|z|.$$

The last inequality follows from concavity of sine on $[0, \pi/2]$. If $q \neq 0$, (7.10) applies with $|b_1| = 2|q| \leq b + 1$, $|b_2| = b$. If $q = 0$, then $z = b\ell \neq 0$, and the one-place bound on $\alpha - 1$, followed by $|e^\ell - 1| \leq e|\ell|$ when $|\ell| \leq 1$, gives

$$\log |z| \geq -D(h(\alpha) + \log 2) - 1.$$

Thus no artificial nonzero-coefficient assumption is needed when reducing a power difference to two logarithms.

A sharper estimate for powers

The following unit-circle estimate follows from the refined determinant theorem proved above, including the torsion base -1 .

Theorem 7.9. Let α have modulus one and not be a root of unity, use its principal logarithm, and let $b \geq 1$ be an integer. Put

$$D_* = \frac{1}{2}[\mathbb{Q}(\alpha) : \mathbb{Q}], \quad a^0 = \log A = \max\{20/D_*, 11|\log \alpha|/D_* + h(\alpha)\},$$

and

$$M = \max\{17/D_*, 1/(10\sqrt{D_*}), \log(b/25) + 2.35 + 5.1/D_*\}.$$

Then

$$\log |\alpha^b - 1| \geq -9D_*^3 a^0 M^2. \quad (7.34)$$

Proof. Write $D = D_*$, $a = Da^0$, $h_0 = DM$. A real number of modulus one is a root of unity, so α is nonreal and the selected local weight is two. Thus $D \geq 1$, and

$$a \geq 20, \quad h_0 \geq 17, \quad h_0^2 \geq D/100.$$

The target magnitude is $T = 9ah_0^2$. If $\delta = |\alpha^b - 1| \geq 1/3$, the assertion is immediate. The weighted product formula for a unit-circle number gives

$$\log \delta \geq -Dbh(\alpha) - (D - 1)\log 2 \geq -ab - D\log 2.$$

When $b \leq 4h_0^2$, the first term uses at most $4T/9$. Since $T \geq 1.8D$, the second uses at most $T \log 2/1.8$; the sum is less than T . This settles this entire range, including $b = 1, 2$.

Henceforth $b > 4h_0^2$ and $0 < \delta < 1/3$. Set $u_0 = \log 25 - 2.35 < 0.869$. The cutoff implies

$$h_0 \geq D(\log b - u_0) + 5.1, \quad D < h_0/6,$$

because $b > 4 \cdot 17^2 = 1156$ and $\log 1156 - u_0 > 6$. Invert α if necessary so $\log \alpha = i\theta$, $0 < \theta < \pi$. Choose an integer q such that

$$z = i(b\theta - 2\pi q), \quad |z| \leq \pi, \quad e^z = \alpha^b.$$

Since $\delta < 1/3$, the logarithm comparison gives $|z| \leq 2\delta < 2/3$. Consequently $0 \leq 2q \leq b$: a larger $2q$ would give $|b\theta - 2\pi q| \geq \pi$. If $q = 0$, monotonicity of sine on $[0, \pi/2]$ gives $\delta \geq |\alpha - 1|$, and the preceding product-formula comparison proves the result. We can therefore write

$$b_1 = 2q \in [1, b], \quad b_2 = b, \quad \Lambda = b_2 \log \alpha - b_1 \log(-1) = z \neq 0.$$

Apply Theorem 7.3 with

$$\rho = 22, \quad \mu = 3/5, \quad \sigma = 23/25, \quad \lambda = \sigma \log 22, \quad a_1 = 22\pi, \quad a_2 = 2a.$$

These height cutoffs hold because $a \geq 11\theta + Dh(\alpha)$. Let $c = 23/5$, $A = 2a \geq 40$, and choose

$$\begin{aligned} L &= \lfloor 2h_0/\lambda \rfloor \geq 11, & K &= 1 + \lfloor cAL \rfloor, \\ R_1 &= 2, & S_1 &= \lceil L/2 \rceil, \\ R_2 &= 1 + \lfloor \sqrt{(K-1)LA/a_1} \rfloor, & S_2 &= 1 + \lfloor \sqrt{(K-1)La_1/A} \rfloor. \end{aligned}$$

The numbers $(-1)^r \alpha^s$, $0 \leq r < 2$, $0 \leq s < S_1$, are all distinct, since α is not torsion. Their number is at least L . Also $R_2 S_2 > (K-1)L$.

First suppose an additive collision occurs in the $R_2 \times S_2$ grid. With $g_0 = \gcd(b_1, b_2)$, it implies

$$b_2/g_0 \leq S_2 - 1 \leq \sqrt{ca_1} L < 18L.$$

The number $\gamma = \alpha^{b_2/g_0} (-1)^{-b_1/g_0}$ is not one. Since it has modulus one and $e^{\Lambda/g_0} = \gamma$, the product formula and $|e^{it} - 1| \leq |t|$ give

$$\log |\Lambda| \geq -a(b_2/g_0) - D \log 2 > -18ah_0 - D \log 2 > -3ah_0^2.$$

Here $L < h_0$, $h_0 \geq 17$, $D < h_0/6$, $a \geq 20$. Together with $\delta \geq |\Lambda|/2$, this is stronger than the claimed estimate. We may assume the additive grid is injective, so both rank conditions hold.

We check the strict numerical determinant condition uniformly. Its factorial parameter, denoted $b_{\det}$ to distinguish it from the exponent, satisfies Lemma 7.2's factor-five bound, with the additional factor $1/2$ in Section 6. Write

$$r = \sqrt{c/a_1}, \quad s = \sqrt{ca_1}.$$

The rounded lengths obey

$$R \leq rAL + 2, \quad S \leq sL + L/2 + 1/2.$$

Since $b_1 \leq b_2 = b$ and $K-1 \geq cAL-1$,

$$\frac{b_{\det}}{b} \leq \frac{5}{2} \frac{r + s/A + 1/(2A) + 3/(2AL)}{c - 1/(AL)} < 0.392 < e^{-0.9}. \quad (7.35)$$

The expression decreases in A, L ; its maximum is at $A = 40, L = 11$, and the exact certificate checks the displayed constants. Thus

$$D(K - 1) \log b_{\det} \leq K(h_0 - 5.1).$$

This follows by first replacing $\log b_{\det}$ by the positive upper bound $\log b - 0.9$; no sign assumption on the actual logarithm is needed.

The function

$$\frac{L}{4}(Ra_1 + SA) - \frac{KL^2}{12}(a_1/S + A/R)$$

increases in R, S . Inserting their upper bounds and $K > cAL$ gives

$$\frac{gL(Ra_1 + SA)}{AL^2} \leq \frac{s}{2} + \frac{1}{8} + \frac{a_1}{2AL} + \frac{1}{8L} - \frac{c}{12} \left\{ \frac{a_1}{s + 1/2 + 1/(2L)} + \frac{1}{r + 2/(AL)} \right\} < 6.23. \quad (7.36)$$

The right side increases in $1/A, 1/L$, so again only $A = 40, L = 11$ needs to be checked. On the other hand $h_0 < \lambda(L + 1)/2$, and

$$5.1 - \log 22 - \lambda/2 > 0, \quad c\lambda/2 > 6.54.$$

Therefore

$$\frac{K(\lambda L - \log 22) - D(K - 1) \log b_{\det}}{AL^2} > 6.54.$$

Finally $N = KL \leq (c + 1)AL^2$ and $D < h_0/6 < \lambda(L + 1)/12$, so

$$\frac{(D + 1) \log N}{AL^2} \leq \frac{(\lambda(L + 1)/12 + 1) \log((c + 1)AL^2)}{AL^2} < 0.009.$$

This expression decreases in A, L : each occurrence of $\log((c + 1)AL^2)/A$, or of that logarithm divided by L or L^2 , decreases in the relevant variable, since the logarithm exceeds two. The exact certificate evaluates its maximum at 40, 11. We already proved $\epsilon(N) < 0.4$ for $N \geq 32$; here $AL^2 \geq 4840$, so $\epsilon(N)/(AL^2) < 0.0001$. The margin $6.54 - 6.23 - 0.009 - 0.0001 > 0$ verifies (7.20) strictly.

Theorem 7.3 yields

$$\log |\Lambda'| > -\mu(\log 22)KL \geq -\frac{8\mu c}{\sigma^2 \log 22} ah_0^2 - \frac{2\mu}{\sigma} h_0.$$

The first coefficient is less than 8.44, and $2\mu/\sigma < 1.305$. The length bounds give $\max\{LR, LS\} < AL^2$. If $|\Lambda| \geq 1/\max\{LR, LS\}$, a stronger bound is immediate. Otherwise the exponentials in Λ' are at most e , and

$$\log |\Lambda| \geq -8.44ah_0^2 - 1.305h_0 - 1 - \log(AL^2).$$

For $A = 2a \geq 40$, $h_0 \geq 17$, $L < h_0$, use $\log A \leq A/10$ and $2 \log h_0 \leq h_0/2$. They imply

$$1.305h_0 + 1 + \log(AL^2) < 0.01ah_0^2.$$

Indeed the ratio of the resulting upper bound $1.805h_0 + 1 + 0.2a$ to ah_0^2 decreases in both variables and is less than 0.007 at $a = 20, h_0 = 17$. Thus $\log |\Lambda| > -8.45ah_0^2$. The loss $\log 2$ in $\delta \geq |\Lambda|/2$ is less than $0.01ah_0^2$, giving $\log \delta > -8.46ah_0^2 > -9ah_0^2$. Since $ah_0^2 = D^3 a^0 M^2$, the proof is complete. $\square$

14. Adding derivatives to the interpolation matrix

The squared logarithm comes from a one-variable Taylor order. Gouillon's Schneider method with multiplicity adds a second additive coordinate. Its rows are indexed by a triangular set of monomials, rather than a rectangle. Before estimating such a determinant, we must explain why changing its derivative basis preserves rank and why its coefficients remain integral. The arguments below supply these arithmetic ingredients of [Gouillon 2006].

Write

$$\mathcal{I}_K = \{(k_0, k_1) \in \mathbb{Z}_{\geq 0}^2 : k_0 + k_1 \leq K\}, \quad C_K = \frac{(K+1)(K+2)}{2}, \quad N = C_K(L+1).$$

For the derivation $\mathcal{D} = \partial_{X_0} + Y\partial_Y$, Leibniz's rule gives

$$\mathcal{D}^t \left(\frac{X_0^{k_0}}{k_0!} X_1^{k_1} Y^l \right) \Big|_{(0,x,y)} = x^{k_1} \binom{t}{k_0} l^{t-k_0} y^l. \quad (7.37)$$

The expression is zero for $t < k_0$; at $t = k_0, l = 0$, the remaining power is one. Thus the columns at the points $(0, rb_2 + sb_1, \alpha_1^r \alpha_2^s)$, with derivative orders $0 \leq t \leq T$, have entries

$$M_{(k_0, k_1, l), (r, s, t)} = (rb_2 + sb_1)^{k_1} \binom{t}{k_0} l^{t-k_0} \alpha_1^r \alpha_2^s.$$

Here $0 \leq l \leq L$. Rank preservation below does not assert that this rank is N ; that assertion requires a zero estimate for this triangular degree condition.

An integral polynomial basis for the derivatives

For integers $b \geq 1, t \geq 0$, write $t = bq + r$, $0 \leq r < b$, and set

$$\delta_b(Z; t) = \binom{Z}{b}^q \binom{Z}{r}, \quad \nu(b) = \text{lcm}(1, \dots, b), \quad \delta_b(Z; t, k) = \frac{d^k}{dZ^k} \delta_b(Z; t).$$

The binomial symbols here are polynomials. In particular $\binom{Z}{0} = 1$. Each $\delta_b(Z; t)$ has degree exactly t and a nonzero leading coefficient, so $\delta_b(Z; 0), \dots, \delta_b(Z; T)$ form a basis of the polynomials of degree at most T .

Lemma (integral divided derivatives). For every integer l , and all $t, k \geq 0$,

$$\frac{\nu(b)^k}{k!} \delta_b(l; t, k) \in \mathbb{Z}. \quad (7.38)$$

Proof. The polynomial Vandermonde identity says

$$\binom{l+Z}{n} = \sum_{j=0}^n \binom{l}{n-j} \binom{Z}{j}.$$

It follows by taking the coefficient of U^n in the formal identity $(1+U)^{l+Z} = (1+U)^l (1+U)^Z$; the coefficients up to any fixed degree are polynomial in Z , and generalized integer binomial coefficients $\binom{l}{m}$ are integral even for negative l . For $j \geq 1$,

$$\binom{Z}{j} = \frac{(-1)^{j-1} Z^{j-1}}{j} \prod_{i=1}^{j-1} (1 - Z/i).$$

Every contribution to its coefficient of Z^k has a denominator dividing a product of k integers from $1, \dots, j$. If $j \leq b$, multiplying by $\nu(b)^k$ makes that contribution integral. The constant coefficient is also integral. Consequently the coefficient of Z^k in $\binom{l+Z}{n}$, for $n \leq b$, has the same property. Products preserve it: in a term of total degree k , the factors $\nu(b)^{k_i}$ multiply to $\nu(b)^k$. Apply this to $\delta_b(l+Z; t)$, a product of binomial polynomials of orders b and $r \leq b$. Its coefficient of Z^k is exactly $\delta_b(l; t, k)/k!$. $\square$

Lemma (change of derivative basis). Replace the entries of M by

$$\widetilde{M}_{(k_0, k_1, l), (r, s, t)} = \binom{rb_2 + sb_1}{k_1} \frac{\nu(b)^{k_0}}{k_0!} \delta_b(l; t, k_0) \alpha_1^r \alpha_2^s. \quad (7.39)$$

The two matrices have the same rank. Their coefficient polynomials in α_1, α_2 have integral coefficients when b_1, b_2, r, s are integers.

Proof. Write $\delta_b(Z; t) = \sum_{v=0}^T q_{v,t} Z^v$. Its degree and leading coefficient make $Q = (q_{v,t})$ invertible. Differentiating and dividing by $k_0!$ gives

$$\frac{\delta_b(l; t, k_0)}{k_0!} = \sum_{v=0}^T q_{v,t} \binom{v}{k_0} l^{v-k_0}.$$

For each fixed pair (r, s) , multiply the block of its $T + 1$ derivative columns by Q . All these block operations are invertible. Next, for each fixed (k_0, l) , replace the rows with $X_1^{k_1}$ by those with $\binom{X_1}{k_1}$. This is a triangular invertible transformation on $0 \leq k_1 \leq K - k_0$, because the leading coefficient is $1/k_1!$. Finally multiply the row by the nonzero number $\nu(b)^{k_0}$. These operations produce exactly (7.39), proving equality of ranks. The binomial coefficient at an integer is integral, as is the divided derivative by (7.38). Expanding any minor therefore gives an integral polynomial in the two bases. $\square$

A factorial estimate for the triangular rows

Each coordinate has the same sum over $\mathcal{I}_K$:

$$\sum_{\mathcal{I}_K} k_0 = \sum_{\mathcal{I}_K} k_1 = \frac{KC_K}{3}.$$

Indeed $k_0, k_1, K - k_0 - k_1$ have identical distributions and sum to K . The following estimate retains the constant needed for the triangular determinant.

Lemma (triangular factorial product). For every integer $K \geq 1$,

$$\sum_{\mathcal{I}_K} \log(k_0!) \geq \frac{KC_K}{3} \left(\log K - \frac{11}{6} \right). \quad (7.40)$$

Proof. Put $S_K = KC_K/3 = \sum_{k=1}^K k(K+1-k)$. The elementary integral estimate $\log(k!) \geq k \log k - k$ reduces the assertion to

$$\frac{1}{S_K} \sum_{k=1}^K k(K+1-k) \log(k/K) \geq -5/6.$$

Here is a convenient exact comparison of this discrete average with an integral. Choose $u \in (0, 1)$ with density $6u(1-u)$, and, conditionally on u , let J be a binomial random variable with parameters $K-1, u$. The probability that $1+J = k$ is

$$6 \binom{K-1}{k-1} \int_0^1 u^k (1-u)^{K+1-k} du = \frac{k(K+1-k)}{S_K}.$$

For nonnegative integers a, c , the integral used here is $a!c!/(a+c+1)!$, obtained by repeated integration by parts. The conditional mean of the reciprocal satisfies

$$\mathbb{E} \left(\frac{K}{1+J} \middle| u \right) = \frac{1 - (1-u)^K}{u} \leq \frac{1}{u}.$$

To verify the equality, use $1/(j+1) = \int_0^1 z^j dz$ in the binomial sum. Concavity of the logarithm now gives

$$\mathbb{E} \left(\log \frac{1+J}{K} \middle| u \right) \geq -\log \mathbb{E} \left(\frac{K}{1+J} \middle| u \right) \geq \log u.$$

This application of concavity follows directly from $\log x \leq x - 1$, by normalizing a finite weighted mean; no probabilistic limit theorem is involved. Finally,

$$\int_0^1 6u(1-u) \log u \, du = 6(-1/4 + 1/9) = -5/6.$$

The identity $\int_0^1 u^a \log u \, du = -1/(a+1)^2$ follows by integration by parts, with the boundary term at zero equal to zero. Integrating the conditional inequality gives the required discrete inequality and then (7.40). The argument includes $K = 1$. $\square$

Counting derivative orders and bounding their product

Let $q \geq 1, T \geq 0$, and select $N \leq q(T+1)$ derivative orders $t_i \in \{0, \dots, T\}$, each used at most q times. Write $N = qm + r$, $0 \leq r < q$. Filling the largest available orders first maximizes their sum, so

$$\sum_i t_i \leq NT - \frac{qm(m-1)}{2} - rm \leq NT - \frac{N^2}{2q} + \frac{N}{2}. \quad (7.41)$$

For the first inequality, if a selected order is smaller than an available larger order, replacing it increases the sum. Repeating this finite operation gives q copies of $T, T-1, \dots, T-m+1$, followed by r copies of $T-m$. For the second, the difference between its right side and the first right side is $r(q-r)/(2q) \geq 0$. Define

$$\omega = 1 - \frac{N}{2q(T+1)}, \quad \omega_0 = \frac{2q(T+1)}{N}.$$

Then (7.41) implies $\sum_i t_i \leq N(\omega T + \omega_0)$. To check this, set $z = N/[q(T+1)] \in (0, 1]$. The difference between $\omega T + \omega_0$ and $T - N/(2q) + 1/2$ is $z/2 + 2/z - 1/2 > 0$. This direct proof also covers $T = 0$.

Take $K, L, T, b \geq 1$, and suppose the selected columns of an N -row minor of (7.39) satisfy this occupancy condition. Its rows contain every $(k_0, k_1, l) \in \mathcal{I}_K \times \{0, \dots, L\}$ once. The block size b may exceed T : in that case $\delta_b(Z; t) = \binom{Z}{t}$ for every column order $t \leq T$, and the same basis and integrality arguments apply. Put

$$H_b = \max\{L, b-1\}, \quad U = \omega T + \omega_0.$$

For any integer $l \in [0, L]$, differentiating the product of t linear factors defining $\delta_b(Z; t)$ gives

$$\frac{|\delta_b(l; t, k)|}{k!} \leq \binom{t}{k} \frac{H_b^{t-k}}{b^{\lfloor t/b \rfloor} (t \bmod b)!} \leq \binom{t}{k} \frac{H_b^{t-k}}{b^t} e^{t+b}.$$

The left side is zero for $k > t$. For the last inequality, the factorial integral bound gives, with $r = t \bmod b$,

$$\log(b^{\lfloor t/b \rfloor} r!) \geq t \log b - t - r \log(b/r) \geq t \log b - t - b.$$

At $r = 0$ the last term is zero; otherwise $r \log(b/r) \leq b/e < b$, by differentiating $x \log(b/x)$.

Consequently, writing t_i for the derivative order assigned to row i ,

$$\begin{aligned} \log \prod_i \frac{|\delta_b(l_i; t_i, k_{0,i})|}{k_{0,i}!} &\leq \frac{KN}{3} \log \frac{T}{KL} + \frac{11KN}{18} \\ &\quad + N \left\{ b + U \left(1 + \log \frac{H_b}{b} \right) \right\}. \end{aligned} \quad (7.42)$$

If a factor vanishes, read the left side as $-\infty$. Otherwise use $\binom{t_i}{k_{0,i}} \leq T^{k_{0,i}}/k_{0,i}!$, sum $k_{0,i} = KN/3$, apply (7.40) to its factorials, and replace H_b by L only in the negative term $-(KN/3) \log H_b$. The coefficient of $\sum_i t_i$ is $1 + \log(H_b/b) > 0$: for $b \geq 2$, $H_b/b \geq 1/2$, while for $b = 1$ it is at least one. Thus replacing that sum by NU is legitimate even when $\log(H_b/b)$ is negative.

The arithmetic length before specializing the parameters

Let $\Delta \neq 0$ be an N -row minor, with columns (r_j, s_j, t_j) , $0 \leq r_j \leq R$, $0 \leq s_j \leq S$. Put $W = R|b_2| + S|b_1| > 0$, and use $q = (R+1)(S+1)$ in the preceding occupancy bounds. There is an integral polynomial $P(X, Y)$ with $\Delta = P(\alpha_1, \alpha_2)$. Its coefficient length satisfies

$$\begin{aligned} \log \mathcal{L}(P) \leq & \log(N!) + \frac{KN}{3} \log \frac{W}{2K} + \frac{11KN}{9} + \frac{KN}{3} \log \frac{T}{KL} \\ & + \frac{KN}{3} \log \nu(b) + N \left\{ b + U \left(1 + \log \frac{H_b}{b} \right) \right\}. \end{aligned} \quad (7.43)$$

Proof. Center the additive coordinate at the midpoint of its range. For nonnegative b_1, b_2 take $\eta = W/2$; with arbitrary signs take the midpoint of $\{rb_2 + sb_1\}$, whose range has length at most W . For each fixed (k_0, l) , replacing $\binom{x}{k_1}$ by $(x - \eta)^{k_1}/k_1!$ is a triangular row operation with diagonal one. Hence it preserves this minor exactly, including its polynomial in X, Y . All centered arguments have absolute value at most $W/2$. Expand the resulting determinant by permutations. In each product the first additive coordinate contributes at most

$$(W/2)^{KN/3} \prod_i \frac{1}{k_{1,i}!} \leq \left(\frac{W}{2K} \right)^{KN/3} e^{11KN/18},$$

by (7.40). The factor $\nu(b)^{\sum_i k_{0,i}}$ is exactly $\nu(b)^{KN/3}$. The remaining derivative product is bounded by (7.42), for every permutation; its derivative orders still satisfy the same occupancy condition. There are $N!$ products. Triangle inequality for their coefficient lengths proves (7.43). Integrality was established before centering, so the possibly rational midpoint does not weaken the arithmetic conclusion. $\square$

The selected-place arithmetic lower bound

The derivative columns change the occupancy, but the centered-exponent proof in Section 5 still applies. For the distinct triples of a minor put

$$\begin{aligned} g &= \frac{1}{4} - \frac{N}{12(R+1)(S+1)(T+1)}, \\ G_1 &= \frac{N(L+1)(R+1)g}{2}, \quad G_2 = \frac{N(L+1)(S+1)g}{2}, \\ M_1 &= \frac{L}{2} \sum_j r_j, \quad M_2 = \frac{L}{2} \sum_j s_j. \end{aligned}$$

Then every permutation satisfies

$$\left| \sum_i (l_i - L/2) r_{\tau(i)} \right| \leq G_1, \quad \left| \sum_i (l_i - L/2) s_{\tau(i)} \right| \leq G_2. \quad (7.44)$$

Indeed, for the first bound encode (s, t) as the integer $s + (S+1)t$. The distinct triples become distinct points in a rectangle with side lengths $R+1$ and $(S+1)(T+1)$. Apply the centered-exponent lemma with its row repetition count K replaced by C_K and its number of levels L replaced by $L+1$. The list of centered levels is precisely $-L/2, \dots, L/2$, each repeated C_K times. Its value of g and first-coordinate bound are exactly those above. Encode (r, t) instead to obtain the second bound. There is no assumption that the projections (r, s) alone are distinct.

Let F denote the right side of (7.43). At the selected embedding suppose $|\alpha_j| \geq 1$, and write $D_* = d/\chi$ as in Section 6. The nonzero minor has the arithmetic bound

$$\begin{aligned} \log |\Delta| \geq & -(D_* - 1)F + (M_1 + G_1) \log |\alpha_1| + (M_2 + G_2) \log |\alpha_2| \\ & - 2D_* \{G_1 h(\alpha_1) + G_2 h(\alpha_2)\}. \end{aligned} \quad (7.45)$$

Proof. At a finite place, use the original integral entries (7.39) and expand the determinant. The coefficient factors have absolute value at most one. Equation (7.44) bounds the exponents in each product, so

$$\log |\Delta|_v \leq \sum_{j=1}^2 \{M_j \log |\alpha_j|_v + G_j |\log |\alpha_j|_v|\}.$$

At an archimedean place center the additive coordinate as in the proof of (7.43). That proof bounds the coefficient product of every permutation by $e^F/N!$. Insert its powers of the bases and use (7.44). Triangle inequality over the $N!$ permutations therefore adds just F to the preceding local bound. These bounds concern the same nonzero algebraic number Δ . Apply the product formula, excluding the selected embedding. The M_j terms sum to zero over all places, while the sums of absolute logarithms are $2dh(\alpha_j)$. The remaining archimedean weight is $d - \chi$. Divide by χ , and restore the excluded terms. Since $|\alpha_j| \geq 1$ at this embedding, its absolute logarithms equal its logarithms. The result is (7.45). $\square$

Taylor orders for the derivative determinant

Here the triangular monomials also control the analytic zero order. Fix a nonzero linear form w on $\mathbb{C}^2$, and let $f_i(X) = p_i(X)e^{l_i w(X)}$, where $\deg p_i \leq K$. Write $f_{i,d}$ for its homogeneous Taylor part of degree d . All the $f_{i,d}$, for a fixed d , lie in a vector space of dimension at most $K + 1$. For $d \leq K$, the entire space of homogeneous polynomials has dimension $d + 1$. For $d > K$, the Taylor expansion

$$f_{i,d} = \sum_{h=0}^K p_{i,h} \frac{l_i^{d-h}}{(d-h)!} w^{d-h}$$

is divisible by w^{d-K} ; the remaining homogeneous polynomial has degree K , so its space has dimension $K + 1$.

Select a set I of m rows of an N -square matrix. In these rows put the entries $\partial_{X_0}^{v_j} f_i(z\xi_j)$, with $0 \leq v_j \leq T$, and multiply entries of each remaining row by arbitrary constants $\theta_{i,j}$. The unselected row functions have the same entire form; their particular values will not matter. The determinant is entire in z and has order at least

$$\frac{m^2}{2(K+1)} - \frac{m}{2} - TN \geq \frac{m}{2} \left(\frac{m+1}{K+1} - \frac{K}{2} - 1 \right) - TN. \quad (7.46)$$

A negative lower bound places no restriction on an entire function.

Proof. Multiply column j by z^{v_j} . Its entries are analytic, and in a selected row its expansion is

$$z^{v_j} \partial_{X_0}^{v_j} f_i(z\xi_j) = \sum_{d \geq 0} z^d \partial_{X_0}^{v_j} f_{i,d}(\xi_j).$$

The differentiated homogeneous part is zero when $d < v_j$, so the formula has no negative powers. Expand the selected rows by multilinearity, leaving the other rows analytic. For any fixed d , the vectors $(\partial_{X_0}^{v_j} f_{i,d}(\xi_j))_{j=1}^N$ belong to an image of a space of dimension at most $K + 1$. A term with more than $K + 1$ selected rows of that degree therefore vanishes, even in the presence of the unselected rows. Among m nonnegative degrees with each value used at most $c = K + 1$ times, the least sum is obtained by using $0, 1, \dots$ in order. If $m = cq + r$, $0 \leq r < c$, it is

$$\frac{cq(q-1)}{2} + qr = \frac{m^2}{2c} - \frac{m}{2} + \frac{r(c-r)}{2c} \geq \frac{m^2}{2c} - \frac{m}{2}.$$

Undo the column factors; their total degree is $\sum_j v_j \leq TN$. This proves the first inequality of (7.46). The difference between its two displayed degree bounds is $m\{K/4 - 1/[2(K+1)]\} \geq 0$ for $K \geq 1$, proving the second. Convergence on compact sets justifies the Taylor expansions and the finite determinant operations. $\square$

We now apply this count to the minor of (7.39). Use chosen logarithms ℓ_j , put $\Lambda = b_2\ell_2 - b_1\ell_1$, and assume $b_1, b_2 > 0$ and $b_1|\ell_1| \leq b_2|\ell_2|$. Write $\beta = b_1/b_2$, $\lambda_i = l_i - L/2$, and center the additive coordinate at $\eta = (Rb_2 + Sb_1)/2$. Set

$$x_j = r_j + \beta s_j - \eta/b_2, \quad \Lambda_* = \Lambda \max \left\{ \frac{LS}{2b_2} e^{LS|\Lambda|/(2b_2)}, \frac{LR}{2b_1} e^{LR|\Lambda|/(2b_1)} \right\}.$$

The first term in the maximum gives $|e^{\lambda_i s_j \Lambda/b_2} - 1| \leq |\Lambda_*|$. Hence write this perturbation as $\Lambda_* \theta_{i,j}$, with $|\theta_{i,j}| \leq 1$; for $\Lambda = 0$, take $\theta_{i,j} = 0$.

Define an entire matrix with entries

$$C_{i,j}(z) = \frac{(zb_2x_j)^{k_{1,i}}}{k_{1,i}!} \frac{\nu(b)^{k_{0,i}}}{k_{0,i}!} \delta_b(l_i; t_j, k_{0,i}) e^{z\lambda_i x_j \ell_1}.$$

Let $\Phi_I(z)$ be its determinant with entries in rows outside I multiplied by $\theta_{i,j}$. Expanding the perturbations row by row, and using $\sum_i \lambda_i = 0$, gives

$$\Delta = e^{M_1\ell_1 + M_2\ell_2} \sum_{I \subseteq \{1, \dots, N\}} \Lambda_*^{N-|I|} \Phi_I(1).$$

This identity follows directly from $r_j\ell_1 + s_j\ell_2 = x_j\ell_1 + \eta\ell_1/b_2 + s_j\Lambda/b_2$. The midpoint factor is a row factor whose product is one.

The coefficient identity for δ_b expresses Φ_I as a finite sum of determinants of column derivatives of

$$\frac{(b_2X_1)^{k_{1,i}} \nu(b)^{k_{0,i}} X_0^{k_{0,i}}}{k_{1,i}! k_{0,i}!} e^{l_i(X_0 + \ell_1 X_1)} e^{-L\ell_1 X_1/2}$$

at $z(0, x_j)$. The last exponential is a common column factor, independent of X_0 , and is nonzero at $z = 0$. Apply (7.46) with $w = X_0 + \ell_1 X_1$ to each summand. Thus Φ_I has at least the second order in (7.46).

For $|z| = E > 1$, the coefficient-product proof of (7.43), with $|\theta_{i,j}| \leq 1$, gives

$$|\Phi_I(z)| \leq e^F E^{KN/3} \exp\{E(G_1|\ell_1| + G_2|\ell_2|)\}. \quad (7.47)$$

Indeed $|b_2x_j| \leq (Rb_2 + Sb_1)/2$, and each permutation's exponent is bounded by

$$|z| \left| \ell_1 \sum_i \lambda_i x_{\tau(i)} \right| \leq |z| \{G_1|\ell_1| + G_2\beta|\ell_1|\} \leq |z| \{G_1|\ell_1| + G_2|\ell_2|\},$$

using (7.44), the zero sum of the λ_i , and the ordering of the logarithm magnitudes.

Completing the comparison for a nonzero minor

Put $u = \log E$ and

$$V = \frac{K+2}{4} (L + \sqrt{L^2 - 1}) u.$$

Suppose $|\Lambda_*| \leq e^{-V}$. Schwarz's estimate at radii 1, E , applied to the order in (7.46), bounds a summand's logarithm, apart from the common right side of (7.47), by

$$-(N-m)V - \left\{ \frac{m^2}{2(K+1)} - \left(\frac{K+2}{4} - \frac{1}{2(K+1)} \right) m - TN \right\} u.$$

If the displayed order is negative, the same bound follows from maximum modulus, since replacing zero by a negative order only enlarges the upper bound. To maximize over m , let $v = V/u$, $c = K+1$, and $a = (K+2)/4 - 1/(2c)$. Completing the square gives an upper bound

$$-NV + \frac{cu}{2} (v+a)^2 + TNu.$$

With $a_0 = (K + 2)/4$, the definition of V satisfies $(v + a_0)^2 = (N/c)v$. Since $0 < a < a_0$, the preceding expression is at most $-NV/2 + TNu$. Summing the 2^N subsets therefore gives

$$\log |\Delta| \leq M_1 \log |\alpha_1| + M_2 \log |\alpha_2| + F + N \log 2 - NV/2 + (T + K/3)Nu + E(G_1|\ell_1| + G_2|\ell_2|).$$

Combine this with (7.45). If positive numbers a_j satisfy $a_j \geq E|\ell_j| - \log |\alpha_j| + 2D_*h(\alpha_j)$, then

$$\frac{V}{2} \leq \frac{D_*F}{N} + \log 2 + (T + K/3) \log E + \frac{g(L+1)}{2} \{(R+1)a_1 + (S+1)a_2\}. \quad (7.48)$$

Consequently, a strict reverse inequality in (7.48) proves $|\Lambda_*| > e^{-V}$, provided a nonzero full-row minor exists. This last proviso is precisely the separate multiplicity estimate; the analytic comparison itself is now complete for arbitrary chosen complex logarithms.

Keeping $\log \nu(b)$ exact makes these estimates directly computable without assuming a prime-distribution estimate. The derivative basis, its denominators, triangular factorial constant, occupancy bound, selected-place arithmetic estimate and analytic comparison all retain their explicit hypotheses. The exact-arithmetic checks verify finite derivative, basis, factorial, occupancy and Taylor-space examples, together with the quadratic identity for V . The arguments above supply the general conclusions.

15. The geometric zero estimate and full rank

The arithmetic and analytic estimates in Section 14 concern a nonzero minor. We now prove that the required minor exists. The argument is Gouillon's complete-intersection method, including the finite-torsion improvement of [Gouillon 2006]. We give the degree and derivative arguments explicitly. The proof works for any number of additive coordinates; its two-coordinate specialization is what the interpolation matrix needs.

The Noether normalization theorem proved in *Krull dimension and Noether normalization*, Corollary 3.2, applies to every nonzero finite-type k -algebra, including a nonreduced algebra. It contains an embedded polynomial algebra over which it is finite. Theorem 4.2 and the paragraph following its proof in Section 4 of that lesson show that the number of parameters is its Krull dimension. We use this full form below.

For a module over a Noetherian ring, the union of its associated primes is its zero-divisor set, by Theorem 1.2 of *Associated primes and primary decomposition*. The associated set of a finite module is finite by Theorem 2.2. Its localization is described in Theorem 3.1.

For a nonzero finite module over a Noetherian local ring, quotienting by a regular element in the maximal ideal lowers both its dimension and depth by one. This is Lemma 2.1 and Theorem 2.3 of *Regular sequences, depth and Cohen–Macaulay modules*. Consequently a regular quotient of a Cohen–Macaulay module is again Cohen–Macaulay. Theorem 4.1 proves that its associated primes are precisely its minimal support primes, and all corresponding quotient dimensions equal its support dimension.

The passage from a module over a local ring to the quotient ring acting faithfully on it introduces no additional assumption. Under a local surjection $B \rightarrow C$, every sequence in the maximal ideal of C lifts to one in that of B , with exactly the same successive actions on a C -module; hence their depths agree. Its supports correspond under the closed embedding $\text{Spec } C \hookrightarrow \text{Spec } B$, with the same chains of primes, so their dimensions agree as well. This proves the quotient-ring compatibility of the Cohen–Macaulay condition directly.

The coordinate ring of H described in (8.57) is a finite product of polynomial Laurent rings. A field is Cohen–Macaulay of dimension zero, and polynomial rings over it are Cohen–Macaulay by Theorem 5.3. Laurent rings inherit the property by localization. A prime of a finite product belongs to one factor: the orthogonal factor idempotents sum to one, so exactly one survives in the domain

quotient by that prime. Its local ring is the corresponding factor's local ring. Thus the product is Cohen–Macaulay. The subgroup description also shows that all these factors have dimension $\dim H$.

Let B be an equidimensional finite-type Cohen–Macaulay k -algebra of dimension D , and let f be a nonzerodivisor with $B/fB \neq 0$. Every maximal localization $B_{\mathfrak{m}}$ has dimension D : each component through that closed point has dimension D , and Theorem 6.1 of *Krull dimension and Noether normalization* applies with residue transcendence degree zero. At every maximal ideal containing f , its regular quotient is therefore Cohen–Macaulay of dimension $D - 1$. All prime localizations of that quotient are Cohen–Macaulay by Theorem 5.1, proving that B/fB is Cohen–Macaulay.

For any minimal prime $\mathfrak{p}$ of B/fB , choose a maximal ideal $\mathfrak{m}$ containing it. The localized prime $\mathfrak{p}_{\mathfrak{m}}$ is minimal, so the associated-component theorem gives

$$\dim(B/\mathfrak{p})_{\mathfrak{m}} = D - 1.$$

Here $\mathfrak{p}$ also denotes its inverse image in B . The algebra $B/\mathfrak{p}$ is a finite-type domain, so Corollary 4.4 of *Krull dimension and Noether normalization* identifies this local dimension with $\dim B/\mathfrak{p}$. Every component of B/fB therefore has dimension $D - 1$, proving equidimensionality.

The associated primes of any Cohen–Macaulay ring are globally minimal as well. For $\mathfrak{p} \in \text{Ass}(B)$, choose $\mathfrak{m} \supseteq \mathfrak{p}$. Localization makes $\mathfrak{p}_{\mathfrak{m}}$ associated to the Cohen–Macaulay local ring $B_{\mathfrak{m}}$, hence minimal by the associated-component theorem. A smaller prime of B would remain strictly smaller after localization at $\mathfrak{m}$, a contradiction. These are the exact facts used in selecting the subsequent regular equations.

Starting with a polynomial Laurent ring, apply the preceding regular-quotient argument successively to a regular sequence. Each proper quotient remains Cohen–Macaulay and equidimensional, its dimension falls by one at each step, and its associated primes remain minimal. These are the precise algebraic facts used below.

Two degrees for the polynomial spaces

Put

$$G = \mathbb{C}^m \times \mathbb{C}^{\times}, \quad R = \mathbb{C}[X_0, \dots, X_{m-1}, Y, Y^{-1}], \quad \mathcal{D} = \partial_{X_0} + Y\partial_Y.$$

Translation by $g = (a, b)$ substitutes $X + a, bY$. It commutes with $\mathcal{D}$ and preserves the total degree in X and the degree in Y . Write the group law additively, including its multiplicative last coordinate.

For a proper ideal $I \subset R$, let

$$h_I(u, v) = \dim_{\mathbb{C}} \text{im}\{P \in \mathbb{C}[X, Y] : \deg_X P \leq u, \deg_Y P \leq v\} \longrightarrow R/I.$$

Here only nonnegative powers of Y occur in the filtered space. The ambient ring still allows Y^{-1} , so it describes the open group rather than introducing a component at $Y = 0$.

Lemma (the two leading degrees). If $d = \dim(R/I) \geq 1$, then $h_I(u, v)$ is a polynomial for sufficiently large integers u, v . Its leading homogeneous part is

$$p_{I,d}(u, v) = A(I) \frac{u^{d-1}v}{(d-1)!} + B(I) \frac{u^d}{d!}, \quad A(I), B(I) \in \mathbb{Z}_{\geq 0}. \quad (7.49)$$

At least one coefficient is positive. If $d = 0$, its eventual value is the positive integer $B(I)$; set $A(I) = 0$ in this case. A fixed shift of either filtration bound does not change the leading part.

Proof. Let $R_0 = \mathbb{C}[X, Y]$ and $I_0 = I \cap R_0$. The filtered image is the corresponding space in R_0/I_0 . This algebra has dimension d . Indeed, its localization at Y is R/I , and every minimal prime of I_0 avoids Y . To justify the latter assertion, Y is a nonzerodivisor on R_0/I_0 , since that algebra embeds in R/I , where Y is invertible. A minimal prime is associated, and therefore cannot contain that nonzerodivisor. Localization leaves the fraction field of each minimal-prime domain

unchanged. The finite-type dimension theorem, obtained from Noether normalization, identifies each such dimension with its transcendence degree. Thus localization does not change the maximum component dimension.

Homogenize the additive variables in a block $(U_0, X_0, \dots, X_{m-1})$, and Y in a block (V_0, Y) . For every pair (u, v) , evaluate a bihomogeneous polynomial at $U_0 = V_0 = 1$ and reduce modulo I_0 . Its kernel, summed over all bidegrees, is a bihomogeneous ideal. The quotient's (u, v) -piece has dimension exactly $h_I(u, v)$.

Here is a direct polynomial count for that quotient. Choose a monomial order and the ideal of leading monomials of its homogeneous kernel. This monomial ideal has finitely many minimal generators. The elementary reason is that an infinite sequence of nonnegative integer vectors has an infinite subsequence nondecreasing in every coordinate: in one coordinate either a value repeats infinitely or one can successively choose increasing values, and this can be repeated over the finitely many coordinates. An infinite antichain of minimal exponent vectors is consequently impossible. Choose a homogeneous polynomial for each minimal leading monomial. Division by this finite list preserves bidegree. Monomials outside the leading ideal form a basis: they span by division, and a nonzero relation would have a leading monomial inside that ideal.

The number of monomials of bidegree (u, v) divisible by a fixed monomial of block degrees (a, b) is $\binom{u-a+m}{m}(v-b+1)$ when the bounds are large enough. Inclusion–exclusion over the finite leading-monomial list, using its least common multiples, gives a polynomial in u, v . Its degree in v is at most one and its degree in u at most m , either from this formula or from the ambient bound $h_I(u, v) \leq \binom{u+m}{m}(v+1)$.

Its total degree is d . For completeness, the ordinary generator filtration of R_0/I_0 grows above and below by positive multiples of t^d . Noether normalization embeds a polynomial algebra in d variables over which R_0/I_0 is finite. Its independent monomials give the lower bound, since its generators have bounded ordinary degree. For the upper bound choose finitely many module generators. Multiplication by any original algebra generator is expressed by a finite matrix of polynomial coefficients of bounded degree. A product of t generators has coefficients of degree at most $ct + c'$, and hence spans only $O(t^d)$ dimensions. This argument also applies to nilpotents. For every fixed positive integer pair (u, v) , the filtration at (tu, tv) contains the ordinary filtration at $t \min(u, v)$ and is contained in that at $t(u + v)$. Its growth is therefore of order t^d . A putative higher leading part would vanish on every positive integer pair, and hence identically; a polynomial cannot vanish on that whole grid unless it is zero. A lower degree would contradict the lower growth bound. The top part is positive at positive pairs.

Since its degree in v is at most one, the top part has precisely the form (7.49). Positivity for every positive ratio v/u , first rational and then real, implies $A, B \geq 0$. They are integers: $\Delta_u^{d-1} \Delta_v p_I = A$ and $\Delta_u^d p_I = B$, and these differences are integer combinations of eventual integer values. For dimension zero the algebra is finite-dimensional, and sufficiently large bounds exhaust it. Fixed shifts affect only lower total degrees. $\square$

For an irreducible variety $V \subset G$, abbreviate these coefficients by $A(V), B(V)$, using its vanishing ideal. Translation gives the same coefficients because it is an invertible map of every filtered space. For the whole group, $h_0(u, v) = \binom{u+m}{m}(v+1)$, so $A(0) = 1, B(0) = 0$.

Lemma (regular cuts). Suppose $I_j = (f_1, \dots, f_j) \subset R$ is proper, its generators form a regular sequence, and each f_i has the two degree bounds (K, L) . Then its quotient has dimension $m + 1 - j$, and

$$A(I_j) \leq K^j \quad (0 \leq j \leq m), \quad B(I_j) \leq jK^{j-1}L \quad (1 \leq j \leq m+1). \quad (7.50)$$

Proof. The cited Cohen–Macaulay facts give the dimension and purity. Multiplication by the next regular element $f = f_{j+1}$ injects the filtered image of bounds (u, v) in R/I_j into that of bounds

$(u + K, v + L)$. Its image is in the kernel of the map to $R/(I_j, f)$. Hence

$$h_{I_{j+1}}(u + K, v + L) \leq h_{I_j}(u + K, v + L) - h_{I_j}(u, v).$$

If the old dimension is d , substitute (tu, tv) and compare the degree- $d - 1$ terms. The upper leading polynomial is $K\partial_u p_{I_j, d} + L\partial_v p_{I_j, d}$. For $d \geq 2$, taking the positive ratio v/u arbitrarily large or small gives

$$A(I_{j+1}) \leq KA(I_j), \quad B(I_{j+1}) \leq KB(I_j) + LA(I_j).$$

For $d = 1$, only the second, constant inequality remains. Starting from $A(0) = 1, B(0) = 0$ proves (7.50) by induction. This proves the needed mixed-degree intersection bound without assuming that affine intersections retain every component of a projective closure. $\square$

Counting translates and transverse derivatives

Let $V_1, \dots, V_s$ be distinct irreducible varieties of the same dimension d , and let $J = I(V_1 \cup \dots \cup V_s)$. Then their leading polynomials add:

$$p_{J, d} = \sum_{i=1}^s p_{I(V_i), d}. \quad (7.51)$$

Indeed, restriction injects each filtered quotient for the union into the sum of the component quotients. Conversely, for each i choose a polynomial c_i vanishing on every other component and not identically on V_i . Such a polynomial is a product of choices in $I(V_h) \setminus I(V_i)$, $h \neq i$. The distinct prime ideals are incomparable because their dimensions agree. Laurent choices can be multiplied by powers of Y to make them polynomials. Multiplying representatives of the i -th quotient by c_i embeds their direct sum into the union quotient after a fixed degree shift: on V_i only its own term survives, and c_i is a nonzero element of a domain. The two inequalities have the same leading parts. This proves (7.51), even when the components intersect.

Lemma (one transverse derivative). Suppose every element of an ideal $I \subset R$ vanishes, with all its $\mathcal{D}$ -derivatives through order T , on $Z = V_1 \cup \dots \cup V_s$. Assume $\dim(R/I) = d = \dim V_i$, and that none of the ideals $I(V_i)$ is preserved by $\mathcal{D}$. Then

$$A(I) \geq (T + 1) \sum_i A(V_i), \quad B(I) \geq (T + 1) \sum_i B(V_i). \quad (7.52)$$

For $d = 0$ only the second inequality is needed. Without the transverse assumption the same assertion holds with factor one.

Proof. Choose $a_i \in I(V_i)$ with $\mathcal{D}a_i \notin I(V_i)$, and choose the c_i just used in (7.51). Set $a = \sum_i c_i^2 a_i$. It vanishes on every component. On V_i , $\mathcal{D}a = c_i^2 \mathcal{D}a_i \neq 0$: derivatives of the other summands still contain a factor vanishing there. Thus $\mathcal{D}a$ is a nonzerodivisor modulo the reduced ideal $J = I(Z)$. To see this without any primarity assumption, if $(\mathcal{D}a)P$ vanishes on the union, its restriction to each component domain forces $P = 0$ there, and hence $P \in J$.

Choose a complement E to J in the polynomial space of bounds (u, v) . The $(T + 1) \dim E$ products Pa^k , P in a basis of E , $0 \leq k \leq T$, are independent modulo I . For otherwise choose the smallest index k with nonzero coefficient $P_k \in E$ in a relation $\sum_k P_k a^k \in I$, and apply $\mathcal{D}^k$. Modulo J , every higher-index term vanishes, whereas the index k term is $k! P_k (\mathcal{D}a)^k$. Every Leibniz term other than this one contains an undifferentiated a . The derivative of the relation lies in J by the hypothesis on I . Characteristic zero and the nonzerodivisor property give $P_k \in J$, a contradiction.

If (a_X, a_Y) bounds the degrees of a , this proves

$$h_I(u + Ta_X, v + Ta_Y) \geq (T + 1)h_J(u, v).$$

Compare leading terms at positive dilates and use (7.51). Their difference is nonnegative for every positive ratio v/u , so each of its two coefficients is nonnegative. In dimension zero the same

comparison is between constants. With no derivatives, the filtered restriction surjection proves the factor-one assertion. $\square$

In applications it suffices to check this derivative hypothesis for generators of I . Leibniz's rule extends it to their polynomial Laurent multiples and sums. This proof measures multiplicity in one transverse direction; it does not assume vanishing in every normal direction to a component.

Stabilizers and the finite torus factor

For an irreducible nonempty $V \subset G$, its stabilizer $H_V = \{g : g + V = V\}$ is a closed algebraic subgroup. It is closed because the condition $g + V \subset V$ is the common vanishing of $P(g + v)$, $P \in I(V)$, $v \in V$; the equal-dimensional irreducible inclusion is equality. The group properties follow from translation and inversion.

Lemma (subgroups of one torus factor). Every closed subgroup of $\mathbb{C}^m \times \mathbb{C}^\times$ is $W \times \mathbb{C}^\times$ or $W \times \mu_n$, where W is a vector subspace and μ_n consists of all n -th roots of unity, $n \geq 1$.

Proof. The Zariski closure W of its additive projection is an additive subgroup. Addition and inversion preserve the closure by applying density successively in each variable. For $w \in W$, every defining polynomial vanishes on tw for integer t . It therefore vanishes for every complex t . Closure under sums makes W a vector subspace.

Let $\Phi = \{a \in \mathbb{Z} : Y^a|_H = 1\}$, either 0 or $n\mathbb{Z}$. Combine a finite Laurent polynomial's exponents modulo Φ , so its terms are $Y^a p_a(X)$ with distinct characters on H . Such terms cannot cancel on H . To prove this, let h bound their polynomial degrees. Translation T_g on block a is the character y_g^a times translation of the polynomial. Its difference from that scalar is nilpotent of order $h+1$, because finite differences reduce degree. Fix block a . For each other block b , choose $g_b \in H$ distinguishing their characters and apply $\prod_{b \neq a} (T_{g_b} - y_{g_b}^b)^{h+1}$. The commuting factors annihilate all other blocks and are invertible on block a : a nonzero scalar plus a nilpotent operator has a finite geometric-series inverse. These operators and that inverse are polynomials in translations, and preserve vanishing on H . Thus every p_a vanishes on the additive projection, hence on W . It follows that $I(H)$ is generated by $I(W)$ and the character relations $Y^a - 1$, $a \in \Phi$. Its zero set is exactly one of the two stated products. $\square$

If $\mathcal{D}I(V) \subseteq I(V)$, then $(te_0, e^t) + V = V$ for every $t \in \mathbb{C}$, where $e_0 = (1, 0, \dots, 0)$. Indeed, the Taylor derivatives of $P(X + te_0, e^t Y)$ at zero are $\mathcal{D}^r P$, which vanish on V ; the function is entire. Its zero Taylor series makes it zero for all t , and translation by $-t$ gives equality. The Zariski closure of this one-parameter group is $\mathbb{C}e_0 \times \mathbb{C}^\times$. For if a Laurent polynomial vanished on it, it would give $\sum_a p_a(t) e^{at} = 0$. The largest exponent's nonzero polynomial cannot tend to zero as real $t \rightarrow +\infty$, while after division by its exponential every smaller-exponent term does. Removing that term successively proves that all coefficients were zero. Consequently

$$\mathcal{D}I(V) \subseteq I(V) \iff H_V = W \times \mathbb{C}^\times \text{ with } e_0 \in W. \quad (7.53)$$

The converse follows by differentiating the translations by $(te_0, 1)$ and $(0, e^t)$, which both preserve V .

If $d = \dim V$ and $H_V = W \times \mathbb{C}^\times$, then $V = U \times \mathbb{C}^\times$ for an irreducible additive variety U . Indeed, invariance under all scalings of Y makes each Laurent coefficient of a defining polynomial vanish separately; its ideal is extended from $\mathbb{C}[X]$. Thus $\dim U = d - 1$, $h_V(u, v) = h_U(u)(v + 1)$, and $A(V) \geq 1$. The single-block version of the Hilbert argument gives the positive integer leading degree of U . Also $\dim W \leq d - 1$, since a translate of H_V lies in V .

If $H_V = W \times \mu_n$, then

$$\dim W \leq d, \quad B(V) \geq n. \quad (7.54)$$

Here is the degree proof, including the full finite-torsion factor. Let U be the closure of the additive projection and $F = \mathbb{C}(U)$. The field of V is $F(Y)$. If Y were transcendental over F , the vanishing ideal would have no relation beyond $I(U)$, so $V = U \times \mathbb{C}^\times$. That would contradict its finite torus

stabilizer. Hence Y is algebraic of degree $q = [F(Y) : F]$, and $\dim U = d$. Every $\zeta \in \mu_n$ gives a field automorphism fixing F and carrying Y to ζY . These are distinct roots of its minimal polynomial, so $q \geq n$. For any fixed sufficiently large $v \geq q - 1$, the powers $1, Y, \dots, Y^{q-1}$ are independent over F . Multiplying them by an additive polynomial basis on U gives $h_V(u, v) \geq qh_U(u)$. Compare the coefficient of u^d : by (7.49) it is $B(V)/d!$, while the right side has coefficient $q \deg(U)/d!$. The positive integer $\deg(U)$ is at least one, proving (7.54). The dimension assertion again follows from a stabilizer orbit in V . In dimension zero, V is a point, its stabilizer is trivial, and $B(V) = 1$; the same conclusions hold directly.

For example, the curve $X_1 = 0$, $X_0 = Y^n$ in $\mathbb{C}^2 \times \mathbb{C}^\times$, $n \geq 2$, has stabilizer $\{0\} \times \mu_n$. Its filtered monomials are powers Y^{na+c} , $0 \leq a \leq u, 0 \leq c \leq v$. When $v \geq n - 1$, their exponent intervals join, giving $h_V(u, v) = nu + v + 1$. Thus $A(V) = 1, B(V) = n$: the finite-torsion factor is attained, rather than an artefact of the bound.

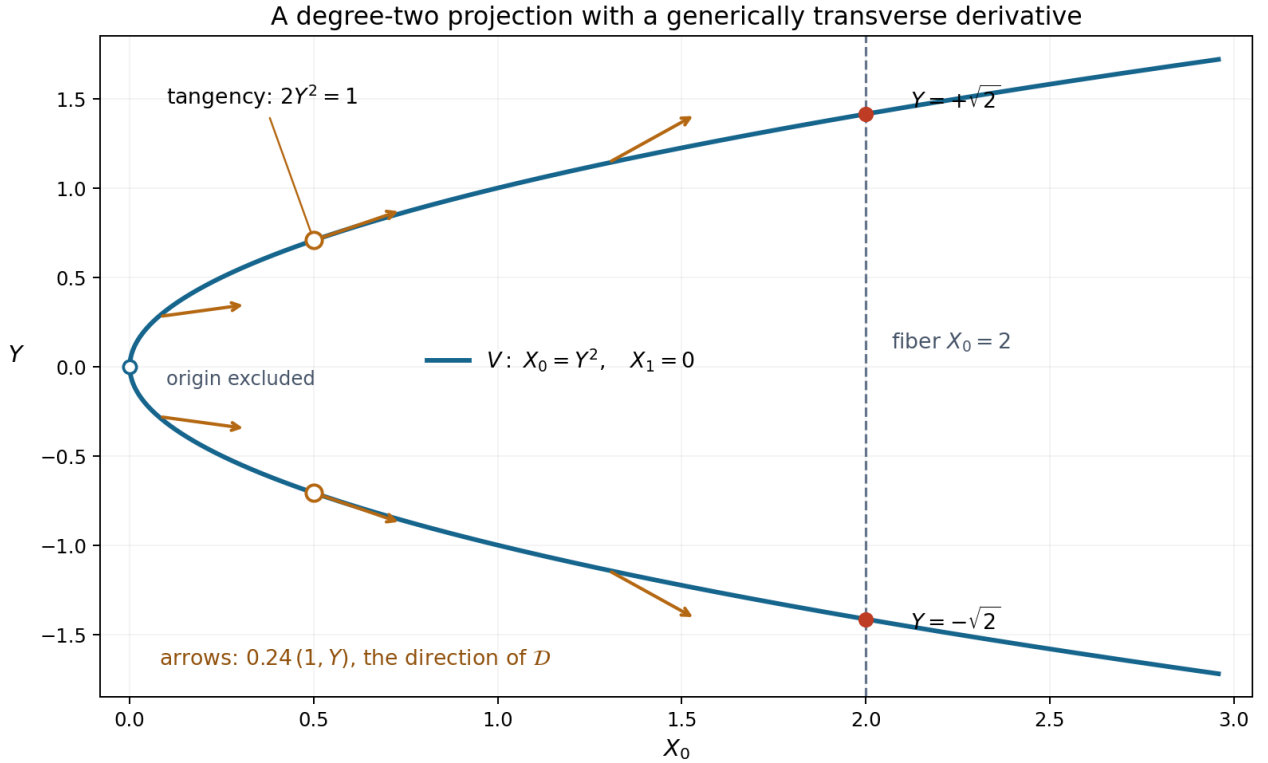


Figure.

A real slice of $X_1 = 0, X_0 = Y^2, Y \neq 0$, the example in (7.54). The projection to $X_0 = 2$ has two points, interchanged by $Y \mapsto -Y$; its complex stabilizer has torus factor μ_2 . The orange arrows are exactly $0.24(1, Y)$, a positive multiple of $\mathcal{D} = \partial_{X_0} + Y\partial_Y$ in this plane. For $Q = X_0 - Y^2$, $\mathcal{D}Q = 1 - 2Y^2$, so tangency occurs at the two marked points with $Y^2 = 1/2$. Elsewhere the direction is transverse. The proof of (7.52) needs a derivative nonzero on each component as a polynomial function; exceptional points do not alter its leading-degree count. The hollow origin is excluded. Original figure by GPT-6.1 Sol (OpenAI), Codex, Ultra; CC0. Reproducible figure source.

The zero theorem

For a subgroup H , $\#(\Sigma/H)$ means the number of its cosets meeting the finite set Σ .

Theorem 7.10 (Gouillon's zero estimate). Let $m, K, L \geq 1$, let $T_1, \dots, T_{m+1} \geq 0$ be integers, and let $\Sigma_1, \dots, \Sigma_{m+1} \subset G$ be finite and nonempty. For every $1 \leq j \leq m$ and every vector subspace $W \subset \mathbb{C}^m$ with $\dim W \leq m - j$, suppose

$$(T_j + 1)^{\epsilon(W)} \#(\Sigma_j / (W \times \mathbb{C}^\times)) > K^j, \quad \epsilon(W) = \begin{cases} 0, & e_0 \in W, \\ 1, & e_0 \notin W. \end{cases}$$

For every $1 \leq j \leq m+1$ and every such subspace with $\dim W \leq m+1-j$, suppose

$$(T_j + 1)\#(\Sigma_j/(W \times \{1\})) > jK^{j-1}L.$$

Then a polynomial of total additive degree at most K and Y -degree at most L , vanishing on $\Sigma_1 + \dots + \Sigma_{m+1}$ with all $\mathcal{D}$ -derivatives through order $T_1 + \dots + T_{m+1}$, is identically zero.

Proof. First ensure that every Σ_j contains the identity. Choose $a_j \in \Sigma_j$, replace the sets by $\Sigma_j - a_j$, and replace P by its translate by $a_1 + \dots + a_{m+1}$. This changes neither degree bounds, derivative orders nor any quotient cardinality in the hypotheses. It preserves the proposed vanishing and nonvanishing of P . We may therefore use the nested tail sets $\Sigma_j + \dots + \Sigma_{m+1}$.

Suppose $P \neq 0$. We construct polynomials $f_1 = P, f_2, \dots, f_{m+1}$ of degree bounds (K, L) , forming a regular sequence in R , such that f_i vanishes on its tail set with derivatives through $T_i + \dots + T_{m+1}$. The first ideal is proper because P has a zero in the group; P is a nonzerodivisor because R is a domain.

Suppose we have reached $I_j = (f_1, \dots, f_j)$, $j \leq m$, and consider the finite family

$$\mathcal{F}_j = \{(\mathcal{D}^t f_i) \circ T_\tau : i \leq j, 0 \leq t \leq T_j, \tau \in \Sigma_j\}.$$

No irreducible component V of the zero set of I_j can contain all of $\mathcal{F}_j$. To prove this, assume the contrary for one component of I_j , and put $H = H_V$. All f_i and their derivatives through T_j vanish on the translates $V + \tau$, $\tau \in \Sigma_j$. These are components of I_j : they are contained in its zero set and have its pure dimension $d = m+1-j$. Distinct translates are indexed by the $s = \#(\Sigma_j/H)$ cosets.

If $H = W \times \mathbb{C}^\times$, we have $\dim W \leq m-j$ and $A(V) \geq 1$. When $e_0 \in W$, the factor-one part of (7.52), together with (7.50), gives $s \leq K^j$. When $e_0 \notin W$, (7.53) makes every translate transverse, and gives $(T_j + 1)s \leq K^j$. Each contradicts the corresponding first hypothesis.

If $H = W \times \mu_n$, it is transverse by (7.53), $\dim W \leq m+1-j$, and $B(V) \geq n$. Thus $(T_j + 1)sn \leq B(I_j) \leq jK^{j-1}L$. Each H -coset contains at most n distinct $W \times \{1\}$ -cosets, so $\#(\Sigma_j/(W \times \{1\})) \leq sn$. This contradicts the second hypothesis. These are all subgroup cases.

Choose a complex linear combination f_{j+1} of $\mathcal{F}_j$ outside every component prime of I_j . Such a choice exists: for each of the finitely many primes the bad coefficient choices form a proper linear subspace, and a finite union of proper linear subspaces does not cover a vector space over an infinite field. One direct proof uses coefficient vectors $(1, t, \dots, t^{a-1})$; each prime excludes only finitely many values of t . There are no embedded associated primes, by the regular-sequence Cohen–Macaulay property. Thus f_{j+1} is a nonzerodivisor.

It has the required tail vanishing. In fact $i \leq j$, the preceding sets contain the identity, and differentiation by at most T_j leaves at least $T_{j+1} + \dots + T_{m+1}$ of the original order budget. Translating by Σ_j sends the remaining tail into the tail of f_i . The combination therefore vanishes on the remaining tail. This nonempty set also keeps I_{j+1} proper. Its dimension and degrees are controlled by (7.50), so induction continues.

Finally I_{m+1} has dimension zero and $B(I_{m+1}) \leq (m+1)K^mL$. Every generator and all its derivatives through T_{m+1} vanish on Σ_{m+1} . A point's ideal is transverse: if its first additive coordinate is a_0 , then $\mathcal{D}(X_0 - a_0) = 1$. Apply (7.52) to these distinct points, whose B -degrees are one:

$$(T_{m+1} + 1)\#\Sigma_{m+1} \leq B(I_{m+1}) \leq (m+1)K^mL.$$

This contradicts the second hypothesis with $j = m+1, W = 0$. The supposed nonzero polynomial cannot exist. $\square$

The identity normalization is what permits the nested tails in the induction. It places no additional restriction on the theorem's finite sets. The finite-torsion degree in (7.54) is precisely what permits $W \times \{1\}$ in the second hypothesis.

Applying the theorem to the triangular matrix

Theorem 7.11 (full row rank). Let $K, L \geq 1$ and let $R_j, S_j, T_j \geq 0$, $j = 1, 2, 3$, be integers. Put $R = \sum_j R_j, S = \sum_j S_j, T = \sum_j T_j$. In the following sets, r, s range over the integers $0 \leq r \leq R_j, 0 \leq s \leq S_j$:

$$\begin{aligned}\mathcal{X}_j &= \{rb_2 + sb_1\}, \\ \mathcal{Y}_j &= \{\alpha_1^r \alpha_2^s\}.\end{aligned}$$

with $\mathcal{Z}_j$ the corresponding set of pairs (x, y) . Assume

$$\begin{aligned}T_1 &\geq K, \\ \#\mathcal{X}_1 &\geq K + 1, \\ (T_1 + 1)\#\mathcal{Y}_1 &\geq L + 1, \\ (T_2 + 1)\#\mathcal{Y}_2 &\geq 2KL + 1, \\ (T_2 + 1)\#\mathcal{X}_2 &\geq K^2 + 1, \\ (T_3 + 1)\#\mathcal{Z}_3 &\geq 3K^2L + 1.\end{aligned}\tag{7.55}$$

Then the matrix in (7.37), and its integral-basis version (7.39), have row rank $C_K(L + 1)$.

Proof. Take $m = 2$ and $\Sigma_j = \{(0, x, y) : (x, y) \in \mathcal{Z}_j\}$. These sets contain the identity, and their sum is contained in the full rectangle of evaluation points with bounds (R, S) . For the first hypothesis of Theorem 7.10 at $j = 1$, a line W containing e_0 is exactly $\mathbb{C}e_0$; its quotient distinguishes all $\mathcal{X}_1$. If W does not contain e_0 , it either equals $\mathbb{C}e_1$, in which case $T_1 + 1 \geq K + 1$ suffices, or meets $\mathbb{C}e_1$ trivially, in which case its quotient distinguishes the $\mathcal{X}_1$ values. The same argument covers $W = 0$. At $j = 2$ only $W = 0$ is allowed, and the fifth inequality in (7.55) gives the required $(T_2 + 1)\#\mathcal{X}_2 > K^2$. For the second hypothesis, any $W \times \{1\}$ -quotient retains the multiplicative coordinate, so its cardinality is at least $\#\mathcal{Y}_j$. The third and fourth inequalities suffice at $j = 1, 2$. At $j = 3, W = 0$, the quotient retains the whole pair, and the final inequality supplies its strict bound. Thus every condition of the zero theorem is verified.

A linear dependence of the rows would give a nonzero polynomial $P = \sum c_{k_0, k_1, l} X_0^{k_0} X_1^{k_1} Y^l / k_0!$ of the required degrees, with all derivatives through T zero on the rectangle. The zero theorem forces $P = 0$, contradicting independence of its monomials. The first matrix has full rank, and the derivative-basis lemma transfers that rank to (7.39). $\square$

Theorem 7.12 (a determinant criterion with an exact denominator). Use the algebraic cancellation setting of Section 14: $b_1, b_2 > 0$, $|\alpha_j| \geq 1$, chosen logarithms $b_1|\ell_1| \leq b_2|\ell_2|$, and $D_* = [\mathbb{Q}(\alpha_1, \alpha_2) : \mathbb{Q}]/\chi$, where $\chi = 1$ for the chosen real place and $\chi = 2$ for the chosen complex place. Suppose (7.55) holds. For any integer $b \geq 1$, use the exact $\nu(b)$ and F in (7.43). Let $E > 1$, and use g, V, Λ^* from Section 14. Choose $a_j \geq E|\ell_j| - \log |\alpha_j| + 2D_*h(\alpha_j)$. If

$$\frac{V}{2} > \frac{D_*F}{N} + \log 2 + (T + K/3) \log E + \frac{g(L + 1)}{2}((R + 1)a_1 + (S + 1)a_2),\tag{7.56}$$

then $|\Lambda^*| > e^{-V}$.

Proof. Theorem 7.11 supplies a nonzero full-row minor. In particular $N \leq (R + 1)(S + 1)(T + 1)$, as used in the centered counts. All hypotheses of the arithmetic and analytic comparison (7.48) are now established. If $|\Lambda^*| \leq e^{-V}$, that comparison contradicts (7.56). $\square$

This criterion retains the denominator exactly and can be checked with integer lcm arithmetic. Its full-rank assertion requires no multiplicative independence beyond the actual cardinality conditions. The exact geometric examples check the finite-torsion degree, a transverse thickening, crossing components, translation normalization and a full-rank matrix with dependent bases. These examples supplement the general proofs above.

An exponential bound for least common multiples

Lemma. For every integer $H \geq 1$,

$$\text{lcm}(1, \dots, H) < 3^H.$$

Proof. Write $d_H = \text{lcm}(1, \dots, H)$. The factorial mechanism below is due to [Hanson 1972].

Set $a_1 = 2$ and $a_{j+1} = a_j(a_j - 1) + 1$. Induction gives

$$a_{k+1} - 1 = \prod_{j=1}^k a_j, \quad \sum_{j=1}^k \frac{1}{a_j} = 1 - \frac{1}{a_{k+1} - 1} < 1.$$

For every real $x \geq 1$,

$$\sum_{j=1}^k \left\lfloor \frac{x}{a_j} \right\rfloor = \sum_{j=1}^k \left\lfloor \frac{\lfloor x \rfloor}{a_j} \right\rfloor < \lfloor x \rfloor.$$

The last sum is an integer, so it is at most $\lfloor x \rfloor - 1$. For $n \geq 2$, choose k by $a_k \leq n < a_{k+1}$, and put

$$m_j = \lfloor n/a_j \rfloor, \quad C_n = \frac{n!}{\prod_{j=1}^k m_j!}.$$

This is an integer, since $\sum m_j < n$. Legendre's formula and the preceding floor inequality show that $v_p(C_n) \geq 1$ for each power $p^b \leq n$, counted separately. Thus $v_p(C_n) \geq v_p(d_n)$, and $d_n \mid C_n$.

Let $r = n - \sum m_j \geq 1$. The term of the multinomial expansion with exponents $m_1, \dots, m_k, r$, applied to $n = m_1 + \dots + m_k + r$, gives

$$C_n \leq \frac{n^n r!}{r^r \prod_j m_j^{m_j}} \leq \frac{n^n}{\prod_j m_j^{m_j}}.$$

For $y \geq 1$, the derivative of $y \log y$ gives

$$y \log y - \lfloor y \rfloor \log \lfloor y \rfloor \leq \log y + 1.$$

Apply this with $y = n/a_j$. Using $a_{k+1} - 1 \geq n$, we obtain

$$\log C_n \leq n \sum_{j=1}^k \frac{\log a_j}{a_j} + (k+1) \log n + k.$$

Also $a_j \geq 2^{j-1} + 1$ for $j \geq 2$, by induction in the recurrence, so $k \leq 1 + (\log n)/\log 2$.

The convergent entropy sum $w = \sum_{j \geq 1} (\log a_j)/a_j$ has a particularly short exact bound. Since $a_{j+1} < a_j^2$ and $a_{j+1} > a_j(a_j - 1)$, its successive terms have ratio less than $2/(a_j - 1)$. The first six entries are

$$2, 3, 7, 43, 1807, 3263443.$$

Thus the first five terms plus

$$\frac{(\log 3263443)/3263443}{1 - 2/(3263443 - 1)}$$

bound the entire sum from above. The exact rational calculation below puts this bound below 1.082389424. For every $n \geq 10000$, it consequently proves

$$\frac{\log d_n}{n} - \log 3 \leq w + \frac{(\log n)^2/\log 2 + (2 + 1/\log 2) \log n + 1}{n} - \log 3 < 0.$$

Indeed, each of $e^{-t}, te^{-t}, t^2e^{-t}$ decreases for $t \geq 9$. Since $\log 10000 > 9$, the displayed upper bound is largest at $n = 10000$, where its certified value is less than -0.0007135 . The same certificate computes

d_n and 3^n with integers for every $1 \leq n < 10000$, and verifies $d_n < 3^n$ in all 9,999 cases. This covers the entire range. $\square$

For completeness, the numerical and finite comparisons in this proof have the following exact certificate. For $0 \leq t \leq 1/3$, integrating the geometric series gives

$$\log \frac{1+t}{1-t} = 2 \sum_{j=0}^{31} \frac{t^{2j+1}}{2j+1} + R, \quad 0 \leq R \leq \frac{2t^{65}}{65(1-t^2)}.$$

Indeed, the remaining positive terms are bounded by replacing each remaining denominator by 65 and summing the geometric tail. For an integer $x \geq 2$, write $x = 2^k u$ with $1 \leq u < 2$, so $t = (u-1)/(u+1) < 1/3$, and use the same series at $t = 1/3$ for $\log 2$. The following program uses only integers and rational numbers; every asserted inequality is therefore exact.

```
from fractions import Fraction as Q
from math import gcd

def series_bounds(t):
    value = sum((2*t**(2*j+1)/Q(2*j+1) for j in range(32)), Q(0))
    error = 2*t**65/(65*(1-t**2))
    return value, value+error

def log_bounds(x):
    x = Q(x)
    k = 0
    while x >= 2:
        x /= 2
        k += 1
    lo2, hi2 = series_bounds(Q(1, 3))
    lo, hi = series_bounds((x-1)/(x+1))
    return k*lo2+lo, k*hi2+hi

a = [2]
for _ in range(5):
    a.append(a[-1]*(a[-1]-1)+1)
assert a == [2, 3, 7, 43, 1807, 3263443]
w_upper = sum((log_bounds(x)[1]/x for x in a[:5]), Q(0))
w_upper += (log_bounds(a[5])[1]/a[5])/(1-Q(2, a[5]-1))
assert w_upper < Q(1082389424, 1000000000)
lo2, hi2 = log_bounds(2)
lo3, hi3 = log_bounds(3)
loT, hiT = log_bounds(10000)
assert loT > 9
margin_upper = w_upper + (hiT**2/lo2+(2+1/lo2)*hiT+1)/10000-lo3
assert margin_upper < -Q(7135, 10000000)
d = 1
power = 1
for n in range(1, 10000):
    d = d//gcd(d,n)*n
    power *= 3
    assert d < power
if __name__ == '__main__':
    print('Exact finite prefix: 1 <= n <= 9999 passed')
    print('Entropy upper:', float(w_upper))
    print('Infinite margin upper:', float(margin_upper))
```

A uniform denominator budget

The exponential least-common-multiple lemma just proved gives $\nu(b) < 3^b$ for every integer $b \geq 1$, without a prime-distribution estimate. The same denominator bound appears in [Yu 1989], Lemma 2.3; the complete proof used here is the preceding factorial and entropy argument.

Corollary (uniform determinant criterion). Retain all the hypotheses and notation of Theorem 7.12, and put

$$q = (R+1)(S+1), \quad \omega = 1 - \frac{N}{2q(T+1)}, \quad \omega_0 = \frac{2q(T+1)}{N}, \quad U = \omega T + \omega_0.$$

Set

$$c = \log 3, \quad J_K = 1 + \frac{cK}{3}, \quad b = 1 + \left\lfloor \frac{U}{J_K} \right\rfloor,$$

and define the real-valued budget

$$\begin{aligned} \mathcal{B} = & \log(N/2) + \frac{K}{3} \log \frac{W}{2K} + \frac{K}{3} \log \frac{T}{KL} + \frac{11K}{9} + J_K + 2U \\ & + U \log \max \left\{ 1, \frac{J_K L}{U} \right\}, \quad W = Rb_2 + Sb_1. \end{aligned} \quad (7.57)$$

If

$$\frac{V}{2} > D_* \mathcal{B} + \log 2 + (T + K/3) \log E + \frac{g(L+1)}{2} ((R+1)a_1 + (S+1)a_2), \quad (7.58)$$

then $|\Lambda^*| > e^{-V}$.

Proof. The cardinality hypotheses give $N \leq q(T+1)$, hence $\omega \geq 1/2$, $\omega_0 \geq 2$, and $U > 0$. They also give $T \geq K \geq 1$, $W > 0$, and $N \geq 6$. The estimate

$$N! \leq (N/2)^N \quad (N \geq 6)$$

has a short integer proof. At $N = 6$, it is $720 < 729$. The induction step follows from $(1 + 1/N)^N > 2$, obtained by retaining the first three terms of the binomial expansion. Thus $\log(N!)/N \leq \log(N/2)$.

Use $\log \nu(b) \leq cb$ in (7.43). The two contributions involving the block size become

$$\frac{K}{3} \log \nu(b) + b + U \left(1 + \log \frac{H_b}{b} \right) \leq J_K b + U \left(1 + \log \frac{H_b}{b} \right), \quad H_b = \max\{L, b-1\}.$$

By the chosen rounding,

$$\frac{U}{J_K} < b \leq \frac{U}{J_K} + 1, \quad \frac{H_b}{b} \leq \max \left\{ 1, \frac{J_K L}{U} \right\}.$$

The latter inequality covers both $H_b = L$ and $H_b = b-1$. It follows that the displayed block-size cost is at most

$$J_K + 2U + U \log \max \left\{ 1, \frac{J_K L}{U} \right\}.$$

Together with the factorial estimate this proves $F/N \leq \mathcal{B}$. Consequently (7.58) implies (7.56), so Theorem 7.12 applies. $\square$

The maximum inside the logarithm is essential to this rounding argument. When the block size exceeds $L+1$, the derivative-product bound uses $H_b = b-1$, rather than L . The criterion therefore retains all positive block sizes, including the small-degree cases. For instance, the full-rank parameters in Exercise 8 give $N = 18, q = 16, T = 10, U = 23003/792$. The rounded choice is $b = 17 > T$, and $H_b = 16 > L$. The integral polynomial basis is still valid. The integer and rational interval certificate checks both branches of this maximum and the rounded cost, including examples with $b > T$.

Controlling the point count in a numerical specialization

The determinant budget also depends on how densely its rows occupy the three-dimensional box. Here is an adaptation of the rounded construction of [Gouillon 2003], Section 5.1, with a uniform proof of its point-count estimates. The integer parts in all three directions matter.

Lemma (rounded boxes and occupancy). Let $D \geq 1, \lambda \geq 1, Q > 0$, and suppose

$$DQ \geq \lambda, \quad a_1, a_2 \geq 3\lambda, \quad 300 \leq c_0 \leq 5000, \quad c_1 \geq 5.1, \quad h \geq \max\{265\lambda/D, 150Q\}.$$

Put $A = a_1 a_2$, $g_0 = 0.241$, $\gamma = 1.309$, and set

$$K = \lfloor c_0 A D Q \lambda^{-3} \rfloor, \quad L = \lfloor c_1 D h \lambda^{-1} \rfloor, \quad \Gamma = \min\{(K+1)/2, L+1\}, \quad \eta = \Gamma^{-1/3}.$$

Define the positive quantities

$$\begin{aligned} x &= \sqrt{K+1}, \\ y &= \frac{(K+1)^{2/3} (2\gamma DQ)^{1/3}}{g_0^{1/3} A^{1/6}}, \\ z &= \frac{g_0^{2/3} (L+1) A^{1/3} (K+1)^{2/3}}{(2\gamma DQ)^{2/3}}. \end{aligned}$$

Take componentwise integer parts in

$$\begin{aligned} (R_1, R_2, R_3) &= \left\lfloor \sqrt{a_2/a_1} (x, \eta y, 3^{1/3} y) \right\rfloor, \\ (S_1, S_2, S_3) &= \left\lfloor \sqrt{a_1/a_2} (x, \eta y, 3^{1/3} y) \right\rfloor, \\ (T_1, T_2, T_3) &= \left(\max\{\lfloor (L+1)/(K+1) \rfloor, K\}, \lfloor \eta z \rfloor, \lfloor 3^{1/3} z \rfloor \right). \end{aligned} \tag{7.59}$$

As before, let $R = \sum R_j, S = \sum S_j, T = \sum T_j, q = (R+1)(S+1)$, and $N = (K+1)(K+2)(L+1)/2$. Then

$$\frac{q(T+1)}{N} < 9.224, \quad g < 0.241, \quad \omega < 0.946, \quad \omega_0 < 20, \tag{7.60}$$

for the exact g, ω, ω_0 used above. Also $T/(KL) < 1/20$, and the block chosen in (7.57) satisfies $b < L$. If the bases are multiplicatively independent and the additive map $(r, s) \mapsto rb_2 + sb_1$ is injective on the full rectangle, all six full-rank conditions (7.55) hold.

Proof. Write $k = c_0 A D Q \lambda^{-3}$. The assumptions give $k \geq 2700$, $K \geq 2700$, $K+1 > k$, $K+1 \leq k(1+1/2700)$, and $L \geq 1351$. In particular $\Gamma \geq 1350$. Put

$$C = 3^{1/3} + \Gamma^{-1/3}, \quad f = 0.452c_0^{-1/6} + 0.151c_0^{-2/3}, \quad t_0 = 0.00642c_0^{1/3}.$$

The definitions immediately give $R \leq \sqrt{a_2/a_1}(x + yC)$ and the symmetric bound for S . The extra unit in $R+1$ is controlled as follows:

$$\begin{aligned} \frac{x}{y} &\leq \left(\frac{g_0}{2\gamma}\right)^{1/3} c_0^{-1/6} \sqrt{\frac{\lambda}{DQ}} < 0.452c_0^{-1/6}, \\ \frac{\sqrt{a_1/a_2}}{y} &\leq \left(\frac{g_0}{2\gamma}\right)^{1/3} \frac{\lambda^2}{c_0^{2/3} a_2 DQ} < 0.151c_0^{-2/3}. \end{aligned}$$

The second inequality uses both $a_2 \geq 3\lambda$ and $DQ \geq \lambda$; it holds uniformly even when λ is large. Consequently

$$R+1 \leq \sqrt{a_2/a_1} y(C+f), \quad S+1 \leq \sqrt{a_1/a_2} y(C+f).$$

For the derivative direction,

$$T + 1 \leq zC + \max\{(L + 1)/(K + 1) + 1, K + 1\}.$$

Divide the last maximum by z . Its second entry is at most

$$\left(\frac{2\gamma}{g_0}\right)^{2/3} \frac{c_0^{1/3} Q}{c_1 h} (1 + 1/2700)^{1/3} < 0.00642 c_0^{1/3}.$$

Its first entry is at most

$$\frac{(2\gamma/g_0)^{2/3}}{9c_0^{2/3}} \left(\frac{1}{1350} + \frac{1}{2700} \right) < 0.0007 c_0^{-2/3} < t_0.$$

Here we used $A \geq 9\lambda^2$ and $K + 1 > k$. It follows that $T + 1 \leq z(C + t_0)$. Since $y^2 z = (K + 1)^2(L + 1)$, we obtain

$$\frac{q(T + 1)}{N} \leq 2(C + f)^2(C + t_0). \quad (7.61)$$

Replace C by its larger constant $C_* = 3^{1/3} + 1350^{-1/3}$. For $r = c_0^{1/6}$, the right side is bounded by

$$F(r) = 2(C_* + 0.452r^{-1} + 0.151r^{-4})^2(C_* + 0.00642r^2).$$

Expanding gives a finite sum of powers of r with positive coefficients. Therefore $F(e^u)$ is convex in u : each summand has nonnegative second derivative. On the interval from $(\log 300)/6$ to $(\log 5000)/6$, its maximum is at an endpoint. Outward rational intervals give the endpoint values less than 9.223640 and 8.862917, respectively. Thus (7.61) is less than $9.224 < 250/27$. Substitution in the definitions of the three occupancy constants proves (7.60).

To check the block size, first use (7.59) and the same rounding bounds:

$$\frac{T}{KL} \leq \frac{1}{1350} + \left(\frac{g_0}{2\gamma}\right)^{2/3} 300^{-1/3} (1 + 1/2700)(1 + 1/1350)C_* < \frac{1}{20}. \quad (7.62)$$

Indeed the contribution of T_1 is at most $\max\{1/L, (L + 1)/(K(K + 1)L)\} \leq 1/1350$. For $T_2 + T_3$, cancel powers using $K + 1 > k$ and $\lambda/(DQ) \leq 1$ to obtain the second term in (7.62). The third box alone gives $q(T + 1) > 3y^2 z > N$, so $U > 0$. With $J_K \geq K \log 3/3$, (7.60) now implies

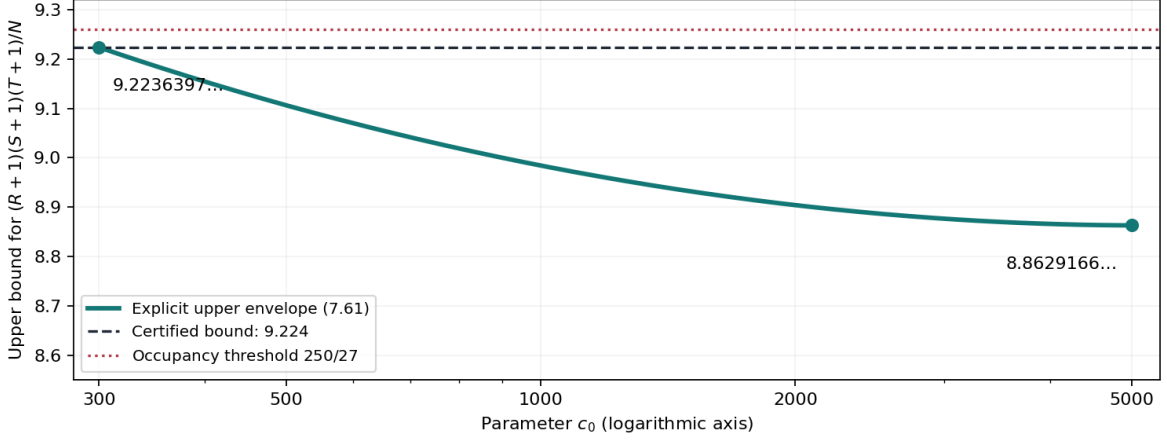
$$b \leq 1 + \frac{U}{J_K} < 1 + \frac{3(0.946)}{20 \log 3} L + \frac{60}{K \log 3} < \frac{L}{7} + 2 < L.$$

Thus this family uses the branch $H_b = L$, with the restriction established rather than presumed.

Finally consider the rank conditions. Under the two injectivity hypotheses, every additive, multiplicative and paired box cardinality equals $(R_j + 1)(S_j + 1)$. The first product exceeds $x^2 = K + 1$, and $T_1 + 1 > (L + 1)/(K + 1)$. For the second box the product with $T_2 + 1$ exceeds $\eta^3 y^2 z = (K + 1)^2(L + 1)/\Gamma$, which exceeds both $K^2 + 1$ and $2KL + 1$. For the third it exceeds $3y^2 z = 3(K + 1)^2(L + 1) > 3K^2 L + 1$. Together with $T_1 \geq K$, these are (7.55). $\square$

Rounded boxes stay below the occupancy threshold

$$C_* = \sqrt[3]{3} + 1350^{-1/3}, \quad r = c_0^{1/6}, \quad F(r) = 2(C_* + 0.452/r + 0.151/r^4)^2(C_* + 0.00642r^2)$$



Positive Laurent coefficients make $F(e^u)$ convex. Its maximum on the interval is at an endpoint.

The certified bound implies $g < 0.241$ and $\omega < 0.946$.
 Adapted from Nicolas Gouillon (2003), thesis §§5.1, 5.3.2. Full proof: Lesson 7, (7.59)–(7.62).
 Curve: numerical samples; proof: exact rational interval endpoint bounds and logarithmic convexity.

Figure 7.3.

The curve samples the explicit function in the proof of (7.61). Convexity in the logarithm of $c_0^{1/6}$ proves the endpoint bound throughout $300 \leq c_0 \leq 5000$; the samples alone are not the proof. The horizontal threshold 250/27 gives both $g < 0.241$ and $\omega < 0.946$. The construction follows Gouillon's rounded boxes; the precise adapted choices, residual bounds and convexity verification are supplied above. Exact integer and interval checks and reproducible figure source are included.

The coefficient cutoff for the rounded boxes

We can now bound the coefficient contribution with the same parameters. This follows the grouping in [Gouillon 2003], Lemma 5.2, using the proved $T/(KL) < 1/20$ and the denominator constant $\log 3$.

Lemma (coefficient and factorial budget). Under the hypotheses of the rounded-box lemma, let $b_1, b_2 > 0$, and put

$$B_0 = \frac{b_1}{a_2} + \frac{b_2}{a_1}, \quad B_{\det} = \frac{Rb_2 + Sb_1}{2K}.$$

Then

$$\begin{aligned} \frac{R}{K} &< \frac{0.566\lambda}{a_1}, \\ \frac{S}{K} &< \frac{0.566\lambda}{a_2}, \\ B_{\det} &< 0.283\lambda B_0. \end{aligned} \tag{7.63}$$

Consequently the arithmetic bracket satisfies

$$\begin{aligned} &\log B_{\det} + \frac{11}{3} + \log 3 \\ &+ \frac{6}{K} \log \frac{K+2}{2} + \log \frac{T}{KL} + \frac{\lambda}{D} \\ &< \log B_0 + \log \lambda + \frac{\lambda}{D} + 0.524. \end{aligned} \tag{7.64}$$

If h is at least the last right side, the uniform budget (7.57) therefore obeys

$$D\mathcal{B} + \frac{\lambda K}{3} < \frac{DKh}{3} + D\log(L+1) + D + DU \left(2 + \log \frac{J_K L}{U} \right). \quad (7.65)$$

Proof. The upper bound for R in the preceding proof gives

$$\frac{R}{K} \leq \frac{\lambda}{a_1} (1 + 1/K) \left\{ \sqrt{\frac{\lambda}{c_0 D Q}} + \left(\frac{2\gamma}{g_0 c_0} \right)^{1/3} C \right\}.$$

Use $K \geq 2700$, $DQ \geq \lambda$, $c_0 \geq 300$, and $C \leq C_*$. The factor in braces, including $1 + 1/K$, is at most

$$\frac{2701}{2700} \left\{ 300^{-1/2} + \left(\frac{2\gamma}{300 g_0} \right)^{1/3} C_* \right\} < 0.565024 < 0.566.$$

The symmetric argument proves the S bound; adding the two coefficient contributions proves the $B_{\det}$ bound.

The function $6 \log((K+2)/2)/K$ decreases for $K \geq 2700$. Its derivative has the sign of $K/(K+2) - \log((K+2)/2)$, which is negative in this range. Its endpoint value is less than 0.0161. Substituting (7.63) and $T/(KL) < 1/20$ now leaves the constant

$$\log(0.283) + \frac{11}{3} + \log 3 + 0.0161 + \log(1/20) < 0.523339 < 0.524,$$

proving (7.64). The supplied rational interval certificate verifies these strict scalar margins, as well as the rounded-length bounds.

Finally,

$$\log(N/2) \leq 2 \log((K+2)/2) + \log(L+1), \quad J_K = 1 + K \log 3/3.$$

The previous lemma proves $b < L$, so $U/J_K < b < L$ and $J_K L/U > 1$. Insert these facts into (7.57), and group the terms proportional to $K/3$ using (7.64). The remaining terms are precisely those displayed in (7.65). $\square$

Completing the rounded determinant bound

The derivative cost contains a logarithm of the ratio of the polynomial degree to the number of derivatives. We control this ratio before combining it with the height term. The following explicit family follows the grouping in [Gouillon 2003], Section 5.3, with the denominator $\log 3$, the adapted boxes above, and the field-to-radius ratio retained through the final remainder estimate.

Corollary (a complete rounded family). Retain the rounded-box and coefficient-budget hypotheses, including $h \geq \log B_0 + \log \lambda + \lambda/D + 0.524$. Use the algebraic cancellation setting of Theorem 7.12, with $D = D_*$, $E = e^\lambda$, and

$$a_j \geq E|\ell_j| - \log |\alpha_j| + 2Dh(\alpha_j).$$

Suppose the bases are multiplicatively independent. Put $\mu_0 = 0.946$, $C_* = 3^{1/3} + 1350^{-1/3}$, and assume

$$Q \geq \frac{\lambda}{D} + 2.16 + \frac{\mu_0}{3} \log c_0 + \mu_0 \log \frac{D}{\lambda}. \quad (7.66)$$

Finally require

$$\begin{aligned}\Psi(c_0, c_1) &= \frac{1/3 + \gamma/150}{c_1} + \frac{g_0}{\sqrt{c_0}} \\ &\quad + \frac{g_0}{3c_0} \\ &\quad + \frac{3g_0^{2/3}\gamma^{1/3}C_*}{2^{2/3}c_0^{1/3}} < 0.249.\end{aligned}\tag{7.67}$$

Then, for $\Lambda = b_1\ell_1 - b_2\ell_2$,

$$\log |\Lambda| > -0.501c_0c_1AD^2hQ\lambda^{-3}.\tag{7.68}$$

There is no additive-injectivity restriction in this conclusion. For example $(c_0, c_1) = (400, 5.1)$ satisfies (7.67), giving the coefficient 1022.04 in (7.68). All the other hypotheses remain in force.

Proof. First consider the case in which the additive map is injective on the full rectangle. Its rank conditions are already proved. Write $t = \lambda/D$. The minimum of $t - \mu_0 \log t$, for $t > 0$, is $\mu_0(1 - \log \mu_0)$. Thus (7.66) and $c_0 \geq 300$ imply $Q > 4.9$. The function $\log Q/Q$ decreases for $Q > e$, so

$$\frac{\mu_0 \log Q}{Q} < 0.309 = \gamma - 1.$$

The strict scalar margins used here are verified by outward rational intervals in the supplied certificate.

We need a lower bound for T . From (7.59),

$$T \geq K + (\eta + 3^{1/3})z - 2 > 3^{1/3}z.$$

Together with $K + 1 \leq k(1 + 1/2700)$, this gives

$$\begin{aligned}\frac{J_K L}{T} &\leq M_0 c_0^{1/3} \frac{DQ}{\lambda}, \\ M_0 &= \left(\frac{\log 3}{3} + \frac{1}{2700} \right) \\ &\quad \cdot \frac{(2\gamma)^{2/3}}{3^{1/3}g_0^{2/3}} \left(1 + \frac{1}{2700} \right)^{1/3}.\end{aligned}$$

Indeed, first bound $(K + 1)(L + 1)/T$ using the definition of z , and then use $J_K/(K + 1) \leq (\log 3)/3 + 1/2700$. The certificate proves $\mu_0(2 + \log(M_0/\mu_0)) < 2.16$. For $Z = J_K L/(\mu_0 T)$, (7.66) therefore yields

$$\begin{aligned}\frac{\lambda}{D} + \mu_0(2 + \log Z) &< Q + \mu_0 \log Q \\ &< \gamma Q.\end{aligned}\tag{7.69}$$

We now separate the two parts of U without enlarging a logarithm unnecessarily. Let $F(y) = y(2 + \log(J_K L/y))$, for $y > 0$. It is concave, and increasing on $(0, J_K L]$. By (7.60) and (7.62),

$$U < \mu_0 T + 20 < J_K L.$$

For the second inequality use $T < KL/20$, $J_K > K/3$, and $KL \geq 2700 \cdot 1351$. The tangent line at $\mu_0 T$ gives

$$F(U) < \mu_0 T(2 + \log Z) + 20(1 + \log Z).$$

Write its height contribution as

$$\begin{aligned}H_{\text{ht}} &= \frac{g_0(L + 1)}{2} \\ &\quad \cdot ((R + 1)a_1 + (S + 1)a_2).\end{aligned}$$

Consequently (7.65) and (7.69) bound the right side of (7.58) by

$$\begin{aligned} & \frac{DKh}{3} + \gamma DQT + H_{\text{ht}} \\ & + D \log(L+1) + D \\ & + 20D(1 + \log Z) + \log 2. \end{aligned}$$

The definitions of x, y, z give

$$\begin{aligned} R & \leq \sqrt{a_2/a_1}(x + Cy), \\ S & \leq \sqrt{a_1/a_2}(x + Cy), \\ T & \leq K + \frac{L+1}{K+1} + Cz, \end{aligned}$$

Here $C = \eta + 3^{1/3} \leq C_*$. Adding their height and derivative costs therefore gives

$$\gamma DQT + H_{\text{ht}} \leq \Phi + \frac{\gamma DQ(L+1)}{K+1}.$$

Here

$$\begin{aligned} \Phi & = \gamma DQK \\ & + g_0(L+1)\sqrt{(K+1)A} \\ & + \frac{g_0(L+1)}{2}(a_1 + a_2) \\ & + \frac{3g_0^{2/3}(\gamma ADQ)^{1/3}}{2^{2/3}} \\ & \cdot C(L+1)(K+1)^{2/3}. \end{aligned} \tag{7.70}$$

For the last term, $\gamma DQ Cz$ is one third of the combined contribution, and $g_0(L+1)Cy\sqrt{A}$ is the other two thirds.

Divide $D(K+1)h/3 + \Phi$ by $(K+1)(L+1)\lambda$. The resulting five terms are respectively less than or equal to

$$\begin{aligned} & \frac{1}{3c_1}, \quad \frac{\gamma}{150c_1}, \quad \frac{g_0}{\sqrt{c_0}}, \\ & \frac{g_0}{3c_0}, \quad \frac{3g_0^{2/3}\gamma^{1/3}C_*}{2^{2/3}c_0^{1/3}}. \end{aligned}$$

Here use $L+1 > c_1 Dh/\lambda$, $h \geq 150Q$, $K+1 > c_0 ADQ/\lambda^3$, $DQ \geq \lambda$, and $a_j \geq 3\lambda$. This proves the bound (7.67).

Put

$$\theta = \frac{1}{8} \left(1 - \frac{1}{L+1} + \sqrt{1 - \frac{2}{L+1}} \right).$$

Since $L \geq 1351$, $\theta > 0.249$, and $V/2 = \theta(K+2)(L+1)\lambda$. After allocating $\theta(K+1)(L+1)\lambda$ to the five terms just bounded, it suffices to show that the two remainders

$$\begin{aligned} \Omega_1 & = -\theta(L+1)\lambda \\ & + \frac{\gamma DQ(L+1)}{K+1} \\ & + D \log(c_1 h) \\ & + D(1 + 1/1351) + \log 2, \\ \Omega_2 & = -Dh/3 + 20D(1 + \log Z) \\ & + D \log(D/\lambda) \end{aligned}$$

are negative. These include the full $D \log(L+1)$ cost, because

$$\log(L+1) \leq \log(D/\lambda) + \log(c_1 h) + 1/1351.$$

Keeping D/λ together is essential when the radius is large.

For Ω_2 , (7.69) gives $1 + \log Z < \gamma Q/\mu_0$, and (7.66) gives $\log(D/\lambda) < Q/\mu_0$. Hence

$$\frac{\Omega_2}{D} < \left(-50 + \frac{20\gamma + 1}{\mu_0}\right) Q < 0.$$

For Ω_1 , use $K+1 > c_0 ADQ/\lambda^3$ and $A \geq 9\lambda^2$. Then $\gamma DQ/(K+1) < \gamma\lambda/2700$, and $0.249 - \gamma/2700 > 0.2485$. Writing $s = c_1 h > 5.1 \cdot 150 \cdot 4.9 > 3700$, we obtain

$$\frac{\Omega_1}{D} < -0.2485s + \log s + 2.001 < 0.$$

The last step follows from $\log s < s/8$ for $s \geq 100$: it holds at 100, and $s/8 - \log s$ increases in that range. Thus (7.58) holds strictly, proving $|\Lambda^*| > e^{-V}$.

To convert back, set

$$M = \max \left\{ \frac{LS}{2b_2}, \frac{LR}{2b_1} \right\}, \quad x = M|\Lambda|.$$

Then $|\Lambda^*| = xe^x$. Since $V > 0$, $xe^x > e^{-V}$ implies $x > e^{-V-1}$. Also (7.63) and $a_j \geq 3\lambda$ give $M < KL$. For $KL \geq 2700 \cdot 1351$,

$$\log(KL) + 1 < 10^{-5}KL.$$

Use the decreasing function $(1 + \log w)/w$ and its certified endpoint. Finally

$$\begin{aligned} V + \log M + 1 &< 0.50001(K+2)(L+1)\lambda \\ &< 0.501c_0c_1AD^2hQ\lambda^{-3}, \end{aligned} \tag{7.71}$$

using $K+2 \leq k(1+2/2700)$, $L+1 \leq (c_1 Dh/\lambda)(1+1/1351)$. This proves (7.68) in the injective case.

Suppose instead that two additive labels coincide. There is a primitive integer pair $(r, s) \neq (0, 0)$, with $|r| \leq R$, $|s| \leq S$, and $rb_2 + sb_1 = 0$. Both coordinates are nonzero, and b_1/r is an integer up to sign. Consequently

$$|\Lambda| = |b_1/r| |r\ell_1 + s\ell_2| \geq |r\ell_1 + s\ell_2|.$$

Multiplicative independence gives $\beta = \alpha_1^r \alpha_2^s \neq 1$. Let $d = [\mathbb{Q}(\alpha_1, \alpha_2) : \mathbb{Q}] \leq 2D$. If $|r\ell_1 + s\ell_2| \geq 1$, the conclusion is immediate. Otherwise the exponential comparison and the product formula give

$$\begin{aligned} -\log |\Lambda| &\leq 1 + (d-1) \log 2 + dh(\beta) \\ &< 1 + 2D \log 2 + Ra_1 + Sa_2. \end{aligned}$$

Here $a_j \geq 2Dh(\alpha_j)$, since $E|\ell_j| - \log |\alpha_j| \geq 0$. By (7.63), $Ra_1 + Sa_2 < 1.132K\lambda$, whereas

$$D < \frac{(K+1)\lambda}{2700Q} < \frac{(K+1)\lambda}{13230}.$$

Thus the displayed cost is less than $1.134K\lambda$. The magnitude of the exponent in (7.68) is $0.501k(c_1 Dh/\lambda)\lambda$, which exceeds $0.501KL\lambda > 1.134K\lambda$. This completes the collision case and the proof. $\square$

The rational interval certificate checks every scalar margin and complete abstract determinant budgets, including $D = \lambda = 10^{30}$ and 10^{1000} . Those examples supplement the uniform proof and do not replace its field, rank or logarithm hypotheses. The full family has linear dependence on $\log B_0$ through the explicit choice of h .

Gouillon's fixed-radius estimates

At a fixed radius we can retain the larger height cutoffs, giving these constants of [Gouillon 2006], Corollaries 2.2–2.3.

Corollary. Let α_1, α_2 be multiplicatively independent nonzero algebraic numbers with chosen logarithms ℓ_j . Use $D = D_*$, and choose

$$a_j^0 = \log A_j \geq \max\{h(\alpha_j), |\ell_j|/D, 1/D\}.$$

For integers n_1, n_2 , not both zero, put

$$\Gamma = n_1 \ell_1 + n_2 \ell_2, \quad b = \frac{|n_1|}{Da_2^0} + \frac{|n_2|}{Da_1^0}.$$

Define

$$\begin{aligned} Q_{\mathbb{C}} &= 3.317 + 1.888/D + 0.946 \log D, \\ h_{\mathbb{C}} &= \max\{\log b + 3.1, 1000/D, \\ &\quad 498 + 284/D + 142 \log D\}. \end{aligned}$$

Then

$$\begin{aligned} \log |\Gamma| &\geq -9400D^4 a_1^0 a_2^0 \\ &\quad \cdot h_{\mathbb{C}} Q_{\mathbb{C}}. \end{aligned} \tag{7.72}$$

For positive-real bases and their real logarithms, use $D = [\mathbb{Q}(\alpha_1, \alpha_2) : \mathbb{Q}]$ and put

$$\begin{aligned} Q_{\mathbb{R}} &= 3.409 + 1.705/D + 0.946 \log D, \\ h_{\mathbb{R}} &= \max\{\log b + 3.1, 1000/D, \\ &\quad 512 + 256/D + 142 \log D\}. \end{aligned}$$

The stronger real estimate is

$$\begin{aligned} \log |\Gamma| &\geq -7200D^4 a_1^0 a_2^0 \\ &\quad \cdot h_{\mathbb{R}} Q_{\mathbb{R}}. \end{aligned} \tag{7.73}$$

Both assertions include every coefficient sign and a zero coefficient.

Proof. First enter the cancellation setting by the reductions already proved. In the rounded boxes (7.59), use these parameters:

Case	E	c_0	c_1	m	q_0	q_1	δ
Complex	6.6	317	5.378	8.6	3.317	1.888	0.039
Real	5.5	313	5.386	6.5	3.409	1.705	0.0395

Set $\lambda = \log E$, $a_j = mDa_j^0$, $Q = q_0 + q_1/D + \mu_0 \log D$, and take the corresponding h . Then $a_j \geq m > 3\lambda$, $Dh \geq 1000 > 265\lambda$, $DQ \geq q_0 + q_1 > \lambda$, and $h \geq 150Q$. For example, $150Q_{\mathbb{C}} = 497.55 + 283.2/D + 141.9 \log D$. The complex height cost is at most $(E+2)Da_j^0 = a_j$. For positive-real logarithms it is at most $(E-1)|\ell_j| + 2Dh(\alpha_j) \leq (E+1)Da_j^0 = a_j$. Moreover $B_0 = b/m$, and

$$-\log m + \log \lambda + \lambda + 0.524 < 3.1.$$

Thus $h \geq \log B_0 + \log \lambda + \lambda/D + 0.524$, and every rounded-box and coefficient-budget hypothesis holds.

Write

$$\begin{aligned} q_* &= q_0 + \mu_0(1 + \log(q_1/\mu_0)), \\ d_* &= q_0 + q_1, \quad k_* = c_0 m^2 d_* \lambda^{-3}, \\ l_* &= 1000c_1/\lambda. \end{aligned}$$

Minimizing $q_1/D + \mu_0 \log D$ gives $Q \geq q_* > 4.9$. Also $DQ \geq d_*$, $K + 1 > k_*$, $L + 1 > l_*$. The certificate verifies $k_*/2 > l_*$, so

$$C = \eta + 3^{1/3} < C_0 := 3^{1/3} + l_*^{-1/3},$$

$$\theta > \theta_0 := \frac{1}{8} \left(1 - \frac{1}{l_*} + \sqrt{1 - \frac{2}{l_*}} \right).$$

Keep the $\log 3$ derivative denominator. With M_0 defined above (7.69), put

$$\mathcal{E} = \frac{\lambda}{D} + \mu_0(2 + \log Z) - \gamma Q.$$

The proved ratio bound gives

$$\mathcal{E} \leq d_0 + \frac{\lambda - q_1}{D} + \mu_0 \log Q - (\gamma - 1)Q.$$

Here

$$d_0 = \mu_0 \left(2 + \log \frac{M_0 c_0^{1/3}}{\mu_0 \lambda} \right) - q_0.$$

Here $\lambda < q_1$, and $\mu_0 \log Q - (\gamma - 1)Q$ decreases for $Q > \mu_0/(\gamma - 1)$. Since q_* exceeds this threshold, its value at q_* bounds the right side from above. Outward rational intervals give upper bounds $0.038826700 < 0.039$ in the complex case and $0.039461582 < 0.0395$ in the real case. The complete derivative grouping therefore retains the extra cost δDT , in addition to γDQT .

Normalize by $(K + 1)(L + 1)\lambda$. The five terms in $D(K + 1)h/3 + \Phi$ have sum at most

$$P = \frac{1/3 + \gamma/150}{c_1} + \frac{g_0}{\sqrt{c_0 d_*/\lambda}} + \frac{g_0 \lambda^2}{c_0 m d_*} + \frac{3g_0^{2/3} \gamma^{1/3} C_0}{2^{2/3} c_0^{1/3}}.$$

These bounds use $a_j \geq m$, $DQ \geq d_*$, $Q/h \leq 1/150$, and the rounded lower bounds on $K + 1, L + 1$. The three terms in the upper bound for T give

$$\frac{DT}{(K + 1)(L + 1)\lambda} \leq W_*,$$

where

$$W_* = \frac{1}{150 c_1 q_*} + \frac{\lambda^5}{c_0^2 m^4 q_*^2} + \frac{C_0 g_0^{2/3}}{(2\gamma)^{2/3} c_0^{1/3} q_*}.$$

For the middle term use $K + 1 > c_0 m^2 DQ \lambda^{-3}$, $D \geq 1$. For the last, substituting z cancels the factor A . The exact certificates prove the positive margins

case	$\theta_0 - P - \delta W_*$	(7.74)
complex	> 0.000099564877	
real	> 0.000002585938	

This allocates every main term.

The remainders have the same form as in the preceding proof. For Ω_1 , $K \geq 2700$, $L \geq 1351$, and $c_1 h > 3700$ give the previously proved negative bound. For Ω_2 , use $1 + \log Z < (\gamma Q + \delta)/\mu_0$, and $\log(D/\lambda) < Q/\mu_0$; the latter follows from $\lambda > 1$ and the formula for Q . Thus

$$\frac{\Omega_2}{D} < \left(-50 + \frac{20\gamma + 1}{\mu_0}\right) Q + \frac{20\delta}{\mu_0} < 0.$$

The full strict determinant inequality (7.58) is verified.

The prefactor argument (7.71), retaining the larger minima k_*, l_* , gives the coefficient

$$0.50001 \left(1 + \frac{2}{k_*}\right) \left(1 + \frac{1}{l_*}\right) c_0 c_1 m^2 \lambda^{-3}.$$

Its interval upper bounds are $9386.248604217 < 9400$ and $7191.782010965 < 7200$. In the collision case, the already proved cost is less than $1.134K\lambda$; each target magnitude exceeds $\frac{1}{2}KL\lambda$, which is larger. Thus no additive-injectivity restriction remains.

For a single coefficient or a remaining same-sign case, the elementary cost used in Corollary 7.5 is at most $\log 4 + 4D^2(2a_j^0 + \log 2)$. A target magnitude $CD^4 a_1^0 a_2^0 hQ$, with $C \geq 7200$, is at least $CD^3 a_j^0 hQ$ and $CD^2 hQ$. The height and constant parts have ratios at most $8/(CDhQ)$ and $6/(ChQ)$; their sum is less than one. Every stated coefficient case is therefore covered. $\square$

The fixed-radius certificate checks all the uniform scalar margins and the full prefactor. For positive-real bases with arbitrary logarithm branches, the imaginary part of Γ is an integer multiple of 2π . If it is nonzero, (7.73) is immediate. If it vanishes, apply (7.73) to the real logarithms, using that the original cutoffs dominate the real-logarithm cutoffs. The period cases thus have the same bound.

A real-radius estimate with the full cutoff range

The radius gain can also be kept when the logarithms are small. We spell out the field-to-radius ratio inside the logarithmic cutoff.

Corollary. Let $\alpha_1, \alpha_2 > 1$ be multiplicatively independent real algebraic numbers, use real logarithms, and set $D = [\mathbb{Q}(\alpha_1, \alpha_2) : \mathbb{Q}]$. Choose

$$a_j^0 = \log A_j \geq \max\{h(\alpha_j), \log \alpha_j/D, 1/D\}.$$

Let n_1, n_2 be integers, not both zero, and put

$$\Gamma = n_1 \log \alpha_1 + n_2 \log \alpha_2, \quad b = \frac{|n_1|}{Da_2^0} + \frac{|n_2|}{Da_1^0}.$$

Define

$$E = 1 + \min_j \frac{Da_j^0}{\log \alpha_j}, \quad v = \log E,$$

and suppose $E \leq \min\{A_1^{D/2}, A_2^{D/3}\}$. In particular $E \geq 2$. Set

$$\begin{aligned} Q &= \max\{v/D, \\ &\quad v/D + 0.946 \log(D/v) + 3.965\}, \\ h &= \max\{\log b + v - \log v - 2.27, \\ &\quad 265v/D, 150Q\}. \end{aligned}$$

Then

$$\log |\Gamma| \geq -8550D^4 a_1^0 a_2^0 \cdot hQv^{-3}. \tag{7.75}$$

This uses Gouillon's real-radius mechanism and constant. The explicitly stated $\log(D/v)$ is defined throughout the advertised $E \geq 2$ range. We retain that range and the coefficient, 265, and 150 cutoffs in the proof.

Proof. In the cancellation case choose $\lambda = v$, $a_j = 3Da_j^0$, $c_0 = 368$, $c_1 = 5.141$, and use the rounded boxes (7.59). The definition of E gives the required height cost, since

$$(E - 1) \log \alpha_j + 2Dh(\alpha_j) \leq 3Da_j^0 = a_j.$$

The two upper cutoffs on E give $a_1 \geq 6v$, $a_2 \geq 9v$. Consequently

$$A \geq 54v^2, \quad v(1/a_1 + 1/a_2) \leq 5/18.$$

These symmetric inequalities survive any permutation in the cancellation reduction. Put $t = v/D$, $r = Q/t \geq 1$. Minimizing $t - \mu_0 \log t$ gives

$$Q \geq q_* := 3.965 + \mu_0(1 - \log \mu_0) > 4.9635.$$

We have $K + 1 > 54c_0r$, $K \geq 19872$, and $L + 1 > c_1 \max\{265, 150r\}$. Also $(K + 1)/2 > 27c_0r > c_1 \max\{265, 150r\}$. Thus Γ , the minimum of these two box lengths, exceeds the latter lower bound. All occupancy, coefficient-length and rank arguments (7.59)–(7.64) remain valid for $\log 2 \leq v < 1$: their proofs use only $DQ/v \geq 1$, $A/v^2 \geq 9$, $a_j/v \geq 3$, $Dh/v \geq 265$, $Q/h \leq 1/150$, and the resulting minima on K, L . No lower bound $v \geq 1$ is used in those arguments.

Write $C_0 = 3^{1/3} + (265c_1)^{-1/3}$, and let $\theta_0 = (1 - (265c_1)^{-1} + \sqrt{1 - 2/(265c_1)})/8$. The five-term normalized height/arithmetical budget is at most

$$\begin{aligned} P(r) = & \frac{1/3 + \gamma \min\{r/265, 1/150\}}{c_1} \\ & + \frac{g_0}{\sqrt{c_0r}} + \frac{5g_0}{36c_0r} \\ & + \frac{3g_0^{2/3}\gamma^{1/3}}{2^{2/3}c_0^{1/3}} \\ & \cdot \left(3^{1/3} + [c_1 \max\{265, 150r\}]^{-1/3}\right). \end{aligned}$$

The actual θ is at least θ_0 . For $1 \leq r \leq 265/150$, the only increasing term in P has derivative $\gamma/(265c_1)$, which is smaller than $g_0/(2\sqrt{c_0}(265/150)^{3/2})$. The square-root term alone therefore makes P decrease. Beyond that breakpoint every term is nonincreasing and the square-root term strictly decreases. Thus P decreases on the whole domain.

There are two further costs. First, the derivative ratio argument gives

$$\frac{v}{D} + \mu_0(2 + \log Z) \leq \gamma Q + \delta, \quad \delta = 0.034.$$

Indeed its excess is at most

$$\begin{aligned} & \mu_0 \left(2 + \log(M_0 c_0^{1/3}/\mu_0)\right) - 3.965 \\ & + \mu_0 \log q_* - (\gamma - 1)q_* < 0.033103084. \end{aligned}$$

Here we used $Q \geq t + 3.965 - \mu_0 \log t$ and the decreasing function of Q from the fixed-radius proof. If $r \leq 1.1$, this excess is negative: the decreasing ratio $1 + (3.965 - \mu_0 \log t)/t$, on $0 < t \leq 14$, at $t = 14$ exceeds 1.1, so $t > 14$; then its excess is at most

$$\begin{aligned} & -14(\gamma - 1) \\ & + \mu_0 \left(2 + \log(1.1M_0 c_0^{1/3}/\mu_0)\right) \\ & < 0. \end{aligned}$$

Otherwise the extra δDT , normalized as before, is at most δW_* , where

$$W_* = \frac{1}{150c_1q_*} + \frac{1}{c_0^2 54^2 q_*} + \frac{C_0 g_0^{2/3}}{(2\gamma)^{2/3} c_0^{1/3} q_*}.$$

The middle term now uses $A \geq 54v^2$ and $t \leq Q$.

Second, retain the coefficient cutoff exactly. Put $c = 2.794 - \log 3$, and $\varepsilon = \max\{0, 2 \log v + t - v + c\}$. Since $B_0 = b/3$, the specified h implies

$$\log B_0 + \log v + t + 0.524 \leq h + \varepsilon.$$

Use (7.64) directly with this upper bound. It adds $DK\varepsilon/3$ to the budget, while every box is still chosen using the original h . Its normalized cost is at most $\varepsilon/(3c_1h)$. For $D \geq 2$, $t - v \leq -v/2$, so $\varepsilon \leq 2 \log 4 - 2 + c < 2.468$. For $D = 1$, $t = v$, and $(2 \log t + c)/t$ decreases for $t \geq 8$. Its global maximum is $2e^{c/2-1}$. Also $r \leq 1.2$ implies $t > 8$, by the same ratio test at 8. Using $h \geq 265t$ and $h \geq 150q_*$, these inequalities prove

$$\frac{\varepsilon}{3c_1h} < \begin{cases} 0.000123, & r \leq 1.1, \\ 0.000180, & 1.1 < r \leq 1.2, \\ 0.000421, & r > 1.2. \end{cases}$$

The bounds include both $D = 1$ and every integer $D \geq 2$.

Monotonicity of P now reduces the three complete main allocations to the following exact scalar margins:

$$\begin{aligned} \theta_0 - P(1) - 0.000123 &> 0.000029665768, \\ \theta_0 - P(1.1) - \delta W_* - 0.000180 &> 0.000161969904, \\ \theta_0 - P(1.2) - \delta W_* - 0.000421 &> 0.000341730440. \end{aligned}$$

The first uses the negative derivative excess. The other two allocate the full extra δDT and coefficient cost. The Ω_1 argument remains valid, and

$$\frac{\Omega_2}{D} < \left(-50 + \frac{20\gamma + 1}{\mu_0}\right) Q + \frac{20\delta}{\mu_0} < 0.$$

Indeed $\mu_0 \log(D/v) \leq Q - t - 3.965 < Q$. This verifies the whole determinant inequality.

For $\log 2 \leq v < 1$, the conversion cost $\log(KL) + 1 < 10^{-5}KL$ is at most $0.000015KLv$. Thus throughout the stated range

$$\begin{aligned} V + \log M + 1 &< 0.500015(K + 2)(L + 1)v. \end{aligned}$$

Its normalized coefficient satisfies

$$\begin{aligned} C_{\text{conv}} &= 0.500015 \left(1 + \frac{2}{54c_0}\right) \\ &\quad \cdot \left(1 + \frac{1}{265c_1}\right) c_0 c_1 \cdot 9 \\ &< 8520.858136944 < 8550. \end{aligned}$$

Since $A = 9D^2a_1^0a_2^0$, this gives (7.75). The primitive-collision cost is again below $1.134Kv$: here $Kv > 13770$, so its additive unit is also absorbed. The target exceeds $\frac{1}{2}KLv$, proving the collision case. For a same-sign or single-coefficient form, the elementary logarithm cost is at most $Da_j^0 + D \log 2$. The target divided by Da_j^0 is at least $8550 \cdot 150 \cdot 2$, using $h \geq 150Q$, $DQ \geq v$, and $Da_{3-j}^0 \geq 2v$. Its ratio to D is at least $8550 \cdot 150 \cdot 6Q > 10^7$, using both radius height cutoffs. Thus those elementary costs are smaller too, covering all coefficients. $\square$

The real-radius interval certificate checks the three uniform allocations, both degree ranges, and conversion down to $v = \log 2$. All field, independence and height hypotheses are part of the proof.

16. A first application: differences of powers

For the real logarithms of 2, 3, take

$$D = 1, \quad a_1 = 1, \quad a_2 = \log 3, \quad B' = \frac{m}{\log 3} + n.$$

The cutoff $1/D$ exceeds $\log 2$; replacing a_1 by $\log 2$ would violate the theorem's hypothesis.

Corollary 7.13. For all positive integers m, n ,

$$|2^m - 3^n| \geq 2^m \exp\{-24000(\max\{10, \log(em)\})^2\}. \quad (7.11)$$

Proof. Put $\delta = |3^n 2^{-m} - 1|$. If $\delta \geq 1/2$, the assertion is immediate. Otherwise

$$n \log 3 < m \log 2 + \log(3/2) \leq m \log 3,$$

where the last inequality uses $m \geq 1$. In particular $n \leq m$, and $B' < 2m < em$. Unique prime factorization shows $n \log 3 - m \log 2 \neq 0$. Writing $T = \max\{10, \log(em)\}$, Theorem 7.1 gives

$$|n \log 3 - m \log 2| \geq e^{-21600(\log 3)T^2}.$$

For real $|u| < 1/2$, integration of $1/(1+u)$ gives $|\log(1+u)| \leq 2|u|$. Hence

$$\delta \geq \frac{1}{2}e^{-21600(\log 3)T^2} \geq e^{-24000T^2},$$

since $\log 3 < 1.1$ and $T \geq 10$. Multiply by 2^m . $\square$

This bound concerns all positive m, n , rather than only the nearest powers. It is weak at modest exponents but becomes an effective finiteness argument.

Corollary 7.14. Fix an integer $k \neq 0$. Every positive integer solution of $2^m - 3^n = k$ satisfies

$$\begin{aligned} m &\leq M_k, \\ n &\leq \frac{M_k \log 2 + \log(|k| + 1)}{\log 3}, \end{aligned}$$

where

$$M_k = \max \left\{ 22027, \left(\frac{24000 + \sqrt{24000^2 + 4(\log 2) \log |k|}}{2 \log 2} \right)^2 \right\}. \quad (7.12)$$

Thus only a finite, explicitly specified rectangle must be searched. For $k = 0$ there are no positive solutions.

Proof. From (7.11),

$$m \log 2 \leq \log |k| + 24000(\max\{10, 1 + \log m\})^2.$$

When $m \geq e^{10}$, the squared maximum is $(1 + \log m)^2 \leq \sqrt{m}$. Indeed $e^{s/2}/(1+s)^2$ increases for $s \geq 10$, and $e^5 > 121$. The resulting quadratic inequality in $\sqrt{m}$ gives the second entry in (7.12). If $m < e^{10} < 22027$, the first entry covers it. Finally

$$3^n = 2^m - k \leq 2^m + |k| \leq 2^m(|k| + 1),$$

which gives the displayed bound for n . $\square$

Comparing the bounds with actual small forms

Let p/q run through convergents of $\xi = \log 3 / \log 2$. For the form $\Lambda = p \log 2 - q \log 3$, both Theorem 7.1 and Corollary 7.5 apply with

$$D = D_* = 1, \quad a_1^0 = 1, \quad a_2^0 = \log 3, \quad B' = p / \log 3 + q.$$

Prime factorization verifies nonvanishing and multiplicative independence. The following values are decimal approximations to certified rational intervals:

Convergent p/q	$ p \log 2 - q \log 3 $	Actual $\log \Lambda $	Theorem 7.1 lower bound	Corollary 7.5 lower bound
8/5	0.0521160	-2.95428	$-2.37300 \cdot 10^6$	-11074.0
1054/665	$4.36541 \cdot 10^{-5}$	-10.0392	$-2.37300 \cdot 10^6$	-11074.0
125743/79335	$3.66472 \cdot 10^{-6}$	-12.5168	$-3.51725 \cdot 10^6$	-11074.0
301994/190537	$6.45075 \cdot 10^{-8}$	-16.5565	$-4.04172 \cdot 10^6$	-11074.0
17087915/10781274	$1.22070 \cdot 10^{-8}$	-18.2213	$-6.92788 \cdot 10^6$	-11074.0

The rational interval certificate evaluates $\log 2, \log 3$ by the convergent series

$$\log x = 2 \sum_{j=0}^m \frac{t^{2j+1}}{2j+1} + \text{a remainder in } \left[0, \frac{2t^{2m+3}}{(2m+3)(1-t^2)} \right], \quad t = \frac{x-1}{x+1},$$

after reduction to $1 \leq x \leq 2$. It certifies every continued-fraction digit by checking that the two interval endpoints have the same integer part; it also certifies the signs of the forms and both displayed inequalities. Sixteen convergents are included in the accompanying data.

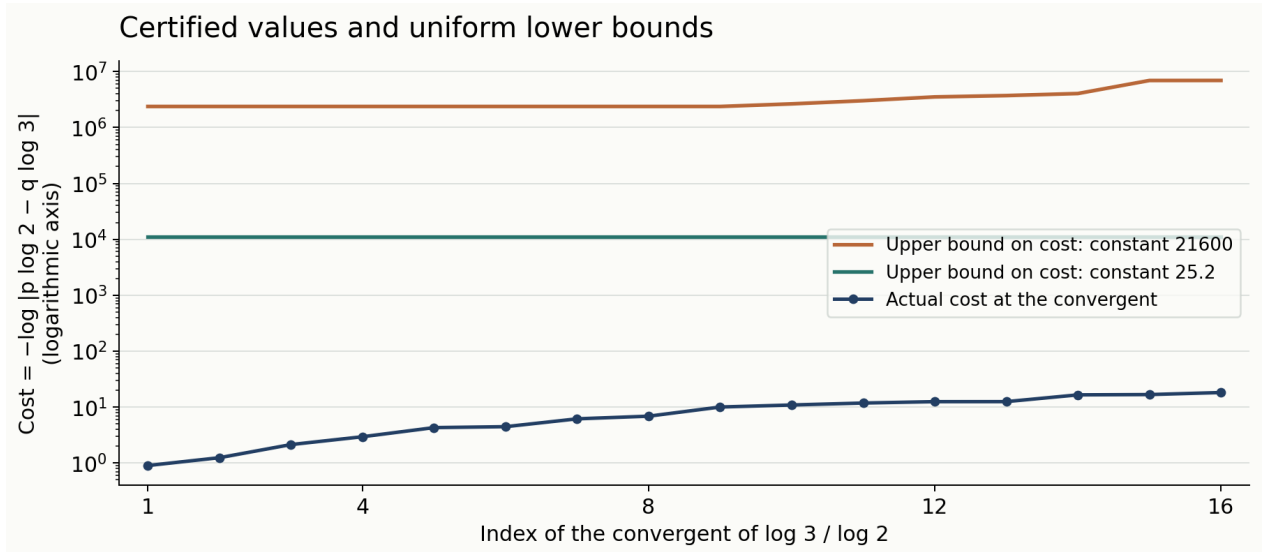


Figure.

The cost is $-\log |\Lambda|$. The vertical axis uses a logarithmic scale, so the uniform estimates and the actual values fit on one plot. The two upper curves bound the cost from above; their plateaus come from the cutoffs 10 and 20. Exact rational intervals underlie the plotted decimal approximations. Original figure and certificate by GPT-6.1 Sol (OpenAI), in Codex, Ultra; CC0. Reproducible figure source.

17. What the squared logarithm does and does not imply

Let $\xi = \log 3 / \log 2$. For rational approximations p/q with p comparable to q , (7.1) gives a lower bound of the shape

$$\left| \xi - \frac{p}{q} \right| \geq \frac{1}{q \log 2} \exp\{-C(\log q)^2\}.$$

Its right side is essentially $q^{-1-C\log q}$. The exponent increases without bound, so this inequality alone supplies no fixed effective irrationality exponent. This is a limitation of the estimate, not a claim that ξ has infinite irrationality exponent.

The effective Baker theorem already proved in *Effective lower bounds II: proof of Baker's theorem* does supply a finite effective exponent for these fixed bases: its lower bound is a fixed power of $\max\{|p|, |q|, 2\}$. Dividing by $q \log 2$, and treating p/q outside a fixed interval around ξ separately, gives a fixed power lower bound for rational approximation. The many-logarithm estimates in the following lesson make this dependence explicit with useful uniform constants. Gouillon's work also supplies two-logarithm bounds with linear dependence on $\log B'$, rather than its square; its numerical constants and hypotheses must be compared alongside this asymptotic advantage.

18. Exercises

1. **Easy.** For the real logarithms of 2, 3, compute admissible a_1, a_2, B' , and write (7.1) explicitly for $m \log 2 - n \log 3$.
2. **Medium.** Deduce Theorem 7.8 from Theorem 7.1, including the nonvanishing argument and the comparison of constants.
3. **Medium.** Give an explicit upper bound for $\max\{m, n\}$ over the positive solutions of $2^m - 3^n = k$, in terms of $|k|$.
4. **Hard.** Explain why a lower estimate with $(\log B)^2$ does not supply a fixed effective irrationality exponent. Show precisely how a lower estimate with $C \log B$, with C fixed for the bases, does.
5. **Medium.** For $b = 2$, write the six polynomials $\delta_b(Z; t)$, $0 \leq t \leq 5$. Compute the determinant of their coefficient matrix in the monomial basis, and all the integral divided derivatives $2^k \delta_2(1; 5, k)/k!$, $0 \leq k \leq 5$.
6. **Medium.** Choose eight orders from $\{0, 1, 2, 3, 4\}$, with each value used at most three times. Determine their largest sum and compare it with (7.41). Explain why the Taylor count in (7.46) uses $K + 1$, rather than the triangular number C_K .
7. **Hard.** In $\mathbb{C}^2 \times \mathbb{C}^\times$, put $Q = X_0 - Y^n$ and $I = (X_1, Q^M)$, with integers $n, M \geq 1$. Show that every element of I and its derivatives through order $M - 1$ vanish on $V = \{X_1 = 0, Q = 0\}$. Prove, for $u \geq M - 1, v \geq nM - 1$, that

$$h_I(u, v) = M(nu + v + 1) - nM(M - 1).$$

Deduce that the factor $T + 1$ in (7.52) can be attained exactly.

8. **Medium.** In Theorem 7.11 take $K = L = 2$, $b_1 = 2, b_2 = 3$, $\alpha_1 = 2, \alpha_2 = 4$, $R_j = S_j = 1$ for all three sets, and $(T_1, T_2, T_3) = (2, 2, 6)$. Verify every cardinality condition. Explain why full rank here is compatible with multiplicative dependence of the bases. Is the numerical determinant inequality (7.56) automatic?
9. **Medium.** For the parameters in Exercise 8, compute ω, ω_0, U and the block size in the uniform criterion. Explain why replacing H_b by L in the logarithm would fail. Verify directly that the basis $\delta_b(Z; t)$, $0 \leq t \leq T$, is unchanged when $b = 17$ is replaced by any larger integer.
10. **Medium.** Let $D \geq 1$ and $\beta = (3/2)^{1/D}$. Show that $\deg \beta = D$ and $Dh(\beta) = \log 3$. For consecutive powers $\alpha_1 = \beta^m, \alpha_2 = \beta^{m+1}$, choose $m \geq \max\{2, \log E / \log 3\}$, where $E = 1 + D \log 3 / \log(3/2)$. Verify every hypothesis of (7.77), compute H and $\Gamma = \log \alpha_2 - \log \alpha_1$, and apply (7.79). Explain why nonintegrality of β causes no problem.

19. Solutions

Solution 1. The Weil heights are $h(2) = \log 2$, $h(3) = \log 3$ and $D = 1$. Thus $a_1 = 1$, $a_2 = \log 3$ are admissible; the integer coefficients are $m, -n$. We obtain

$$\log |m \log 2 - n \log 3| \geq -21600(\log 3) \left(\max \left\{ 10, \log \left(\frac{m}{\log 3} + n \right) \right\} \right)^2.$$

The form is nonzero because $2^m \neq 3^n$.

Solution 2. The field generated by $-1, \alpha$ is $\mathbb{Q}(\alpha)$, of degree D . With $a_1 = \pi/D$, all three lower cutoffs for the first base hold, and the normalized coefficient sum becomes $|b_1|/(Da) + |b_2|/\pi$. If the form vanished, $\alpha^{2b_2} = 1$, contradicting the hypothesis. The constant is $21600\pi < 68000$, yielding exactly (7.10); no independence of $-1, \alpha$ was used.

Solution 3. Let M_k be (7.12) and $U_k = \log(|k| + 1)$. An explicit real upper bound is

$$\max\{m, n\} \leq \max \left\{ M_k, \frac{M_k \log 2 + U_k}{\log 3} \right\}.$$

Taking its ceiling gives an integer search limit. The proof is Corollary 7.14; it includes $|k| = 1$, when $\log |k| = 0$, and both signs of k . For $k = 0$, prime factorization excludes a solution.

Solution 4. Suppose p/q lies in a fixed bounded interval around ξ ; then $\max\{|p|, q, 2\} \leq C_0 q$. A bound $\log |p \log 2 - q \log 3| \geq -C(\log(C_0 q))^2$ gives a variable exponent of order $C \log q$. For any fixed μ , its lower bound is eventually smaller than $q^{-\mu}$, so it does not entail such a fixed power bound. By contrast, $\log |p \log 2 - q \log 3| \geq -C \log(C_0 q)$ implies

$$|\xi - p/q| \geq \frac{C_0^{-C}}{\log 2} q^{-(C+1)}.$$

Outside that interval a fixed positive distance suffices. This proves a finite effective irrationality exponent after enlarging the constant to cover small denominators.

Solution 5. The polynomials are

$$1, \quad Z, \quad (Z^2 - Z)/2, \quad (Z^3 - Z^2)/2, \quad (Z^4 - 2Z^3 + Z^2)/4, \quad (Z^5 - 2Z^4 + Z^3)/4.$$

The coefficient matrix is triangular, with diagonal $1, 1, 1/2, 1/2, 1/4, 1/4$, so its determinant is $1/64 \neq 0$. This is the invertible column transformation used in the derivative-basis lemma. At $Z = 1 + U$, the last polynomial is $U^2(1 + U)^3/4 = (U^2 + 3U^3 + 3U^4 + U^5)/4$. Multiply its coefficient of U^k by 2^k ; the requested integers, in order, are $0, 0, 1, 6, 12, 8$. The factorial normalization is essential: Taylor coefficients are divided derivatives.

Solution 6. Fill the largest orders first: three copies of four, three copies of three and two copies of two. Their sum is 25. With $q = 3, T = 4, N = 8$, the second right side of (7.41) is $32 - 64/6 + 4 = 76/3$, so the bound holds, with loss $1/3$. This is exactly $r(q - r)/(2q)$ for $m = 2, r = 2$. For the Taylor count, fix a homogeneous degree d . If $d \leq K$, its space has dimension $d + 1 \leq K + 1$. If $d > K$, every homogeneous Taylor part is divisible by the same w^{d-K} , and the quotient is a homogeneous binary form of degree K , whose dimension is $K + 1$. The C_K original monomials do not give C_K independent directions at each fixed Taylor degree. This is the analytic advantage of the triangular row indexing.

Solution 7. Since $\mathcal{D}X_1 = 0$, all derivatives of the first generator vanish on V . In a derivative of Q^M of order less than M , at least one factor Q remains. Leibniz's rule extends this vanishing to all elements of I . The quotient is $\mathbb{C}[Y, Y^{-1}, Q]/(Q^M)$, with $X_0 = Y^n + Q$. For a filtered monomial,

$$X_0^a Y^c = \sum_{i=0}^{M-1} \binom{a}{i} Q^i Y^{n(a-i)+c}, \quad 0 \leq a \leq u, \quad 0 \leq c \leq v,$$

where terms with $i > a$ are zero. Group the monomials by weight $w = na + c$. Distinct weights occupy disjoint basis elements, because the basis element $Q^i Y^j$ has weight $j + ni$. For fixed w , let c_w be the number of integers a with $0 \leq a \leq u, 0 \leq w - na \leq v$. Its columns are $((\binom{a}{i})_{0 \leq i < M})$ for distinct such a , and have rank $\min(M, c_w)$. Indeed, binomial polynomials of degrees $0, \dots, r - 1$ have nonzero triangular leading coefficients, so their evaluation determinant at r distinct integers is a nonzero multiple of the Vandermonde determinant. The possible rows with negative Y -exponent have zero coefficients and do not change this rank: $r - 1 \leq \lfloor w/n \rfloor$. Thus $h_I(u, v) = \sum_w \min(M, c_w)$.

For $1 \leq i \leq M$, the condition $c_w \geq i$ means that w is covered by a block of i consecutive intervals $[na, na + v]$. Equivalently it belongs to one of

$$[n(a + i - 1), na + v], \quad 0 \leq a \leq u - i + 1.$$

Because $v \geq nM - 1 \geq ni - 1$, these integer intervals join without gaps. Their union contains $nu + v + 1 - 2n(i - 1)$ integers. Summing over $i = 1, \dots, M$ gives the required formula. Its leading part is $Mn, u + Mv$, so $A(I) = MA(V)$ and $B(I) = MB(V)$, with $A(V) = 1, B(V) = n$. Taking $T = M - 1$ attains both bounds in (7.52).

Solution 8. All three sets have additive values $\{0, 2, 3, 5\}$, multiplicative values $\{1, 2, 4, 8\}$, and four distinct pairs. The six conditions are respectively $2 \geq 2, 4 \geq 3, 12 \geq 3, 12 \geq 9, 12 \geq 5, 28 \geq 25$. Consequently the full 18-row matrix with $R = S = 3, T = 10$ has rank 18. Its 176 columns are also checked by exact rational elimination in the geometric certificate. The theorem requires these actual cardinalities, so the relation $\alpha_2 = \alpha_1^2$ causes no conflict. For the chosen real logarithms, the form in Section 14 is $4 \log 2$, which is nonzero. Full rank establishes existence of a nonzero minor. The separate size inequality (7.56) still requires its own parameter check.

Solution 9. Here $q = 16, N = 18, T = 10$, so

$$\omega = \frac{167}{176}, \quad \omega_0 = \frac{176}{9}, \quad U = \frac{23003}{792}.$$

The bounds $1.098 < \log 3 < 1.099$ give $16J_2 < U < 17J_2$. Thus the rounded block size is $b = 17$, and $H_b = 16$. Using $L = 2$ instead would replace $\log(16/17)$ by the smaller $\log(2/17)$, which does not give the required upper bound for the product of derivatives. For every $b \geq 17$ and $0 \leq t \leq 10$, division of t by b has quotient zero and remainder t . Consequently $\delta_b(Z; t) = \binom{Z}{t}$ throughout the column block. The common integer multiplier $\nu(b)$ can change, but the polynomial basis and its rank do not.

Solution 10. The primitive polynomial $2X^D - 3$ is irreducible by the same reduction argument as in the radical example, now modulo three. In a nontrivial integer factorization, neither leading coefficient is divisible by three, and both constant coefficients would be divisible by three. Their product cannot be -3 . Every conjugate has modulus $(3/2)^{1/D}$; thus its Mahler measure is $2(3/2) = 3$, and the height formula gives $Dh(\beta) = \log 3$. The consecutive powers generate β by their ratio. Set $a_1 = m \log 3/D$ and $a_2 = (m + 1) \log 3/D$. Height homogeneity makes these valid height bounds. Since $\log(3/2) < \log 3$ and $m \log 3 \geq 2 \log 3 > 1$, the other height cutoffs hold. The exact radius is the stated E , and $m \log 3 \geq \log E$ implies both exponential radius caps. We have $H = \log 3, k = 1$, and

$$\Gamma = \frac{\log(3/2)}{D} = \frac{H}{E - 1}.$$

Also $B' \leq 5/(6 \log 3) < 1$, so $W_0 = 3/D$. The generator is not an algebraic integer, because its primitive irreducible polynomial is not monic; in particular it is not an algebraic unit. The first condition of the proposition therefore proves (7.79). Its proof uses the primitive integer polynomial and applies to nonintegral algebraic numbers as well.

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- [Gouillon 2003] Nicolas Gouillon, *Minorations explicites de formes linéaires en deux logarithmes*, doctoral thesis, Université de la Méditerranée, 2003, HAL version tel-00003964v1. Section 5.1 supplies the rounded-box construction; Section 5.3.2 discusses its occupancy. The precise adapted choices, estimates and full-rank verification used here are proved in the lesson.

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