Two p-adic logarithms: determinant bounds

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A large valuation of $2^a - 3^b$ means that the two powers agree to many p-adic digits. The determinant method measures how far this agreement can continue. It places products of powers in a single residue class, obtains a nonzero algebraic determinant, and compares a height bound with the large valuation forced by near agreement.

We use the exponential, logarithm, power laws and cyclotomic distances proved in p-adic analysis for linear forms in logarithms. The polynomial zero argument is the method proved in Two logarithms I: interpolation determinants. The weighted product formula is Places of number fields in extensions and the product formula, Theorem 5.2. The height convention and its power law are proved in Heights of algebraic numbers. The necessary determinant estimates are written out below.

Throughout, $v = v_p$ is normalized by $v(p) = 1$, and ln is the real natural logarithm. All algebraic numbers have a specified embedding into $\overline{\mathbb{Q}}_p$. We distinguish the global degree d, local ramification index e, and local residue degree f.

1. Residues and principal units

Put $K_0 = \mathbb{Q}(\alpha_1, \alpha_2)$, $d = [K_0 : \mathbb{Q}]$, and $F = \mathbb{Q}_p(\alpha_1, \alpha_2)$. Its residue field has p^f elements and $[F : \mathbb{Q}_p] = ef \le d$. Assume $v(\alpha_i) = 0$.

The multiplicative group of the residue field is cyclic. Here is an elementary proof. In a finite abelian group, elements whose orders have prescribed prime power parts can be multiplied to obtain an element of order equal to the least common multiple m of all element orders. To see this, for each prime choose an element whose order contains the largest occurring power of that prime, and raise it to its complementary part; the resulting elements have pairwise coprime orders. Their product has the product of those orders, since a power that kills the product must kill each component. In a finite subgroup of a field's multiplicative group, every element is a root of $X^m - 1$, so the group has at most m elements. The element just constructed already has m distinct powers. It generates the group.

Every nonzero residue has a unique lift to a $(p^f - 1)$-st root of unity in F. For completeness, take any unit a with that residue and set $g = p^f - 1$. Since $a^g \in 1 + \mathfrak{m}_F$, the powers $(a^g)^{p^n}$ tend to one: the power laws of Lesson 9 first enter $v(x - 1) > 1/(p - 1)$, and then add one at each further p-th power. Consequently

$$a^{p^{f(n+1)}} - a^{p^{fn}} = a^{p^{fn}}\left((a^g)^{p^{fn}} - 1\right) \longrightarrow 0.$$

The differences have valuations tending to infinity, so the sequence is Cauchy. Its limit η has the original residue and satisfies $\eta^{p^f} = \eta$. Thus $\eta^g = 1$. If two such lifts have the same residue, their quotient w is a principal unit with $w^g = 1$. The prime-to-p power law in Lesson 9 gives $v(w^g - 1) = v(w - 1)$ when $w \neq 1$, which rules this out. This proves uniqueness.

Lift a generator of the residue group to η. We may write

$$\alpha_i = \eta^{m_i}\epsilon_i, \qquad \epsilon_i \in 1 + \mathfrak{m}_F, \qquad g = p^f - 1. \tag{10.1}$$

We use the full residue group here; no condition $\gcd(m_1, m_2, g) = 1$ is required. Choose the least integer $t \geq 0$ such that

$$p^t > \frac{e}{p-1}, \qquad c_0 = \frac{p^t}{e}. \tag{10.2}$$

Lesson 9 proves $v(\epsilon_i^{p^t} - 1) \geq c_0 > 1/(p-1)$. Also

$$\frac{p^t}{e} \leq \frac{p}{p-1} \leq 2. \tag{10.3}$$

If $t > 0$, minimality gives the first inequality. If $t = 0$, it follows from $e \geq 1$.

2. Three integral determinant estimates

Lemma 10.1 (factorials and alternants). For $n \geq 0$,

$$\frac{n}{p-1} - \frac{\ln(n+1)}{\ln p} \leq v(n!) \leq \frac{n}{p-1}. \tag{10.4}$$

For $z_1, \ldots, z_n \in \mathbb{Z}_p$,

$$v\left(\prod_{i<j}(z_j - z_i)\right) \geq \sum_{j=0}^{n-1} v(j!) \geq \frac{n(n-1)}{2(p-1)} - \frac{n \ln n}{\ln p}. \tag{10.5}$$

Use $0 \ln 0 = 0$. If $0 \leq \nu_1 < \cdots < \nu_n$, the determinant $\det(z_i^{\nu_j})$ satisfies the same lower bound.

Proof. The digit formula $v(n!) = (n - s_p(n))/(p-1)$ was proved in Lesson 9. At a fixed digit sum $s = a(p-1) + c$, $0 \leq c < p-1$, the smallest nonnegative integer is $(c+1)p^a - 1$: moving one digit unit from a higher position to an unfilled lower position decreases the number. Concavity of $\ln(1+x)$ on $[0, p-1]$ gives $\ln(c+1) \geq c \ln p/(p-1)$. Hence $s_p(n) \leq (p-1)\ln(n+1)/\ln p$, proving (10.4).

The polynomial $\binom{Z}{j}$ is p-adically integral on $\mathbb{Z}_p$. It is integral on nonnegative integers, which are dense in $\mathbb{Z}_p$: every residue modulo p^h has such a representative. Continuity gives the assertion on the completion. Replacing the powers in the Vandermonde determinant by falling factorials gives

$$\det\left(\binom{z_i}{j-1}\right)_{i,j=1}^{n} = \frac{\prod_{i<j}(z_j - z_i)}{\prod_{j=0}^{n-1} j!}.$$

The determinant on the left is integral. Sum (10.4), and use $\ln(n!) \leq n \ln n$, to obtain (10.5).

Finally $\det(Z_i^{\nu_j})$ is alternating, so it vanishes on $Z_i = Z_j$. Division by the monic polynomial $Z_j - Z_i$ takes place over $\mathbb{Z}$. Each such factor is prime in $\mathbb{Z}[Z_1, \ldots, Z_n]$, because its quotient ring is a polynomial ring over $\mathbb{Z}$, an integral domain. Distinct pair factors are not associates. They therefore all divide the alternant, with an integer polynomial quotient. Evaluating that quotient in $\mathbb{Z}_p$ and applying (10.5) proves the last assertion. □

Lemma 10.2 (centering without an occupancy hypothesis). Let $N = KL$, and let the row labels l_i consist of K copies of each integer $0, \ldots, L-1$. For any ordering of integers $0 \leq r_i \leq R-1$,

$$\left|\sum_i l_i r_i - \frac{L-1}{2}\sum_i r_i\right| \le \frac{NL(R-1)}{8}. \tag{10.6}$$

Proof. The weights $l_i-(L-1)/2$ sum to zero. Their positive sum is $K\lfloor L^2/4\rfloor/2$: sum the arithmetic progression on either side of the midpoint. A weighted sum with $r_i \in [0, R-1]$ has maximum at $R-1$ on the positive weights and zero on the negative weights, and minimum at the reverse choice. Both absolute values are at most $K(R-1)L^2/8 = NL(R-1)/8$. □

Lemma 10.3 (a polynomial zero lemma). Let $a_1, a_2 \neq 0$ lie in a field of characteristic zero. Let $E_1, E_2 \subset \mathbb{Z}^2$ be finite and suppose

$$\#\{a_1^r a_2^s : (r,s) \in E_1\} \ge L, \qquad \#\{b_2 r + b_1 s : (r,s) \in E_2\} > (K-1)L. \tag{10.7}$$

A nonzero polynomial of partial degrees at most $K-1, L-1$ cannot vanish at all points

$$(b_2(r+r') + b_1(s+s'), a_1^{r+r'} a_2^{s+s'}), \quad (r,s) \in E_1,\ (r',s') \in E_2.$$

Proof. Write $P(X,Y) = \sum_{j=1}^{q} Q_j(X) Y^{e_j}$, with distinct $0 \le e_j < L$ and nonzero Q_j. Evaluation of these monomials on the at least L distinct available values has row rank q: otherwise a nonzero polynomial of degree less than L would have at least L roots. Choose q labels $(r_i, s_i) \in E_1$ giving an invertible matrix $(y_i^{e_j})$, where $x_i = b_2 r_i + b_1 s_i$ and $y_i = a_1^{r_i} a_2^{s_i}$.

The polynomial $D(X) = \det(Q_j(X+x_i) y_i^{e_j})_{i,j}$ has nonzero leading coefficient: it is the product of the leading coefficients of the Q_j's times $\det(y_i^{e_j})$. Its degree is $\sum_j \deg Q_j \le (K-1)L$. Proposed vanishing gives, at each point (x,y) from E_2, a homogeneous linear system with coefficient determinant $D(x)$ and nonzero vector (y^{e_j}). Thus $D(x) = 0$. The second cardinality condition supplies more roots than its degree. □

3. A close root and the analytic determinant

Lemma 10.4 (removing the common p-part). Suppose ϵ_1, ϵ_2 are principal units in F, $b_i > 0$, and $u = v_p(\gcd(b_1, b_2))$. If

$$v(\epsilon_1^{b_1} - \epsilon_2^{b_2}) > u + \frac{1}{p-1},$$

there is a root of unity $\zeta \in F$, of order dividing both p^u and p^t, such that, for $b_i' = b_i/p^u$,

$$\delta = \epsilon_1^{b_1'} - \zeta \epsilon_2^{b_2'}, \qquad v(\delta) = v(\epsilon_1^{b_1} - \epsilon_2^{b_2}) - u. \tag{10.8}$$

Proof. Set $\sigma = \epsilon_1^{b_1'}/\epsilon_2^{b_2'}$. Then $v(\sigma^{p^u} - 1)$ equals the valuation in the hypothesis. Choose a p^u-th root ζ maximizing $v(\sigma - \zeta)$. For every other root ξ, maximality and the ultrametric inequality give $v(\sigma - \xi) \le v(\zeta - \xi)$. The cyclotomic product identity proved in Lesson 9 gives

$$\sum_{\xi^{p^u}=1,\ \xi \neq \zeta} v(\zeta - \xi) = u.$$

Factoring $\sigma^{p^u} - 1$ therefore yields $v(\sigma - \zeta) > 1/(p-1)$. Distinct p-power roots have distance valuation at most $1/(p-1)$, again by Lesson 9. Thus all the other factors have exactly their root distance valuation, giving $v(\sigma - \zeta) = v(\sigma^{p^u} - 1) - u$.

The chosen root is closer than any of its other $\mathbb{Q}_p$-conjugates, so Krasner's lemma, in Extensions of complete valued fields, Theorem 4.1, gives $\zeta \in \mathbb{Q}_p(\sigma) \subset F$. If its order is p^w with $w > 0$, the cyclotomic ramification theorem proved in Lesson 9 gives $p^{w-1}(p-1) \le e$. Definition (10.2) implies $w \le t$. Multiplying $\sigma - \zeta$ by the unit $\epsilon_2^{b_2'}$ proves (10.8). $\square$

Take a matrix with rows (k, l), $0 \le k < K$, $0 \le l < L$, and distinct selected columns (r_j, s_j), $0 \le r_j < R$, $0 \le s_j < S$, all in one class $m_1 r_j + m_2 s_j \equiv c \pmod{g}$. Its entries are

$$\binom{b_2 r_j + b_1 s_j}{k} \alpha_1^{p^t l r_j} \alpha_2^{p^t l s_j}. \tag{10.9}$$

Lemma 10.5 (analytic valuation). Let Δ be a determinant of order $N = KL$ extracted in this way. Write $\Lambda = \alpha_1^{b_1} - \alpha_2^{b_2}$. If

$$v(\Lambda) \ge c_0 \left(N - \frac{1}{2} \right) + u,$$

then

$$v(\Delta) \ge \frac{p^t N K (L-1)}{2e} - \frac{N \ln N}{\ln p}. \tag{10.10}$$

Proof. Substitute (10.1). Since all columns have one residue class, the root-of-unity factor in each row is constant across that row; removing these factors changes no valuation. The positive valuation of Λ also forces $m_1 b_1 \equiv m_2 b_2 \pmod{g}$, so the principal-unit difference has the same valuation. We may therefore work with ϵ_1, ϵ_2.

Lemma 10.4 applies. Interchange the two bases, coefficients and column coordinates if necessary so that $p \nmid b_2'$. Put

$$\theta = b_1'/b_2' \in \mathbb{Z}_p, \qquad z_j = r_j + \theta s_j \in \mathbb{Z}_p, \qquad \omega = \log(\epsilon_1^{p^t}), \qquad v(\omega) \ge c_0.$$

The logarithm here is the p-adic logarithm. The deep-unit power laws of Lesson 9, together with $\zeta^{p^t} = 1$, give

$$\epsilon_2^{p^t l s_j} = \epsilon_1^{p^t \theta l s_j} \big(1 - \epsilon_1^{-b_1'} \delta\big)^{p^t l s_j / b_2'} = \epsilon_1^{p^t \theta l s_j} (1 + \delta q_{ij}), \qquad v(q_{ij}) \ge 0. \tag{10.11}$$

Indeed the final power has p-adic integral exponent, and the logarithm and exponential are isometries on their deep domains. Its difference from one has valuation at least $v(\delta)$. All the substitutions take place in those domains, since $v(\delta) \ge c_0(N - 1/2) > 1/(p-1)$.

If $\delta = 0$, the power factor in (10.11) is exactly one; set every $q_{ij} = 0$. This treats a possible zero difference without dividing by zero.

The factor q_{ij} in (10.11) depends on l_i, s_j, not on k_i. Within each fixed l, the binomial polynomials $\binom{X}{k}$ may be replaced by $X^k/k!$ by a triangular row transformation with diagonal entries one. Factoring $b_2^{k_i}$ from each row, define

$$\phi_i(z) = \frac{z^{k_i}}{k_i!} \exp(l_i \omega z), \qquad \Delta' = b_2^{\sum_i k_i} \det\big(\phi_i(z_j)(1 + \delta q_{ij})\big). \tag{10.12}$$

Here $v(\Delta') = v(\Delta)$, and the extracted scalar has nonnegative valuation.

Expand the determinant by choosing its ordinary part in a set I of n rows and its error part in the other $N-n$ rows. Expand that summand by the columns chosen for those n rows. Every summand then contains a factor δ^{N-n}, a pure n-row minor A of $\phi_i(z_j)$, and a complementary error minor B. We claim

$$v(A) \geq c_0 \left(\frac{n(n-1)}{2} - \sum_{i \in I} k_i \right) - \frac{n \ln n}{\ln p}, \qquad v(B) \geq - \frac{\sum_{i \notin I} k_i}{p-1}. \tag{10.13}$$

The bound for B follows directly from $z_j \in \mathbb{Z}_p$, unit exponential values, integral q_{ij}, and $v(k_i!) \leq k_i/(p-1)$.

For A, expand

$$\phi_i(z) = \sum_{\nu \geq k_i} \binom{\nu}{k_i} l_i^{\nu - k_i} \omega^{\nu - k_i} \frac{z^\nu}{\nu!}.$$

Its determinant is a sum over distinct orders $0 \leq \nu_1 < \cdots < \nu_n$. The node determinant is $\det(z_j^{\nu_h})$; Lemma 10.1 bounds its valuation below by $n(n-1)/(2(p-1)) - n \ln n / \ln p$. The coefficient determinant is an integer polynomial in ω, homogeneous of degree $\sum_h \nu_h - \sum_{i \in I} k_i$. If it is nonzero this degree is nonnegative. The factorial denominators lose at most $\sum_h \nu_h/(p-1)$. Since $c_0 > 1/(p-1)$ and $\sum_h \nu_h \geq n(n-1)/2$, the resulting lower bound is exactly the bound for A in (10.13). This argument includes $\omega = 0$; terms of positive degree then vanish.

These series may be regrouped: ϕ_i is analytic on a disc of radius greater than one, because $v(\omega) > 1/(p-1)$. In a finite product of its convergent coefficient series only finitely many terms exceed any fixed positive absolute size. The summation lemma of Lesson 9 justifies the multilinear expansions and the regrouping by distinct orders.

Now $\sum_i k_i = N(K-1)/2$. As $c_0 > 1/(p-1)$, each complete summand has valuation at least

$$(N-n) v(\delta) + c_0 \left(\frac{n(n-1)}{2} - \frac{N(K-1)}{2} \right) - \frac{N \ln N}{\ln p}.$$

For $0 \leq n \leq N$,

$$(N-n)\left(N - \frac{1}{2}\right) + \frac{n(n-1)}{2} = \frac{N(N-1)}{2} + \frac{(N-n)^2}{2} \geq \frac{N(N-1)}{2}. \tag{10.14}$$

Apply the ultrametric inequality to the finite row expansion, then subtract $N(K-1)/2$. This proves (10.10). $\square$

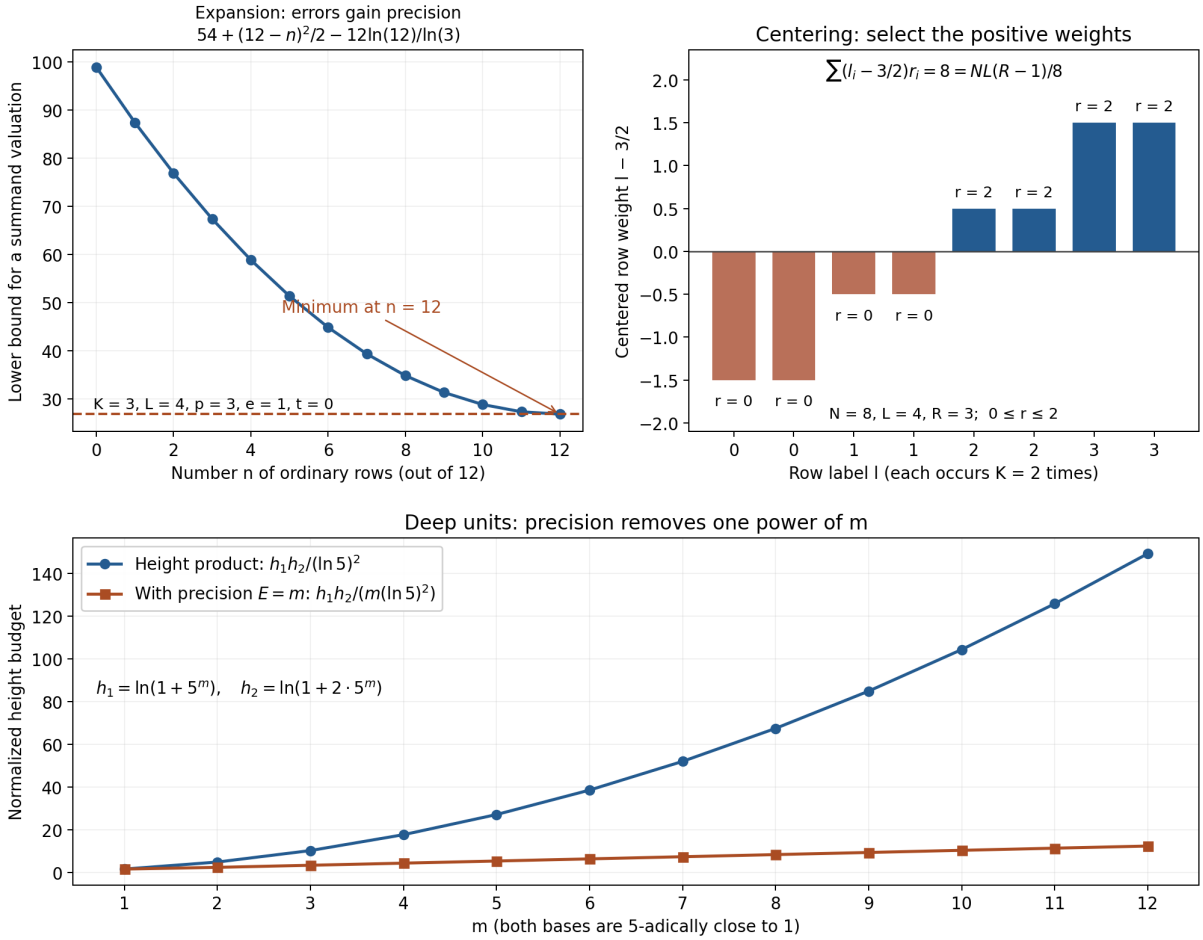


Figure 1. Left: Lemma 10.5 with $K = 3, L = 4, N = 12, p = 3, e = 1, t = 0, u = 0$. Under its precision hypothesis, a summand using n ordinary rows has valuation at least $54 + (12 - n)^2/2 - 12 \ln 12/\ln 3$; the dots are these lower bounds, not measured determinant valuations. The minimum is attained in the bound at $n = 12$. Right: Lemma 10.2 with $K = 2, L = 4, N = 8, R = 3$. The weights are $l_i - 3/2$; selecting $r_i = 2$ on positive weights and zero on negative weights gives the exact extremum $8 = NL(R - 1)/8$. Equations (10.6) and (10.14) prove these two mechanisms. Bottom: the family $\alpha_1 = 1 + 5^m, \alpha_2 = 1 + 2 \cdot 5^m$ from Theorem 10.11, with $1 \le m \le 12$. The curves show the calculated height budgets $h_1 h_2/(\ln 5)^2$ and $h_1 h_2/(m(\ln 5)^2)$, where $h_i = h(\alpha_i)$. They illustrate the precision divisor in (10.43); the curves are height factors, while the exact difference valuation is m.

4. The arithmetic determinant and a parameter theorem

For the selected columns define

$$W = (R - 1)b_2 + (S - 1)b_1, \qquad \beta = \frac{W}{2} \left(\prod_{k=1}^{K-1} k! \right)^{-2/(K(K-1))}. \tag{10.15}$$

Lemma 10.6 (height bound). If $\Delta \neq 0$, then

$$v(\Delta) \le \frac{dN}{2ef \ln p} \left\{ \ln N + (K - 1) \ln \beta + \frac{p^t L}{2} ((R - 1)h(\alpha_1) + (S - 1)h(\alpha_2)) \right\}. \tag{10.16}$$

Proof. The polynomial

$$P(X,Y)=\det\left(\binom{b_2r_j+b_1s_j}{k_i}X^{l_ir_j}Y^{l_is_j}\right)\in\mathbb{Z}[X,Y]$$

satisfies $\Delta = P(\alpha_1^{p^t},\alpha_2^{p^t})$. Its coefficients are integers because each binomial is evaluated at a nonnegative integer.

On $|X|=|Y|=1$, replace the binomial basis in each l-block by $(Z-W/2)^k/k!$, again without changing the determinant. Each row has Euclidean norm at most $\sqrt{N}(W/2)^k/k!$. Hadamard's inequality gives

$$\|P\|_{\mathbb{T}^2}\le N^{N/2}\beta^{N(K-1)/2}. \tag{10.17}$$

One can prove the inequality used here by successively subtracting orthogonal projections of each row onto the span of earlier rows. These operations preserve the determinant; the resulting perpendicular row lengths multiply to its absolute value, and each is no larger than its original row length. If the rows are dependent the determinant is zero.

Every monomial in the determinant expansion has first exponent $\sum_i l_i r_{\pi(i)}$. Lemma 10.2 places all such exponents in an interval of length at most $NL(R-1)/4$, and places all second exponents in an interval of length at most $NL(S-1)/4$. Divide P by its smallest actual powers of X,Y, obtaining an integer polynomial Q of these partial degrees. Multiplication by those unit powers changes neither its torus norm nor the valuation at the selected embedding.

We spell out the torus form of Liouville's inequality. For a complex polynomial of partial degrees a,b, maximum modulus applied inside the unit disc, and to the reciprocal polynomial outside it, gives

$$|Q(x,y)|\le\|Q\|_{\mathbb{T}^2}\max(1,|x|)^a\max(1,|y|)^b.$$

Apply the one-variable argument successively. At finite places an integer polynomial satisfies the same bound without the torus factor. The weighted product formula, applied to the nonzero value, and the definition $h(x)=d^{-1}\sum_w n_w\ln^+|x|_w$, therefore give

$$v(Q(x,y))\le\frac{d}{ef\ln p}\left(\ln\|Q\|_{\mathbb{T}^2}+ah(x)+bh(y)\right).$$

Here the selected local weight is ef. A nonzero integer polynomial has torus norm at least one: any nonzero coefficient is the corresponding Fourier coefficient on the torus and has absolute value at most that norm. Thus the archimedean factor used in the product formula has nonnegative logarithm. Insert (10.17), the two degree widths, and $h(\alpha_i^{p^t})=p^th(\alpha_i)$. This proves (10.16). □

Theorem 10.7 (p-adic determinant parameter theorem). Let $K\ge3, L\ge2$, and let R_1,R_2,S_1,S_2 be positive integers. Put $R=R_1+R_2-1$, $S=S_1+S_2-1$, $N=KL$, and define β by (10.15). Suppose two residue classes $c_1,c_2 \pmod g$ satisfy

$$\begin{aligned}&\#\{\alpha_1^{p^tr}\alpha_2^{p^ts}:0\le r<R_1,\ 0\le s<S_1,\ m_1r+m_2s\equiv c_1 \pmod g\}\ge L,\\ &\#\{b_2r+b_1s:0\le r<R_2,\ 0\le s<S_2,\ m_1r+m_2s\equiv c_2 \pmod g\}>(K-1)L.\end{aligned} \tag{10.18}$$

If

$$\begin{aligned}p^tK(L-1)\ln p>{}&(d/f+2e)\ln N+(d/f)(K-1)\ln\beta\\ &+\frac{dp^tL}{2f}\big((R-1)h(\alpha_1)+(S-1)h(\alpha_2)\big),\end{aligned} \tag{10.19}$$

then

$$v(\alpha_1^{b_1} - \alpha_2^{b_2}) < \frac{p^t}{e}\left(N - \frac{1}{2}\right) + u. \tag{10.20}$$

Proof. The two rectangles restricted to their indicated classes are sets E_1, E_2. Their sum lies in the larger rectangle and class $c_1 + c_2$. Apply Lemma 10.3 with $a_i = \alpha_i^{p^t}$. A row dependence in (10.9) would be a nonzero polynomial of the stated partial degrees vanishing on the sum set. Hence the matrix restricted to that class has row rank N, and contains a nonzero minor Δ.

If (10.20) failed, Lemmas 10.5 and 10.6 would both apply. Multiply their comparison by $2e \ln p/N$. The result is the opposite weak inequality to (10.19). This contradiction proves the theorem. □

5. A bound with the height floor $1/d$

Theorem 10.8 (two p-adic logarithms). Suppose α_1, α_2 are multiplicatively independent p-adic units, and b_1, b_2 are nonzero integers. Let $d = [\mathbb{Q}(\alpha_1, \alpha_2) : \mathbb{Q}]$, and choose real $A_i > 1$ satisfying

$$\ln A_i \geq \max\{h(\alpha_i), 1/d\}.$$

Define

$$\begin{aligned} B' &= \frac{|b_1|}{d \ln A_2} + \frac{|b_2|}{d \ln A_1}, \\ H &= \max\left\{\ln B' + \ln\ln p, \frac{10 \ln p}{d}\right\}. \end{aligned} \tag{10.21}$$

Then

$$v(\alpha_1^{b_1} - \alpha_2^{b_2}) \leq 2500 \, \frac{p^d - 1}{(\ln p)^4} \, d^4 \ln A_1 \ln A_2 \, H^2. \tag{10.22}$$

In particular, the same inequality holds with 132000 in place of 2500.

Proof. Inverting a base when its exponent is negative makes both exponents positive, preserving the difference, heights, degree, and independence. We assume this has been done. Set

$$G = p^d - 1, \quad a_i = \frac{d \ln A_i}{\ln p}, \quad z = G a_1 a_2, \quad B = \frac{dH}{\ln p}, \quad k = 256. \tag{10.23}$$

We have $B \geq 10$, $a_i \geq 1/\ln p$, and $z \geq d^2 \geq 1$. For the last assertion use $\mathrm{e}^x - 1 \geq x^2$ for $x \geq 0$, with $x = d \ln p$. Its derivative is $\mathrm{e}^x - 2x > 0$, since $\mathrm{e}^x \geq 1 + x + x^2/2 > 2x$, and it vanishes at $x = 0$.

Choose

$$\begin{aligned} L &= \lfloor 2B \rfloor + 2, & K &= \lfloor kzL \rfloor + 1, \\ R_1 &= \left\lfloor \sqrt{GLa_2/a_1} \right\rfloor + 1, & S_1 &= \left\lfloor \sqrt{GLa_1/a_2} \right\rfloor + 1, \\ R_2 &= \left\lfloor \sqrt{G(K-1)La_2/a_1} \right\rfloor + 1, & S_2 &= \left\lfloor \sqrt{G(K-1)La_1/a_2} \right\rfloor + 1. \end{aligned} \tag{10.24}$$

Thus $2B < L \leq (11/5)B$, $K \geq kzL$, and

$$R_1S_1 > GL \geq gL, \qquad R_2S_2 > G(K-1)L \geq g(K-1)L. \tag{10.25}$$

Multiplicative independence makes the products in the first rectangle distinct. Pigeonhole among the g classes gives the first condition in (10.18).

We first suppose the map $(r,s) \mapsto b_2r + b_1s$ is injective within every class of the second rectangle. Pigeonhole gives the second condition. We verify the remaining parameter inequality.

Let $Q_K = (\prod_{j=1}^{K-1} j!)^{-2/(K(K-1))}$. Elementary integration of $\ln x$ gives $\ln(j!) \geq j\ln j - j$. Since $x\ln x$ increases for $x \geq 1$,

$$\sum_{j=1}^{K-1} j\ln j \geq \int_0^{K-1} x\ln x\,dx = \frac{(K-1)^2}{2}\ln(K-1) - \frac{(K-1)^2}{4}.$$

For the first interval the integral is negative and its endpoint value is zero; for every later unit interval the increasing-function comparison applies. Consequently

$$\ln Q_K \leq -\ln(K-1) + \frac{3}{2} + \frac{\ln(K-1) - 1/2}{K} \leq -\ln(K-1) + 2. \tag{10.26}$$

The last step uses $\ln m \leq (m+1)/2$ for $m \geq 2$, obtained by differentiating from $m = 2$. Also $K - 1 \geq 255zL$. Formula (10.24) gives

$$\begin{aligned} W &\leq \sqrt{zL}(1+\sqrt{K-1})\left(\frac{b_1}{a_2} + \frac{b_2}{a_1}\right), \\ \beta &\leq \frac{\mathrm{e}^2}{2\sqrt{255}}\left(1 + \frac{1}{\sqrt{K-1}}\right)\left(\frac{b_1}{a_2} + \frac{b_2}{a_1}\right) < \frac{b_1}{a_2} + \frac{b_2}{a_1} = B'\ln p. \end{aligned} \tag{10.27}$$

Indeed $\mathrm{e} < 3$, $\sqrt{255} > 15$, and the parenthesis is at most two. The elementary bound $\mathrm{e} < 3$ follows by bounding the tail of $\sum 1/j!$ by a geometric series. Thus $\ln\beta \leq H$. Using $d/f \leq d$, $e \leq d$, $p^t \geq 1$, and the height budgets, a sufficient condition for (10.19) is

$$K(L-1) > \frac{3d}{\ln p}\ln N + (K-1)B + L\big((R-1)a_1 + (S-1)a_2\big). \tag{10.28}$$

This sufficient comparison is valid even when $\ln\beta < 0$, because $H > 0$ and $\ln\beta \leq H$.

The left side minus $(K-1)B$ is at least $kzLB$. For the height term,

$$L\big((R-1)a_1 + (S-1)a_2\big) \leq 2\sqrt{k}\,zL^2 + 2\sqrt{z}\,L^{3/2} < 72zLB. \tag{10.29}$$

To check the last strict inequality, divide by zLB. The first term is at most $32 \cdot 11/5 = 70.4$; the second is at most $2\sqrt{(11/5)/10} < 1$.

Also $N \leq 257zL^2$. Since $z \geq d^2$, we have $d/z \leq 1$ and $d\ln z/z \leq 1$: the latter follows from $d \leq \sqrt{z}$ and $\ln z/\sqrt{z} \leq 2/\mathrm{e} < 1$, found by differentiation. Using $1/\ln p \leq 1/\ln 2 < 2$, we obtain

$$\frac{3d\ln N}{zLB\ln p} \leq \frac{6(\ln 257 + 1 + 2\ln L)}{LB} \leq \frac{6(\ln 257 + 1 + 2\ln 20)}{200} < 1. \tag{10.30}$$

The middle comparison uses $B \geq 10, L \geq 20$ and the decreasing function $(\ln 257 + 1 + 2\ln L)/L$. For the last comparison the elementary bounds $\ln 257 < 6$, $\ln 20 < 3$ already suffice. Hence the right side

of (10.28), after removing $(K-1)B$, is less than $73zLB$; its left side is at least $256zLB$. Theorem 10.7 applies and gives $v(\Lambda) < 2N + u$.

It remains to treat a collision in the second rectangle. Subtract the colliding labels. With $n = \gcd(b_1, b_2)$, $r_0 = b_1/n$, $s_0 = b_2/n$, the differences are $(\ell r_0, -\ell s_0)$ for a nonzero integer ℓ. Put $\sigma = \alpha_1^{r_0}\alpha_2^{-s_0}$. Its residue has an order $q \mid g$. The class congruence gives $q \mid \ell$, so

$$qr_0 \le R_2 - 1, \qquad qs_0 \le S_2 - 1. \tag{10.31}$$

If $q \nmid n$, then $v(\Lambda) = 0$. Otherwise $n = qp^u n'$, with $p \nmid n'$. The prime-to-p power law, followed by the p-power laws of Lesson 9, shows

$$v(\Lambda) = v(\sigma^n - 1) \le u + v(\sigma^{qp^t} - 1).$$

If $u \ge t$, the deep-domain formula gives equality with $u - t$ in place of u. If $u < t$, raising the principal unit to further p-th powers never decreases its positive valuation, giving the displayed inequality. Independence ensures $\sigma^{qp^t} \ne 1$. Applying the weighted product formula to $X - 1$, as in Lemma 10.6, gives

$$\begin{aligned} v(\Lambda) &\le u + \frac{d}{ef\ln p}\Big(p^t q(r_0 h(\alpha_1) + s_0 h(\alpha_2)) + \ln 2\Big) \\ &\le u + 2\big((R_2 - 1)a_1 + (S_2 - 1)a_2\big) + d \\ &\le u + 64zL + z. \end{aligned} \tag{10.32}$$

Here (10.3) bounds p^t/e, (10.31) bounds the exponents, and $d \le z$.

Finally, $b_1 \le a_2 \mathrm{e}^H$. Since $\ln^+ a_2 \le a_2$ and $a_1 \ln p \ge 1$,

$$u \le \frac{\ln b_1}{\ln p} \le \frac{B}{d} + \frac{\ln^+ a_2}{\ln p} \le B + a_1 a_2 \le B + z \le \frac{11}{100} zB^2. \tag{10.33}$$

In the injective case,

$$2N + u \le \left(514\left(\frac{11}{5}\right)^2 + \frac{11}{100}\right) zB^2 < 2500zB^2.$$

The collision bound (10.32) is at most $14.2zB^2$, since $L \le (11/5)B$ and $B \ge 10$. Thus both cases give (10.22), because

$$zB^2 = \frac{p^d - 1}{(\ln p)^4}\, d^4 \ln A_1 \ln A_2\, H^2.$$

This completes the proof. □

The factor $p^d - 1$ in these parameter choices is an upper budget for the number of residue classes. It is not being asserted to equal the local residue-group order. This distinction allows the height floor $1/d$ without inserting an additional absolute cutoff in H.

6. Coprime powers and a worked example

Corollary 10.9. Let $a > b \geq 1$ be coprime positive integers, and let $p \nmid ab$ be prime. Then

$$v_p(a^{p-1} - b^{p-1}) \leq 250000 \, \frac{p-1}{(\ln p)^2} \, \ln(5a) \ln(5b). \tag{10.34}$$

Proof. First suppose $b > 1$. Coprimality makes the bases a, b multiplicatively independent: a prime dividing either base forces its exponent to vanish in a proposed relation. In Theorem 10.8 take $d = 1$, $b_1 = b_2 = p - 1$, and $\ln A_1 = \ln(5a)$, $\ln A_2 = \ln(5b)$. These exceed their rational heights and one.

For $b = 1$, choose $c \in \{2, 3, 5\}$ such that $c \neq p$ and a, c are multiplicatively independent. Such a prime always exists. Dependence of $a \geq 2$ and a prime c would force $a = c^j$ for an integer $j \geq 1$: taking a prime valuation outside c in the relation makes the exponent of a zero unless every prime divisor is c. Thus at most one of the three primes is dependent with a, and excluding p removes at most one more.

Set $\alpha_1 = a/c, \alpha_2 = 1/c$. A relation $\alpha_1^u \alpha_2^v = 1$ is $a^u c^{-u-v} = 1$, so independence of a, c forces $u = v = 0$. Both bases are p-adic units, their common number field is $\mathbb{Q}$, and

$$h(a/c) \leq \ln \max\{a, c\} \leq \ln(5a), \qquad h(1/c) = \ln c \leq \ln 5.$$

Use $\ln A_1 = \ln(5a)$, $\ln A_2 = \ln 5$, and again $b_1 = b_2 = p - 1$. The exact identity

$$\left(\frac{a}{c}\right)^{p-1} - \left(\frac{1}{c}\right)^{p-1} = c^{-(p-1)}(a^{p-1} - 1)$$

preserves the valuation, since $c \neq p$.

In both cases the height choices are the stated $\ln(5a), \ln(5b)$, and

$$B' \ln p \leq \frac{2(p-1)\ln p}{\ln 5}, \qquad \ln(B' \ln p) \leq 3 \ln p < 10 \ln p.$$

For the second inequality use $\ln \ln p \leq \ln p$, $p - 1 \leq p$, and $\ln(2/\ln 5) < \ln 2 \leq \ln p$. Thus $H = 10 \ln p$. Substitute it into the proved bound (10.22), and use the displayed valuation identity when $b = 1$. This proves (10.34) for every allowed prime, including $p = 2$. □

For positive integers a, b, the independent bases $2, 3$ are 5-adic units. Taking $\ln A_1 = 1$, $\ln A_2 = \ln 3$, Theorem 10.8 gives

$$v_5(2^a - 3^b) \leq \frac{10000 \ln 3}{(\ln 5)^4} \max\left\{ \ln\left(\frac{a}{\ln 3} + b\right) + \ln \ln 5, 10 \ln 5 \right\}^2. \tag{10.35}$$

If $M = \max(a, b)$, its maximum is at most $10 \ln 5 + \ln(3M)$. This is an explicit constant times $(1 + \ln M)^2$. For example $2^3 - 3 = 5$, so its valuation is one; $2^7 - 3 = 125$, so its valuation is three. These are exact illustrations of the quantity being bounded, rather than numerical evidence for the theorem.

Corollary 10.10 (all powers). For coprime $a > b \geq 1$, an odd prime $p \nmid ab$, and $n \geq 1$,

$$v_p(a^n - b^n) \leq 250000 \ln(5a) \ln(5b) \frac{p}{(\ln p)^2} + v_p(n). \tag{10.36}$$

Proof. Let $q \mid p - 1$ be the residue order of a/b. If $q \nmid n$, the valuation is zero. Otherwise write $n = qm$. The base $(a/b)^q$ is a principal unit in $\mathbb{Q}_p$, whose positive valuation is at least one and thus strictly exceeds $1/(p-1)$. The prime-to-p and deep p-power laws proved in Lesson 9 give

$$v_p(a^n - b^n) = v_p(a^q - b^q) + v_p(m) = v_p(a^{p-1} - b^{p-1}) + v_p(n).$$

Here $p \nmid q$ and $p \nmid (p-1)/q$. Apply Corollary 10.9 and $p - 1 \le p$. □

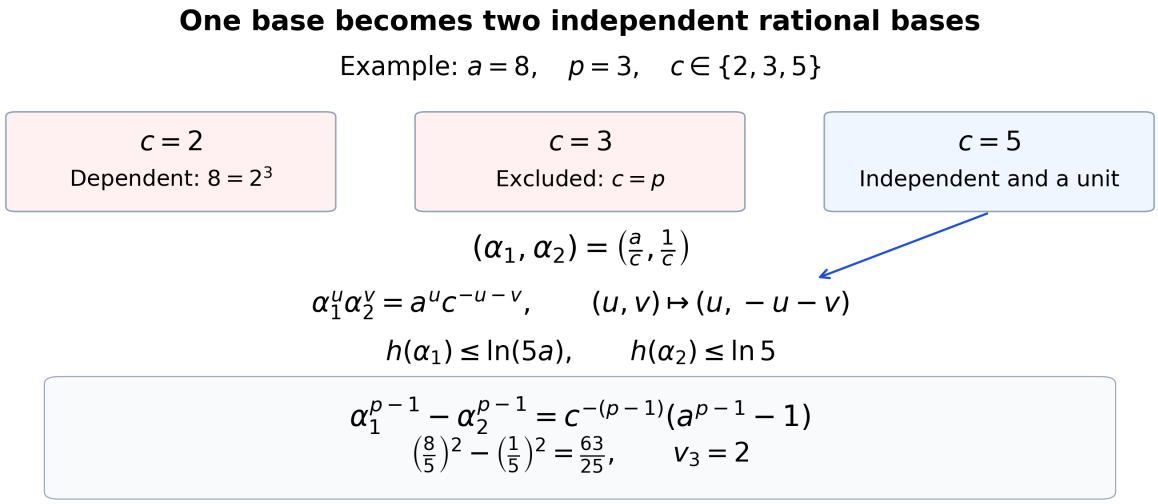


Figure 10.1b. Choose $c \in \{2, 3, 5\}$ with $c \neq p$ and independent of a. At most one candidate is removed by dependence and at most one by the prime p. The invertible exponent change $(u, v) \mapsto (u, -u - v)$ proves that $a/c, 1/c$ are independent. The lower row gives the exact unit-factor identity and the height choices $\ln A_1 = \ln(5a), \ln A_2 = \ln 5$. The example $a = 8, p = 3$ removes $c = 2$ by dependence and $c = 3$ by the unit condition, leaving $c = 5$; both differences have valuation two. The complete argument is Corollaries 10.9–10.10, and Solution 30 verifies the examples and the height cutoff. Human-source context: the determinant estimate from Bugeaud's freely accessible 1999 paper, cited below; the auxiliary-prime embedding here is proved explicitly from Theorem 10.8. Figure program.

7. A deep-unit refinement

The precision of the first base can improve the height factor. We state the coefficient and convergence conditions explicitly.

Theorem 10.11 (an E-parameter estimate). Let α_1, α_2 be multiplicatively independent nonzero rationals with $v_p(\alpha_i) = 0$. Put $g = p - 1$, and suppose

$$v_p(\alpha_1^g - 1) \ge E > \frac{1}{p-1}. \tag{10.37}$$

If $p = 2$, require also $v_2(\alpha_2 - 1) \ge 2$. Let b_1, b_2 be nonzero integers, $u = v_p(\gcd(b_1, b_2))$, and require $p \nmid b_2/p^u$. Choose

$$\begin{gathered} \ln A_i \ge \max\{h(\alpha_i), E \ln p\}, \\ B' = \frac{|b_1|}{\ln A_2} + \frac{|b_2|}{\ln A_1}, \qquad H_E = \max\{\ln B' + \ln(E \ln p), 10E \ln p\}. \end{gathered} \tag{10.38}$$

Then

$$v_p(\alpha_1^{b_1} - \alpha_2^{b_2}) \le 1300 \, \frac{p-1}{E^3 (\ln p)^4} \, \ln A_1 \ln A_2 \, H_E^2. \tag{10.39}$$

Consequently this inequality also holds with 88000 in place of 1300.

Proof. Invert bases with negative exponents as before. Inversion preserves the precision conditions, because the denominators introduced are units. Decompose $\alpha_i = \eta^{m_i}\epsilon_i$ as in (10.1), now in $\mathbb{Q}_p$. Since $p \nmid g$, the prime-to-p power law gives $v(\epsilon_1 - 1) = v(\alpha_1^g - 1) \geq E$. Both principal units are in the exponential domain. For odd p their positive integer valuations are at least one, greater than $1/(p-1)$; for $p = 2$, (10.37) and the second-base condition give valuations at least two.

If the original difference has valuation zero, the conclusion is immediate. Otherwise its residue factors agree, so it has the valuation of $\epsilon_1^{b_1} - \epsilon_2^{b_2}$. For $b_i' = b_i/p^u$, the deep p-power law gives

$$\delta = \epsilon_1^{b_1'} - \epsilon_2^{b_2'}, \qquad v(\delta) = v(\alpha_1^{b_1} - \alpha_2^{b_2}) - u.$$

The matrix (10.9) is used with exponent multiplier one. In Lemma 10.5 take $\omega = \log \epsilon_1$, so $v(\omega) \geq E$. The assumed unit denominator b_2' gives $z_j = r_j + (b_1'/b_2')s_j \in \mathbb{Z}_p$. Formula (10.11) holds with multiplier one and no root of unity in δ. The coefficient of its error part remains integral whenever $v(\delta) > 1/(p-1)$.

We give the resulting comparison, keeping the scale change visible. If $v(\delta) \geq E(N - 1/2)$, the alternant and factorial estimates in (10.13) give

$$v(A) \geq E\left(\frac{n(n-1)}{2} - \sum_{i \in I} k_i\right) - \frac{n \ln n}{\ln p},$$
$$v(B) \geq -\frac{\sum_{i \notin I} k_i}{p-1} \geq -E \sum_{i \notin I} k_i.$$

The positive coefficient $E - 1/(p-1)$ is what permits replacement of the Taylor-order sum by $n(n-1)/2$. Formula (10.14) then gives

$$v(\Delta) \geq \frac{ENK(L-1)}{2} - \frac{N \ln N}{\ln p}.$$

The polynomial height proof of Lemma 10.6 has $d = e = f = 1$ and exponent multiplier one. Thus a sufficient parameter inequality is

$$EK(L-1)\ln p > 3 \ln N + (K-1)\ln \beta + \frac{L}{2}\big((R-1)h(\alpha_1) + (S-1)h(\alpha_2)\big). \tag{10.40}$$

Under the two cardinality conditions (10.18), the zero lemma and these two bounds prove $v(\Lambda) < E(N - 1/2) + u$.

Use (10.24) with

$$G = p - 1, \qquad a_i = \frac{\ln A_i}{E \ln p}, \qquad z = Ga_1a_2, \qquad B = \frac{H_E}{E \ln p}, \qquad k = 256. \tag{10.41}$$

Here $a_i \geq 1$, $z \geq G \geq 1$, and $B \geq 10$. The cardinalities, factorial estimate and height comparison (10.25)–(10.29) apply unchanged, with $B'E \ln p = b_1/a_2 + b_2/a_1$. In particular $\ln \beta \leq H_E$, and the height contribution after division by $E \ln p$ is less than $72zLB$.

We check the remaining logarithmic contribution. From $E > 1/G$ and $z \geq G$,

$$\frac{1}{Ez \ln p} < \frac{1}{\ln p} < 2, \qquad \frac{\ln z}{Ez \ln p} \leq 1.$$

For the second bound, at $p = 2, 3$ use $1/(E \ln p) < 2$ and $\ln z/z \le 1/\mathrm{e}$. For $p \ge 5$, the function $\ln z/z$ decreases on $z \ge G \ge 4$, giving a bound $\ln G/\ln p < 1$. With $N \le 257 z L^2$, therefore,

$$\frac{3 \ln N}{EzLB \ln p} \le \frac{6(\ln 257 + 1 + 2 \ln L)}{LB} < 1.$$

The left side of the sufficient parameter inequality, after removing $(K-1)B$, is at least $256zLB$, while its remaining right side is less than $73zLB$. This verifies (10.40) in the injective case.

In the collision case, take q, r_0, s_0 as in (10.31). If $q \nmid n = \gcd(b_1, b_2)$, the valuation is zero. Otherwise σ^q, with $\sigma = \alpha_1^{r_0} \alpha_2^{-s_0}$, is a deep principal unit: its residue root has disappeared, and its two principal factors were already deep. Hence $v(\Lambda) = u + v(\sigma^q - 1)$. The one-place polynomial bound for $X - 1$ gives

$$\begin{aligned} v(\Lambda) &\le u + \frac{q(r_0 h(\alpha_1) + s_0 h(\alpha_2)) + \ln 2}{\ln p} \\ &\le u + E((R_2 - 1)a_1 + (S_2 - 1)a_2) + 1 \\ &\le u + 32EzL + 1 \le u + Ez(32L + 1). \end{aligned} \tag{10.42}$$

Here $Ez > EG > 1$. Finally $b_1 \le a_2 \mathrm{e}^{H_E}$, so

$$u \le EB + \frac{\ln^+ a_2}{\ln p} \le EB + 2Ez \le \frac{12}{100} EzB^2.$$

The middle bound uses $\ln^+ a_2 \le a_2$, $EGa_1 > 1$, and $1/\ln p < 2$. In the injective case,

$$EN + u \le \left(257 \left(\frac{11}{5}\right)^2 + \frac{12}{100}\right) EzB^2 = 1244 EzB^2 < 1300 EzB^2.$$

The collision bound is at most $7.17 EzB^2$. Substituting (10.41) into EzB^2 proves (10.39). □

The coefficient condition singles out the first base whose logarithm supplies the precision E. Interchanging bases is permissible only if the precision and second-base convergence hypotheses are checked for the newly ordered pair. At $p = 2$, a unit such as three has $v_2(3 - 1) = 1$, on the exponential boundary; its positivity alone does not provide the deep-unit substitution used in this proof.

For a concrete family take

$$\alpha_1 = 1 + 5^m, \qquad \alpha_2 = 1 + 2 \cdot 5^m, \qquad E = m, \qquad m \ge 1.$$

The two integers are coprime, because twice the first minus the second is one. They exceed one, so they are multiplicatively independent. Their fourth powers have distance valuation m from one, by the prime-to-five power law. Both height budgets may be their actual logarithms. For $b_1 = b_2 = 1$, $H_E = 10m \ln 5$ and the exact difference valuation is m. The bound (10.39) is

$$v_5(\alpha_1 - \alpha_2) \le 520000 \, \frac{\ln(1 + 5^m) \ln(1 + 2 \cdot 5^m)}{m (\ln 5)^2}. \tag{10.43}$$

The height factor is linear in m, bounded explicitly by $(m \ln 5 + \ln 2)(m \ln 5 + \ln 3)/(m(\ln 5)^2)$. Without the precision divisor, the product of these heights is quadratic in m.

8. Saturated bases and Kummer degree

We now allow an arbitrary number of bases. A root which already belongs to the number field should be incorporated into the bases before taking an auxiliary extension. The following construction does this and proves the resulting extension degree.

Let K be a number field of degree d, embedded in $\mathbb{C}$. Let q be a prime, and assume that K contains all the q-th roots of unity. For $n \geq 1$ and multiplicatively independent $\alpha_1, \ldots, \alpha_n \in K^*$, define

$$\begin{aligned} M_q(\boldsymbol{\alpha}) = \{\boldsymbol{v} \in \mathbb{Z}[1/q]^n : \text{for some } T = q^t,\ T\boldsymbol{v} \in \mathbb{Z}^n, \\ \alpha_1^{Tv_1} \cdots \alpha_n^{Tv_n} = \beta^T \text{ for some } \beta \in K^*\}. \end{aligned} \tag{10.44}$$

No choice of complex logarithms is involved in this definition.

The complete finite-height count in Heights of algebraic numbers, Theorem 2.6, bounds the number of algebraic numbers of degree at most d and height at most $H \geq 0$ by

$$\mathcal{N}(d, H) = d^2\left(2^{d+1}\mathrm{e}^{dH} + 1\right)^{d+1}. \tag{10.45}$$

We use its polynomial coefficient proof and the height arithmetic of Proposition 2.5 in that earlier lesson.

Lemma 10.12 (finite saturation). The set $M = M_q(\boldsymbol{\alpha})$ is an additive subgroup of $\mathbb{Q}^n$, contains $\mathbb{Z}^n$, and has finite index over it. More precisely,

$$[M : \mathbb{Z}^n] \leq \mathcal{N}\left(d, \sum_{j=1}^{n} h(\alpha_j)\right). \tag{10.46}$$

This index is a power of q. The group M has a basis of n rational vectors over $\mathbb{Z}$.

Proof. Powers T in (10.44) can be enlarged to a common power of q. Multiplying or dividing the corresponding β's proves closure under addition and subtraction. Integral vectors belong to M with $T = 1$.

Every coset modulo $\mathbb{Z}^n$ has a unique representative $\boldsymbol{v}$ with $0 \leq v_j < 1$. Choose one associated β from (10.44). The product inequality and power identity in Heights of algebraic numbers, Proposition 2.5, give

$$Th(\beta) = h\left(\prod_j \alpha_j^{Tv_j}\right) \leq T\sum_j v_j h(\alpha_j) \leq T\sum_j h(\alpha_j).$$

Distinct representatives give distinct chosen β's. Indeed, if two choices were equal, raise their defining equations to a common power T. Their quotient would give $\prod_j \alpha_j^{T(v_j - w_j)} = 1$. Independence forces every $v_j = w_j$. The earlier finite-height count (10.45) therefore bounds the number of cosets as asserted.

There are finitely many representatives, and each has a denominator which is a power of q. Choose a common denominator q^a. Then $\mathbb{Z}^n \subset M \subset q^{-a}\mathbb{Z}^n$. The finite group $M/\mathbb{Z}^n$ is a subgroup of $(q^{-a}\mathbb{Z}/\mathbb{Z})^n$, so its order is a power of q.

For completeness, a subgroup $L \subset \mathbb{Z}^n$ containing $m\mathbb{Z}^n$, with $m > 0$, has an integral basis of size n. Prove this by induction. Its first-coordinate image is $c\mathbb{Z}$ for some $c > 0$: take the least positive first coordinate and apply integer division. Choose $\boldsymbol{x} \in L$ with first coordinate c. The kernel of that projection contains $m\mathbb{Z}^{n-1}$, so has an $(n-1)$-element basis by induction. Any element of L becomes

an element of the kernel after subtracting the appropriate integer multiple of $\boldsymbol{x}$. These n vectors are independent and span L. Apply this to $L = q^a M$ and divide its basis by q^a. $\square$

Lemma 10.13 (the maximal q-primary root). The q-primary roots of unity in K form a finite cyclic group. If α_0 generates it and has order q^u, then $u \geq 1$, and α_0^a is not a q-th power in K when $q \nmid a$.

Proof. Roots of unity have height zero, by the height power identity. Finiteness follows from (10.45) with $H = 0$. The finite-subgroup argument of Section 1 applies over any field: its root count and the existence of an element of maximal exponent prove cyclicity. The assumption that q-th roots are in K gives $u \geq 1$. If $\beta^q = \alpha_0^a$ with $q \nmid a$, then $\beta^{q^{u+1}} = 1$, while its q-th power has order q^u. Thus β has order q^{u+1}, contradicting maximality of u. $\square$

Fix such a generator α_0.

Choose a basis $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ of M. By (10.44), after choosing a common power $T = q^t$, there are $\theta_i \in K^*$ with

$$\theta_i^T = \prod_{j=1}^{n} \alpha_j^{T v_{ji}}, \qquad v_{ji} = (\boldsymbol{v}_i)_j. \tag{10.47}$$

Theorem 10.14 (the saturated Kummer condition). The classes of $\alpha_0, \theta_1, \dots, \theta_n$ in $K^*/(K^*)^q$ are linearly independent over $\mathbb{F}_q$. Consequently

$$[K(\alpha_0^{1/q}, \theta_1^{1/q}, \dots, \theta_n^{1/q}) : K] = q^{n+1}. \tag{10.48}$$

Proof. Suppose $\alpha_0^{a_0} \prod_i \theta_i^{a_i} = \beta^q$, with integers a_i and $\beta \in K^*$. Enlarge T in (10.47) until it is divisible by the order of α_0. Raising the relation to T shows that

$$\boldsymbol{w} = \frac{1}{q} \sum_i a_i \boldsymbol{v}_i \in M : \quad \prod_j \alpha_j^{q T w_j} = \beta^{qT}.$$

Uniqueness of integral coordinates in the basis of M forces $q \mid a_i$ for $1 \leq i \leq n$. Dividing out the corresponding q-th powers, we find that $\alpha_0^{a_0}$ is a q-th power in K. If $q \nmid a_0$, its q-th root would have order q^{u+1}, contradicting the choice of α_0. Thus $q \mid a_0$ as well. This proves independence of all the classes.

We prove the degree consequence directly. More generally, let $c_1, \dots, c_r \in K^*$ have independent classes modulo q-th powers, and choose roots $\rho_i^q = c_i$. The field $L = K(\rho_1, \dots, \rho_r)$ splits the separable polynomials $X^q - c_i$, since all q-th roots of unity are already in K. Fix a primitive q-th root ζ. Every K-automorphism acts by $\rho_i \mapsto \zeta^{t_i} \rho_i$, giving an injective group homomorphism

$$\mathrm{Aut}_K(L) \longrightarrow \mathbb{F}_q^r.$$

Its image is a vector subspace. If it were proper, Gaussian elimination would give a nonzero vector $\boldsymbol{a} \in \mathbb{F}_q^r$ annihilating it. Choose integer representatives $0 \leq a_i < q$. The element $\prod_i \rho_i^{a_i}$ would then be fixed by every automorphism and would belong to K. Its q-th power is $\prod_i c_i^{a_i}$, contradicting the independence of the classes.

Here the fixed-element assertion and the automorphism count need only elementary separability. An embedding of an intermediate field extends across each algebraic generator by choosing a root of the transported minimal polynomial. In characteristic zero these roots are distinct, so the number of embeddings of L equals the product of the successive degrees, namely $[L : K]$. Since L is a splitting field, every such embedding has image L. If an element outside K were fixed, choose a different root

of its separable minimal polynomial and extend that embedding to L; the resulting automorphism would move it. Thus the fixed elements are precisely K.

The automorphism image is therefore all of $\mathbb{F}_q^r$, and $[L : K] = q^r$. Apply this with $r = n + 1$ and the already proved independent classes. □

Lemma 10.15 (the residue index does not depend on the saturated basis). Suppose $q \neq p$, a prime of K above p is fixed, and every α_j is a unit there. The group $\langle \alpha_0, \theta_1, \dots, \theta_n \rangle \subset K^*$ is independent of the basis and of the root choices in (10.47). Its generators are p-adic units. If the residue field has size p^f, the integer

$$\delta_q(\boldsymbol{\alpha}) = \frac{p^f - 1}{|\langle \overline{\alpha_0}, \overline{\theta_1}, \dots, \overline{\theta_n} \rangle|} \tag{10.49}$$

is consequently well defined.

Proof. Replacing a root choice in (10.47) multiplies θ_i by a q-primary root of unity in K, hence by a power of α_0. For a change of basis write $\boldsymbol{v}_i' = \sum_j c_{ji} \boldsymbol{v}_j$, with the integer matrix C invertible over $\mathbb{Z}$. Raising to a common power $T = q^t$ shows that $\theta_i' / \prod_j \theta_j^{c_{ji}}$ is again a q-primary root of unity. The new group is contained in the old one. Applying the inverse integral matrix gives the reverse inclusion.

Taking the valuation in (10.47) gives $v(\theta_i) = 0$; roots of unity are units too. Reduction is therefore defined. Its image is a subgroup of the cyclic group of order $p^f - 1$ proved in Section 1, so its order divides $p^f - 1$, and the quotient in (10.49) is an integer. Basis independence proves the final assertion. □

For example, take $K = \mathbb{Q}$, $q = 2$, $\alpha_1 = 4$, and $\alpha_2 = 9$. Prime valuations show that $M_2(4, 9) = \frac{1}{2}\mathbb{Z}^2$. A saturated basis gives $\theta_1 = 2, \theta_2 = 3$, and $\alpha_0 = -1$. Theorem 10.14 proves

$$[\mathbb{Q}(\sqrt{-1}, \sqrt{2}, \sqrt{3}) : \mathbb{Q}] = 8.$$

Starting with $4, 9$ would instead leave both classes trivial modulo squares. Saturation exposes the arithmetic independence needed for the auxiliary extension.

9. Counting heights and bounding saturation

The index in (10.46) is finite, but its bound uses the sum of the heights inside an exponential. A sharper count gives a product of heights and a small power of the field degree. This is the arithmetic input for reducing a form with many logarithms to independent bases.

Write $\mathcal{H}(\gamma) = \exp h(\gamma)$ for multiplicative height. Let K have degree d. The number field containing a scalar θ below can be larger than K.

Lemma 10.16 (a count at one archimedean place). Fix an archimedean place of K, represented by $\sigma : K \to \mathbb{C}$, and put $f = 1$ for a real place and $f = 2$ for a complex place. Fix $\phi \neq 0$ algebraic and a number field $L \supset K(\phi)$, of degree ℓ. Use the product-formula absolute values of L, and set

$$\Phi_v = \prod_{w \mid v} |\phi|_w^{1/\ell}, \qquad \Phi_{\widehat{v}} = \prod_{w \nmid v} |\phi|_w^{1/\ell},$$
$$A_v(a) = \prod_{w \mid v} \max\{|\phi|_w, |a|_w\}^{1/\ell}, \quad A_{\widehat{v}}(a) = \prod_{w \nmid v} \max\{|\phi|_w, |a|_w\}^{1/\ell}.$$

Thus $\Phi_v \Phi_{\widehat{v}} = 1$, and $A_v(a) A_{\widehat{v}}(a) = \mathcal{H}(a/\phi)$. If $X, Y \geq 1$, $N \geq 2$, and $u = N^{1/(N-1)}$, then

$$\#\{a \in K : A_v(a) \le X\Phi_v,\ A_{\widehat{v}}(a) \le Y\Phi_{\widehat{v}}\} \le (N-1)\left(1 + (uXY^2)^{d/f}\right)^f. \tag{10.50}$$

Proof. Finiteness follows from the earlier finite-height count: the numbers a/ϕ lie in L, have degree at most ℓ, and have height at most $\ln(XY)$. The identity for $A_v A_{\widehat{v}}$ follows by factoring $|\phi|_w$ from each maximum and applying the product formula to ϕ.

The first local inequality implies $|\sigma a| \le R$, where $R = (X\Phi_v)^{d/f}$. This follows because the sum of archimedean weights above v is $f\ell/d$. Indeed, each embedding of K has $[L : K]$ extensions; conjugate complex embeddings contribute weight two, and real embeddings contribute weight one.

Put $r = (uY^2\Phi_{\widehat{v}})^{-d/f}$. Draw around each σa a ball of radius r in $\mathbb{R}^f$. These balls lie in the ball of radius $R + r$. We show that no point is in the interiors of N of them.

Otherwise there are distinct $a_1, \dots, a_N$ whose images lie in a ball of radius strictly less than r, with centre z. Put $Q = N(N-1)/2$, and let $\Delta = \prod_{i<j}(a_j - a_i) \ne 0$. Translating the Vandermonde rows replaces $(\sigma a_i)^k$ by $(\sigma a_i - z)^k$ without changing the determinant. Factoring the radius from its rows and using Hadamard's inequality gives $|\sigma\Delta| < N^{N/2} r^Q = u^Q r^Q$. Here Hadamard's inequality follows by orthogonalizing the columns: this preserves the determinant and weakly decreases their norms, while the determinant of the orthogonal columns has absolute value the product of those norms.

At a finite place $w \nmid v$, expand the determinant into its permutation monomials. Every monomial has total degree Q and degree at most $N-1$ in each a_i. With $\lambda = |\phi|_w$, its absolute value is at most

$$\lambda^{-Q} \prod_i \max\{\lambda, |a_i|_w\}^{N-1}.$$

The ultrametric inequality gives the same bound for Δ. At an archimedean embedding τ outside v, apply Hadamard to the powers of $\tau a_i/\tau\phi$. Each column has norm at most $\sqrt{N}\max\{1, |\tau a_i/\tau\phi|\}^{N-1}$, so the same bound has the extra factor $N^{f_w N/2}$, where $f_w = 1$ or two is its archimedean weight.

Crucially, the sum of these *unselected* archimedean weights is exactly $\ell(d-f)/d$. Multiplying their bounds and the finite-place bounds gives

$$\prod_{w \nmid v} |\Delta|_w \le u^{\ell Q(d-f)/d} Y^{2\ell Q} \Phi_{\widehat{v}}^{\ell Q}.$$

The selected place contributes strictly less than $(ur)^{f\ell Q/d}$. The full product is therefore strictly less than $(r^{f/d} u Y^2 \Phi_{\widehat{v}})^{\ell Q} = 1$, contrary to the product formula. This proves the asserted multiplicity bound for the balls.

Integrate their indicator functions. Their sum is at most $N-1$ outside their boundaries, which have volume zero. Scaling a ball's volume by its radius gives $Mr^f \le (N-1)(R+r)^f$ for the number M of centres. Since $R/r = (uXY^2)^{d/f}$, this is (10.50). □

We need a uniform choice of N. For $d \ge 2$, take

$$m = \lfloor d \ln d \rfloor, \qquad N = m + 1, \qquad u = N^{1/m}, \qquad u^d < 8. \tag{10.51}$$

Here is a proof of the last inequality for every degree. For $d = 2, 3$, the choices are $N = 2, 4$, giving $u^d = 4$ in both cases. For $d \ge 4$, put $x = d \ln d > 4$. Then $m \ge 3x/4$, $N \le 5x/4$, and

$$\frac{d \ln N}{m} \le \frac{4}{3}\left(1 + \frac{\ln\ln d}{\ln d} + \frac{\ln(5/4)}{\ln d}\right) < \frac{4}{3}\left(1 + \frac{3}{8} + \frac{1}{6}\right) = \frac{37}{18} < \ln 8.$$

The first fraction is at most $1/\mathrm{e} < 3/8$, by differentiating $\ln t/t$. The second is less than $1/6$, since $(5/4)^6 < 4$. Finally $\ln 2 = 2\sum_{j\ge 0}(1/3)^{2j+1}/(2j+1) > 56/81$, which proves $\ln 8 > 37/18$. This series follows by integrating the geometric series on $[0, 1/3]$. The elementary bounds $8/3 < \mathrm{e} < 3$ follow directly from its exponential series. They give $1/2 < \ln 2 < 1$ and $1 < \ln 3 < 4/3$; for the last upper bound, use $(8/3)^4 > 3^3$. These also verify the two small choices of N.

Theorem 10.17 (uniform twisted height count). Let $\theta \neq 0$ be algebraic and $H \ge 1$. The following bounds hold:

$$\begin{aligned} \#\{a \in K : \mathcal{H}(\theta a) \le H\} &\le 68 d \ln d\, H^{2d} \quad (d \ge 2), \\ &\le 31 d \ln d\, H^{2d} \quad (d \ge 2,\ \theta \in K), \\ &\le 17 H^2 \qquad\quad (d = 1). \end{aligned} \tag{10.52}$$

Proof. Apply Lemma 10.16 with $\phi = 1/\theta$. Both ratios A_v/Φ_v and $A_{\widehat{v}}/\Phi_{\widehat{v}}$ are at least one, and their product is $\mathcal{H}(\theta a)$. Divide the first ratio into bands $2^{(i-1)/d} \le A_v/\Phi_v < 2^{i/d}$. There are at most $I = 1 + \lfloor d \ln H/\ln 2 \rfloor$ bands. Within band i, set $X_i = 2^{i/d}$, $Y_i = 2^{-(i-1)/d} H$. Then $X_i Y_i^2 = 2^{(2-i)/d} H^2$.

Let $M(H)$ denote the count. Summing (10.50), using $I \le H^d$, $\sum_{i\ge 1} 2^{2-i} = 4$, and $\sum_{i\ge 1} 2^{(2-i)/2} = 2/(\sqrt{2}-1)$, gives

$$\begin{aligned} M(H) &\le m(1 + 4u^d) H^{2d} \qquad\qquad\qquad (f = 1), \\ M(H) &\le m \left[\left(1 + \frac{4u^{d/2}}{\sqrt{2}-1} \right) H^d + 4u^d H^{2d} \right] \quad (f = 2). \end{aligned} \tag{10.53}$$

To see $I \le H^d$, put $t = H^d$; if $k = \lfloor \log_2 t \rfloor$, then $k + 1 \le 2^k \le t$. With (10.51), the first coefficient is less than 33, and the two coefficients in the second line are less than 31 and 32: use $\sqrt{8} < 3$ and $1/(\sqrt{2}-1) = 1 + \sqrt{2} < 5/2$. Since $m \le d \ln d$, both imply the first assertion of (10.52). For $d = 1$, take $N = 2, u = 2$ instead. The first line yields $M(H) \le H + 8H^2 \le 9H^2 < 17H^2$.

If $\theta \in K$, multiplication by θ is a bijection, so reduce to $\theta = 1$. The set contains $0, 1, -1$. All other elements pair as a, a^{-1}, with equal heights, and at least one member of each pair satisfies $|\sigma a| \le 1$. Consequently a subset of cardinality $(M(H) + 3)/2$ satisfies the inequalities of Lemma 10.16 with $X = 1, Y = H, \phi = 1$. At a complex place this gives $M(H) \le 2m(1 + u^{d/2} H^d)^2 - 3 \le 30 d \ln d\, H^{2d}$; at a real place it gives $M(H) \le 2m(1 + u^d H^{2d}) - 3 \le 18 d \ln d\, H^{2d}$. These prove the second assertion. □

Theorem 10.18 (the full saturation index). For independent $\alpha_1, \dots, \alpha_n \in K^*$, define

$$M_K(\boldsymbol{\alpha}) = \{ \boldsymbol{v} \in \mathbb{Q}^n : T\boldsymbol{v} \in \mathbb{Z}^n,\ \prod_j \alpha_j^{T v_j} = \beta^T \text{ for some } T \ge 1,\ \beta \in K^* \}. \tag{10.54}$$

Let w_K be the number of roots of unity in K. This is a lattice containing $\mathbb{Z}^n$, and

$$w_K [M_K : \mathbb{Z}^n] \le C_d \frac{n! \mathrm{e}^n}{n^n} \prod_j h(\alpha_j), \qquad C_d = \begin{cases} 58 d^{n+1} \ln d, & d \ge 2, \\ 17, & d = 1. \end{cases} \tag{10.55}$$

Proof. Common integer powers prove that M_K is a subgroup. The coset argument of Lemma 10.12 applies with arbitrary positive integers T: distinct representatives in $[0, 1)^n$ give distinct elements of K of height at most $\sum h(\alpha_j)$. Thus its index J is finite. A common denominator puts it between $\mathbb{Z}^n$ and $T^{-1}\mathbb{Z}^n$; the integral basis induction there proves it is a lattice. Its fundamental volume is $1/J$, since the unit-volume fundamental domain of $\mathbb{Z}^n$ contains J fundamental domains of M_K.

The group of all roots of unity in K is finite by (10.45) with height zero, and the finite-subgroup argument of Section 1 makes it cyclic. Each $h_j = h(\alpha_j)$ is positive: otherwise all powers of α_j have height zero and lie in K, so the same finite-height count would make two powers equal, contradicting independence.

For $H > 1$, the open diamond

$$S = \{\boldsymbol{x} : \sum_j h_j|x_j| < \ln H\} \quad \text{has volume} \quad V = \frac{2^n(\ln H)^n}{n!\prod_j h_j}. \tag{10.56}$$

In each orthant, scale the coordinates by h_j and integrate the simplex $\sum y_j < \ln H$; induction by its last coordinate gives volume $(\ln H)^n/n!$. This proves the volume formula.

Average the count $\sum_{\boldsymbol{v}\in M_K} 1_S(\boldsymbol{t}+\boldsymbol{v})$ over a fundamental domain. Translating its integrals tiles $\mathbb{R}^n$, so the integral equals V and its average equals JV. Some coset therefore has at least JV points in S. There are only finitely many in the bounded diamond. Since S is open, perturb the translating vector to a rational vector while retaining this finite set of points. Denote the resulting rational points by $P_0, \dots, P_s$; their differences still lie in M_K, and $s + 1 \geq JV$.

Choose determinations $\theta_i = \prod_j \alpha_j^{(P_i)_j}$. They are algebraic because the coordinates are rational. The definition of M_K gives $\theta_i/\theta_0 = \mu_i\beta_i$ for $\beta_i \in K$ and roots of unity μ_i in an algebraic closure. The height power and product identities give $h(\theta_0\beta_i) = h(\theta_i) < \ln H$. Multiplying by a root of unity preserves height: raise to its order and use the power identity.

The β_i are distinct modulo roots of unity in K. If a quotient were a root of unity, then so would θ_i/θ_j; raising to a common denominator and then to its order would contradict independence of the α_j. Consequently their w_K root-of-unity multiples give $w_K(s+1)$ distinct elements of K counted by Theorem 10.17 with twist θ_0. Hence $w_K JV \leq M(H)$.

Take $H = \exp(n/(2d))$. At a complex place, (10.53) divided by $\exp(n)$ is less than $d\ln d(31/\sqrt{\mathrm{e}}+32) < 58d\ln d$, since $\sqrt{\mathrm{e}} > 3/2$. At a real place the coefficient is less than $33d\ln d$. Substitute (10.56) to obtain (10.55). For $d = 1$, the bound $17H^2$ gives exactly its stated constant 17. □

In particular, since $w_K \geq 1$ and the index is at least one,

$$\prod_j h(\alpha_j) \geq \frac{w_K}{C_d}\frac{n^n}{n!\mathrm{e}^n}. \tag{10.57}$$

Corollary 10.19 (the q-primary part). The lattice of Section 8 is exactly $M_q = M_K \cap \mathbb{Z}[1/q]^n$. Therefore (10.55) also bounds $w_K[M_q : \mathbb{Z}^n]$.

Proof. Only the reverse inclusion needs proof. For a vector in the intersection, write a power realizing (10.54) as $T = q^a m$ with $q \nmid m$. Since its coordinates have q-power denominators, $q^a\boldsymbol{v}$ is integral. Put $\gamma = \prod_j \alpha_j^{q^a v_j}$. The defining equation says $\gamma^m = \beta^{q^a m}$, so $\zeta = \gamma/\beta^{q^a} \in K$ has order dividing m. Raising to q^a is a bijection on this finite cyclic group, by Bezout's identity for q^a, m. Choose $\eta \in \langle\zeta\rangle$ with $\eta^{q^a} = \zeta$. Then $\gamma = (\beta\eta)^{q^a}$, which realizes (10.44). The inclusion of finite quotients gives the index inequality. □

Corollary 10.20 (a primitive circuit relation). Suppose $\alpha_0, \dots, \alpha_n \in K^*$ are dependent but every n of them are independent. There are nonzero integers b_i with $\prod_i \alpha_i^{b_i} = 1$ and

$$|b_i| \leq C_d\frac{n!\mathrm{e}^n}{n^n}\prod_{j\neq i} h(\alpha_j) \qquad (0 \leq i \leq n), \tag{10.58}$$

where C_d is the degree factor in (10.55).

Proof. The integer vectors whose products are roots of unity have rational rank one. Two nonproportional such vectors could eliminate a coordinate, producing a nonzero torsion relation on only n bases; raising to the torsion order contradicts their independence. Divide any nonzero relation vector by its coordinate gcd to obtain a primitive vector $\boldsymbol{r}$. Its product is still a root of unity, since a power of that product is one. All $r_i \neq 0$, and every integer vector in this rational line is an integer multiple of $\boldsymbol{r}$, by Bezout's identity for its coordinate gcd.

The vector $(r_1/r_0, \dots, r_n/r_0)$ belongs to $M_K(\alpha_1, \dots, \alpha_n)$: take $T = |r_0|w_K$ and $\beta = \alpha_0^{-1}$ in (10.54). The root of unity in the relation disappears upon raising to T/r_0. Its order modulo $\mathbb{Z}^n$ is exactly $|r_0|$, again by Bezout's identity for the coordinate gcd. Thus $|r_0| \leq [M_K : \mathbb{Z}^n]$. Set $b_i = w_K r_i$; their product is one, and (10.55) proves the bound for b_0. Apply the same argument after omitting each other coordinate. The vector $\boldsymbol{b}$ is the same throughout, so all the bounds hold simultaneously. □

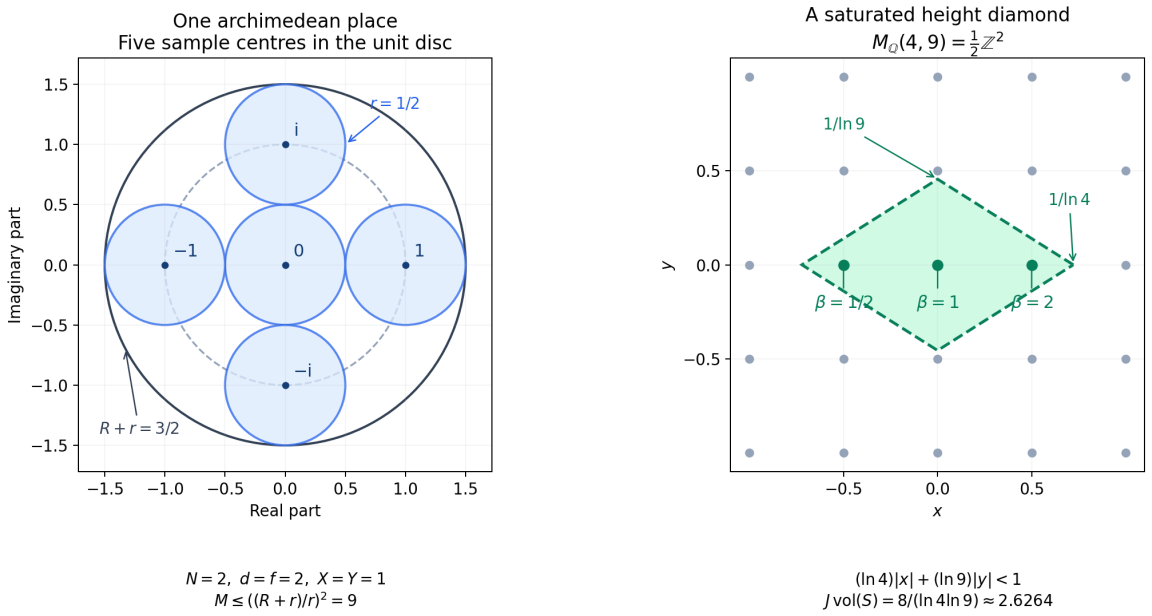


Figure 10.2. Left: the sample $0, \pm 1, \pm i$ in $\mathbb{Q}(i)$ has multiplicative height one and lies in the unit disc. For $N = 2, d = f = 2, X = Y = 1$, Lemma 10.16 uses radius $r = 1/2$; the five small discs have disjoint interiors and lie in the disc of radius $R + r = 3/2$. The area bound is nine centres. Right: for the rational bases $4, 9$, the full lattice is $\frac{1}{2}\mathbb{Z}^2$. The open diamond $(\ln 4)|x| + (\ln 9)|y| < 1$ has volume $2/(\ln 4 \ln 9)$; its coset average is $8/(\ln 4 \ln 9)$, between two and three. The zero coset contains the three displayed points, giving $1/2, 1, 2$. Their two rational root-of-unity multiples are six distinct numbers of height below e. *These are samples of the proved packing and averaging steps (10.50), (10.56), not numerical substitutes for the uniform bounds. Source mechanism: Loher and Masser, cited below. Reproducible source: height_counting_geometry.py.*

10. One logarithm with its residue order

The one-dimensional case needs the order of the original residue class, rather than the size of a saturated residue group. Keeping that order and the local residue degree in the height inequality gives a sharper form of the one-logarithm bound.

Theorem 10.21 (the actual residue order). Let K have degree d, and let $\mathfrak{p}$ be a prime above p, of ramification index e and residue degree f. Normalize $v_p(p) = 1$ and $\operatorname{ord}_{\mathfrak{p}} = ev_p$. Suppose $\alpha \in K^*$ is a unit at $\mathfrak{p}$, and let $g = |\langle \bar{\alpha} \rangle|$ be the order of its nonzero residue class. If $b \in \mathbb{Z} \setminus \{0\}$ and $\alpha^b \neq 1$, then

$$\operatorname{ord}_{\mathfrak{p}}(\alpha^b - 1) \leq ev_p(b) + \frac{d}{f \ln p} \left(\frac{gep}{p-1} h(\alpha) + \ln 2 \right). \tag{10.59}$$

Consequently,

$$\operatorname{ord}_{\mathfrak{p}}(\alpha^b - 1) \le \frac{d}{f \ln p} \left(\ln(2|b|) + g \left(1 + \frac{1}{p-1} \right) e h(\alpha) \right). \tag{10.60}$$

Both bounds include roots of unity, subject to the nonvanishing condition.

Proof. Replacing b by $|b|$ does not change the valuation: for a unit x, $x^{-1} - 1 = -x^{-1}(x-1)$. Hence take $b > 0$. The order g divides $p^f - 1$, so it is prime to p. If $g \nmid b$, reduction makes the left side zero.

For every nonzero $\eta \in K$, the normalized height formula gives

$$\operatorname{ord}_{\mathfrak{p}}(\eta) \ln p \le \frac{d}{f} h(\eta).$$

Indeed, when the valuation is positive, the contribution of $\mathfrak{p}$ to $dh(\eta^{-1})$ is $f \operatorname{ord}_{\mathfrak{p}}(\eta) \ln p$, and all other contributions are nonnegative. Use $h(\eta^{-1}) = h(\eta)$. When it is nonpositive, the inequality is immediate. These are the height and local-degree identities proved in Heights of algebraic numbers and Places of number fields in extensions and the product formula.

First suppose α is not a root of unity and $g \mid b$. Put $\beta = \alpha^g$. Its positive finite valuation $v_p(\beta - 1)$ is at least $1/e$. Let t be the least nonnegative integer with $p^t > e/(p-1)$. The proved entrance-time and power-step statements in p-adic logarithms: power series, heights and the one-logarithm bound give

$$c_t = v_p(\alpha^{gp^t} - 1) > \frac{1}{p-1}, \qquad p^t \le \frac{ep}{p-1}.$$

The second inequality follows from minimality when $t > 0$. When $t = 0$, its right side is greater than one since $e \ge 1$.

Write $b = gp^a m$, with $p \nmid m$. Then $a = v_p(b)$. Prime-to-p powers preserve positive valuation. After step t, each p-power adds one. Earlier positive valuations increase strictly at each step, including a step on the boundary. Thus, whether $a \ge t$ or $a < t$,

$$v_p(\alpha^b - 1) \le a + c_t.$$

Apply the local height inequality to $\eta = \alpha^{gp^t} - 1 \ne 0$. The proved height power and sum inequalities give $h(\eta) \le gp^t h(\alpha) + \ln 2$. Multiplying the preceding valuation bound by e therefore yields

$$\operatorname{ord}_{\mathfrak{p}}(\alpha^b - 1) \le ea + \frac{d}{f \ln p} \left(gp^t h(\alpha) + \ln 2 \right).$$

Use $p^t \le ep/(p-1)$ to obtain (10.59).

Now suppose α is a root of unity. Its height is zero by the height power identity. If the left side is positive, the nontrivial root α^b has p-power order: write its order as $p^u s$, $p \nmid s$, and use the prime-to-p power step to show that a root of order dividing s congruent to one must be one. The completely proved cyclotomic distance formula in the same lesson, Section 5 gives

$$\operatorname{ord}_{\mathfrak{p}}(\alpha^b - 1) \le \frac{e}{p-1} \le \frac{d \ln 2}{f \ln p}.$$

For the second inequality use $ef \le d$ and $\ln p \le (p-1) \ln 2$, which follows by induction from $p \le 2^{p-1}$ for integers $p \ge 2$. This proves (10.59) for torsion as well.

Finally $v_p(b)\ln p \le \ln|b|$ and $ef \le d$, so $ev_p(b) \le d\ln|b|/(f\ln p)$. Substitute in (10.59) and use $p/(p-1) = 1 + 1/(p-1)$ to obtain (10.60). □

For example, take $K = \mathbb{Q}$, $\alpha = 4$, $p = 5$. The original residue has order $g = 2$, while the saturated basis 2, together with -1, generates all four nonzero residues. Thus (10.49) gives saturated index one, whereas the original one-dimensional residue index is two. The parameters serve different bounds.

The exact power laws give valuation zero for odd b, and $1 + v_5(b)$ for even b. For $b = 2 \cdot 5^k$, (10.60) reads

$$1 + k \le k + \frac{7\ln 4}{2\ln 5}.$$

At the torsion boundary, $K = \mathbb{Q}$, $p = 2$, $\alpha = -1$, $b = 1$, the height term vanishes and (10.60) is the equality $1 = \ln 2/\ln 2$.

Corollary 10.22 (a rational one-base reduction). Let $\alpha_1, \dots, \alpha_n \in K^*$ be units at $\mathfrak{p}$, and let $\zeta \in K$ be a root of unity of order w. Let $c_0, c_1, \dots, c_n$ and $b_1, \dots, b_n$ be integers, with $\boldsymbol{b} \neq 0$, and suppose

$$b_j = \frac{u}{v}c_j \quad (1 \le j \le n), \qquad u \neq 0, \quad v \ge 1, \quad \gcd(u, v) = 1.$$

Put $\gamma = \zeta^{c_0}\prod_j \alpha_j^{c_j}$, $\Xi = \prod_j \alpha_j^{b_j}$, and $B_* = \min_{b_j \neq 0}|b_j|$. If γ is not a root of unity and $\Xi \neq 1$, then, with $g_\gamma = |\langle\bar{\gamma}\rangle|$,

$$\operatorname{ord}_{\mathfrak{p}}(\Xi - 1) \le \frac{d}{f\ln p}\left(\ln(2wB_*) + g_\gamma\left(1 + \frac{1}{p-1}\right)eh(\gamma)\right). \tag{10.61}$$

Proof. The coefficient identities give $\Xi^v = \zeta^{-uc_0}\gamma^u$, and hence $\Xi^{vw} = \gamma^{uw} \neq 1$. All these numbers are local units. Factoring $X^{vw} - 1$ by $X - 1$ and evaluating at Ξ shows that $\operatorname{ord}_{\mathfrak{p}}(\Xi - 1) \le \operatorname{ord}_{\mathfrak{p}}(\gamma^{uw} - 1)$: the remaining finite sum consists of integral units and is therefore integral. Apply (10.60) to γ and exponent uw. Since $vb_j = uc_j$ and u, v are coprime, u divides every b_j. Thus $|u| \le B_*$, proving (10.61). □

This reduces a form whose coefficient vector is rationally proportional to one integral vector to a single unit, with an exact exponent budget. Applying it to a many-logarithm estimate also requires a proved bound for $h(\gamma)$ in terms of the original heights; the saturation bounds of Section 9 supply one of the ingredients for that step.

11. Integral derivative bases and a multiplicity reduction

The many-logarithm argument changes the torus coordinates and then differentiates in a rational hyperplane. Two facts matter: the change of derivative basis must preserve p-adic integrality, and the zero estimate must allow different allocations of points at its different dimension steps. We establish both, keeping the coordinate degrees separate.

A minor of minimum valuation

Lemma 10.23 (integral derivative coordinates). Let $2 \le r \le n$. Suppose $L_0 = z_0, L_1, \dots, L_r$ are independent integral linear forms, with $L_i = a_{i0}z_0 + \sum_{j=1}^n a_{ij}z_j$, and $\sum_j b_j z_j = B_0L_0 + \sum_{i=1}^r B_iL_i$, where $b_j \in \mathbb{Z}$, $B_i \in \mathbb{Q}$, $b_n \neq 0$, and $B_r \neq 0$. Set

$$C_{ij} = b_n a_{ij} - b_j a_{in} \quad (1 \le i \le r,\ 1 \le j < n), \qquad E_i = Y_i\frac{\partial}{\partial Y_i}. \tag{10.62}$$

The derivations

$$\delta_j = \sum_{i=1}^{r} C_{ij} E_i = \frac{1}{B_r} \sum_{i<r} C_{ij}(B_r E_i - B_i E_r) \quad (1 \le j < n) \tag{10.63}$$

span the torus hyperplane $\sum_i B_i x_i = 0$, of dimension $r-1$. Let C' be the first $r-1$ rows of C. Choose $r-1$ columns J whose nonzero determinant has minimum v_p among all such minors. Then the corresponding δ_j, $j \in J$, are independent, and every δ_j has an expression

$$\delta_j = \sum_{k \in J} \lambda_{kj} \delta_k, \qquad \lambda_{kj} \in \mathbb{Q} \cap \mathbb{Z}_p. \tag{10.64}$$

Vanishing of all jets of total order at most T for these selected derivations and ∂_{Y_0} is therefore equivalent to vanishing for all the displayed derivations and ∂_{Y_0}. The changes of jets have p-adically integral coefficients.

Proof. Comparing the coefficients of z_j, $j \ge 1$, gives $b_j = \sum_i B_i a_{ij}$. Hence $\sum_i B_i C_{ij} = 0$, proving the equality in (10.63).

The r row vectors $a_i = (a_{i1}, \dots, a_{in})$ are independent. Otherwise a nonzero combination of the L_i would be a multiple of L_0, contrary to the assumed independence. Suppose a combination $a = \sum_{i<r} q_i a_i$ annihilates the vectors $b_n e_j - b_j e_n$, $j < n$. Then $b_n a_j - b_j a_n = 0$ for every $j < n$, so $a = (a_n/b_n)b$. Since $b = \sum_i B_i a_i$ and $B_r \neq 0$, row independence first forces $a_n/b_n = 0$, then all $q_i = 0$. Thus C' has rank $r-1$. Its column images in the full matrix are determined by their first $r-1$ coordinates through $\sum_i B_i C_{ij} = 0$. They span exactly the stated hyperplane.

Solve $C'_j = C'_J \lambda_j$ by Cramer's rule. Each numerator is zero or, up to sign, another $(r-1)$-column minor. Minimality of the chosen denominator's valuation therefore gives $v_p(\lambda_{kj}) \ge 0$. This proves (10.64).

The Euler derivations commute with one another and with ∂_{Y_0}. Expanding a product of the linear combinations (10.64) expresses every jet of order at most T as a combination of selected jets of that order. The coefficients are products and sums of the λ_{kj}, multiplied by integer multinomial coefficients, so they are in $\mathbb{Z}_p$. Conversely the selected derivations are among the original ones. This proves both directions of the vanishing assertion. □

These operators also preserve the coordinate degree bounds and local integral coefficients. In fact, $\delta_j(Y^{\boldsymbol{m}}) = (\sum_i C_{ij} m_i) Y^{\boldsymbol{m}}$. For a polynomial of degrees at most $D_i \ge 1$, put $M_j = \max\{1, \sum_i |C_{ij}| D_i\}$. On a monomial, the multiplier of $\partial_{Y_0}^{t_0} \prod_j \delta_j^{t_j}$ is an integer, and its ordinary absolute value is at most

$$D_0^{t_0} \prod_j M_j^{t_j}. \tag{10.65}$$

Indeed the additive factor is the falling factorial of its exponent, bounded by $D_0^{t_0}$, and each Euler factor has the displayed bound. Nonzero terms retain their torus exponents and lose exactly t_0 in the additive exponent, so distinct input monomials do not merge. At finite places the integral multipliers introduce no new denominators.

Weighted relations with a sharp degree constant

We shall use the complete upper successive-minima inequality and subgroup-minor calculation in Modern estimates: Matveev and Waldschmidt. The following deduction uses the Euclidean radius $1/\sqrt{r}$ in the weighted estimate.

Lemma 10.24 (weighted independent relations). Let $\mathcal{M} \subseteq \mathbb{Z}^r$ have rank $1 \le \nu \le r$, with basis matrix U of size r by ν. Let $A_j > 0$. If $|\det U_J| \prod_{j\in J} A_j \le B$ for every ν-row set J, there are independent $u_1, \dots, u_\nu \in \mathcal{M}$ with

$$\prod_{i=1}^{\nu} \sum_{j} A_j |u_{ij}| \le K_{r,\nu} B, \qquad K_{r,\nu} = 2^\nu (r/\pi)^{\nu/2} \Gamma(1+\nu/2) \binom{r}{\nu}^{1/2} \le r^\nu. \tag{10.66}$$

They span $\mathcal{M}_{\mathbb{Q}}$; an integral basis is not asserted.

Proof. Give its real span the weighted Euclidean metric $\sum_j A_j^2 x_j^2$. The Gram determinant identity gives covolume $(\sum_{|J|=\nu} \det(U_J)^2 \prod_{j\in J} A_j^2)^{1/2} \le \binom{r}{\nu}^{1/2} B$. The unit ball of $\sum_j A_j |x_j|$ contains the weighted Euclidean ball of radius $1/\sqrt{r}$, by Cauchy–Schwarz. Its ν-volume is therefore at least $(\pi/r)^{\nu/2}/\Gamma(1+\nu/2)$. The complete volume and upper successive-minima arguments in the earlier lesson give independent lattice vectors with product of these norms at most 2^ν times the covolume divided by that volume. This proves the first inequality.

For the second use $\binom{r}{\nu} \le r^\nu/\nu!$. It suffices to show $a_\nu = 2^\nu \Gamma(1+\nu/2)/(\pi^{\nu/2}\sqrt{\nu!}) \le 1$. The Gaussian integral gives $a_0 = a_1 = 1$, and integration by parts in the defining gamma integral gives

$$\frac{a_{\nu+2}}{a_\nu} = \frac{2}{\pi}\sqrt{\frac{\nu+2}{\nu+1}} < 1 \quad (\nu \ge 0),$$

since $\pi > 3 > 2\sqrt{2}$. For completeness, $\pi > 3$ follows from $\pi/4 = \int_0^1 (1+x^2)^{-1}\,dx$: the first eight terms of the integrated geometric series give the strict lower bound $\sum_{j=0}^{7} (-1)^j/(2j+1) > 3/4$. Its omitted remainder is the integral of $x^{16}/(1+x^2) > 0$. Thus both parity sequences decrease from one. This proves $K_{r,\nu} \le r^\nu$. $\square$

Unequal allocations in the zero estimate

Let k be algebraically closed of characteristic zero, and put $G = \mathbb{G}_a \times \mathbb{G}_m^r$. Write $g^s = (s, \vartheta_1^s, \dots, \vartheta_r^s)$, where the $\vartheta_i \in k^*$ are multiplicatively independent. For nonzero $\boldsymbol{B} \in \mathbb{Q}^r$, let W be the span of ∂_{Y_0} and the Euler directions in the hyperplane $\boldsymbol{B} \cdot \boldsymbol{x} = 0$. Its dimension is r.

Theorem 10.25 (multiplicity and dimension reduction). Suppose $r \ge 2$, and let S, T be nonnegative integers. Choose nonincreasing nonnegative integer allocations

$$S_0 \ge \cdots \ge S_r, \quad T_0 \ge \cdots \ge T_r, \qquad \sum_{m=0}^{r} S_m \le S, \quad \sum_{m=0}^{r} T_m \le T. \tag{10.67}$$

Let the positive coordinate bounds $D_0, \dots, D_r$ satisfy $D_0 \ge \max\{D_1, \dots, D_r, T_r + r\}$, and assume

$$\begin{aligned} (S_m+1)\binom{T_m+m+1}{m+1} &> (m+1)! D_0 \max_{\substack{J\subseteq\{1,\dots,r\}\\ |J|=m}} \prod_{j\in J} D_j \quad (0 \le m < r), \\ (S_r+1)\binom{T_r+r}{r} &> (r+1)! D_0 \prod_{j=1}^{r} D_j. \end{aligned} \tag{10.68}$$

If a nonzero polynomial of these coordinate bounds vanishes at g^s, $0 \le s \le S$, with all W-jets of total order at most T, then there are $1 \le \nu < r$ and independent integral vectors $u_1, \dots, u_\nu$, whose rational span contains $\boldsymbol{B}$. For any $A_j > 0$, put $R_i = \sum_j A_j |u_{ij}|$ and $C_{r,\nu} = \nu! K_{r,\nu} \le \nu! r^\nu$. At least one of

$$(S_\nu+1)\binom{T_\nu+\nu-1}{\nu-1}\prod_i R_i \le C_{r,\nu}\max_{|J|=\nu}\prod_{j\in J}A_jD_j, \tag{10.69}$$

$$(S_\nu+1)\binom{T_\nu+\nu}{\nu}\prod_i R_i \le (\nu+1)C_{r,\nu}D_0\max_{|J|=\nu}\prod_{j\in J}A_jD_j \tag{10.70}$$

holds.

Proof. We use the fully proved stabilizer, jet and coordinate-degree arguments of the multiplicity subsection of the earlier lesson, explaining the allocation step explicitly. Let $N=r+1$. For $1\le j\le N+1$, let I_j be generated by the translates of all W-derivatives of P of order at most $\sum_{m=0}^{j-2}T_m$, at translating points g^s with $0\le s\le\sum_{m=0}^{j-2}S_m$; empty sums are zero. Write $X_j=Z(I_j)$. Translations and invariant derivatives preserve the coordinate bounds. Hence $G\supsetneq X_1\supseteq\cdots\supseteq X_{N+1}$, and the last set contains the identity by (10.67). Every set is nonempty.

Choose the first j for which X_j, X_{j+1} have the same dimension, and a common component V of that dimension. Such a $j\le N$ exists, since $\dim X_1\le N-1$. Before it each dimension falls, so $\dim V\le N-j$. As in the cited stabilizer proof, define $E=\{x : x+V\subseteq X_j\}$ and $H=\{x : x+V=V\}$. The set E is a finite union of H-cosets. Its defining ideal, obtained by translating the generators of I_j by V, has dimension $\dim H$ and the same coordinate degree bounds. That ideal vanishes through order T_{j-1} on $\{g^s : 0\le s\le S_{j-1}\}+H$, because the added derivatives and translates are among those defining X_{j+1}.

Replace H by its identity component H_0, and put $c=N-\dim H_0$. Since $\dim H_0\le\dim V$, we have $c\ge j$. Monotonicity gives at least $S_{c-1}+1$ point cosets and at least T_{c-1} jets on them. All these cosets are distinct. Indeed, if $g^q\in H_0$ for $q\ne 0$, the nonzero additive coordinate forces the additive part of H_0 to be $\mathbb{G}_a$. Properness then supplies a nonzero torus character u with $\boldsymbol{\vartheta}^{qu}=1$, contrary to multiplicative independence.

Put $\sigma=\dim(W/(W\cap T(H_0)))$. The complete positive-filtration upper and lower estimates (8.72)–(8.76) give, for every codimension-c coordinate set J,

$$(S_{c-1}+1)\binom{T_{c-1}+\sigma}{\sigma}\delta(H_0;J)\le c!\prod_{j\in J}D_j. \tag{10.71}$$

Here $\delta(H_0;J)$ is the nonnegative integral coefficient of the complementary monomial in the subgroup Hilbert polynomial. To extract a coefficient from the lower polynomial inequality, send exactly the coordinates in that complementary monomial to a common infinity, keeping the others one. The polynomials are squarefree of the same degree, so division by that power leaves exactly the desired coefficient. This is the coefficient argument already proved in the earlier lesson.

Since W is a hyperplane, $\sigma=c-1$ when $T(H_0)\subseteq W$, and $\sigma=c$ otherwise. In the second case $c\le r$, and some δ is at least one. The maximum product of c coordinate bounds is $D_0\max_{|J|=c-1}\prod_{j\in J}D_j$, because D_0 is largest. Inequality (10.71) contradicts (10.68) with $m=c-1$. Hence $T(H_0)\subseteq W$.

Write $H_0=H_a\times H_m$, and let $\mathcal{M}$ be the primitive character lattice of H_m, of rank ν. The subgroup and tangent classification in the earlier lesson shows that $\boldsymbol{B}\in\mathcal{M}_{\mathbb{Q}}$. Thus $\nu\ge 1$. If $\nu=r$ and $H_a=\{0\}$, then $c=r+1, \sigma=r$, and (10.71) contradicts the last line of (10.68). If $\nu=r$ and $H_a=\mathbb{G}_a$, then $c=r, \sigma=r-1$. We may decrease the available budgets to S_r, T_r; its unique torus coefficient is one. Thus $(S_r+1)\binom{T_r+r-1}{r-1}\le r!\prod_{j=1}^{r}D_j$. The last line of (10.68), and $\binom{T_r+r}{r}/\binom{T_r+r-1}{r-1}=(T_r+r)/r$, contradict this since $D_0\ge T_r+r$. Therefore $1\le\nu<r$.

Let U be a basis matrix of $\mathcal{M}$. If $H_a=\mathbb{G}_a$, the subgroup coefficients are $|\det U_J|$, $c=\nu, \sigma=\nu-1$. Decrease its budgets to S_ν, T_ν. Multiply (10.71) by $\prod_{j\in J}A_j$, and apply Lemma 10.24 to the resulting

bounds on every weighted minor. The independent vectors supplied by that lemma span $\mathcal{M}_{\mathbb{Q}}$, so their span contains $\boldsymbol{B}$. Its factor $K_{r,\nu}$ times $\nu!$ gives (10.69). If $H_a = \{0\}$, then $c = \nu + 1, \sigma = \nu$; its coefficients on $\{0\} \cup J$ are $|\det U_J|$. The same argument uses $(\nu+1)!D_0$ in (10.71), proving (10.70). This accounts for both additive subgroups and completes the proof. □

For the allocations $S_0 = \lfloor S/3 \rfloor$, $S_i = \lfloor 2S/(3r) \rfloor$ for $i \geq 1$, and $T_i = \lfloor T/(r+1) \rfloor$, (10.67) holds: $r \geq 2$ makes the point allocations nonincreasing, and their sums are at most S, T. Since the T_i are equal and $D_0 \geq T_r + r$, (10.69) also implies (10.70). Multiply (10.69) by $(T_\nu + \nu)/\nu \leq D_0$; its enlarged right side is at most the right side of (10.70). Thus the second bound alone suffices for this choice of allocations.

Corollary 10.26 (new algebraic bases). If the ϑ_j are independent algebraic units in a number field and u_i are the vectors in Theorem 10.25, put $\gamma_i = \prod_j \vartheta_j^{u_{ij}}$. They are independent units and

$$h(\gamma_i) \leq \sum_j |u_{ij}| h(\vartheta_j) \leq R_i \quad \text{if } A_j \geq h(\vartheta_j). \tag{10.72}$$

If $\boldsymbol{b} \in \mathbb{Z}^r$ is in their rational span, there are integers $m \geq 1, c_i$ with $m\boldsymbol{b} = \sum_i c_i u_i$, and $(\prod_j \vartheta_j^{b_j})^m = \prod_i \gamma_i^{c_i}$.

Proof. Integer products preserve local units, and the proved height power and product inequalities give (10.72). A multiplicative relation among the γ_i gives a zero integral combination of the u_i, by independence of the ϑ_j; linear independence of the vectors makes every coefficient zero. Clearing the denominators of a rational span expression proves the last identity. □

Example of a sharp weighted factor. For $r = 2, \nu = 1$, take $\mathcal{M} = \mathbb{Z}(3, -2)$, weights $A_1 = 2, A_2 = 3$, and minor bound $B = 6$. Its primitive vector has weighted norm 12, exactly $K_{2,1}B = 2B$. In the torus put $H_m = \{Y_1^3 = Y_2^2\}$, $H = \mathbb{G}_a \times H_m$, and $g = (1, 2, 3)$. For $s = 0, 1, 2$, its three distinct H-cosets have characters $Y_1^3/Y_2^2 = (8/9)^s$. The polynomial

$$P(Y_1, Y_2) = \prod_{s=0}^{2} (9^s Y_1^3 - 8^s Y_2^2)$$

vanishes on all three. The directions ∂_{Y_0} and $2E_1 + 3E_2$ are tangent to these cosets, so every such derivative vanishes there. Its coordinate bounds are $D_1 = 9, D_2 = 6$, and the corresponding weighted relation inequality is the equality $3 \cdot 12 = 2 \max(2 \cdot 9, 3 \cdot 6) = 36$. There is no transverse multiplicity factor in this case: $\sigma = \nu - 1 = 0$.

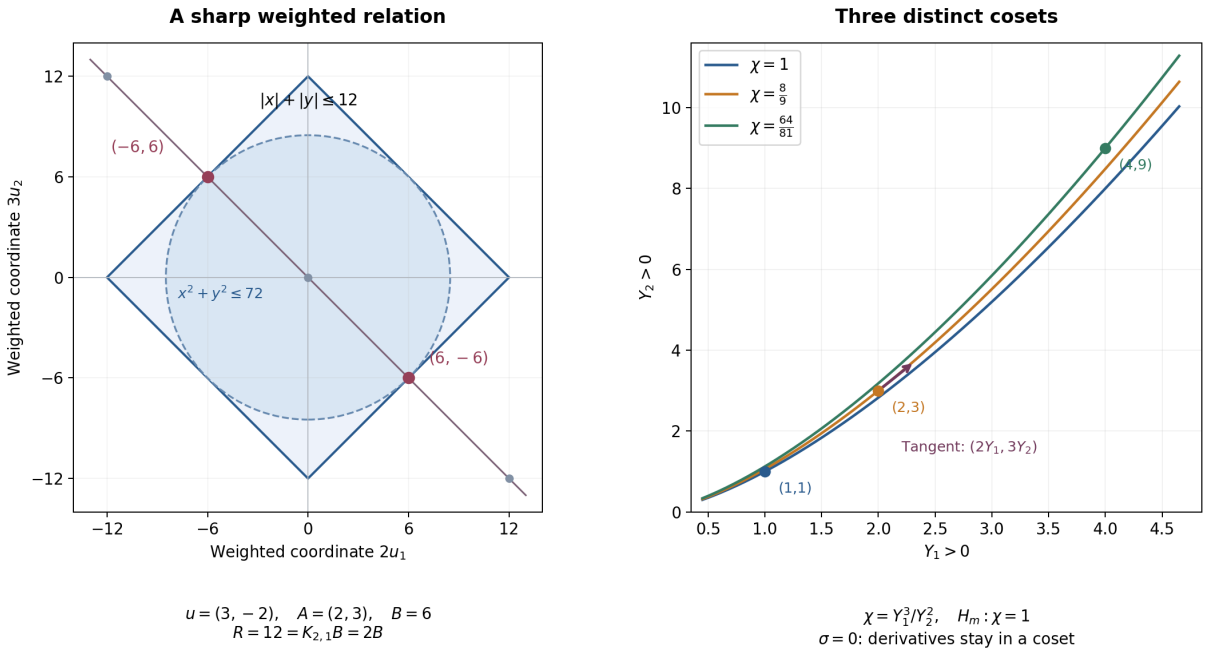


Figure 10.3. Left: in weighted coordinates $(2u_1, 3u_2)$, the relation $(3, -2)$ becomes $(6, -6)$. Its diamond norm is 12; the Euclidean circle of radius $12/\sqrt{2}$ lies in the diamond and meets that relation line at the two marked primitive vectors. This is the equality case of (10.66) for $r = 2, \nu = 1$. Right: a positive real section of the torus projections of the three cosets above. Their quotient character values $1, 8/9, 64/81$ are distinct, and the points $(1,1), (2,3), (4,9)$ lie on their respective curves. The additive coordinates $0, 1, 2$ are omitted from this projection. The arrow is tangent to a coset, in the direction $2E_1 + 3E_2$; it illustrates $\sigma = 0$, not a transverse derivative. The complete subgroup, degree and lattice arguments are (10.66), (10.69)–(10.71) and the example. Mathematical background: the earlier complete multiplicity lesson and Yu's free 2013 article, cited below. Figure program.

12. Closing the one-base branch

The many-logarithm construction may reduce to one new base before the auxiliary polynomial is needed. A one-logarithm inequality alone does not finish that step: its constant term must also be compared with the product of the original heights. We give that comparison with the full residue-order and root-of-unity factors.

Let $n \geq 2$, let K have degree d, and let $\mathfrak{p}$ lie above p, with ramification index $e_{\mathfrak{p}}$ and residue degree f. Put $t = f \ln p$ and $\ln^* d = \max\{1, \ln d\}$. Take multiplicatively independent $\mathfrak{p}$-adic units $\alpha_1, \dots, \alpha_n$, integers b_j not all zero, and write

$$\Xi = \prod_j \alpha_j^{b_j}, \qquad P_h = \prod_j h(\alpha_j), \qquad B_* = \min_{b_j \neq 0} |b_j|. \tag{10.73}$$

Take $q = 2$ for $p > 2$, and $q = 3$ for $p = 2$. In the second case assume $\zeta_3 \in K$. Let $w_q = q^u$ be the order of the q-primary roots of unity in K, and choose their generator α_0. Use the intrinsic saturated residue index $\delta(\boldsymbol{\alpha})$ from Lemma 10.15, and define

$$\begin{aligned} Q_n &= p^f/\delta(\boldsymbol{\alpha}), & M_n &= \max\{Q_n/t^{n+1}, \mathrm{e}^n/n^n\}, \\ L &= \max\{\ln(\mathrm{e}^4(n+1)d), e_{\mathfrak{p}}, t\}, & \varepsilon_n &= 10^{-26}/(2n), \\ a &= \begin{cases} 7(p-1)/(p-2), & p \geq 5,\ e_{\mathfrak{p}} = 1, \\ 14, & p > 2 \text{ in the other cases}, \\ 26, & p = 2, \end{cases} & R_n &= \frac{\mathrm{e}a}{q}(1+\varepsilon_n)(n+1)d. \end{aligned} \tag{10.74}$$

Here $\mathrm{e} = \exp(1)$ is distinct from the ramification index. For any $c \geq 206$, put

$$\mathcal{C}_n = c a^n \frac{n^n (n+1)^{n+1} d^{n+2} \ln^* d}{n! w_q t} M_n L. \tag{10.75}$$

Theorem 10.27 (the one-base numerical closure). Suppose there are independent integral forms $L_0 = z_0, L_1 = a_0 z_0 + \sum_j a_j z_j$, and rationals B_0, B_1, with $B_1 \neq 0$, such that $\sum_j b_j z_j = B_0 L_0 + B_1 L_1$. Set $\gamma = \alpha_0^{a_0} \prod_j \alpha_j^{a_j}$, and let g_γ be its actual residue order at $\mathfrak{p}$. Define

$$\delta_\gamma = \frac{p^f - 1}{g_\gamma}, \qquad M_\gamma = \max\{p^f/(\delta_\gamma t^2), \mathrm{e}\}. \tag{10.76}$$

Suppose a positive σ satisfies

$$h(\gamma) \leq \sigma \leq R_n^{n-1} \frac{M_n}{M_\gamma} P_h. \tag{10.77}$$

If $H \geq \max\{\ln B_*, (n+1)t, \ln(4d)\}$, then

$$\mathrm{ord}_{\mathfrak{p}}(\Xi - 1) \leq \frac{1}{500} \mathcal{C}_n P_h H < \mathcal{C}_n P_h H. \tag{10.78}$$

Proof. Write $B_1 = v_1/v_2$ in lowest terms, with $v_2 > 0$. Coefficient comparison gives $v_2 b_j = v_1 a_j$; hence $v_1 \mid b_j$ and $|v_1| \leq B_*$. Also $v_2 B_0 = -v_1 a_0$ is integral. Raising the resulting multiplicative identity to w_q removes α_0: $\Xi^{v_2 w_q} = \gamma^{v_1 w_q}$.

The vector $(a_1, \ldots, a_n)$ is nonzero, by independence of the two forms. Thus γ is not a root of unity: a torsion relation would, after a further power killing α_0, contradict independence of the α_j. The same argument proves that Ξ is not torsion. All powers just used are therefore different from one. The finite geometric sum and Theorem 10.21 give

$$\mathrm{ord}_{\mathfrak{p}}(\Xi - 1) \leq \frac{d}{t} \left(\ln(2 w_q B_*) + g_\gamma \frac{p}{p-1} e_{\mathfrak{p}} h(\gamma) \right). \tag{10.79}$$

Negative v_1 is covered by that theorem. The geometric sum has integral local-unit terms, so its valuation is nonnegative.

We need two elementary estimates. First,

$$w_q \leq \frac{q}{q-1} d \leq 2d. \tag{10.80}$$

Indeed Proposition 9.7, applied at the prime q, proves that Φ_{q^u} is irreducible over $\mathbb{Q}_q$, of degree $q^{u-1}(q-1)$. It is consequently irreducible over $\mathbb{Q}$ as well. Since its root α_0 belongs to K, that degree is at most d, proving (10.80). Thus $\ln(2 w_q B_*) \leq 2H$.

Second, if w_K is the total number of roots of unity in K, Theorem 10.18 and the index lower bound one give the uniform weakened form

$$P_h \geq \frac{w_K n^n}{58 n! \mathrm{e}^n d^{n+1} \ln^* d} \geq \frac{w_q n^n}{58 n! \mathrm{e}^n d^{n+1} \ln^* d}. \tag{10.81}$$

For $d = 1$, the proved constant 17 is smaller than 58, so the same form holds. There is no lower bound for a single new height hidden in this step.

Combining (10.75), (10.81) and $M_n \geq \mathrm{e}^n/n^n$ gives

$$\mathcal{C}_n P_h \geq \frac{c}{58} a^n \frac{n^n (n+1)^{n+1}}{(n!)^2} \frac{d}{t} L \geq \frac{c}{58} a^n (n+1) \frac{d}{t} L > 2000 \frac{d}{t}. \tag{10.82}$$

Here $n! \leq n^n$, $(n+1)^{n+1}/n^n \geq n+1$, $a \geq 7$, and $L \geq 4 + \ln 3 > 5$. The last numerical comparison is $206 \cdot 49 \cdot 3 \cdot 5 > 2000 \cdot 58$. The logarithmic term in (10.79) is therefore at most $\mathcal{C}_n P_h H/1000$.

For the height term use $g_\gamma/M_\gamma < t^2$, directly from (10.76), and (10.77). Its ratio to $\mathcal{C}_n P_h H$ is at most

$$\frac{n!}{n^n} \left(\frac{\mathrm{e}}{q}\right)^{n-1} (1+\varepsilon_n)^{n-1} \frac{w_q e_{\mathfrak{p}} t^2}{ca(n+1)^2 d^2 \ln^* d\, LH} \frac{p}{p-1}. \tag{10.83}$$

This is an exact cancellation of the M_n, degree and exponential factors. The local degree inequality gives $e_{\mathfrak{p}} \leq d$; also $L \geq t$, $H \geq (n+1)t$, $p/(p-1) \leq 2$, and (10.80) gives $w_q/d \leq 2$.

For completeness, monotonicity of $\ln x$ gives $\ln(n!) \leq \int_1^n \ln x\, dx + \ln n = n \ln n - n + 1 + \ln n$. Consequently $n! \mathrm{e}^{n-1}/n^n \leq n$. Furthermore $(1+\varepsilon_n)^{n-1} < 2$: use $\ln(1+x) \leq x$, which follows by integrating $(1+x)^{-1} \leq 1$, and $\varepsilon_n(n-1) < 1/2$. The exponential series gives $\mathrm{e} < 3$, since $k! \geq 2^{k-1}$ for $k \geq 2$, with strict inequality for $k \geq 3$; hence $\exp(1/2) < 2$. This also proves $\ln 3 > 1$ used above.

Thus (10.83) is at most $8n/(q^{n-1} ca(n+1)^3) \leq 4n/(ca(n+1)^3)$, because $q \geq 2, n \geq 2$. Since $(n+1)^2 \geq 4n$ and $n+1 \geq 3$, this is at most $1/(3ca) \leq 1/(3 \cdot 206 \cdot 7) < 1/1000$. The height term is also at most $\mathcal{C}_n P_h H/1000$. Adding the two terms proves (10.78). □

We check the correspondence with the free 2013 preparation. Let κ be the unique nonnegative integer with $p^{\kappa-1}(p-1) \leq 2e_{\mathfrak{p}} < p^\kappa(p-1)$. Set $c_2 = 7/4$ for $p > 2$, and $c_2 = 13/9$ for $p = 2$. Put $\vartheta = (p-2)/(p-1)$ when $p \geq 5, e_{\mathfrak{p}} = 1$, and $\vartheta = p^\kappa/(2e_{\mathfrak{p}})$ otherwise, and let $\theta = \vartheta/(1+\varepsilon_n)$. Direct substitution gives

$$\mathrm{e} c_2 q \frac{p^\kappa}{e_{\mathfrak{p}} \theta} (n+1) d = R_n. \tag{10.84}$$

In the first case $\kappa = 0$; in the other cases the factors are respectively 7 and $26/3$ before multiplication by $\mathrm{e}(1+\varepsilon_n)(n+1)d$. Thus (10.77) is precisely the one-base height-product hypothesis of that preparation. Formula (10.75) is its first numerical coefficient, with any of its listed choices of c, all at least 206. Its parameter $h^{(1)}$ is at least $\ln B_*$ and $(n+1)t$; its term $(n+1)(a_0^{(1)} n + a_1^{(1)} + \ln(a_0^{(1)} n + a_2^{(1)}) + \ln d)$ is larger than $\ln(4d)$, since the listed constants satisfy $a_0^{(1)} > 3, a_1^{(1)} > 2, a_2^{(1)} > 2$. Theorem 10.27 therefore proves the full numerical conclusion in this branch. The proof retains the original saturated index for the n bases, and the actual unsaturated residue order for the one new base.

13. Binomial preparation of the auxiliary polynomial

The auxiliary polynomial uses shifted powers of binomial polynomials and binomial polynomials in the Euler eigenvalues. We prove their independence, their exact denominator control and their equivalence to ordinary jets. These are algebraic statements; no logarithm estimate is assumed in their proofs.

Taylor coefficients at integral and fractional nodes

For integers $k \geq 1, \ell \geq 1, m \geq 0$, define

$$\begin{aligned}\Delta(z;k) &= \frac{(z+1)\cdots(z+k)}{k!}, \quad \Delta(z;0)=1,\\ v(k) &= \operatorname{lcm}(1,\dots,k),\\ \Theta(z;k,\ell,m) &= \frac{1}{m!}\frac{d^m}{dz^m}\Delta(z;k)^\ell.\end{aligned} \tag{10.85}$$

Lemma 10.28 (prepared Taylor integrality). For every integer s,

$$v(k)^m\Theta(s;k,\ell,m)\in\mathbb{Z}. \tag{10.86}$$

If q is prime, $I\in\mathbb{Z}_{\ge 0}$, and $A,s\in\mathbb{Z}$, then

$$q^{\chi_I\ell(kI+v_q(k!))}v(k)^m\Theta(s/q^I+A;k,\ell,m)\in\mathbb{Z}, \qquad \chi_0=0, \quad \chi_I=1\ (I>0). \tag{10.87}$$

In particular $v(k)^m\Theta(s/q^I+A;k,\ell,m)$ belongs to $\mathbb{Z}[1/q]$, before multiplication by the displayed q-power.

Proof. Generalized binomial coefficients $\binom{N}{j}$ are integral for every integer N: for $N<0$, their product formula is $(-1)^j\binom{-N+j-1}{j}$. Vandermonde's polynomial identity gives $\Delta(s+Z;k)=\sum_{j=0}^{k}\binom{s+k}{k-j}\binom{Z}{j}$. One proof first counts subsets for nonnegative integer arguments and then uses polynomial equality to extend to all arguments. For $j\ge 1$,

$$\binom{Z}{j}=\frac{(-1)^{j-1}Z}{j}\prod_{a=1}^{j-1}(1-Z/a).$$

Each coefficient of Z^m has denominators consisting of j and $m-1$ numbers among $1,\dots,j-1$. Every factor divides $v(k)$. Hence multiplication by $v(k)^m$ makes that coefficient integral. The constant coefficient is integral too. In the ℓ-th power, a coefficient of order m is a sum of products whose orders sum to m; the same scaling therefore clears all of them. This proves (10.86), including vanishing coefficients beyond the degree.

Fix $F(Z)=v(k)^m\Theta(Z;k,\ell,m)$. It is a rational polynomial, integral at every integer. At $x=s/q^I+A$, it has no denominator at a prime $r\neq q$. To see this without a limiting argument, choose B bounding the r-adic denominators of its finitely many coefficients, and an integer $N\equiv x \pmod{r^{B+1}}$; the inverse of q^I modulo that power supplies N. Both x,N are r-integral. Factoring x^j-N^j shows that $F(x)-F(N)$ is r-integral, so $F(x)$ is r-integral.

The coefficients of $\Delta(Z;k)^\ell$ have denominator dividing $(k!)^\ell$. Dividing an ordinary derivative of order m by $m!$ multiplies each monomial coefficient by an integer binomial coefficient and decreases its degree by m. Thus its value at x has q-adic denominator at most $q^{\ell v_q(k!)+Ik\ell}$; the integer factor $v(k)^m$ cannot worsen this bound. Together with the preceding assertion at every other prime this proves (10.87) for $I>0$. For $I=0$, (10.86) already gives an integer and no q-factor is needed. □

Independence of the shifted powers

Lemma 10.29 (a binomial polynomial basis). For $k,L\ge 1$, the kL polynomials

$$F_{a,\ell}(X)=\Delta(X+a;k)^\ell, \qquad 0\le a<k, \quad 1\le\ell\le L, \tag{10.88}$$

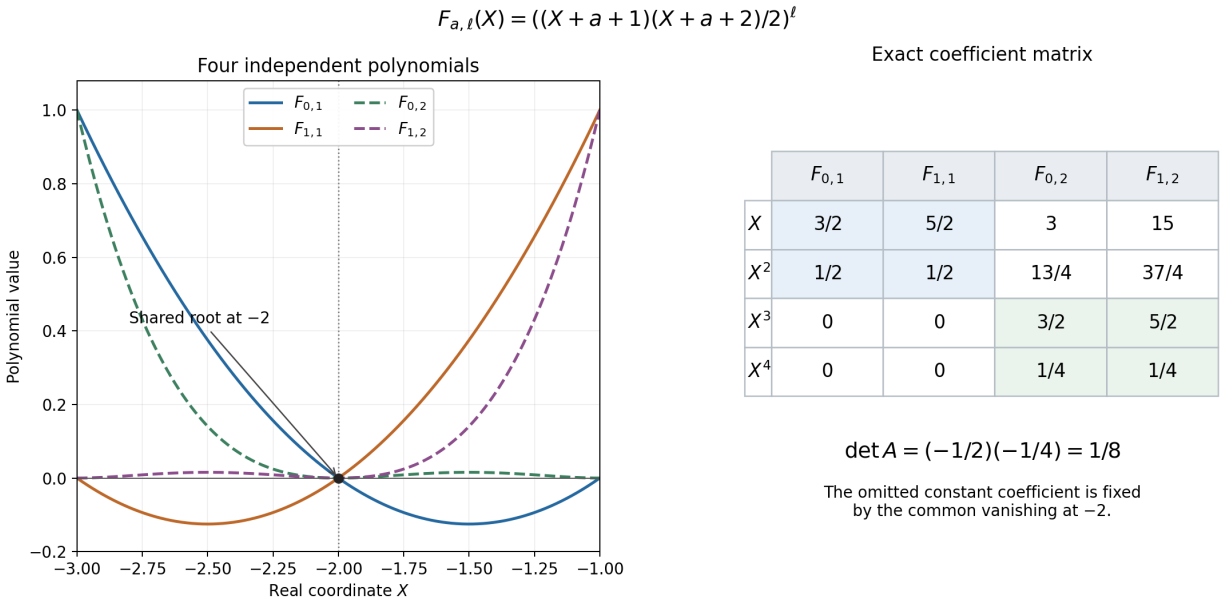
are a basis of the polynomials of degree at most kL vanishing at $X=-k$. For fixed ℓ, their top k coefficient matrix satisfies

$$\det\Big([X^{k\ell-j}]F_{a,\ell}\Big)_{0\le j,a<k} = \frac{\prod_{j=0}^{k-1}\binom{k\ell}{j}}{(k!)^{k\ell}} \prod_{0\le a<b<k}(b-a) \neq 0. \tag{10.89}$$

Proof. The coefficient in row j, regarded as a polynomial in a, has degree j and leading coefficient $\binom{k\ell}{j}/(k!)^{\ell}$. Indeed expand the degree-$k\ell$ polynomial $\Delta(X;k)^{\ell}$ after substituting $X+a$; its highest term gives that leading coefficient, and lower terms give smaller powers of a. Changing these row polynomials to monomials is triangular. The monomial evaluation determinant at $0,\dots,k-1$ is the Vandermonde product in (10.89). It follows either by subtracting columns and induction, or by observing its zeros at equal columns and its leading coefficient one. This proves the formula.

In a proposed relation, take the largest ℓ occurring. Lower levels have degree at most $k(\ell-1)$. The top k coefficients and (10.89) force every coefficient at this largest level to vanish. Descending proves independence. Every polynomial in (10.88) vanishes at $-k$, because the factor with index $k-a$ is zero there. The space with this one vanishing condition has dimension kL: evaluation at $-k$ is a nonzero linear functional on the $kL+1$-dimensional polynomial space. Thus the independent family is a basis. An invertible affine substitution preserves its independence. □

For $k=L=2$, Figure 10.4 shows the four basis polynomials and their exact coefficient matrix. A shared root does not give a relation among the polynomials.



	$F_{0,1}$	$F_{1,1}$	$F_{0,2}$	$F_{1,2}$
X	3/2	5/2	3	15
X^2	1/2	1/2	13/4	37/4
X^3	0	0	3/2	5/2
X^4	0	0	1/4	1/4

Figure 10.4. Left: the restrictions to the real interval $[-3,-1]$ of $F_{0,1},F_{1,1},F_{0,2},F_{1,2}$ in (10.88) for $k=2$. All vanish at -2; the squared polynomials have a double zero there. Right: their coefficients of X,X^2,X^3,X^4, in the same column order. The diagonal two-by-two block determinants are $-1/2,-1/4$, so the full determinant is $1/8$. Their constant coefficients are determined by their common vanishing at -2. This is the exact independence mechanism in Lemma 10.29 and (10.89), applied to the auxiliary polynomial below. Human-source background: the binomial preparation in Yu's free 2013 article, cited below. Figure program.

Affine changes of binomial jets

Lemma 10.30 (Newton reparametrization). Over a characteristic-zero field, for $h\neq 0$ and c,

$$\Delta(hZ+c;t) = \sum_{j=0}^{t} d_j \Delta(Z;j), \qquad d_j = \sum_{i=0}^{j}(-1)^i \binom{j}{i}\Delta(c-h(i+1);t), \qquad d_t = h^t. \tag{10.90}$$

If $h,c\in\mathbb{Z}[1/q]$, then all d_j belong to that ring. If moreover $h=q^b$, $b\in\mathbb{Z}$, the triangular change and

its inverse are p-adically integral for every $p \neq q$. Products in several commuting variables preserve the total-order filtration.

Proof. The polynomials $\Delta(Z; j)$, $0 \le j \le t$, have degrees j and nonzero leading coefficients, so form a basis. At $Z = -i - 1$, their values are $(-1)^j \binom{i}{j}$, zero when $j > i$. Inverting this triangular evaluation matrix gives (10.90). Explicitly the inner binomial inversion sum is $\binom{a}{b} \sum_{i=b}^{a} (-1)^{i-b} \binom{a-b}{i-b}$, which is zero for $a > b$ and one for $a = b$, by $(1-1)^{a-b}$. Comparing leading coefficients gives $d_t = h^t$.

Lemma 10.28 with $\ell = 1, m = 0$ makes each displayed value at a $\mathbb{Z}[1/q]$ argument belong to that ring; the sum for d_j does too. When $h = q^b$, the diagonal h^t is a unit in $\mathbb{Z}[1/q]$. Successive substitution therefore also puts the inverse matrix in that ring, hence in $\mathbb{Z}_p$ for $p \neq q$. In products, every new index is coordinatewise at most the old one, so its total order cannot increase. □

The integrality assertion here concerns changes between binomial families. Conversion between a binomial family and ordinary powers has leading coefficient $1/t!$; we use that conversion only for algebraic equivalence of vanishing.

From prepared equations to the ordinary zero estimate

Proposition 10.31 (the auxiliary polynomial and its jets). Let $\mathcal{M} \subset \mathbb{Z}^r$ be finite, with distinct exponent vectors $\boldsymbol{m}$, and let $A_i \ge 0$ bound $|m_i|$. Put $d_i = \lfloor A_i \rfloor$. Fix q, I, k, L as above, and coefficients $c_{\boldsymbol{m},a,\ell}$ in a characteristic-zero field, not all zero. Form

$$Q(X, Y) = \sum_{\boldsymbol{m} \in \mathcal{M}} \sum_{0 \le a < k,\, 1 \le \ell \le L} c_{\boldsymbol{m},a,\ell} \Delta(q^{-I} X + a; k)^{\ell} Y^{\boldsymbol{m}}, \qquad P = Y^{\boldsymbol{d}} Q. \tag{10.91}$$

Then P is nonzero, is an ordinary polynomial, and has coordinate bounds $kL, 2A_1, \dots, 2A_r$.

Let $D_j = \sum_i C_{ij} Y_i \partial_{Y_i}$ be commuting Euler derivations, and put $\omega_j(\boldsymbol{m}) = \sum_i C_{ij} m_i$, $1 \le j \le r-1$. For a torus point $\boldsymbol{y}$ define the prepared value

$$\mathcal{Q}(s; \boldsymbol{t}) = \sum_{\boldsymbol{m},a,\ell} c_{\boldsymbol{m},a,\ell} v(k)^{t_0} \Theta(q^{-I} s + a; k, \ell, t_0) \prod_{j=1}^{r-1} \Delta(\omega_j(\boldsymbol{m}); t_j) \boldsymbol{y}^{\boldsymbol{m}}. \tag{10.92}$$

For any T, vanishing of all these values with $|\boldsymbol{t}| \le T$ is equivalent to vanishing of all ordinary jets of Q, and hence of P, at $(s, \boldsymbol{y})$ for $\partial_X, D_1, \dots, D_{r-1}$ through total order T.

Proof. Distinct Laurent monomials are linearly independent. For each fixed $\boldsymbol{m}$, Lemma 10.29, with the invertible substitution $X \mapsto q^{-I} X$, proves independence of its coefficient polynomials. Thus not all coefficients can cancel, and $Q \neq 0$. Since m_i is integral and $m_i \ge -A_i$, it satisfies $m_i \ge -\lfloor A_i \rfloor$. Hence $0 \le m_i + d_i \le 2A_i$. This proves the degree and nonvanishing assertions for P.

On its monomial terms, (10.92) is exactly the value of the commuting operator $(q^I v(k))^{t_0} \partial_X^{t_0} / t_0!$ times $\prod_j \Delta(D_j; t_j)$, applied to Q. The chain rule supplies q^{-It_0}, canceled by q^{It_0}. Each $\Delta(D_j; t_j)$ has leading term $D_j^{t_j} / t_j!$ and only lower powers. All diagonal scalars are nonzero. Induction on the torus part of the total order therefore makes these operators and the ordinary jet operators span exactly the same filtered space. This proves the first equivalence, in both directions.

Put $c_j = \sum_i C_{ij} d_i$. The product rule gives

$$\partial_X^{t_0} \prod_j D_j^{t_j} P = Y^{\boldsymbol{d}} \partial_X^{t_0} \prod_j (D_j + c_j)^{t_j} Q. \tag{10.93}$$

Expanding is triangular with diagonal one and preserves total order. The inverse uses $-c_j$ and $Y^{-\boldsymbol{d}}$. Since every torus coordinate is nonzero, evaluation proves the second equivalence. □

When the C_{ij} are integers, s is integral and $q \neq p$, Lemma 10.28 and integer-valued Δ give

$$v(k)^{t_0}\Theta(q^{-I}s + a; k, \ell, t_0) \prod_j \Delta(\omega_j(\boldsymbol{m}); t_j) \in \mathbb{Z}[1/q] \subseteq \mathbb{Z}_p.$$

The prepared product $v(k)^{t_0}\Theta$, rather than Θ alone, needs at most the q-denominator in (10.87). Integrality of the whole value additionally requires locally integral $c_{\boldsymbol{m},a,\ell}$ and local-unit torus coordinates.

For the application to Theorem 10.25, take $D_1, \dots, D_{r-1}$ to be the selected integral Euler directions in Lemma 10.23, renumbered. They span the Euler hyperplane in that theorem's space W. Apply the proposition at $(s, \vartheta_1^s, \dots, \vartheta_r^s)$. All prepared equations then give precisely the ordinary W-jets required there, with

$$D_0 = kL, \qquad D_i = 2A_i, \qquad A_i = q^{\nu - I} D_i^{\mathrm{old}}, \quad k = D_{-1}^{\mathrm{old}} + 1, \quad L = D_0^{\mathrm{old}} + 1. \tag{10.94}$$

Here $\nu \geq 0$ is an exponent clearing the denominators of the q-saturated basis, supplied by Lemma 10.12; it is distinct from the subgroup rank used in Theorem 10.25. The Euler shift adds $\sum_i C_{ij} d_i$ to its eigenvalue; Lemma 10.30 supplies the corresponding reversible binomial change. Thus the nonzero-polynomial and prepared-to-ordinary steps require no omitted derivative lemma. Theorem 10.25 still requires its separate independence, degree and allocation inequalities.

14. Constructing the initial auxiliary coefficients

The prepared equations are a finite homogeneous linear system. The weighted Siegel lemma has a complete proof in the earlier multiplicity lesson, equation (8.40), including dependent rows and integral solutions. We apply it with all weights one and bound every row explicitly. Figure 10.4 and Lemma 10.29 ensure that a nonzero coefficient vector produces a nonzero polynomial.

For a number field K of degree d, use all archimedean embeddings, counting a complex pair twice. At a finite prime ideal $\mathfrak{p}$, put $|x|_{\mathfrak{p}} = (N\mathfrak{p})^{-v_{\mathfrak{p}}(x)}$. These are the product-formula conventions of the earlier lesson. For a nonzero vector $c \in K^N$, define its Euclidean projective height by

$$h_2(c) = \frac{1}{d} \left(\sum_{\sigma \mid \infty} \log\Big(\sum_j |\sigma(c_j)|^2 \Big)^{1/2} + \sum_{\mathfrak{p}} \log \max_j |c_j|_{\mathfrak{p}} \right).$$

The product formula makes this invariant under multiplication of the whole vector by any nonzero scalar in K. Replacing the archimedean Euclidean norm by the maximum defines $h_\infty(c) \leq h_2(c)$.

Theorem 10.32 (initial prepared-jet kernel). Let $r, k, L \in \mathbb{Z}_{\geq 1}$, $S, T \in \mathbb{Z}_{\geq 0}$, and let $\mathcal{M} \subset \mathbb{Z}^r$ be a finite set of distinct vectors with $|m_i| \leq A_i$. Take $\vartheta_i \in K^\times$ and integer Euler coefficients C_{ij}, $1 \leq j < r$. Write $\omega_j(\boldsymbol{m}) = \sum_i C_{ij} m_i$, and put

$$\begin{aligned} N &= kL|\mathcal{M}|, \qquad M = (2S+1)\binom{T+r}{r}, \\ \Omega &= \max\big(\{0\} \cup \{|\omega_j(\boldsymbol{m})| : j < r, \boldsymbol{m} \in \mathcal{M}\}\big), \\ \mathcal{B} &= \left(\frac{(S + 2k - 1 + v(k))^k}{k!} \right)^L \max\{1, \Omega + T\}^T. \end{aligned} \tag{10.95}$$

If $N > M$, there are algebraic integers $c_{\boldsymbol{m},a,\ell}$, not all zero, for which every prepared value (10.92), with $I = 0$, vanishes at $(s, \vartheta_1^s, \dots, \vartheta_r^s)$ for $s \in [-S, S] \cap \mathbb{Z}$ and $|\boldsymbol{t}| \leq T$. They may be chosen with

$$h_\infty(c) \leq h_2(c) \leq \frac{1}{2}\log N + \frac{\log|\Delta_K|}{2d} + \frac{M}{N - M}\left(\frac{1}{2}\log N + \log \mathcal{B} + 2S \sum_i A_i h(\vartheta_i) \right). \tag{10.96}$$

The resulting $P = Y^{\boldsymbol{d}}Q$ is nonzero and has the degree and ordinary-jet properties of Proposition 10.31. When the directions are the selected basis in Lemma 10.23, these are all the required W-jets at the stated points.

Proof. There are $\binom{T+r}{r}$ nonnegative r-tuples of total order at most T. Append a slack coordinate to give total T; distributing T objects among $r+1$ slots amounts to choosing the r divider positions among $T+r$. Thus the prepared values give at most M linear equations in the N coefficients. Their entries lie in K. Rows that vanish or depend on others cause no difficulty in (8.40).

We bound a row entry. The coefficient of Z^{t_0} in $\Delta(s+a+v(k)Z;k)^{\ell}$ is exactly $v(k)^{t_0}\Theta(s+a;k,\ell,t_0)$. For $|s| \le S$, $0 \le a < k$, the sum of the absolute values of all coefficients of this polynomial is at most

$$\frac{\prod_{b=1}^{k}(|s+a+b|+v(k))^{\ell}}{(k!)^{\ell}} \le \left(\frac{(S+2k-1+v(k))^{k}}{k!}\right)^{\ell}.$$

The base on the right is at least one, so $\ell \le L$ gives its first factor in $\mathcal{B}$. For an integer ω with $|\omega| \le \Omega$, $|\Delta(\omega;t)| \le (\Omega+t)^{t}$ when $t > 0$, directly from its product formula and $t! \ge 1$. The product over Euler indices is therefore at most $\max\{1,\Omega+T\}^{T}$, also when all indices are zero. Every prepared scalar has absolute value at most $\mathcal{B}$ and is an integer, by Lemma 10.28 and integer-valued Δ.

At an archimedean embedding, the Euclidean row norm is consequently at most

$$\sqrt{N}\,\mathcal{B}\prod_{i}\max\{|\sigma(\vartheta_i)|,|\sigma(\vartheta_i)|^{-1}\}^{SA_i}.$$

At a finite place its maximum norm is at most the analogous product, since its prepared scalar is an integer. The exponents satisfy $|sm_i| \le SA_i$, including negative s and negative m_i. The product formula and the proved height formula give

$$\frac{1}{d}\sum_{v}\log\max\{|\vartheta_i|_v,|\vartheta_i|_v^{-1}\} = 2h(\vartheta_i)\,:$$

the sums of the positive and negative parts of $\log|\vartheta_i|_v$ are equal. Thus every nonzero row has adelic norm at most $\sqrt{N}\mathcal{B}\exp(2S\sum_i A_i h(\vartheta_i))$, which is at least one. Set all weights in (8.40) to one, with $J = N, I = M$. Its finite conditions give $c \in \mathcal{O}_K^N$; taking logarithms gives (10.96). Proposition 10.31 proves the nonzero-polynomial and derivative assertions. □

For a fixed finite place $\mathfrak{p}$ above p, select a nonzero coefficient c_* of minimum p-adic valuation. The scalar normalization

$$\widetilde{c} = c/c_*, \qquad \min_j v_{\mathfrak{p}}(\widetilde{c}_j) = 0, \qquad h_2(\widetilde{c}) = h_2(c) \tag{10.97}$$

preserves all equations and makes the coefficients locally integral with one local unit. It need not preserve integrality at other finite places; the unnormalized vector supplied by the theorem is globally integral.

This construction supplies an explicit initial coefficient-height budget. Applying it in a numerical logarithm estimate still requires comparing (10.96) with that estimate's parameters and supplying the jet precision required by the following extrapolation bounds.

15. P-adic interpolation at integer nodes

We prove the analytic extrapolation estimates directly. The power-series algebra and its norm are those of Convergence and legitimate substitution, Lemmas 9.1–9.2. We use $v_p(p) = 1$ and $v_p(0) = +\infty$. A *normal series* means $F(Z) = \sum_{n\ge 0} a_n Z^n \in \mathcal{A}_1$, with $v_p(a_n) \ge 0$ and $a_n \to 0$. Its divided jet is $F_j(a) = F^{(j)}(a)/j!$.

Division and the Hermite remainder

Lemma 10.33 (division inside the unit disc). If $v_p(a) > 0$ and F is normal, then

$$F(Z) = (Z-a)G(Z) + F(a), \qquad G(Z) = \sum_{n\geq 0}\left(\sum_{j\geq 0} a_{n+1+j}a^j\right)Z^n \tag{10.98}$$

has a normal quotient G. Consequently, for a finite set of distinct such points a_s and an integer $\mu \geq 1$, there are unique H, G with $F = H + WG$, $W(Z) = \prod_s (Z-a_s)^\mu$, $\deg H < \mu|E|$, and G normal. The polynomial H has the same jets as F through order $\mu - 1$ at every a_s.

Proof. The inner series converges because $|a| < 1$. Its coefficients have absolute value at most one, and tend to zero, since their absolute values are at most $\sup_{m\geq n+1}|a_m|$. The coefficient recurrence verifies (10.98); the constant coefficient uses the convergent identity for $F(a)$. Repeating division by the listed linear factors gives $F = H + WG$, with a normal quotient and the stated remainder degree. Every partial remainder is a polynomial with integral coefficients: the successive values and the roots are integral, and products and sums preserve that condition.

Translation of a normal series is legitimate on the unit disc: the coefficient of T^j in $F(a+T)$ is $\sum_{n\geq j}\binom{n}{j}a_n a^{n-j}$, of absolute value at most $\sup_{n\geq j}|a_n|$. The full binomial family has only finitely many terms of any fixed minimum size, since $a_n \to 0$. Lemma 9.1 therefore justifies regrouping, and these Taylor coefficients also tend to zero. Since W contains $(Z-a_s)^\mu$, the first μ Taylor coefficients of $F - H$ vanish there. Conversely a polynomial with those zero coefficients is divisible by that factor. Distinct factors may be removed successively: a factor nonzero at another point has an invertible Taylor series there, so removing it preserves the other vanishing order. Their product therefore divides any polynomial with all these zero jets. A polynomial of degree below $\deg W$ with those jets zero is zero. This proves uniqueness of H. Then G is unique because multiplication by the nonzero polynomial W is injective on formal power series: the product of the first nonzero coefficients cannot vanish. □

The exact integer separation cost

Fix a prime p, an integer $R \geq 1$, and let $B = \lfloor \log_p(2R) \rfloor$. We use either the full node set $E = [-R, R] \cap \mathbb{Z}$, or, for a prime $q \neq p$, its subset of nodes not divisible by q. Put $n = |E|$, and

$$L_s(X) = \prod_{t\in E\setminus\{s\}} \frac{X-t}{s-t}, \qquad \kappa = \begin{cases} 0 & \text{for the full set,} \\ 1 & \text{for the q-deleted set.} \end{cases} \tag{10.99}$$

Theorem 10.34 (normal-series Hermite extrapolation). Choose $\rho \in \mathbb{C}_p$ with $v_p(\rho) = \theta > 0$, let $\mu \in \mathbb{Z}_{\geq 1}$, and suppose that the normal series F satisfies

$$v_p(F_j(\rho s)) + j\theta \geq \Lambda \qquad (s \in E,\ 0 \leq j < \mu). \tag{10.100}$$

Then every $x \in \mathbb{Z}_p \subset \mathbb{Q}_p$ satisfies

$$v_p(F(\rho x)) \geq \min\{n\mu\theta,\ \Lambda - ((\kappa+1)\mu - 1)B\}. \tag{10.101}$$

Proof. First $v_p(L_s(x)) \geq -\kappa B$. At a node, its value is zero or one, so assume $x \notin E$. For each integer $a \geq 1$, count the elements of E in each residue class modulo p^a. For the full interval these counts differ by at most one. For the q-deleted interval they differ by at most two: the full counts differ by at most one, and so do the counts of deleted points qt, because q is invertible modulo p^a and $-\lfloor R/q \rfloor \leq t \leq \lfloor R/q \rfloor$.

Write these counts as $N_a(u)$ for a residue class u. The contribution at level a to the numerator valuation minus the denominator valuation of $L_s(x)$ is $N_a(x) - N_a(s) + 1 - \mathbf{1}_{x \equiv s \bmod p^a}$. If the classes agree it is zero. Otherwise it is at least $-\kappa$. For $a > B$, there is no denominator contribution, since nonzero differences of integer nodes have ordinary absolute value at most $2R < p^a$. The remaining contributions are nonnegative. Summing proves the bound. Each valuation is an integer, so the identity $v_p(u) = \sum_{a \geq 1} \mathbf{1}_{u \equiv 0 \bmod p^a}$ applies; numerator valuations are finite because $x \notin E$.

The Taylor inverse of L_s at its own node has the form

$$A_s(U) = L_s(s+U)^{-\mu} = \prod_{t \neq s} (1 + U/(s-t))^{-\mu}.$$

The coefficient of U^h has valuation at least $-hB$. Indeed the coefficients of $(1+V)^{-\mu}$ are the integers $(-1)^h \binom{\mu+h-1}{h}$, obtained by multiplying μ geometric series; every denominator $(s-t)^h$ has valuation at most hB. Products preserve the bound.

Let $[A_s]_{\leq b}$ mean truncation through degree b. Lemma 10.33's unique Hermite remainder is explicitly

$$H(\rho X) = \sum_{s \in E} L_s(X)^{\mu} \sum_{j=0}^{\mu-1} \rho^j F_j(\rho s)(X-s)^j [A_s(X-s)]_{\leq \mu-1-j}.$$

Its degree is at most $\mu(n-1) + \mu - 1 = n\mu - 1$. At each other node the factor L_s^{μ} gives order μ; at its own node, multiplication by the truncated inverse reproduces the required jets. This proves the formula by uniqueness, without an assumption about interpolation matrices.

For $x \in \mathbb{Z}_p$, $x - s$ is integral. Each displayed summand has valuation at least $\Lambda - \mu\kappa B - (\mu-1)B$, by (10.100) and the two bounds just proved. Hence $H(\rho x)$ has that valuation bound. On the other hand, $v_p(W(\rho x)) = \mu \sum_s (\theta + v_p(x-s)) \geq n\mu\theta$, and its normal quotient has integral value. The ultrametric inequality for $F = H + WG$ proves (10.101). $\square$

Corollary 10.35 (the two integer-node steps). For the full node set, it is enough to have

$$\Lambda \geq (2R+1)\mu\theta + \mu \frac{\log(2R+1)}{\log p} \quad \Longrightarrow \quad v_p(F(\rho x)) \geq (2R+1)\mu\theta. \tag{10.102}$$

For the q-deleted node set, if $q \mid R$, it is enough to have

$$\Lambda \geq 2(1-1/q)R\mu\theta + 2\mu \frac{\log(2R)}{\log p} \quad \Longrightarrow \quad v_p(F(\rho x)) \geq 2(1-1/q)R\mu\theta. \tag{10.103}$$

Both conclusions hold for every $x \in \mathbb{Z}_p$, including $x = s/q$ for any integer s. In (10.102), the case $R = 0$ is also valid: Taylor truncation at zero gives $v_p(F(\rho x)) \geq \min\{\mu\theta, \Lambda\}$.

Proof. The full count is $2R+1$. The deleted count is $2R - 2\lfloor R/q \rfloor$, equal to $2(1-1/q)R$ when $q \mid R$. Also $B \leq \log(2R)/\log p$, so the stated costs dominate $(\mu-1)B$ and $(2\mu-1)B$. Apply the theorem. The separate Taylor proof covers $R = 0$. Since $q \neq p$, division of an integer by q stays in $\mathbb{Z}_p$. $\square$

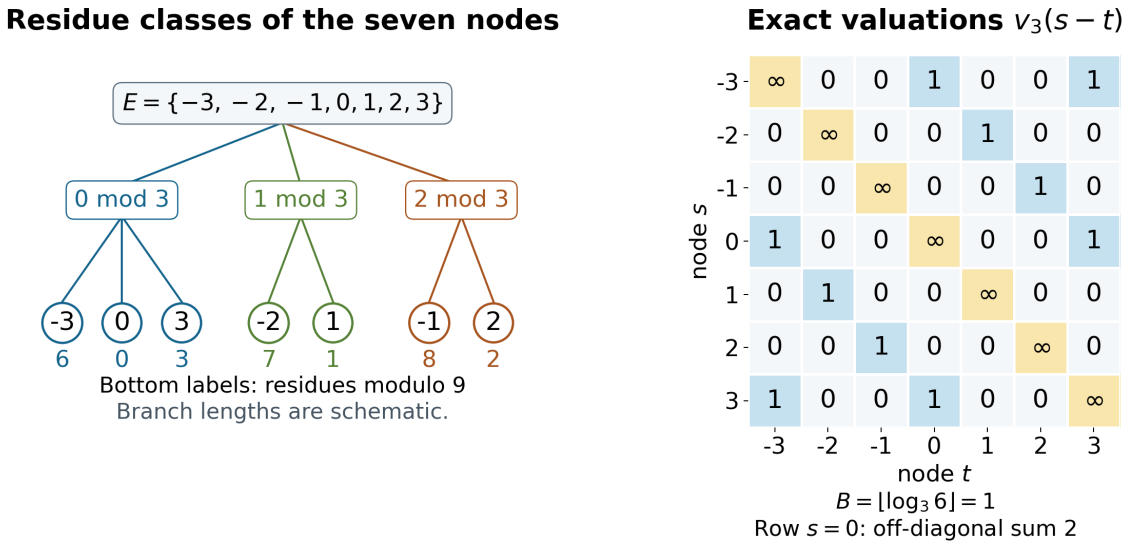


Figure 10.5. The seven nodes $-3,\dots,3$, grouped by their residues modulo three, and the exact matrix $v_3(s-t)$. The diagonal is $+\infty$. Nodes in the same indicated class have difference valuation one; all other nonzero differences have valuation zero. The groups split into singletons modulo nine. The drawing is a residue-class schematic: its line lengths are not p-adic distances. For $R=3, p=3$, the exact separation cost in Theorem 10.34 is $B=1$. The off-diagonal entries in the row at zero sum to two, matching $v_3((3!)^2)=2$, but this factorial total cancels in the integral cardinal factors for the full node set. The complete mechanism is proved in (10.99)–(10.103). Human-source context: the normal-series extrapolation in Yu's freely accessible 1990 and 2013 articles, cited below. Figure program.

For an application with jet order M, put $\mu = M+1$. Replacing either logarithmic cost above by a larger maximum with a coefficient-height parameter only strengthens the hypothesis. The conclusion then follows once normality and the input jet bound (10.100) hold.

The error from projecting a logarithmic curve

Lemma 10.36 (uniform projection error). Let $q \neq p$ be a prime, $I \geq 0$ be an integer, and use the prepared family (10.92) with integer Euler coefficients and a finite coefficient vector $c \in \mathbb{C}_p^N \setminus \{0\}$. Put $\delta = \min_j v_p(c_j)$. Take $b \in \mathbb{Z} \setminus \{0\}$, $a_i \in \mathbb{Z}_p$, and $u_i, \mathcal{L} \in \mathbb{C}_p$, satisfying $v_p(u_i) > 1/(p-1)$, $v_p(\mathcal{L}) \geq U$, and $U - v_p(b) > 1/(p-1)$. For $y \in \mathbb{Z}_p$, define

$$\begin{aligned} \Phi(y;\boldsymbol{t}) &= \mathcal{Q}(y;\boldsymbol{t})\big|_{Y_i=\exp(yu_i)}, \\ f(y;\boldsymbol{t}) &= \mathcal{Q}(y;\boldsymbol{t})\big|_{Y_i=\exp(y(u_i-a_i\mathcal{L}/b))}. \end{aligned} \tag{10.104}$$

Then for every nonnegative index $\boldsymbol{t}$,

$$v_p(f(y;\boldsymbol{t}) - \Phi(y;\boldsymbol{t})) \geq \delta + U - v_p(b). \tag{10.105}$$

Proof. Here the integer $v(k) = \operatorname{lcm}(1,\dots,k)$ retains its earlier meaning, distinct from the valuation v_p. The prepared Taylor factor is p-integral at $q^{-I}y + a$. Here is the slight extension of Lemma 10.28 needed for $y \in \mathbb{Z}_p$: the rational polynomial $v(k)^m\Theta(Z;k,\ell,m)$ is integral at integers. Choose an integer approximating any given p-integral argument modulo a sufficiently high power of p, exceeding all its coefficient denominator valuations. Factoring the differences of monomials proves its value is p-integral, exactly as in the proof of Lemma 10.28. The Euler binomial factors are integers. Thus every prepared scalar in (10.92) is p-integral.

All displayed exponentials exist. In each monomial term the ratio of the projected exponential to the original one is $\exp(-y\mathcal{L}\sum_i m_i a_i/b)$, by Proposition 9.4 in The exponential and the logarithm.

Its exponent has valuation at least $U - v_p(b) > 1/(p-1)$, because y, m_i and a_i are p-integral. The exact first-term exponential valuation in Proposition 9.3 of that subsection gives that the ratio minus one has valuation at least $U - v_p(b)$. The original exponential is a local unit. Multiplying by the prepared scalar and its coefficient, then summing, proves (10.105). □

In logarithmic applications take $\mathcal{L} = p^\kappa \log \Xi$, where $\kappa \in \mathbb{Z}_{\geq 0}$. If $v_p(\Xi - 1) \geq U > 1/(p-1)$, Proposition 9.3 gives $v_p(\mathcal{L}) \geq U$. The coefficient a_i may include a denominator that is a q-power, since it remains p-integral. The precise domain and coefficient valuation cost of the projection estimate are thus visible in (10.104)–(10.105).

Normalizing the prepared series

Proposition 10.37 (a normal prepared auxiliary function). Retain the integer data of (10.92), the finite nonzero coefficient vector c, and $\delta = \min_j v_p(c_j)$. Choose a coefficient c_* of that valuation and $\rho \in \mathbb{C}_p$ with $v_p(\rho) = \theta > 0$. Suppose that every logarithmic slope w_i satisfies $v_p(w_i) > \theta + 1/(p-1)$. Define $f(y; \boldsymbol{t})$ by substituting $Y_i = \exp(yw_i)$ in (10.92), and put

$$\begin{aligned} D = kL, \qquad \gamma = \rho^D (k!)^L / c_*, \\ \Gamma = v_p(\gamma) = D\theta + L v_p(k!) - \delta, \\ F(Z; \boldsymbol{t}) = \gamma f(Z/\rho; \boldsymbol{t}). \end{aligned} \tag{10.106}$$

For every nonnegative prepared index $\boldsymbol{t}$, the function F is normal. At an integer node s, its divided jets satisfy

$$F_j(\rho s; \boldsymbol{t}) = \gamma \rho^{-j} f_j(s; \boldsymbol{t}), \qquad v_p(F_j(\rho s; \boldsymbol{t})) + j\theta = \Gamma + v_p(f_j(s; \boldsymbol{t})). \tag{10.107}$$

Consequently, if $v_p(f_j(s; \boldsymbol{t})) \geq P$ at the node set of Theorem 10.34 for $0 \leq j < \mu$, then

$$v_p(f(x; \boldsymbol{t})) \geq \min\{n\mu\theta - \Gamma,\ P - ((\kappa+1)\mu - 1)B\} \qquad (x \in \mathbb{Z}_p). \tag{10.108}$$

Proof. Consider one prepared polynomial factor after $y = Z/\rho$. Its coefficient of Z^h has valuation at least $-h\theta - \ell v_p(k!)$, and its degree is at most $k\ell - t_0$. To see this without a coefficient estimate from another source, multiply it by $(k!)^\ell$: it is the coefficient of U^{t_0} in

$$\prod_{b=1}^{k} (q^{-I} Z/\rho + a + b + v(k)U)^\ell.$$

All constants and $v(k)$ are integers, while $q \neq p$. A term of degree h in Z therefore has valuation at least $-h\theta$. Multiplying by $\rho^D (k!)^L$ makes every coefficient integral, because $h \leq k\ell \leq D$ and $\ell \leq L$. The Euler binomial factors are integers, and c_j/c_* is locally integral. Thus the normalized polynomial part of every summand has Gauss norm at most one.

The exponential part of that summand is $\exp(\beta Z/\rho)$, where $\beta = \sum_i m_i w_i$. Either $\beta = 0$, or $v_p(\beta/\rho) > 1/(p-1)$, since the exponents m_i are integers. By (9.4), its coefficient of degree $h \geq 1$ has valuation $hv_p(\beta/\rho) - v_p(h!) \geq h(v_p(\beta/\rho) - 1/(p-1)) + 1/(p-1) > 0$, tending to infinity. Its constant coefficient is one. It belongs to $\mathcal{A}_1$ with norm one. The product and finite-sum statements of Lemma 9.2 now prove that $F \in \mathcal{A}_1$ with norm at most one, hence is normal.

The slope hypotheses define f on the full closed disc $|y| \leq |\rho|^{-1}$, so the chain rule is valid at every integer node. It gives (10.107), also for zero jets with infinite valuation. Apply Theorem 10.34 with $\Lambda = \Gamma + P$, then subtract Γ, to obtain (10.108). □

The larger scale $D(\theta + 1/(p-1)) - \delta$ also suffices: (9.4) gives $Lv_p(k!) \leq D/(p-1)$, and a scalar of any additional nonnegative valuation preserves normality. For the projected slopes in (10.104), it

suffices that $v_p(u_i) > \theta + 1/(p-1)$ and $U - v_p(b) > \theta + 1/(p-1)$. The ultrametric inequality then supplies the slope hypotheses above. Input derivative precision remains an explicit hypothesis in (10.108), distinct from this proof of normality.

16. Prepared jets and derivative precision

The interpolation argument can keep the precision of each derivative separately. We first obtain that precision from the prepared equations, without changing the selected Euler directions.

Differentiating along the projected curve

Lemma 10.38 (the loss in an ordinary divided jet). Take the integral forms, matrix C, selected column set J, and derivations δ_j of Lemma 10.23. Use those selected derivations as the D_j in (10.92), renumbering them if necessary. Let $q \neq p$ be a prime, $I, \nu \geq 0$ be integers, and $z_j \in \mathbb{C}_p$, $1 \leq j < n$, satisfy $v_p(z_j) > 1/(p-1)$. Put $\beta = v_p(b_n)$, and define

$$\begin{aligned} w_i &= \frac{1}{b_n q^\nu} \sum_{j<n} C_{ij} z_j, \\ \xi_a &= \frac{1}{b_n q^\nu} \sum_{j<n} \lambda_{aj} z_j \quad (a \in J), \\ C_0 &= \max\{v_p(v(k)), \beta\}. \end{aligned} \tag{10.109}$$

Here the coefficients λ_{aj} are those in (10.64), with $\lambda_{aj} = \mathbf{1}_{a=j}$ for selected columns. Suppose that $v_p(w_i) > 1/(p-1)$ for every i. Define $f(y; \boldsymbol{t})$ by substituting $Y_i = \exp(y w_i)$ in the prepared family (10.92). Fix $s \in \mathbb{Z}_p$, an integer $T \geq 0$, and a real P_0. If $v_p(f(s; \boldsymbol{t}')) \geq P_0$ for every $|\boldsymbol{t}'| \leq T$, then

$$v_p(f_j(s; \boldsymbol{t})) \geq P_0 - jC_0 \quad \text{whenever} \quad |\boldsymbol{t}| + j \leq T, \qquad f_j = \frac{1}{j!}\frac{d^j f}{dy^j}. \tag{10.110}$$

Proof. By (10.64), $\sum_i w_i E_i = \sum_{a \in J} \xi_a \delta_a$. Each λ_{aj} is p-integral, and $q \neq p$, so either $\xi_a = 0$ or $v_p(\xi_a) > 1/(p-1) - \beta$. Differentiating along the curve therefore applies the commuting operator $V = \partial_X + \sum_{a \in J} \xi_a \delta_a$. This identity follows directly on a term $P(X)Y^{\boldsymbol{m}}$: its restriction is $P(y)\exp(y\sum_i m_i w_i)$, whose derivative has the stated two terms. The exponential laws and convergence in Propositions 9.3–9.4 justify this differentiation on the finite prepared sum.

Write $A = q^I v(k)$. The additive prepared operator is $\mathcal{B}_{t_0} = A^{t_0}\partial_X^{t_0}/t_0!$. For $h \geq 0$,

$$\frac{\partial_X^h}{h!}\mathcal{B}_{t_0} = A^{-h}\binom{t_0 + h}{h}\mathcal{B}_{t_0+h}.$$

Its scalar has valuation at least $-h v_p(v(k))$.

For an indeterminate Z, the defining product for Δ gives

$$Z\Delta(Z; t) = (t+1)\Delta(Z; t+1) - (t+1)\Delta(Z; t).$$

Repeated use of this identity expresses $Z^h\Delta(Z; t)$ as an integer linear combination of $\Delta(Z; e)$, with $t \leq e \leq t + h$. Substitution of an Euler derivation is legitimate because all the operators commute. Multiplication by $\xi_a^h/h!$ gives coefficients of valuation at least $-h\beta$: for $h > 0$, (9.4) gives $h v_p(\xi_a) - v_p(h!) \geq -h\beta$; for $h = 0$, the coefficient is one. A zero ξ_a causes no problem.

Expand $V^j/j!$ over $h_0 + \sum_a h_a = j$. The multinomial coefficient cancels the denominator $j!$, leaving the product of the divided powers just considered. Every resulting prepared index has total order at most $|\boldsymbol{t}| + j$, and every scalar has valuation at least $-h_0 v_p(v(k)) - \sum_a h_a\beta \geq -jC_0$. Evaluation at the same curve point s, followed by the ultrametric inequality, proves (10.110). This proof retains the factorials; no extra $v_p(j!)$ loss is needed. □

Retaining the precision of each jet

Theorem 10.39 (interpolation with a linear jet loss). Retain the normal function, integer node set, ρ, θ, n, μ, B, and $\kappa \in \{0,1\}$ of Theorem 10.34. Suppose $C \geq 0$ and $v_p(F_j(\rho s)) + j\theta \geq \Lambda - jC$ for every node s and $0 \leq j < \mu$. Then

$$v_p(F(\rho x)) \geq \min\{n\mu\theta,\ \Lambda - \kappa\mu B - (\mu-1)\max\{B,C\}\} \qquad (x \in \mathbb{Z}_p \subset \mathbb{Q}_p). \tag{10.111}$$

Proof. Use the explicit Hermite remainder in the proof of Theorem 10.34. Its term with jet index j contains a coefficient of degree $h \leq \mu - 1 - j$ in A_s. That proof gives valuations at least $-\kappa\mu B$ for $L_s(x)^\mu$, and at least $-hB$ for that coefficient. The remaining power of $x - s$ is integral. The term thus has valuation at least $\Lambda - jC - \kappa\mu B - (\mu - 1 - j)B$. Since $0 \leq j \leq \mu - 1$,

$$jC + (\mu - 1 - j)B \leq (\mu - 1)\max\{B,C\}.$$

The ultrametric inequality gives the second bound in (10.111) for the full Hermite remainder. The normal quotient and its root product give the first bound $n\mu\theta$, exactly as before. Their sum proves the claim. □

For $C = 0$ this is Theorem 10.34. For $C > 0$, first replacing all jets by their least precision would instead give the larger loss $(\mu - 1)(C + B) + \kappa\mu B$. The exact Hermite formula avoids that loss by pairing each jet with only the inverse coefficients it actually uses.

The input precision for extrapolation

Corollary 10.40 (prepared vanishing supplies the required jets). Retain the selected directions and notation of Lemma 10.38, the prepared coefficient vector $c \neq 0$, and $\delta = \min_j v_p(c_j)$. Suppose the unprojected and projected functions Φ and f satisfy Lemma 10.36 with $b = b_n$, projection precision U, and coefficient minimum δ. Suppose also that the slopes of f satisfy Proposition 10.37 for $v_p(\rho) = \theta > 0$. Let Γ be its exact scale in (10.106).

At every node of Theorem 10.34, assume $\Phi(s; \boldsymbol{t}') = 0$ for all $|\boldsymbol{t}'| \leq T$. Fix $\boldsymbol{t}$ with $|\boldsymbol{t}| + \mu - 1 \leq T$, and write $M_0 = \max\{B, C_0\}$. Then, for every $x \in \mathbb{Z}_p$,

$$v_p(f(x; \boldsymbol{t})) \geq \min\{n\mu\theta - \Gamma,\ U + \delta - \beta - \kappa\mu B - (\mu - 1)M_0\}. \tag{10.112}$$

In particular the following explicit budget suffices for $v_p(f(x; \boldsymbol{t})) \geq n\mu\theta - \Gamma$:

$$U + D\theta + Lv_p(k!) \geq n\mu\theta + (\kappa + 1)\mu M_0, \qquad D = kL. \tag{10.113}$$

Under that budget the same lower bound holds for $\Phi(x; \boldsymbol{t})$. For the full interval, $n = 2R + 1$ and $\kappa = 0$; for the q-deleted interval with $q \mid R$, $n = 2(1 - 1/q)R$ and $\kappa = 1$.

These hypotheses have a direct logarithmic model. Take $z_1, \ldots, z_n$ with $v_p(z_j) > \theta + 1/(p-1)$, and put

$$u_i = q^{-\nu} \sum_{j=1}^{n} a_{ij} z_j, \qquad \mathcal{L} = \sum_{j=1}^{n} b_j z_j, \qquad a_i = q^{-\nu} a_{in}.$$

Then $a_i \in \mathbb{Z}_p$, and $w_i = u_i - a_i\mathcal{L}/b_n$ is exactly (10.109). If $v_p(\mathcal{L}) \geq U$ and $U - \beta > \theta + 1/(p-1)$, the hypotheses of both the projection and normality estimates follow from the ultrametric inequality. The chosen-minor argument supplies the integral coefficients used for the ordinary input jets.

Proof. Vanishing of Φ and (10.105) give $v_p(f(s; \boldsymbol{t}')) \geq U + \delta - \beta$ for every required prepared index. Lemma 10.38 therefore gives the ordinary input jets with loss jC_0. By (10.107), the corresponding

normal jets have weighted precision $\Gamma + U + \delta - \beta - jC_0$. Theorem 10.39 and subtraction of Γ prove (10.112).

Here $\beta \le C_0 \le M_0$ and $B \le M_0$. Consequently $\beta + \kappa\mu B + (\mu-1)M_0 \le (\kappa+1)\mu M_0$. Substitute $\Gamma + \delta = D\theta + Lv_p(k!)$ in (10.113) to see that the second entry of the minimum in (10.112) is at least the first. It also follows that $U + \delta - \beta \ge n\mu\theta - \Gamma$. The projection error (10.105) holds at x, so $\Phi = f + (\Phi - f)$ has the same lower bound. The node counts are those proved in Corollary 10.35. □

The coefficient minimum cancels from (10.113); it remains in the valuation conclusion through Γ. One may instead choose the larger normalizing scale $\Gamma' = D(\theta + 1/(p-1)) - \delta$ allowed by Proposition 10.37. Multiplying the normal function by a scalar of valuation $\Gamma' - \Gamma \ge 0$ repeats the proof with Γ' throughout. It gives the sufficient budget and its corresponding conclusion

$$U + D\left(\theta + \frac{1}{p-1}\right) \ge n\mu\theta + (\kappa+1)\mu M_0 \quad \Longrightarrow \quad v_p(f(x;\boldsymbol{t})),\ v_p(\Phi(x;\boldsymbol{t})) \ge n\mu\theta - \Gamma'. \tag{10.114}$$

Such a scalar exists: its required valuation is the nonnegative rational $D/(p-1) - Lv_p(k!)$, and a b-th root of p^a has valuation a/b in $\mathbb{C}_p$. The conclusion with Γ' does not assert the stronger conclusion with Γ.

Finally, $v_p(v(k)) = \lfloor \log_p k \rfloor$: the highest p-power dividing a least common multiple is the highest p-power dividing one of its factors, and the largest such power among $1, \dots, k$ is $p^{\lfloor \log_p k \rfloor}$. Also $\beta \le \ln|b_n| / \ln p$. Thus $M_0 \le \max\{\ln(2R), \ln k, \ln|b_n|\} / \ln p$, a useful real-valued upper bound in (10.113)–(10.114). To deduce an algebraic zero from the valuation conclusion one must additionally compare it with the height of the nonzero prepared value. Those arithmetic and numerical comparisons are separate from this interpolation argument.

17. When extrapolation forces an algebraic zero

An analytic lower bound for a valuation forces an algebraic value to vanish only when it exceeds the upper bound for a nonzero value. We prove that upper bound with the exact local degree and with the denominators of the prepared polynomial retained.

A nonzero prepared value

Use the height conventions of Section 14 and the product formula proved in Places of number fields in extensions and the product formula. The height identities for powers and inverses are proved in Heights of algebraic numbers.

Theorem 10.41 (arithmetic precision of a prepared value). Let K have degree d, and let its chosen prime above p have ramification index e and residue degree f. Take the prepared family (10.92) with a prime $q \ne p$, integers $I, J \ge 0$, $k, L \ge 1$, a nonzero vector $c \in K^N$, and integer Euler arguments $\omega_j(\boldsymbol{m})$. Let $|m_i| \le A_i$, put $N = kL|\mathcal{M}|$, $D = kL$, and take $x = s/q^J$, $s \in \mathbb{Z}$, with $|x| \le X$. Suppose $\eta_i \in K^\times$ are units at the chosen prime. Evaluate (10.92) at $(x, \boldsymbol{\eta})$, and call its value V. Let $|\boldsymbol{t}| \le T$, $T \in \mathbb{Z}_{\ge 0}$, and put

$$\begin{aligned} \delta &= \min_j v_p(c_j), \qquad \Omega = \max\bigl(\{0\} \cup \{|\omega_j(\boldsymbol{m})|\}\bigr), \\ \mathcal{B}_I(X,T) &= \left(\frac{(q^{-I}X + 2k - 1 + v(k))^k}{k!}\right)^L \max\{1, \Omega + T\}^T, \\ \Xi_{I,J,t_0} &= \begin{cases} 0, & I + J = 0, \\ \max\{0, Lv_q(k!) + (I+J)(D - t_0) - t_0 v_q(v(k))\}, & I + J > 0. \end{cases} \end{aligned} \tag{10.115}$$

If $V \ne 0$, then

$$v_p(V)-\delta \le \frac{d}{ef\ln p}\Big(\min\{h_\infty(c)+\ln N,\ h_2(c)+\tfrac12\ln N\} + \ln \mathcal{B}_I(X,T)+\Xi_{I,J,t_0}\ln q+2\sum_i A_i h(\eta_i)\Big). \tag{10.116}$$

If $t_0 > D$, the value is zero automatically. The same estimate holds when the Euler arguments are any assigned integers bounded by Ω; their linear dependence on $\boldsymbol{m}$ is not needed for this arithmetic assertion.

Proof. Write each row entry as its rational prepared scalar times $\boldsymbol{\eta}^{\boldsymbol{m}}$. The additive scalar is the coefficient of Z^{t_0} in $\Delta(q^{-I}x+a+v(k)Z;k)^\ell$. The sum of the absolute values of the coefficients of this polynomial is at most

$$\frac{\prod_{b=1}^{k}(q^{-I}|x|+a+b+v(k))^\ell}{(k!)^\ell} \le \left(\frac{(q^{-I}X+2k-1+v(k))^k}{k!}\right)^\ell.$$

The displayed base is at least one: $2k-1+v(k)\ge k$, and $k^k \ge k!$. Since $\ell \le L$, its L-th power bounds this coefficient. The product formula for $\Delta(\omega;t_j)$ gives the remaining factor in $\mathcal{B}_I$, as in Theorem 10.32. Thus every rational scalar has ordinary absolute value at most $\mathcal{B}_I$.

There is no denominator away from q, by Lemma 10.28 at the argument $q^{-I-J}s+a$; the Euler factors are integers. If $I+J=0$, every scalar is an integer. Otherwise the coefficient of degree h of the ordinary divided derivative has denominator dividing $(k!)^\ell$, because differentiating a monomial and dividing by $t_0!$ multiplies its coefficient by an integer binomial coefficient. Its degree is at most $k\ell - t_0$. Evaluation at $q^{-I-J}s+a$ costs at most $(I+J)(k\ell-t_0)$ in q-adic valuation. Multiplication by $v(k)^{t_0}$ restores $t_0 v_q(v(k))$. Taking $\ell \le L$ gives the clearing factor $q^{\Xi_{I,J,t_0}}$. If $t_0 > D$, every additive derivative is zero.

Choose a coefficient c_* of valuation δ, and put $z = V/c_*$. At the chosen prime the maximum norm of the coefficient vector c/c_* is exactly one. For $V \neq 0$, the product formula therefore gives

$$ef\ln p\,(v_p(V)-\delta) = -\ln|z|_{\mathfrak{p}} = \sum_{v\neq\mathfrak{p}} \ln|z|_v.$$

The factor is ef, since $|z|_{\mathfrak{p}} = p^{-f\,\mathrm{ord}_{\mathfrak{p}}(z)}$ and $\mathrm{ord}_{\mathfrak{p}} = ev_p$.

At each archimedean embedding, the triangle inequality bounds the sum by N times the coefficient maximum times the entry maximum. Alternatively Cauchy–Schwarz gives the coefficient Euclidean norm times $\sqrt{N}$ times the entry maximum. This last inequality follows by applying $(\sum u_j v_j)^2 \le (\sum u_j^2)(\sum v_j^2)$ to the absolute values; its difference is $\sum_{i<j}(u_i v_j - u_j v_i)^2 \ge 0$. At finite places the ultrametric inequality uses maximum norms and introduces no factor N.

Sum these bounds outside the chosen prime. The normalized coefficient norms contribute exactly $dh_\infty(c)$, or $dh_2(c)$, because their missing finite norm is one. There are d archimedean embeddings with a complex pair counted twice. The rational scalar contributes at most $d\ln\mathcal{B}_I$ there. Its clearing factor contributes at most $d\Xi_{I,J,t_0}\ln q$ over the primes above q, by the local-degree identity for the element q.

At every other place, the torus contribution is bounded by $\sum_i A_i \ln\max\{|\eta_i|_v, |\eta_i|_v^{-1}\}$. Its omitted contribution at the chosen prime is zero, since the η_i are units. Summing over all places gives $2d\sum_i A_i h(\eta_i)$, by the inverse-height identity and product formula. Divide by $ef\ln p$, and take the better of the two coefficient norm bounds, to obtain (10.116). Every step used only integrality and the absolute size of the Euler factors, proving the final assertion too. □

Fractional nodes and the coefficient field

Proposition 10.42 (roots and a common local unit). Let $\vartheta_i \in K^\times$ be units at p, and choose q^J-th roots η_i in a fixed algebraic embedding into $\mathbb{C}_p$. Put $F = K(\eta_1, \dots, \eta_r)$. For $x = s/q^J$, consider the prepared value with torus coordinates η_i^s. Theorem 10.41 applies over F. Its torus height term is at most $2X \sum_i A_i h(\vartheta_i)$ when $|x| \le X$, but its local-degree coefficient is

$$\frac{[F:\mathbb{Q}]}{e_F f_F} = \frac{d}{ef} \frac{[F:K]}{[F_{\mathfrak{P}} : K_{\mathfrak{p}}]}. \tag{10.117}$$

One cannot replace this coefficient by $d/(ef)$ without further information.

If every exponent belongs to $\boldsymbol{m}_* + q^J\mathbb{Z}^r$, where $\boldsymbol{m}_*$ is one of the exponent vectors, define $\widehat{A}_i = \max_{\boldsymbol{m}} |m_i - m_{*,i}| \le 2A_i$. Then the common-factor identity is

$$\begin{aligned} V &= \boldsymbol{\eta}^{s\boldsymbol{m}_*} V_0, \\ V_0 &= \sum_{\boldsymbol{m},a,\ell} c_{\boldsymbol{m},a,\ell}\, \mathcal{S}_{\boldsymbol{m},a,\ell}(x;\boldsymbol{t}) \boldsymbol{\vartheta}^{s(\boldsymbol{m}-\boldsymbol{m}_*)/q^J} \in K, \qquad v_p(V) = v_p(V_0). \end{aligned} \tag{10.118}$$

Here $\mathcal{S}$ denotes the rational prepared scalar in (10.92). In this case (10.116) may instead use $d/(ef)$ and the torus height term $2X \sum_i \widehat{A}_i h(\vartheta_i)$, with the same $\mathcal{B}_I$, clearing factor, and coefficient heights. For a logarithmic curve with $\vartheta_i = \exp(u_i)$, choose $\eta_i = \exp(u_i/q^J)$. The exponential laws of Proposition 9.4 give $\eta_i^{q^J} = \vartheta_i$ and $\exp(xu_i) = \eta_i^s$, identifying the fractional analytic value with the stated algebraic value.

Proof. A root of a local unit is a local unit. The height power identity gives $h(\eta_i) = h(\vartheta_i)/q^J$, and hence $h(\eta_i^s) = |x| h(\vartheta_i)$. Heights of the coefficient vector are unchanged by extending the field: every archimedean embedding is repeated $[F:K]$ times, and the corresponding finite-place contribution is multiplied by that same factor, as proved by the local-degree identities in the earlier places lesson. Dividing by the field degree cancels that factor. The relative ramification and residue degrees multiply the original ones, and their product is $[F_{\mathfrak{P}} : K_{\mathfrak{p}}]$. This proves (10.117) and the assertion about applying the preceding theorem.

In the coset case, $(\boldsymbol{m} - \boldsymbol{m}_*)/q^J$ is an integer vector. Factoring its common root monomial gives (10.118), term by term. The common factor is a local unit, so it changes no valuation. Apply the last assertion of Theorem 10.41 to V_0: the Euler arguments stay their original integers, and the new exponent bound is $\widehat{A}_i/q^J$. The torus coordinates are ϑ_i^s, of height $|s| h(\vartheta_i)$. Their contribution is at most $2|s| \sum_i (\widehat{A}_i/q^J) h(\vartheta_i) \le 2X \sum_i \widehat{A}_i h(\vartheta_i)$. The rational scalars and coefficient vector have not changed. □

The two comparisons in Theorem 10.43 exclude every finite valuation in the case illustrated in Figure 10.6.

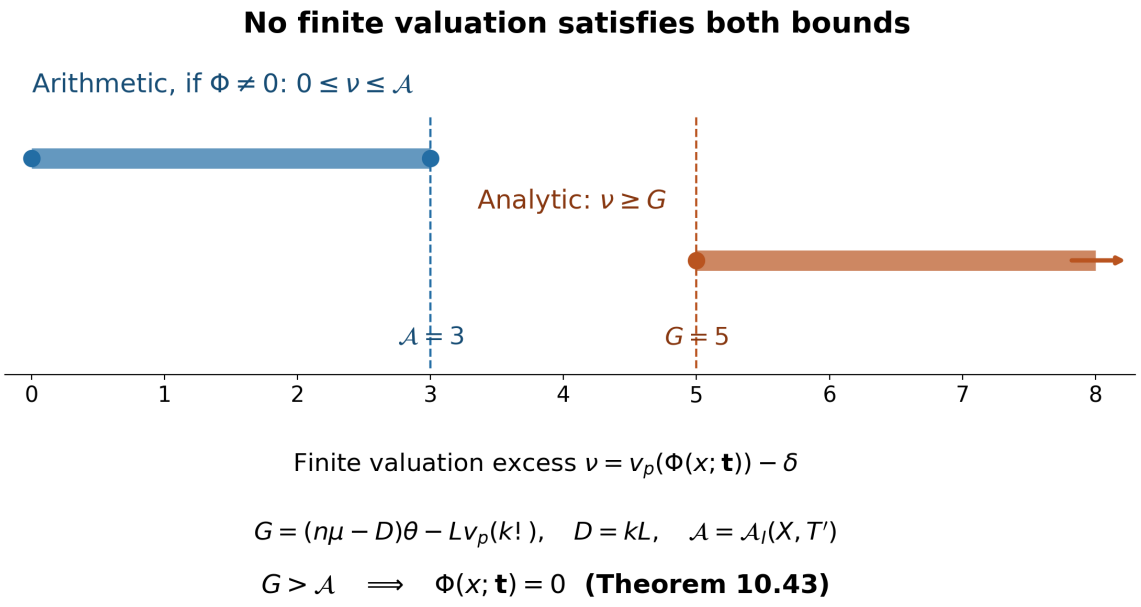


Figure 10.6. The prepared scalars are p-integral and the torus coordinates are local units, so Φ/c_* is p-integral and the valuation excess $\nu = v_p(\Phi(x;\boldsymbol{t})) - \delta$ is nonnegative. If the value is nonzero, the product-formula proof of Theorem 10.41 gives $\nu \le \mathcal{A}_I(X,T')$. The analytic proof gives $\nu \ge (n\mu - D)\theta - Lv_p(k!)$. The drawn endpoints $\mathcal{A} = 3, G = 5$ illustrate these two intervals; they are not parameters of a specified auxiliary function. Their separation is exactly the strict second comparison in (10.120), and forces $\Phi(x;\boldsymbol{t}) = 0$. The zero value has infinite valuation and is outside the finite axis. Human-source context: the product-formula step in Yu's free2013 article, cited below. Figure program.

An explicit zero-forcing step

Theorem 10.43 (extension of the prepared zeros). Retain the hypotheses of Corollary 10.40 over the field K, and suppose that at every integer x the unprojected function is the algebraic prepared value at $(x, \vartheta_1^x, \dots, \vartheta_r^x)$, with $\vartheta_i \in K^\times$ local units. Let $T' = T - \mu + 1 \ge 0$, and let $X \ge 0$ be real. Put

$$Q_I = \begin{cases} 0, & I = 0, \\ L(kI + v_q(k!)), & I > 0, \end{cases}$$
$$\mathcal{A}_I(X,T') = \frac{d}{ef\ln p}\Bigg(\min\{h_\infty(c) + \ln N,\ h_2(c) + \tfrac{1}{2}\ln N\} + \ln \mathcal{B}_I(X,T') + Q_I \ln q + 2X\sum_i A_i h(\vartheta_i)\Bigg). \tag{10.119}$$

The node count n, κ, θ, and $M_0 = \max\{B, C_0\}$ are those of Corollary 10.40. If

$$\begin{aligned} U + D\theta + Lv_p(k!) &\ge n\mu\theta + (\kappa+1)\mu M_0, \\ (n\mu - D)\theta - Lv_p(k!) &> \mathcal{A}_I(X,T'), \end{aligned} \tag{10.120}$$

then $\Phi(x;\boldsymbol{t}) = 0$ for every integer $|x| \le X$ and every $|\boldsymbol{t}| \le T'$. With the larger normalizing scale of (10.114), an alternative sufficient pair is

$$\begin{aligned} U + D\left(\theta + \frac{1}{p-1}\right) &\ge n\mu\theta + (\kappa+1)\mu M_0, \\ (n\mu - D)\theta - \frac{D}{p-1} &> \mathcal{A}_I(X,T'). \end{aligned} \tag{10.121}$$

Proof. The first line of (10.120) supplies the analytic budget of Corollary 10.40. At every stated index its conclusion gives

$$v_p(\Phi(x;\boldsymbol{t})) - \delta \geq (n\mu - D)\theta - Lv_p(k!).$$

If the prepared value were nonzero, Theorem 10.41 with $J = 0$ would give the opposite upper bound $\mathcal{A}_I(X, T')$: its torus coordinates have height $|x|h(\vartheta_i)$, and its clearing exponent is at most Q_I. The strict second line of (10.120) contradicts this. A prepared index with $t_0 > D$ already has zero value. These cases prove every claimed zero. The same argument using (10.114) instead proves (10.121). □

Both comparisons remove δ, so they are invariant under scaling the coefficient vector. The first comparison gives enough analytic precision; the second makes a nonzero algebraic value impossible. At fractional nodes one must additionally use the field or common-factor rule of Proposition 10.42. Applying this step repeatedly requires checking these inequalities, the slope conditions, and the available derivative orders at every stage.

18. Retaining the derivative orders in the height budget

The uniform estimate (10.115) bounds every coefficient by the sum of all coefficients. It also discards the factorials in the Euler binomials. Those bounds suffice for Theorem 10.43, but a numerical induction benefits from retaining the actual derivative order. We now do so, including the average over the symmetric initial nodes. The additive estimate below has the form of Lemma 1.6 in Yu's free 1990 paper; we give its full proof using only an elementary factorial inequality.

Positive polynomial majorants

Let $\Omega_j = \max_{\boldsymbol{m}\in\mathcal{M}} |\omega_j(\boldsymbol{m})|$, for $1 \leq j < r$, and let $\Omega_\Sigma = \sum_{j<r} \Omega_j$. For integers $\Omega, h \geq 0$, define $E_h(\Omega) = \binom{\Omega+h}{h}$. Thus $E_0(\Omega) = E_h(0) = 1$. Put $\Gamma_0 = 1$; for $h > 0$, put $\Gamma_h = 0$ if $r = 1$, and $\Gamma_h = \binom{\Omega_\Sigma+r-2+h}{h}$ if $r \geq 2$. Use empty products and sums when $r = 1$. Define the positive polynomial

$$\begin{aligned}\mathcal{P}_{I,X}(Z) &= \left(\frac{1}{k!}\prod_{b=1}^{k}(q^{-I}X + k - 1 + b + v(k)Z)\right)^L \\ &= \sum_{u=0}^{D}\mathcal{C}_{I,u}(X)Z^u, \qquad D = kL.\end{aligned} \tag{10.122}$$

Theorem 10.44 (an order-dependent scalar bound). For the rational prepared scalar $\mathcal{S}_{\boldsymbol{m},a,\ell}(x;\boldsymbol{t})$ in (10.92), with $0 \leq a < k$, $1 \leq \ell \leq L$, $|x| \leq X$, and $t_0 \leq D$, one has

$$\begin{aligned}|\mathcal{S}_{\boldsymbol{m},a,\ell}(x;\boldsymbol{t})| &\leq \mathcal{H}_I(X;\boldsymbol{t}) := \mathcal{C}_{I,t_0}(X)\prod_{j<r}E_{t_j}(\Omega_j), \\ \mathcal{B}_I^*(X,T) &:= \max_{|\boldsymbol{t}|\leq T,\ t_0\leq D}\mathcal{H}_I(X;\boldsymbol{t}).\end{aligned} \tag{10.123}$$

An index with $t_0 > D$ has zero scalar. If $\Omega_j = 0$, its Euler factor is $\Delta(0; t_j) = 1$, so changing that index to zero leaves the scalar unchanged. The maximum in (10.123) is at least one and satisfies

$$\begin{aligned}&\mathcal{B}_I^*(X,T) \leq \mathcal{B}_I(X,T), \\ &\mathcal{B}_I^*(X,T) \leq \left(\mathrm{e}\left(2 + \frac{q^{-I}X}{k}\right)\right)^D \max_{\substack{u,h\geq 0 \text{ integers} \\ u\leq D,\ u+h\leq T}} v(k)^u\Gamma_h.\end{aligned} \tag{10.124}$$

Here $\mathcal{B}_I$ is (10.115); the second bound retains the division of the total order between the additive and Euler directions. For $h > 0$, one may further bound Γ_h by $[\mathrm{e}(1 + (\Omega_\Sigma + r - 1)/h)]^h$.

Proof. At a given real or complex x, replace each factor $q^{-I}x+a+b$ by its absolute-value upper bound $q^{-I}X+a+b$. Expansion of the product bounds the absolute value of each coefficient by the corresponding coefficient of this polynomial with nonnegative coefficients. Its constant coefficient after division by $k!$ is at least one: each factor $q^{-I}X+a+b$ is at least b. Consequently increasing its power from ℓ to L can only increase every coefficient. Increasing a to $k-1$ has the same property. Taking the coefficient of Z^{t_0} gives $\mathcal{C}_{I,t_0}(X)$.

For an integer ω with $|\omega|\le\Omega$, the product defining $\Delta(\omega;t)$ gives $|\Delta(\omega;t)|\le E_t(\Omega)$. Indeed $|\omega+b|\le\Omega+b$, for $1\le b\le t$, and the factorial $t!$ is retained. When $\Omega=0$, the Euler argument is zero and every such binomial is one. This proves (10.123), the degree-zero case and the duplicate-direction assertion. Every $\mathcal{C}_{I,u}(X)$ is positive. At $X=0$, it is the prepared Taylor coefficient at the integer $k-1$, so it is an integer by Lemma 10.28 and is at least one. The coefficients increase with X. In particular the index $\boldsymbol{t}=0$ proves $\mathcal{B}_I^*\ge 1$.

For the first inequality in (10.124), each $\mathcal{C}_{I,u}(X)$ is at most $\mathcal{P}_{I,X}(1)$. Bounding each of its k factors by $q^{-I}X+2k-1+v(k)$ gives the additive part of (10.115). Put $\Omega=\max_j\Omega_j$, with $\Omega=0$ for an empty index set. The Euler product is at most $\max\{1,\Omega+T\}^T$, as in Theorem 10.32. Hence this sharper maximum never exceeds the earlier one.

We prove the additive estimate used for the second inequality. For every integer $k\ge1$, monotonicity of $\log x$ on each interval $[j-1,j]$ gives $\log k!\ge\int_1^k\log x\,dx=k\log k-k+1$. Thus $k!\ge\mathrm{e}(k/\mathrm{e})^k$, including $k=1$. At $y\ge0$, coefficient comparison with $(Z+k)^{k\ell}/(k!)^\ell$ gives

$$\begin{aligned}|\Theta(z;k,\ell,u)|&\le\frac{\binom{k\ell}{u}(|z|+k)^{k\ell-u}}{(k!)^\ell}\\&\le\frac{\ell^u}{u!}\,\frac{(|z|+k)^{k\ell}}{(k!)^\ell}\le\left(\frac{\mathrm{e}(|z|+k)}{k}\right)^{k\ell}.\end{aligned}\tag{10.125}$$

The first comparison takes absolute values in the differentiated product. The second uses $\binom{k\ell}{u}\le(k\ell)^u/u!$ and $|z|+k\ge k$. The last uses the factorial bound and $\ell^u/u!\le\mathrm{e}^\ell$, since this is one nonnegative term of the exponential series. Derivatives above degree $k\ell$ are zero. Apply (10.125) with $z=q^{-I}x+a$, multiply by $v(k)^u$, and increase ℓ to L. The base $\mathrm{e}(2+q^{-I}X/k)$ is greater than one. The same bound applies to $\mathcal{C}_{I,u}(X)$, by using $x=X,a=k-1,\ell=L$.

Finally, $E_h(\Omega)$ is the coefficient of Z^h in the product of $\Omega+1$ copies of $1+Z+Z^2+\cdots$. Distributing h objects among these slots gives $\binom{\Omega+h}{h}$. Multiplying these series over the Euler directions uses $\Omega_\Sigma+r-1$ slots, so every fixed Euler product of total order h is at most Γ_h: it is one of the nonnegative summands of that coefficient. For no Euler directions only order zero occurs. This proves the second inequality in (10.124). When $h>0,r\ge2$, the numerator product for Γ_h is at most $(\Omega_\Sigma+r-1+h)^h$; the factorial bound gives the further estimate. When $r=1$, its left side is zero. □

Averaging the initial rows

Keep the data of Theorem 10.32, and write $H=\sum_i A_ih(\vartheta_i)$. Let r_0 be the number of Euler indices with $\Omega_j>0$. The representative prepared indices after removing duplicate zero directions, and their row count, are

$$\begin{aligned}&\mathcal{J}_T=\{\boldsymbol{t}\ge0:|\boldsymbol{t}|\le T,\ t_0\le D,\ t_j=0\text{ if }\Omega_j=0\},\\&J_T=|\mathcal{J}_T|=\sum_{u=0}^{\min(D,T)}\binom{T-u+r_0}{r_0},\qquad M_*=(2S+1)J_T.\end{aligned}\tag{10.126}$$

Corollary 10.45 (a sharper initial coefficient budget). If $N>M_*$, the integral kernel in Theorem 10.32 exists with every originally specified zero and with

$$h_\infty(c) \le h_2(c) \le \tfrac{1}{2}\log N + \frac{\log|\Delta_K|}{2d} + \frac{1}{N-M_*}\sum_{s=-S}^{S}\sum_{\boldsymbol{t}\in\mathcal{J}_T}\left(\tfrac{1}{2}\log N + \log\mathcal{H}_0(|s|;\boldsymbol{t}) + 2|s|H\right). \tag{10.127}$$

A simpler consequence is

$$h_2(c) \le \tfrac{1}{2}\log N + \frac{\log|\Delta_K|}{2d} + \frac{M_*}{N-M_*}\left(\tfrac{1}{2}\log N + \log\mathcal{B}_0^*(S,T) + \frac{2S(S+1)}{2S+1}H\right). \tag{10.128}$$

When $N > M$ in Theorem 10.32, the right side of (10.128) is no larger than the right side of (10.96).

Proof. An index above the additive degree has zero scalar. An index in a zero Euler direction duplicates the row with that index set to zero, by Theorem 10.44. Removing it therefore preserves every original vanishing condition. Fixing $t_0 = u$ leaves r_0 Euler indices of total order at most $T-u$. The divider argument from Theorem 10.32 counts them as $\binom{T-u+r_0}{r_0}$, also when $r_0 = 0$. This proves (10.126).

At each retained row $(s,\boldsymbol{t})$, the same adelic row-norm proof as in Theorem 10.32 now uses $\mathcal{H}_0(|s|;\boldsymbol{t})$. At finite places the prepared scalars are still integers. The row norm is therefore at most $\sqrt{N}\mathcal{H}_0(|s|;\boldsymbol{t})\exp(2|s|H)$. The scalar majorant is at least one: its positive additive coefficient at integer $|s|+k-1$ is an integer by Lemma 10.28, and every retained Euler factor is a positive integer. Hence the displayed row bound is at least one, as required by the maximum in (8.40). A row that vanishes for additional reasons has norm zero and contributes the factor one there. Apply that proved weighted Siegel lemma with all weights one, retaining the separate row factors. Its logarithm is exactly (10.127), and its finite conditions give algebraic-integer coordinates. Dependent rows are already included in that lemma.

The uniform scalar bound in (10.123) and the identity $\sum_{s=-S}^{S}|s| = S(S+1)$ give (10.128). Finally $M_* \le M$, $\mathcal{B}_0^* \le \mathcal{B}$, and $S(S+1)/(2S+1) \le S$. All summands are nonnegative. The function $m/(N-m)$ increases for $0 \le m < N$, as cross multiplication shows. These facts prove the comparison with (10.96). The nonzero polynomial and its required ordinary jets follow from Proposition 10.31, exactly as before. □

The smaller nonzero-value budget

For the integer target nodes in Theorem 10.43, define, at each $\boldsymbol{t} \in \mathcal{J}_{T'}$,

$$\begin{aligned}\mathcal{A}_{I,\boldsymbol{t}}^*(X) &= \frac{d}{ef\ln p}\Big(\min\{h_\infty(c)+\ln N, h_2(c)+\tfrac{1}{2}\ln N\} \\ &\qquad + \ln\mathcal{H}_I(X;\boldsymbol{t}) + \Xi_{I,0,t_0}\ln q + 2XH\Big), \\ \mathcal{A}_I^*(X,T') &= \max_{\boldsymbol{t}\in\mathcal{J}_{T'}}\mathcal{A}_{I,\boldsymbol{t}}^*(X).\end{aligned} \tag{10.129}$$

Corollary 10.46 (zero forcing with the retained orders). Under the hypotheses of Theorem 10.43, its first analytic comparison and

$$(n\mu - D)\theta - Lv_p(k!) > \mathcal{A}_I^*(X,T') \tag{10.130}$$

force every zero asserted there. The analogous assertion holds with its larger normalizing scale and lower bound $(n\mu - D)\theta - D/(p-1)$. Moreover $\mathcal{A}_I^*(X, T') \le \mathcal{A}_I(X, T')$.

Proof. Repeating the product-formula proof of Theorem 10.41 uses the individual scalar bound $\mathcal{H}_I(X; \boldsymbol{t})$ instead of $\mathcal{B}_I$, with its same exact clearing exponent. Thus a nonzero value at a retained index has valuation excess at most $\mathcal{A}^*_{I,\boldsymbol{t}}$. The analytic lower bound contradicts (10.130). Every removed index either has zero scalar by additive degree or duplicates a retained index by setting its zero Euler directions to zero. The larger normalizing scale gives the stated second conclusion by the same argument. Finally $\mathcal{H}_I(X; \boldsymbol{t}) \le \mathcal{B}_I(X, T')$ and $\Xi_{I,0,t_0} \le Q_I$. Taking the maximum proves the comparison with (10.119). Fractional targets still require the field or common-factor conditions of Proposition 10.42. □

These refinements retain the exact factorials, lcm, derivative orders and mean node size. To obtain a numerical logarithm estimate, their bounds and the analytic slope and order conditions must all be checked for its chosen induction parameters.

19. Passing to one exponent coset

The next auxiliary function is obtained by retaining one coset of the exponent set and dividing its exponent differences by q. Two facts justify this step: independent root monomials separate the cosets, and an invertible affine change preserves the filtered Euler jets. The saturated Kummer theorem and the Newton reparametrization have already been proved in Theorem 10.14 and Lemma 10.30. We now combine them, retaining the coefficient field and the condition on the integer node. This is the algebraic mechanism of the coset passage in Yu's free 2013 paper, Lemma 5.4.

Separating the root monomials

Theorem 10.47 (exact coset extraction). Let F be a characteristic-zero field, q a prime, and $\vartheta_i \in F^\times$, $1 \le i \le r$. Choose $\eta_i^q = \vartheta_i$, and suppose $[F(\eta_1, \dots, \eta_r) : F] = q^r$. Take a finite set $\mathcal{M} \subset \mathbb{Z}^r$, coefficients $A_{\boldsymbol{m}} \in F$, and an integer s coprime to q. If $\sum_{\boldsymbol{m}} A_{\boldsymbol{m}} \boldsymbol{\eta}^{s\boldsymbol{m}} = 0$, then for each residue vector $\boldsymbol{\rho} \in \{0, \dots, q-1\}^r$,

$$\sum_{\boldsymbol{m} \equiv \boldsymbol{\rho} \ (q)} A_{\boldsymbol{m}} \boldsymbol{\vartheta}^{s(\boldsymbol{m}-\boldsymbol{\rho})/q} = 0. \tag{10.131}$$

Proof. The q^r monomials $\boldsymbol{\eta}^{\boldsymbol{\rho}}$ span the root field over F: reduce every exponent modulo q, using $\eta_i^q = \vartheta_i \ne 0$, also for negative exponents. The span is closed under multiplication. Every nonzero element of this finite-dimensional span has an inverse in it: multiplication by that element is injective in the surrounding field and thus surjective on the span. Hence it is the whole root field. Its degree is q^r, so these spanning monomials are a basis.

Group the given sum by $\boldsymbol{\rho}$. Its coefficient after factoring $\boldsymbol{\eta}^{s\boldsymbol{\rho}}$ is exactly the expression in (10.131), an element of F. Let $\boldsymbol{\rho}'$ be the least nonnegative residues of $s\boldsymbol{\rho}$, and put $\boldsymbol{b} = (s\boldsymbol{\rho} - \boldsymbol{\rho}')/q$. Then $\boldsymbol{\eta}^{s\boldsymbol{\rho}} = \boldsymbol{\vartheta}^{\boldsymbol{b}} \boldsymbol{\eta}^{\boldsymbol{\rho}'}$. Multiplication by s permutes the residue vectors because $q \nmid s$. Thus grouping gives a relation among distinct basis monomials, with each grouped coefficient multiplied by the nonzero scalar $\boldsymbol{\vartheta}^{\boldsymbol{b}}$. Every coefficient vanishes, proving (10.131). □

This argument extracts exact zeros. A large finite valuation of the sum does not by itself supply that valuation for each coset.

The next prepared family

Theorem 10.48 (coset restriction and affine jet transfer). Take a number field K, a prime $q \ne p$, local units $\vartheta_i \in K^\times$, and the integer Euler forms $\omega_j(\boldsymbol{m}) = \sum_i C_{ij} m_i$ in (10.92). Let $c \in K^N \setminus \{0\}$. Suppose that a field $F \supset K$ and chosen roots $\eta_i^q = \vartheta_i$ satisfy the degree condition of

Theorem 10.47. Let $\mathcal{E}$ be any set of integers coprime to q. Assume that all original prepared values at $(s/q, \eta_1^s, \dots, \eta_r^s)$, $s \in \mathcal{E}$, vanish through total order T.

Choose $\boldsymbol{m}_*$ at which some coefficient attains $\delta = \min v_p(c_j)$. Retain its exponent coset, and put

$$\begin{aligned} \mathcal{M}' &= \{(\boldsymbol{m} - \boldsymbol{m}_*)/q : \boldsymbol{m} \in \mathcal{M},\ \boldsymbol{m} \equiv \boldsymbol{m}_*\ (q)\}, \\ c'_{\boldsymbol{n},a,\ell} &= c_{\boldsymbol{m}_* + q\boldsymbol{n},a,\ell}, \\ Q'(X,Y) &= \sum_{\boldsymbol{n},a,\ell} c'_{\boldsymbol{n},a,\ell} \Delta(q^{-(I+1)}X + a; k)^\ell Y^{\boldsymbol{n}}. \end{aligned} \tag{10.132}$$

Then $Q' \neq 0$. Its new prepared family, with stage $I+1$ and the same Euler matrix C, vanishes at $(s, \boldsymbol{\vartheta}^s)$, $s \in \mathcal{E}$, through the same total order T.

The original prepared values restricted to this coset and the new prepared values are related by the affine changes

$$\begin{aligned} \Delta(qZ + \omega_j(\boldsymbol{m}_*); t_j) &= \sum_{u_j=0}^{t_j} d^{(j)}_{t_j,u_j} \Delta(Z; u_j), \qquad d^{(j)}_{t_j,t_j} = q^{t_j}, \\ \min_{|\boldsymbol{t}| \le T} v_p(\text{original coset value at } s/q) &= \min_{|\boldsymbol{t}| \le T} v_p(\text{new prepared value at } s). \end{aligned} \tag{10.133}$$

The minimum identity includes ∞ when all values are zero. The coefficients satisfy

$$\min v_p(c'_j) = \delta, \qquad h_\infty(c') \le h_\infty(c), \qquad h_2(c') \le h_2(c). \tag{10.134}$$

Global integrality of c is retained by c'. For any convex body $\mathcal{C}$ and vector x_0 with $\mathcal{M} \subset q^{-I}\mathcal{C} - x_0$, one has

$$\begin{aligned} \mathcal{M}' &\subset q^{-(I+1)}\mathcal{C} - (x_0 + \boldsymbol{m}_*)/q, \\ \deg_{Y_i}(\text{ordinary monomial shift of } Q') &\le \left\lfloor \frac{\max_{\boldsymbol{m} \in \mathcal{M}} m_i - \min_{\boldsymbol{m} \in \mathcal{M}} m_i}{q} \right\rfloor. \end{aligned} \tag{10.135}$$

Its additive degree is at most $D = kL$.

Proof. Fix $s \in \mathcal{E}$ and a prepared index. The rational additive and Euler factors, multiplied by the coefficients c, belong to $K \subset F$. Theorem 10.47 therefore extracts a zero separately in every exponent coset. For the chosen one, factoring $\boldsymbol{\eta}^{s\boldsymbol{m}_*}$ gives a zero with torus monomials $\boldsymbol{\vartheta}^{s\boldsymbol{n}}$.

The additive scalar at s/q is exactly $v(k)^{t_0}\Theta(q^{-(I+1)}s + a; k, \ell, t_0)$. The old Euler argument is $\omega_j(\boldsymbol{m}_* + q\boldsymbol{n}) = q\omega_j(\boldsymbol{n}) + \omega_j(\boldsymbol{m}_*)$. Lemma 10.30 gives the first line of (10.133), and both its matrix and inverse have coefficients in $\mathbb{Z}[1/q]$. The product changes only to indices with $u_j \le t_j$, keeping t_0 fixed. On the finite space of indices of total order at most T, it is triangular with nonzero diagonal $q^{\sum_{j<r} t_j}$; its inverse preserves the same filtration. Every new jet is therefore a linear combination of the extracted old coset jets, proving all the claimed zeros without reducing T.

The factored root monomial is a local unit, since every ϑ_i is one. The affine matrix and inverse are p-integral because $q \neq p$. Applying the ultrametric inequality to each change shows in both directions that the minimum valuation cannot decrease. This proves the second line of (10.133). It compares the jets of an individual coset; extraction from the whole sum used exact vanishing.

The selected coefficient vector is nonzero and contains a coefficient of the original minimum valuation. This proves the first assertion of (10.134). At every place its maximum norm is at most the old one; at each archimedean embedding the same is true for its Euclidean norm. Summing logarithms in the projective height definitions proves the remaining two inequalities. Coordinates of an integral

vector remain integral. Distinct exponents give distinct $\boldsymbol{n}$, so Lemma 10.29 proves $Q' \neq 0$, as in Proposition 10.31.

For the geometric inclusion, divide $\boldsymbol{m} + x_0 \in q^{-I}\mathcal{C}$ by q and subtract $(x_0 + \boldsymbol{m}_*)/q$. Each coordinate width of the retained exponent set is divided by q. Multiplying Q' by the inverse of its coordinatewise least Laurent monomial makes its torus exponents nonnegative, with exactly those widths. They are integers, proving (10.135). The additive degree is unchanged. □

Figure 10.7 shows the exponent change in Theorem 10.48 for a square support at $q = 2$.

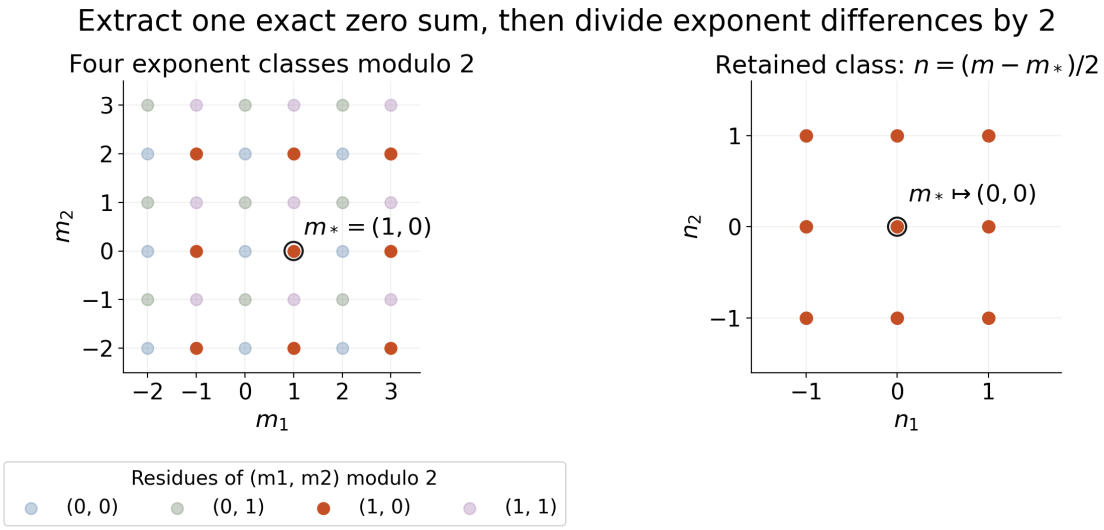


Figure 10.7. The exact support is $\mathcal{M} = \{-2, -1, 0, 1, 2, 3\}^2$. The four colors mark its four classes modulo two. Selecting $\boldsymbol{m}_* = (1, 0)$ retains the nine orange points $m_1 \in \{-1, 1, 3\}, m_2 \in \{-2, 0, 2\}$; translating by $-\boldsymbol{m}_*$ and dividing by two gives $\{-1, 0, 1\}^2$. The original coordinate width five becomes retained width four and then new width two, exactly as in (10.135). For odd s, full root-field degree separates the four root monomials in Theorem 10.47; their exact zero sums can then be treated separately. In the selected class the common local unit is $\boldsymbol{\eta}^{s\boldsymbol{m}_*}$, and the remaining monomials are $\boldsymbol{\vartheta}^{s\boldsymbol{n}}$, by (10.132)–(10.133). The displayed support illustrates that algebraic transformation; it does not assert a particular coefficient vector or numerical logarithm bound. Human-source context: Yu's free2013 Lemma5.4, cited below. Figure program.

Root-of-unity phases and the coefficient field

Corollary 10.49 (q-primary phases). Let $\alpha_0, \theta_1, \ldots, \theta_r$ be the saturated generators of Theorem 10.14. Choose $\xi^q = \alpha_0$ and $\rho_i^q = \theta_i$, and put $F = K(\xi)$. For an integer $P \geq 1$ coprime to q and integers d_i, set

$$\vartheta_i = \theta_i^P \alpha_0^{d_i}, \qquad \eta_i = \rho_i^P \xi^{d_i}, \qquad [F(\boldsymbol{\eta}) : F] = q^r, \qquad h(\vartheta_i) = Ph(\theta_i). \tag{10.136}$$

Thus Theorems 10.47–10.48 apply to these phased torus coordinates. If the θ_i are local units and $q \neq p$, all the displayed roots are local units. In particular one may take $P = p^a$, $a \geq 0$.

Proof. Theorem 10.14 gives total root degree q^{r+1}, and the same proved Kummer argument for α_0 alone gives $[F : K] = q$. Hence $[F(\boldsymbol{\rho}) : F] = q^r$. Clearly $F(\boldsymbol{\eta}) \subset F(\boldsymbol{\rho})$. Choose integers A, B with $AP + Bq = 1$; integer division gives such a pair because P and q are coprime. Then $\rho_i = (\eta_i \xi^{-d_i})^A \theta_i^B$, giving the reverse inclusion and the degree in (10.136). The q-th power identity follows directly. A root of unity has height zero; applying the product-height inequality in both directions shows that multiplying by it preserves height. The proved power identity then gives $h(\vartheta_i) = Ph(\theta_i)$. Finally taking valuations in the defining power equations proves every unit assertion. □

The coefficient field F holds the root-of-unity phase, while the selected new coefficients and torus bases in Theorem 10.48 remain in K. The next zero set consists of the stated integer nodes coprime to q; extension to further nodes uses the extrapolation results, with their slope and precision conditions.

20. Retaining the phase congruence in the next support

The phase used to make a local unit close to one need not belong to the global field K. A support congruence controls its powers. After division by q, the congruence has either one class or q classes; in the latter case the q-primary root separates them. We prove both cases, keeping all the new coefficients in K. This supplies the phase and support passage of Yu's free 2013 paper, equations (5.51)–(5.61), before the separate analytic extension to further nodes.

The congruence and its phase classes

Lemma 10.50 (division of a phase congruence). Let q be prime, $G_0 \geq 1$ an integer, and $a \in \mathbb{Z}$. The congruence $qu + a \equiv 0 \pmod{G_0}$ has the following solutions:

$$\begin{aligned} &u \equiv -q^{-1}a \pmod{G_0}, && \gcd(q, G_0) = 1, \\ &u \equiv -a/q + bG_0/q \pmod{G_0}, && 0 \leq b < q, \quad q \mid G_0,\ q \mid a. \end{aligned} \tag{10.137}$$

In the second case there is no solution if $q \nmid a$, and the displayed q classes are distinct. Suppose additionally that $\alpha_0 \in K^\times$ is not a q-th power, $\mu_q \subset K$, and $\beta^q = \alpha_0$. Then, for every integer s prime to q, the q powers β^{sb}, $0 \leq b < q$, are linearly independent over K.

Proof. If q and G_0 are coprime, integer division gives integers A, B with $Aq + BG_0 = 1$. Multiplication by A solves the congruence, uniquely modulo G_0. If $q \mid G_0$, reduction modulo q first forces $q \mid a$. Put $H = G_0/q$. The original congruence is equivalent to $u + a/q \equiv 0 \pmod{H}$, so $u = -a/q + bH + G_0 v$ for a unique b modulo q. This proves the second formula and its distinctness.

The complete Kummer argument of Theorem 10.14 applied to the one independent class α_0 proves $[K(\beta) : K] = q$. Equivalently $1, \beta, \dots, \beta^{q-1}$ are a basis, since their span is the field and has that degree, as proved in Theorem 10.47. Multiplication by s permutes exponents modulo q; reducing β^{sb} by $\beta^q = \alpha_0$ multiplies those basis elements by nonzero scalars in K. This proves the asserted independence, also for negative s. □

The phased prepared values

Take the saturated generators $\alpha_0, \theta_1, \dots, \theta_r$ of Theorem 10.14, with α_0 of order q^u. Let $G_0 \geq 1$, put $G = q^u G_0$, and choose roots of unity and algebraic roots in a fixed embedding such that

$$\begin{aligned} &\zeta^{G_0} = \alpha_0, \quad \zeta \text{ has order } G, \quad \xi^q = \zeta, \quad \beta = \xi^{G_0}, \quad \beta^q = \alpha_0, \\ &\rho_i^q = \theta_i, \qquad \gamma_i = \theta_i^P \zeta^{d_i}, \qquad \eta_i = \rho_i^P \xi^{d_i}, \qquad \eta_i^q = \gamma_i, \end{aligned} \tag{10.138}$$

where $P \geq 1$ is prime to q, and $d_i \in \mathbb{Z}$. The existence of ζ with the specified order is a hypothesis here; Lemma 10.53 constructs it for the local residue modulus. Write $d \cdot m = \sum_i d_i m_i$, and let $F = K(\beta)$. Theorem 10.14 gives $[F(\boldsymbol{\rho}) : F] = q^r$.

Let $I \geq 0, k, L \geq 1, T \geq 0$ be integers. Let a finite nonempty $\Lambda \subset \mathbb{Z}^r$ satisfy $d \cdot \iota \equiv \varepsilon \pmod{G_0}$. Fix $\iota_0 \in \Lambda$, an integer Euler matrix C, and a nonzero coefficient vector $c \in K^N$, with $N = kL|\Lambda|$. The fractional prepared values are

$$\Psi(s/q; \boldsymbol{t}) = \sum_{\iota, a, \ell} c_{\iota,a,\ell} v(k)^{t_0} \Theta(q^{-(I+1)} s + a; k, \ell, t_0) \prod_{j<r} \Delta(C_j(\iota - \iota_0); t_j) \boldsymbol{\eta}^{s(\iota - \iota_0)}. \tag{10.139}$$

Here $C_j(m) = \sum_i C_{ij} m_i$, the indices a, ℓ retain $0 \le a < k, 1 \le \ell \le L$, and the sum ranges over $\iota \in \Lambda$.

Theorem 10.51 (complete phase and support passage). Suppose that (10.139) is zero for every s in a set $\mathcal{E} \subset \mathbb{Z}$ prime to q, and every $|\boldsymbol{t}| \le T$. Put $\delta = \min_j v_p(c_j)$, where $q \ne p$ and the θ_i are units in a fixed embedding into $\mathbb{C}_p$.

Choose $\iota_{\min}$ at which some coefficient has valuation δ, and let $\lambda_* \in \{0, \dots, q-1\}^r$ represent $\iota_{\min} - \iota_0$ modulo q. First retain the exponents in this q-coset, and write

$$\iota - \iota_0 = q\lambda + \lambda_*, \qquad a_* = d \cdot \lambda_*, \qquad q\, d \cdot \lambda + a_* \equiv 0 \pmod{G_0}. \tag{10.140}$$

If $\gcd(q, G_0) = 1$, retain this whole coset and put $\varepsilon' = -q^{-1} a_* \pmod{G_0}$. If $q \mid G_0$, then $q \mid a_*$. Put $H = G_0/q$, $a_0 = -a_*/q$, and choose the unique $b_{\min} \in \{0, \dots, q-1\}$ for which the λ corresponding to $\iota_{\min}$ satisfies $d \cdot \lambda \equiv a_0 + b_{\min} H \pmod{G_0}$. Retain only that subclass and put $\varepsilon' = a_0 + b_{\min} H$.

The resulting nonempty set Λ' has $d \cdot \lambda \equiv \varepsilon' \pmod{G_0}$. Select $\lambda_0 \in \Lambda'$ corresponding to $\iota_{\min}$, retain the same coefficients $c'_{\lambda,a,\ell} = c_{\iota,a,\ell}$, and form

$$\begin{aligned} Q'(X, Y) &= \sum_{\lambda,a,\ell} c'_{\lambda,a,\ell} \Delta(q^{-(I+1)} X + a; k)^\ell Y^{\lambda - \lambda_0}, \\ \Psi'(s; \boldsymbol{t}) &= \sum_{\lambda,a,\ell} c'_{\lambda,a,\ell} v(k)^{t_0} \Theta(q^{-(I+1)} s + a; k, \ell, t_0) \prod_{j<r} \Delta(C_j(\lambda - \lambda_0); t_j) \boldsymbol{\gamma}^{s(\lambda - \lambda_0)}. \end{aligned} \tag{10.141}$$

Then $Q' \ne 0$, and $\Psi'(s; \boldsymbol{t}) = 0$ for every $s \in \mathcal{E}, |\boldsymbol{t}| \le T$. All those values belong to K, although the individual γ_i need not belong to K. The coefficients retain global integrality when present, satisfy $\min v_p(c'_j) = \delta$, and have $h_\infty(c') \le h_\infty(c)$, $h_2(c') \le h_2(c)$.

If $\Lambda \subset q^{-I}\mathcal{C} - x$, the uncentered support satisfies

$$\begin{aligned} \Lambda' &\subset q^{-(I+1)}\mathcal{C} - x', \qquad x' = (x + \iota_0 + \lambda_*)/q, \\ \deg_{Y_i}(\text{ordinary monomial shift of } Q') &\le \left\lfloor \frac{\max_{\iota \in \Lambda} \iota_i - \min_{\iota \in \Lambda} \iota_i}{q} \right\rfloor, \qquad \deg_X Q' \le kL. \end{aligned} \tag{10.142}$$

Proof. Since $d \cdot (\iota - \iota_0)$ is divisible by G_0, let $w_\iota = d \cdot (\iota - \iota_0)/G_0 \in \mathbb{Z}$. The torus part of an old fractional term is $\boldsymbol{\rho}^{Ps(\iota - \iota_0)} \beta^{s w_\iota}$. The latter factor belongs to F, while its rational prepared scalar and coefficient belong to K. Over F, the monomials $\boldsymbol{\rho}^{Ps\lambda_*}$, one for each q-residue vector, are independent: $[F(\boldsymbol{\rho}) : F] = q^r$ and Ps is prime to q, so Theorem 10.47 supplies exactly that basis permutation. Thus every original zero separates into a zero for each first q-coset.

In the selected coset, substituting (10.140) into the zero and removing the common nonzero factor $\boldsymbol{\rho}^{Ps\lambda_*} \xi^{sa_*}$ gives

$$\sum_{\iota \text{ in the selected coset}, a, \ell} c_{\iota,a,\ell} \mathcal{S}_\iota(s; \boldsymbol{t}) \boldsymbol{\gamma}^{s\lambda} = 0,$$

where $\mathcal{S}_\iota$ contains the additive scalar and the original Euler arguments. This equality holds for every specified $s, \boldsymbol{t}$. Lemma 10.50 proves that the divided support has exactly the stated phase congruence when q and G_0 are coprime.

Suppose $q \mid G_0$. Lemma 10.50 gives the q subclasses $d \cdot \lambda = a_0 + bH + G_0 v_\lambda$. Since $q \mid G_0$, (10.138) implies $\zeta^H = \xi^{qH} = \beta$. Multiply the last zero by ζ^{-sa_0}. The contribution of subclass b now belongs to $\beta^{sb} K$: each term is its coefficient and rational scalar times $\boldsymbol{\theta}^{Ps\lambda} \alpha_0^{s v_\lambda} \beta^{sb}$. The q powers β^{sb} are independent by Lemma 10.50, so every subclass sum is zero. In particular the subclass containing $\iota_{\min}$ is zero for all the specified indices, not just at one node.

Remove the common local unit $\boldsymbol{\gamma}^{s\lambda_0}$. The old Euler arguments are

$$C_j(\iota - \iota_0) = qC_j(\lambda - \lambda_0) + C_j(q\lambda_0 + \lambda_*).$$

The shift is an integer. Applying the invertible affine binomial change of Lemma 10.30 in every Euler direction converts the extracted zeros into the new ones in (10.141). Both changes preserve the total-order cutoff, as proved in Theorem 10.48. The additive scalar is already the stage $I+1$ scalar and is unchanged.

For $\lambda \in \Lambda'$, the integer $v'_\lambda = d \cdot (\lambda - \lambda_0)/G_0$ exists. Each new torus term is $\boldsymbol{\theta}^{Ps(\lambda-\lambda_0)}\alpha_0^{sv'_\lambda} \in K$. This proves the coefficient-field assertion. All roots of unity are local units, since their finite powers are one; roots of the local units θ_i are units too. Thus every removed common torus factor is a local unit. The coefficient subset contains a coefficient of valuation δ; the projective-height and integrality assertions follow from the proved subset inequalities in Theorem 10.48. Distinct new exponents and the binomial basis in Lemma 10.29 give $Q' \neq 0$.

Finally divide $\iota + x \in q^{-I}\mathcal{C}$ by q and use $\iota = \iota_0 + \lambda_* + q\lambda$ to obtain the inclusion in (10.142). Centering at λ_0 changes no coordinate width. Every retained width is at most the original width divided by q, and is an integer. The monomial shift and additive bound follow as in Theorem 10.48. □

Figure 10.8 gives the exact two-class split in the case $q \mid G_0$.

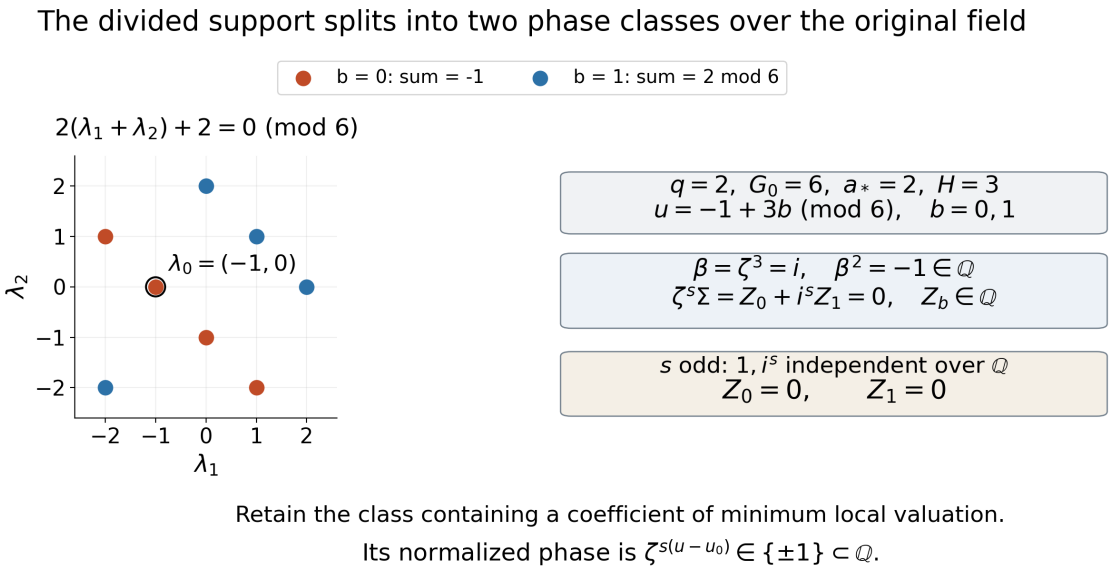


Figure 10.8. Here $q = 2, G_0 = 6, d = (1,1)$, and the first residue vector is $\lambda_* = (1,1)$, so $a_* = 2$. The divided support is the eight points $\lambda \in \{-2,-1,0,1,2\}^2$ with $2(\lambda_1+\lambda_2)+2 \equiv 0 \pmod 6$. The original support may be taken as $\{(0,0)\} \cup \{2\lambda + (1,1)\}$, with reference $\iota_0 = (0,0)$; every original point has coordinate sum divisible by six. Put $K = \mathbb{Q}, \alpha_0 = -1$, choose a primitive twelfth root ζ with $\zeta^6 = -1$, and write $\beta = \zeta^3 = i$ in a complex embedding. The normalized phase sum at a fixed odd node is $Z_0 + i^s Z_1$, with each $Z_b \in \mathbb{Q}$. When that sum is zero, independence in Lemma 10.50 forces both coefficient sums to vanish. Red points have coordinate sum -1, blue points have sum -4 or 2. The circled reference $\lambda_0 = (-1,0)$ illustrates retaining the subclass containing a coefficient of minimum valuation; its normalized phase is one. Choosing a blue reference instead gives phases in $\{\pm 1\}$, also in $\mathbb{Q}$. These are exactly the congruence, extraction and field-return mechanisms of Theorem 10.51 and (10.140)–(10.142). The plotted coefficient sums are formal variables. Human-source context: Yu's free2013 equations5.51–5.59, cited below. Figure program.

The analytic branch and arithmetic height

Corollary 10.52 (the phase does not enlarge the height field). If $v_p(\gamma_i - 1) > 1/(p-1)$, choose the fractional branch

$$\eta_i = \exp(q^{-1}\log\gamma_i), \qquad \rho_i^q = \theta_i, \qquad \rho_i^P \xi^{d_i} = \eta_i. \tag{10.143}$$

Such a choice of the ρ_i exists. It identifies (10.139) with the fractional values of the corresponding logarithmic curve. At an integer node $|s| \le X$, a nonzero new value in (10.141) satisfies

$$\begin{aligned} v_p(\Psi'(s;\boldsymbol{t})) - \delta \le \frac{[K:\mathbb{Q}]}{ef\ln p}\Big(& \min\{h_\infty(c') + \ln N', h_2(c') + \tfrac{1}{2}\ln N'\} \\ & + \ln \mathcal{H}_{I+1}(X;\boldsymbol{t}) + \Xi_{I+1,0,t_0}\ln q + 2PX\sum_i A_i' h(\theta_i)\Big), \end{aligned} \tag{10.144}$$

where $N' = kL|\Lambda'|$, $A_i' = \max_{\lambda\in\Lambda'}|\lambda_i - \lambda_{0,i}|$, and $\mathcal{H}$ uses the new integer Euler arguments in Theorem 10.44. An index above the additive degree is already zero. Every local degree in (10.144) is that of K, not the degree of a field containing all individual phases.

Proof. The exponential laws proved in Proposition 9.4 give the q-th power of the chosen η_i as γ_i, since $q \ne p$ changes no slope valuation. Begin with any root $\rho_i^q = \theta_i$. The quotient $\eta_i/(\rho_i^P\xi^{d_i})$ is in μ_q. Raising to P permutes that cyclic group because $q \nmid P$; multiplication of ρ_i by the appropriate q-th root of unity therefore gives (10.143). This proves the branch identification without a further root-existence assertion.

Theorem 10.51 writes each new torus term over K as a power of α_0 times powers of the θ_i. At every place, a root of unity has absolute value one, because its order-th power is one. Thus these phases contribute zero to every local norm bound, including the chosen p-adic place. The prepared scalar has the order-dependent archimedean bound $\mathcal{H}_{I+1}$ and the rational clearing factor of Theorem 10.41. Apply its complete normalized product-formula argument over K, retaining the coefficient norms and using $|Ps(\lambda_i - \lambda_{0,i})| \le PXA_i'$. The signed torus-height sum is $2[K:\mathbb{Q}]PX\sum_i A_i' h(\theta_i)$. This proves (10.144). □

Constructing the local phase

Lemma 10.53 (lifting the residue phase and its depth). Let $K_{\mathfrak{p}}$ be the completion at the chosen prime above p, with ramification index e and residue field of order p^f. Suppose $q \ne p$, and let $\alpha_0 \in K$ have order q^u, $u \ge 1$. Then $q^u \mid G = p^f - 1$, and there is a primitive G-th root $\zeta \in K_{\mathfrak{p}}$ such that

$$G_0 = G/q^u, \qquad \zeta^{G_0} = \alpha_0, \qquad \theta_i\zeta^{-r_i} \equiv 1 \pmod{\mathfrak{p}} \tag{10.145}$$

for suitable integers r_i, whenever the $\theta_i \in K^\times$ are local units. Given a real depth target $\vartheta \ge 0$, let t be the least nonnegative integer with $p^t > e/(p-1)$. Choose an integer $j \ge 0$ with $p^t/e + j > \vartheta + 1/(p-1)$, put $h = t + j$, and set

$$\begin{aligned} P = p^h, \qquad d_i = -Pr_i, \qquad \gamma_i = \theta_i^P\zeta^{d_i}, \\ v_p(\gamma_i - 1) \ge p^t/e + j > \vartheta + \frac{1}{p-1}. \end{aligned} \tag{10.146}$$

Thus these phases supply the required analytic branch and supernormal slope depth. This is a sufficient explicit choice of P; using a smaller prescribed P in a numerical estimate requires checking its depth separately.

Proof. We first prove the lifting assertion directly. In a complete discretely valued field, take a polynomial f with integral coefficients and an integral x_0 satisfying $v_p(f(x_0)) > 0$, $v_p(f'(x_0)) = 0$. Define $x_{n+1} = x_n - f(x_n)/f'(x_n)$. A finite polynomial expansion gives $f(x+y) = f(x) + f'(x)y +$

$y^2R(x,y)$ with R integral when x, y are integral. The linear terms cancel at the Newton step, so $v_p(f(x_{n+1})) \geq 2v_p(f(x_n))$. Also $f'(x_{n+1}) - f'(x_n)$ has positive valuation, leaving the derivative a unit. Every step remains integral and in the original residue class. The successive differences have valuations tending to infinity, hence form a Cauchy sequence. Completeness gives its limit, and the polynomial difference identity proves that the limit is a root. If x, y are roots in that residue class, then $0 = f(y) - f(x) = (y-x)(f'(x) + (y-x)R(x,y))$. The factor in parentheses is a unit. Therefore $x = y$. This proves existence and uniqueness of a simple-residue lift, including the case of a zero residual error where the iteration has already stopped.

Apply this to $X^m - 1$, for $p \nmid m$. Its derivative at a unit is a unit. In particular a prime-to-p root of unity reducing to one is one, by uniqueness in that residue class. Reduction is consequently injective on all prime-to-p roots of unity. The reduction of α_0 has its exact order q^u: a smaller order would make a nontrivial prime-to-p root reduce to one. Since the finite multiplicative residue group has order G, we get $q^u \mid G$.

The group of G-th roots in $K_{\mathfrak{p}}$ maps bijectively to that residue group: every nonzero residue solves $X^G - 1$ and has its unique lift. Multiplication commutes with reduction, and the finite-field cyclicity proved in Section 1 therefore gives a primitive root z. Write $\alpha_0 = z^{G_0 a}$ with $q \nmid a$, since its order is q^u. Let $n = G/q^{v_q(G)}$, so $\gcd(n, q) = 1$. Choose $a' \equiv a \pmod{q^u}$ and $a' \equiv 1 \pmod{n}$: Bezout's identity gives a solution by varying a' among $a + q^u b$. Then $\gcd(a', G) = 1$. The primitive root $\zeta = z^{a'}$ has $\zeta^{G_0} = \alpha_0$. Its reduction generates the residue group, so every $\overline{\theta_i}$ is $\overline{\zeta}^{r_i}$. This proves (10.145).

Put $g_i = \theta_i \zeta^{-r_i}$. Its positive valuation distance from one is at least $1/e$, unless it is already one. The proved p-power inequality of Lesson 9 is

$$v_p(g_i^{p^{a+1}} - 1) \geq \min\{p\, v_p(g_i^{p^a} - 1),\ v_p(g_i^{p^a} - 1) + 1\}.$$

The function on the right increases with its argument. For $0 \leq a < t$, the lower bound p^a/e is at most $1/(p-1)$, by the definition of t. Induction therefore gives $v_p(g_i^{p^t} - 1) \geq p^t/e > 1/(p-1)$. Each further p-power adds one by the exact power law already proved in Lesson 9; if a power is one its infinite valuation satisfies the same bound. After j further steps we obtain (10.146), because $g_i^P = \gamma_i$. Finally P is prime to q, so all the root-field and phase arguments above apply. Proposition 9.4 gives $v_p(\log \gamma_i) = v_p(\gamma_i - 1)$, including the infinite value when $\gamma_i = 1$; hence these logarithmic slopes have the stated depth. □

Filling the missing integer nodes

Theorem 10.51 gives the new zeros at nodes prime to q. We can now prove a precise condition for filling the remaining nodes, while keeping the height comparison over K.

Corollary 10.54 (phase-sensitive q-deleted-node closure). Take the new family (10.141), with $\gamma_i = \exp(u_i)$, and retain the slope, projection and selected integer Euler-basis hypotheses of Corollary 10.40 at stage $I+1$. In particular $v_p(w_i) > \theta + 1/(p-1)$, $v_p(\mathcal{L}) \geq U$, and $U - v_p(b_n) > \theta + 1/(p-1)$. Let $R \geq 1, \mu \geq 1, T \geq 0$ be integers with $T' = T - \mu + 1 \geq 0$. Suppose that $\Psi'(s; \boldsymbol{t}) = 0$ for every $s \in [-R, R] \cap \mathbb{Z}$ prime to q and every $|\boldsymbol{t}| \leq T$. Put $n = 2R - 2\lfloor R/q \rfloor$, $B = \lfloor \log_p(2R) \rfloor$, $C_0 = \max\{v_p(v(k)), v_p(b_n)\}$, $M_0 = \max\{B, C_0\}$, and $D = kL$. For $X \geq 0$, define the nonzero-value budget

$$\mathcal{A}_{\mathrm{ph}}(X, T') = \max_{\substack{|\boldsymbol{t}| \leq T' \\ t_0 \leq D}} \frac{[K:\mathbb{Q}]}{ef \ln p} \Big(\min\{h_\infty(c') + \ln N', h_2(c') + \tfrac{1}{2} \ln N'\} + \ln \mathcal{H}_{I+1}(X; \boldsymbol{t}) + \Xi_{I+1,0,t_0} \ln q + 2PX \sum_i A_i' h(\theta_i) \Big). \tag{10.147}$$

If

$$\begin{aligned} U + D\theta + Lv_p(k!) &\geq n\mu\theta + 2\mu M_0, \\ (n\mu - D)\theta - Lv_p(k!) &> \mathcal{A}_{\mathrm{ph}}(X, T'), \end{aligned} \tag{10.148}$$

then $\Psi'(x; \boldsymbol{t}) = 0$ for every integer $|x| \leq X$ and every $|\boldsymbol{t}| \leq T'$. In particular $X = R$ fills all the previously omitted integer nodes in the interval. With the larger normalizing scale, the alternative sufficient pair is

$$\begin{aligned} U + D\left(\theta + \frac{1}{p-1}\right) &\geq n\mu\theta + 2\mu M_0, \\ (n\mu - D)\theta - \frac{D}{p-1} &> \mathcal{A}_{\mathrm{ph}}(X, T'). \end{aligned} \tag{10.149}$$

If $q \mid R$, the node count is exactly $n = 2(1 - 1/q)R$. For a desired order $0 \leq O \leq T$, the choice $\mu = T - O + 1$ makes the output order exactly $T' = O$.

Proof. The q-deleted input set has the stated exact count and separation cost $\kappa = 1$, by Corollary 10.35. The complete analytic proof of Corollary 10.40 at the new stage applies to the local torus coordinates γ_i. Its normality, prepared integrality, ordinary input jets and projection estimates use those local units and their logarithmic slopes; they do not require each γ_i to lie in K. The first line of (10.148) is its exact sufficient precision budget, with $\kappa + 1 = 2$. Consequently

$$v_p(\Psi'(x; \boldsymbol{t})) - \delta \geq (n\mu - D)\theta - Lv_p(k!)$$

at every stated integer and derivative index.

The support congruence in Theorem 10.51 does put each of these integer values in K. If a value were nonzero, Corollary 10.52 would bound its valuation excess above by the corresponding summand of (10.147), and hence by $\mathcal{A}_{\mathrm{ph}}$. The second line of (10.148) is a strict contradiction. An index with $t_0 > D$ is already zero. This proves the full conclusion. Using the larger scale and its proved lower bound (10.114) instead proves (10.149); the two scales have their respective conclusions. Finally $q \mid R$ gives $\lfloor R/q \rfloor = R/q$, and the asserted order choice is the identity $T - (T - O + 1) + 1 = O$. □

The phase and support passage, the local phase construction and the conditional closure of the missing nodes now have complete proofs. A numerical Yu estimate still requires a verified coefficient construction and a check of the slope, order and strict height comparisons at every stage with its actual parameters.

21. Constructing coefficients for the phased values

The initial coefficient construction can also be performed over K when the separate phased torus bases lie outside K. What matters is the field of each normalized row entry. The support congruence determines that field and removes the root-of-unity phase from every absolute-value estimate.

Theorem 10.55 (an initial kernel over the original field). Let K have degree d and discriminant Δ_K. Take nonzero $\theta_i \in K$, a root of unity $\alpha_0 \in K$, an integer $G_0 \geq 1$, and a root of unity ζ with $\zeta^{G_0} = \alpha_0$. Let $P \in \mathbb{Z}_{\geq 1}$, $d_i \in \mathbb{Z}$, and set $\gamma_i = \theta_i^P \zeta^{d_i}$. The separate γ_i need not lie in K.

Take integers $r, k, L \geq 1$, $S, T \geq 0$, and a nonempty finite set $\Lambda \subset \mathbb{Z}^r$ of distinct vectors satisfying $\boldsymbol{d} \cdot \lambda \equiv \varepsilon \pmod{G_0}$, where $\boldsymbol{d} = (d_1, \ldots, d_r)$. Fix $\lambda_0 \in \Lambda$, put $A_i = \max_{\lambda \in \Lambda} |\lambda_i - \lambda_{0i}|$, and take integer Euler forms $\omega_j(\lambda) = C_j(\lambda - \lambda_0)$, $1 \leq j < r$. Write $D = kL$, $N = kL|\Lambda|$, and use $\mathcal{J}_T, J_T, M_* = (2S+1)J_T$ from (10.126), with these new Euler arguments. Define $H_{\mathrm{ph}} = P \sum_i A_i h(\theta_i)$.

If $N > M_*$, there is a nonzero vector $c \in \mathcal{O}_K^N$ for which

$$
\begin{aligned}
Q(X,Y) &= \sum_{\lambda,a,\ell} c_{\lambda,a,\ell}\Delta(X+a;k)^{\ell}Y^{\lambda-\lambda_0},\\
\Psi_0(s;\boldsymbol{t}) &= \sum_{\lambda,a,\ell} c_{\lambda,a,\ell}v(k)^{t_0}\Theta(s+a;k,\ell,t_0)\prod_{j<r}\Delta(\omega_j(\lambda);t_j)\gamma^{s(\lambda-\lambda_0)} = 0
\end{aligned}
\tag{10.150}
$$

for every integer $|s| \le S$ and every $|\boldsymbol{t}| \le T$, with $0 \le a < k$, $1 \le \ell \le L$. The Laurent polynomial Q is nonzero. Every normalized monomial in these equations belongs to K, since

$$
v_\lambda = \frac{\boldsymbol{d}\cdot(\lambda-\lambda_0)}{G_0} \in \mathbb{Z}, \qquad \gamma^{s(\lambda-\lambda_0)} = \alpha_0^{sv_\lambda}\prod_i \theta_i^{Ps(\lambda_i-\lambda_{0i})} \in K. \tag{10.151}
$$

The vector may be chosen with the separate-row budget

$$
\begin{aligned}
h_\infty(c) \le h_2(c) \le{}& \tfrac{1}{2}\ln N + \frac{\ln|\Delta_K|}{2d}\\
&+ \frac{1}{N-M_*}\sum_{s=-S}^{S}\sum_{\boldsymbol{t}\in\mathcal{J}_T}\left(\tfrac{1}{2}\ln N + \ln\mathcal{H}_0(|s|;\boldsymbol{t}) + 2|s|H_{\mathrm{ph}}\right),
\end{aligned}
\tag{10.152}
$$

and hence with the mean-node budget

$$
\begin{aligned}
h_2(c) \le{}& \tfrac{1}{2}\ln N + \frac{\ln|\Delta_K|}{2d}\\
&+ \frac{M_*}{N-M_*}\left(\tfrac{1}{2}\ln N + \ln\mathcal{B}_0^*(S,T) + \frac{2S(S+1)}{2S+1}H_{\mathrm{ph}}\right).
\end{aligned}
\tag{10.153}
$$

These estimates involve K and the original θ_i, even when the individual phases require a larger global field.

Proof. Subtracting the two support congruences gives the integer v_λ. Raising $\zeta^{G_0} = \alpha_0$ to sv_λ proves (10.151), for negative s and negative exponent differences as well. This identity puts every matrix entry of (10.150) in K.

The prepared scalar in each entry is an integer: Lemma 10.28 proves the additive assertion at every integer $s+a$, and each rising Euler binomial at an integer is an integer. An index with $t_0 > D$ has zero row. If ω_j vanishes throughout the support, its factor is $\Delta(0;t_j) = 1$; setting $t_j = 0$ gives the same row and a no larger total order. Thus imposing the representative rows imposes every stated equation. Counting them exactly as in (10.126) gives M_* rows and N columns, including any further zero or dependent rows.

We bound each representative row over K. At an archimedean embedding its Euclidean norm is at most

$$
\sqrt{N}\,\mathcal{H}_0(|s|;\boldsymbol{t})\prod_i \max\{|\sigma(\theta_i)|,|\sigma(\theta_i)|^{-1}\}^{P|s|A_i}.
$$

At a finite place its maximum norm is at most the analogous product without the first two factors. Indeed the scalar is an integer, while the root of unity $\alpha_0^{sv_\lambda}$ in (10.151) has absolute value one at every place. Theorem 10.44 supplies the scalar bound. The product formula, with the height conventions of Section 14, gives the exact identity

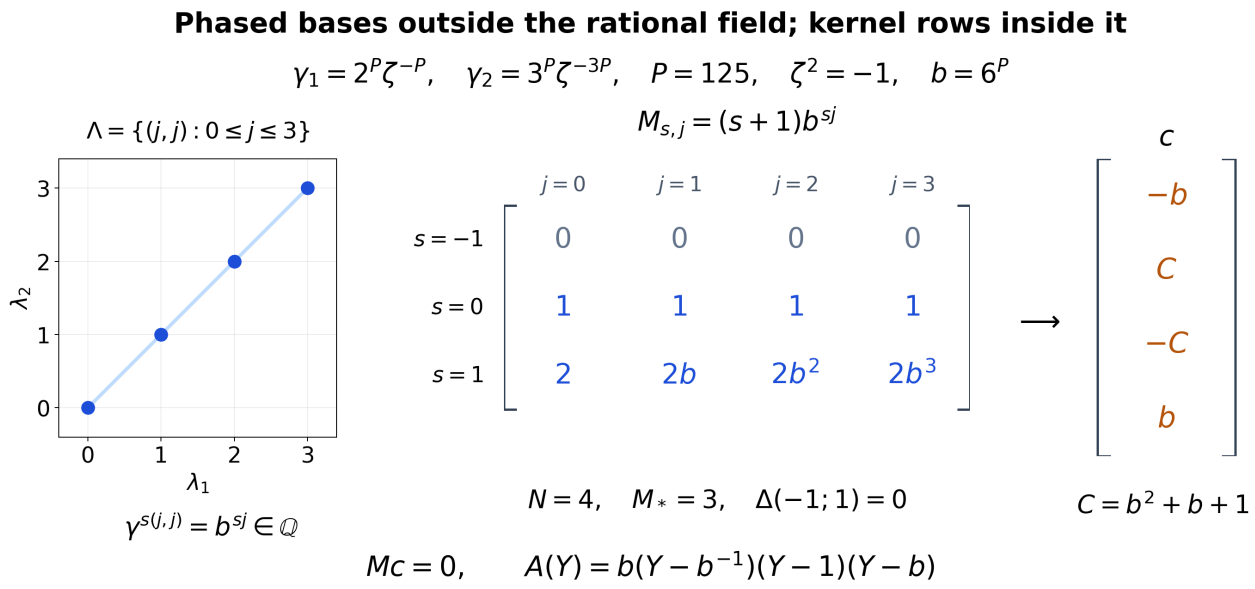
$$
\frac{1}{d}\sum_v \ln\max\{|\theta_i|_v,|\theta_i|_v^{-1}\} = 2h(\theta_i).
$$

Consequently the logarithm of the row-norm bound is $\frac{1}{2}\ln N + \ln\mathcal{H}_0(|s|;\boldsymbol{t}) + 2|s|H_{\mathrm{ph}}$. This bound is nonnegative, since its scalar majorant is at least one by Theorem 10.44. A zero row has the allowed row factor one.

Apply the complete weighted Siegel lemma (8.40) from Small solutions over the coefficient field, with all weights one and with these actual K-rows. Its finite-place conditions give $c \in \mathcal{O}_K^N \setminus \{0\}$. Taking logarithms of its bound proves (10.152); no discriminant or degree of a field containing the individual γ_i enters. The monomials $Y^{\lambda-\lambda_0}$ are distinct, and Lemma 10.29 gives the independence of the additive polynomials within each monomial block. Hence the nonzero coefficient vector gives $Q \neq 0$.

Finally $\mathcal{H}_0(|s|;\boldsymbol{t}) \leq \mathcal{B}_0^*(S,T)$, and $\sum_{s=-S}^{S} |s| = S(S+1)$. Summing the separate bounds proves (10.153). □

Multiplying Q by a monomial clears its negative exponents. The complete prepared-to-ordinary jet argument of Proposition 10.31 applies when the γ_i are nonzero elements of the chosen local field and the Euler directions are the selected integral basis. Each monomial factor is nonzero at the evaluation point, so the order of vanishing is preserved. A coefficient of minimum local valuation can also be divided out as in (10.97), preserving the heights and equations; this last normalization is locally integral and need not remain globally integral.



Theorem 10.55 and equations (10.150)–(10.153); exact support, nodes and kernel.

Figure 10.9. In $\mathbb{Q}_5$, choose the root $\zeta^2 = -1$ with $\zeta \equiv 2 \pmod 5$, and put $P = 125$, $G_0 = 2$, $\alpha_0 = -1$, $\theta = (2,3)$, $d = (-P,-3P)$. Both $\gamma_1 = 2^P\zeta^{-P}$ and $\gamma_2 = 3^P\zeta^{-3P}$ are outside $\mathbb{Q}$, but $\gamma_1\gamma_2 = 6^P = b \in \mathbb{Q}$. For support $\Lambda = \{(j,j) : 0 \le j \le 3\}$, reference $(0,0)$, $k = L = 1$, $S = 1$, $T = 0$, the normalized torus column is b^{sj}. The drawn matrix includes the additive factor $\Delta(s;1) = s+1$, so its first row is zero. With $C = b^2 + b + 1$, its integer vector $(-b, C, -C, b)$ is a kernel vector. Equations (10.150)–(10.153) prove the general field and height mechanism; Solution 28 checks this exact example and its extra zero. The plotted support points and matrix entries are exact. Human-source context: the field reduction and initial kernel in Lemma 4.2 of Yu's free 2013 paper, cited below. Figure program.

22. The field and budget at the first fractional nodes

At a fractional node the value need not be in K, even if its chosen completion is still $K_\mathfrak{p}$. We now determine the smaller field generated by the support, and then prove the arithmetic comparison needed before the next coset extraction.

The support-generated root field

Theorem 10.56 (the field of the normalized fractional monomials). Retain the saturated Kummer and phase data of Section 20: in particular $\mu_q \subset K$, $q \neq p$, $q \nmid P$, and $[K(\beta, \rho_1, \dots, \rho_r) : K] = q^{r+1}$, with $\beta^q = \alpha_0$, $\rho_i^q = \theta_i$. Suppose $\gamma_i \in K_\mathfrak{p}$ have $v_p(\gamma_i - 1) > 1/(p-1)$, and use the compatible principal roots $\eta_i = \exp(q^{-1}\log\gamma_i) = \rho_i^P \xi^{d_i}$ of Corollary 10.52.

Let Λ be a nonempty finite support with $\boldsymbol{d}\cdot\lambda\equiv\varepsilon \pmod{G_0}$, and fix $\lambda_0\in\Lambda$. Define

$$\begin{aligned} m_\lambda &= \lambda-\lambda_0, & w_\lambda &= \frac{\boldsymbol{d}\cdot m_\lambda}{G_0}\in\mathbb{Z},\\ g &= \gcd(G_0,d_1,\dots,d_r), & G_* &= G_0/g, \qquad d_{*i}=d_i/g,\\ v_\lambda &= (w_\lambda, Pm_{\lambda1},\dots,Pm_{\lambda r})\in\mathbb{F}_q^{r+1}, & h_\Lambda &= \dim_{\mathbb{F}_q}\langle v_\lambda:\lambda\in\Lambda\rangle,\\ U_\lambda &= \eta^{m_\lambda}, & E_\Lambda &= K(U_\lambda:\lambda\in\Lambda). \end{aligned} \tag{10.154}$$

Then

$$PG_*(v_\lambda)_0-\sum_i d_{*i}(v_\lambda)_i=0, \qquad h_\Lambda\le r, \qquad [E_\Lambda:K]=q^{h_\Lambda}, \qquad (E_\Lambda)_{\mathfrak{P}}=K_{\mathfrak{p}} \tag{10.155}$$

for the prime $\mathfrak{P}$ selected by these principal roots. The first relation is a nonzero linear equation over $\mathbb{F}_q$. In particular the global degree can increase while the chosen local degree stays one.

Proof. Cancelling the positive gcd gives $G_*w_\lambda=\sum_i d_{*i}m_{\lambda i}$, and $\gcd(G_*,d_{*1},\dots,d_{*r})=1$. Multiply by P and reduce modulo q to obtain the relation in (10.155). If $q\nmid G_*$, its first coefficient is nonzero, since $q\nmid P$. If $q\mid G_*$, at least one d_{*i} is nonzero modulo q. Thus it defines a hyperplane of dimension r, proving $h_\Lambda\le r$. Cancelling the gcd is essential: the unreduced equation can otherwise be the identity $0=0$.

The phase congruence and the compatible roots give

$$U_\lambda=\beta^{w_\lambda}\prod_i\rho_i^{Pm_{\lambda i}}, \qquad U_\lambda^q=\alpha_0^{w_\lambda}\prod_i\theta_i^{Pm_{\lambda i}}\in K.$$

Choose $\lambda_1,\dots,\lambda_h$, $h=h_\Lambda$, whose residue vectors form a basis of the support span. Express any v_λ as $\sum_j a_jv_{\lambda_j}$, with $0\le a_j<q$. The corresponding integer exponent vectors differ by $q(z_0,z_1,\dots,z_r)$. The displayed exact monomial identity therefore gives

$$U_\lambda=\alpha_0^{z_0}\prod_i\theta_i^{z_i}\prod_{j=1}^{h}U_{\lambda_j}^{a_j}.$$

Consequently the h selected roots generate E_Λ, and their q-th powers lie in K, so its degree is at most q^h.

For the reverse bound, the monomials $\prod_j U_{\lambda_j}^{a_j}$, $0\le a_j<q$, have distinct residue vectors $\sum_j a_jv_{\lambda_j}$. Reduce their exponents in the full root field $K(\beta,\rho_1,\dots,\rho_r)$. Its degree is q^{r+1}, so the root monomials with exponents in $\{0,\dots,q-1\}^{r+1}$ are a basis by Theorem 10.47. Our q^h monomials are nonzero K-multiples of distinct elements of that basis, and are therefore independent. This proves the exact degree, including $h=0$.

Each η_i lies in $K_{\mathfrak{p}}$ by the exponential proof in Lesson 9, hence $E_\Lambda\subset K_{\mathfrak{p}}$. Its closure contains the closure $K_{\mathfrak{p}}$ of K, and is contained in $K_{\mathfrak{p}}$. Thus its chosen completion is precisely $K_{\mathfrak{p}}$. The absolute ramification index and residue degree at $\mathfrak{P}$ are the original e,f. □

A nonzero fractional prepared value

Corollary 10.57 (the phase-sensitive fractional arithmetic budget). Take the support of Theorem 10.56, integers $I\ge0,k,L\ge1$, a nonzero coefficient vector $c\in K^N$, $N=kL|\Lambda|$, and integer Euler arguments $C_j(m_\lambda)$. Put $D=kL$, $A_i=\max_\lambda|m_{\lambda i}|$, $\delta=\min_j v_p(c_j)$, and take $x=s/q$, $s\in\mathbb{Z}$, with $|x|\le X$. Let $V=\Psi_I(s/q;\boldsymbol{t})$ be the prepared value whose torus term is U_λ^s and whose additive argument is $q^{-I}x+a$. If $t_0>D$, it is zero. Otherwise it belongs to E_Λ, and if it is nonzero,

$$v_p(V)-\delta\le \mathcal{A}_{\mathrm{frac},\boldsymbol{t}}(X):=\frac{q^{h_\Lambda}[K:\mathbb{Q}]}{ef\ln p}\Big(\min\{h_\infty(c)+\ln N,h_2(c)+\tfrac12\ln N\} \\ +\ln\mathcal{H}_I(X;\boldsymbol{t})+\Xi_{I,1,t_0}\ln q+2PX\sum_i A_i h(\theta_i)\Big). \tag{10.156}$$

Here $\mathcal{H}_I$ and $\Xi_{I,1,t_0}$ are the already proved scalar majorant and q-clearing exponent of (10.123) and (10.115).

Proof. Every torus term lies in E_Λ, and every prepared scalar is rational, so $V\in E_\Lambda$. Lemma 10.28 and Theorem 10.41 clear each scalar's denominator by $q^{\Xi_{I,1,t_0}}$, with no denominator at any other prime; Theorem 10.44 bounds its ordinary absolute value by $\mathcal{H}_I(X;\boldsymbol{t})$.

At every place of E_Λ, taking the absolute value of U_λ^q removes the root-of-unity phase. Its signed torus contribution is bounded by

$$\sum_i \frac{P|s|A_i}{q}\ln\max\{|\theta_i|_v,|\theta_i|_v^{-1}\}.$$

The torus terms are units at $\mathfrak{P}$. Summing this contribution outside that prime is therefore at most $2[E_\Lambda:\mathbb{Q}]PX\sum_i A_i h(\theta_i)$, by the same inverse-height identity as in Theorem 10.41. The coefficient heights retain their values over K: each archimedean embedding is repeated $[E_\Lambda:K]$ times, and the local degrees over a finite place sum to that same factor.

Now repeat the full coefficient-normalized product-formula argument of Theorem 10.41 over E_Λ, retaining its Euclidean and maximum coefficient bounds and the new scalar majorant. Theorem 10.56 gives $[E_\Lambda:\mathbb{Q}]=q^{h_\Lambda}[K:\mathbb{Q}]$ and the unchanged e,f at the selected prime. These are exactly the factor and all terms in (10.156). □

The unchanged completion does not remove q^{h_Λ}. Nor does a high valuation at the principal-root prime give that valuation at the other primes of the global field.

Closing the first fractional-node extension

Corollary 10.58 (a proved first fractional extension). Retain the phase family of Corollary 10.57 and all selected-basis, slope, normality and projection hypotheses of Corollary 10.40 for its local curve $\gamma^x=\exp(x\log\gamma)$. Suppose its prepared values vanish at all integers $|s|\le R$ through total order T, where $R\ge1, T\ge0$ are integers. Choose an integer $\mu\ge1$ with $T'=T-\mu+1\ge0$, and put $n=2R+1$, $B=\lfloor\log_p(2R)\rfloor$, $C_0=\max\{v_p(v(k)),v_p(b_n)\}$, $M_0=\max\{B,C_0\}$. Let $\mathcal{A}_{\mathrm{frac}}(X,T')=\max_{|\boldsymbol{t}|\le T',\,t_0\le D}\mathcal{A}_{\mathrm{frac},\boldsymbol{t}}(X)$, using (10.156). If

$$\begin{aligned} U+D\theta+Lv_p(k!)&\ge n\mu\theta+\mu M_0,\\ (n\mu-D)\theta-Lv_p(k!)&>\mathcal{A}_{\mathrm{frac}}(X,T'),\end{aligned} \tag{10.157}$$

then every first fractional prepared value $\Psi_I(s/q;\boldsymbol{t})$ is zero for integers $|s|\le qX$ and $|\boldsymbol{t}|\le T'$. With the larger normalizing scale, an alternative sufficient pair is

$$\begin{aligned} U+D\left(\theta+\frac{1}{p-1}\right)&\ge n\mu\theta+\mu M_0,\\ (n\mu-D)\theta-\frac{D}{p-1}&>\mathcal{A}_{\mathrm{frac}}(X,T').\end{aligned} \tag{10.158}$$

Proof. The full integer input interval has separation cost $\kappa=0$, as proved in Corollary 10.35. The first inequality of (10.157) is consequently the sufficient analytic budget (10.113). Its complete proof,

retaining the loss from the selected integer Euler basis, gives

$$v_p(\Psi_I(x;\boldsymbol{t})) - \delta \geq (n\mu - D)\theta - Lv_p(k!)$$

for every $x \in \mathbb{Z}_p$ and each output index. The local curve satisfies the same proof when its separate torus bases lie outside K, since that argument uses their local logarithms, prepared scalars and projection data.

For $x = s/q$, the assumption $q \neq p$ gives $x \in \mathbb{Z}_p$. Proposition 9.4 identifies the local exponential torus factors with the principal roots η^s, so this analytic value is exactly the algebraic value of Corollary 10.57. If it were nonzero, (10.156) would give the opposite upper bound $\mathcal{A}_{\mathrm{frac}}(X, T')$, contradicting the strict second inequality. Indices with $t_0 > D$ are already zero. This proves every asserted fractional zero. Repeating the same comparison with the proved larger-scale conclusion (10.114) gives (10.158). □

These hypotheses separate the two tasks in a numerical descent: constructing the integer input zeros and verifying the precision and strict height comparisons. Once the fractional zeros are available at nodes prime to q, Theorem 10.51 performs the exact phase and support passage.

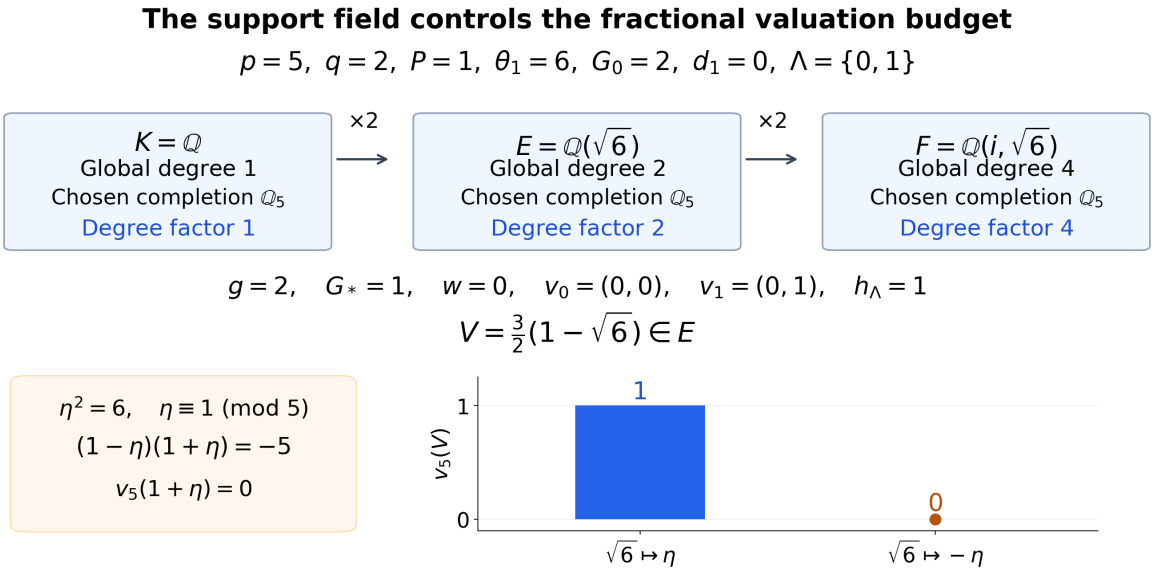


Figure 10.10. The exact example has $K = \mathbb{Q}, p = 5, q = 2, \alpha_0 = -1, \theta_1 = 6, P = 1, G_0 = 2, d_1 = 0, \Lambda = \{0, 1\}$. The cancelled relation is $w = 0$, and the support vectors span $(0, 1)$ over $\mathbb{F}_2$, so $h_\Lambda = 1$. The support field $E = \mathbb{Q}(\sqrt{6})$ has degree two; the full root field $F = \mathbb{Q}(i, \sqrt{6})$ has degree four. Choose the completion sending $\sqrt{6}$ to $\eta \equiv 1 \pmod 5$. Both these roots and $i \equiv 2 \pmod 5$ lie in $\mathbb{Q}_5$, so all three chosen completions in the diagram are $\mathbb{Q}_5$. Their global-to-local degree factors are $1, 2, 4$. The value $V = \frac{3}{2}(1 - \sqrt{6})$ has valuation one at this root and zero at the other root $-\eta$; the two bars display these exact valuations. Theorem 10.56 and (10.154)–(10.155) prove the field mechanism, Corollary 10.57 proves its arithmetic budget, and Solution 29 verifies every number in this example. Human-source context: the first fractional step and product-formula field discussion in Yu's free 2013 paper, printed pp. 352 and 365–366, cited below. Figure program.

23. Heights of the actual exponent support

The torus contribution can be measured before enclosing the exponents in a coordinate box. This keeps the geometry of a support through a change of multiplicative bases and through division by q. Positive and negative nodes require separate heights unless the support is symmetric.

A projective height from a support function

Let K have degree d, and use the absolute values and place multiplicities of Section 14. Thus $\sum_v \ln |z|_v = 0$ for $z \in K^\times$; a complex place is counted with multiplicity two. Take $\theta = (\theta_1, \ldots, \theta_r) \in$

$(K^\times)^r$, and put $\ell_v = (\ln|\theta_1|_v, \dots, \ln|\theta_r|_v)$. Only finitely many finite-place vectors ℓ_v are nonzero. For a nonempty compact set $C \subset \mathbb{R}^r$, and a nonempty finite set $M \subset \mathbb{Z}^r$, define

$$\begin{aligned} \mathfrak{H}_\theta(C) &= \frac{1}{d}\sum_v \sup_{u\in C} u\cdot\ell_v, \\ \mathfrak{H}_\theta(M) &= \mathfrak{H}_\theta(\operatorname{conv} M) = h_\infty((\theta^m)_{m\in M}). \end{aligned} \tag{10.159}$$

The finite-set equality, and all the properties needed below, are proved next.

Theorem 10.59 (the support height). The quantities in (10.159) are finite and nonnegative. For a real vector a, a real $t \ge 0$, and an integer s,

$$\begin{gathered} \mathfrak{H}_\theta(C+a) = \mathfrak{H}_\theta(C), \qquad \mathfrak{H}_\theta(tC) = t\mathfrak{H}_\theta(C), \\ C\subset C' \Longrightarrow \mathfrak{H}_\theta(C) \le \mathfrak{H}_\theta(C'), \quad h_\infty((\theta^{sm})_{m\in M}) = \begin{cases} s\mathfrak{H}_\theta(M), & s\ge 0, \\ |s|\mathfrak{H}_\theta(-M), & s<0. \end{cases} \end{gathered} \tag{10.160}$$

If $|m_i| \le A_i$ for every $m \in M$, then

$$\mathfrak{H}_\theta(M),\ \mathfrak{H}_\theta(-M) \le 2\sum_i A_i h(\theta_i). \tag{10.161}$$

These heights are unchanged by extending K. Root-of-unity factors attached separately to the monomial coordinates change neither height.

Proof. The support function of a finite convex hull is the maximum over its original points: a convex combination cannot exceed their maximum, and those points belong to the hull. Since $\ln|\theta^m|_v = m\cdot\ell_v$, its normalized sum is exactly the maximum projective height in (10.159). All sums are finite. For a compact C, translating by a adds $a\cdot\ell_v$ at each place. The product formula for each θ_i makes the sum of these additions zero. Translating by the negative of any point of C places zero in the set; every resulting support function is then nonnegative. This proves nonnegativity.

Nonnegative scaling multiplies each support function by t, including $t = 0$; inclusion increases it. Taking $s \ge 0$ in the monomial identity multiplies the local maximum by s, while $s < 0$ replaces M by $-M$. This proves (10.160). At any place, the box bounds both support functions by $\sum_i A_i|\ln|\theta_i|_v|$. The sum of the positive logarithms for one element is $dh(\theta_i)$, and the product formula says the sum of the negative logarithms has the same absolute value. Their total is $2dh(\theta_i)$, proving (10.161).

For completeness, an extension E/K repeats each archimedean embedding $[E:K]$ times. At a finite place v, the logarithm of the norm absolute value at a place w of E restricting to v is multiplied by $[E_w : K_v]$. The local degrees sum to $[E:K]$, by the places identities proved in Absolute values and places. The support function is homogeneous for these positive multipliers. Thus its unnormalized place sum is multiplied by $[E:K]$; dividing by $[E:\mathbb{Q}] = [E:K]d$ preserves its value. Every root of unity has absolute value one at every place, by its finite-order equation, so its coordinate factors make no contribution. □

Symmetry is a sufficient reason for the two signs to agree; it is not automatic. Over $\mathbb{Q}$, for $\theta = (2,3)$ and $M = \{(0,0),(1,0),(0,1)\}$, the monomial vector is $(1,2,3)$, of height $\ln 3$. The inverse vector $(1,1/2,1/3)$ becomes the primitive integer vector $(6,3,2)$, of height $\ln 6$. These are $\mathfrak{H}_\theta(M)$ and $\mathfrak{H}_\theta(-M)$.

Contraction and the original multiplicative bases

Theorem 10.60 (a height envelope for the divided support). Let $C \subset \mathbb{R}^r$ be nonempty and compact. If a finite nonempty support M_I satisfies $M_I \subset q^{-I}C - x_I$, then

$$\mathfrak{H}_\theta(M_I) \le q^{-I}\mathfrak{H}_\theta(C), \qquad \mathfrak{H}_\theta(-M_I) \le q^{-I}\mathfrak{H}_\theta(-C). \tag{10.162}$$

The support passage $M_{I+1} = \{(m - m_*)/q : m \text{ in a retained coset of } M_I\}$ retains this form of containment. A further restriction of the support and any choice of reference point retain the bound.

Suppose also that $\alpha_1, \dots, \alpha_n \in K^\times$, that $B = (b_{ji})$ is a real n-by-r matrix of rank r, and that a positive integer τ satisfies $\tau b_{ji} \in \mathbb{Z}$ and $\theta_i^\tau = \prod_j \alpha_j^{\tau b_{ji}}$. For finite $D_j \ge 0$, put $C_B = \{u : |(Bu)_j| \le D_j,\ 1 \le j \le n\}$. Then C_B is compact and centrally symmetric, and

$$\mathfrak{H}_\theta(C_B) = \mathfrak{H}_\theta(-C_B) \le 2\sum_{j=1}^{n} D_j h(\alpha_j), \qquad \mathfrak{H}_\theta(\pm M_I) \le 2q^{-I}\sum_j D_j h(\alpha_j) \quad (M_I \subset q^{-I}C_B - x_I). \tag{10.163}$$

Proof. Translation, scaling and inclusion in Theorem 10.59 give (10.162). The next retained support is contained in $q^{-(I+1)}C - (x_I + m_*)/q$, so induction preserves exactly this envelope. Restriction uses inclusion; changing reference uses translation.

Choose an invertible r-by-r row minor of B. Its inverse bounds each coordinate of $u \in C_B$ by a fixed finite linear combination of the D_j. Thus C_B is bounded; its defining inequalities make it closed. To see directly that its support maximum is attained, take a sequence approaching the supremum of a linear function. Successive bisection of a bounded coordinate interval chooses an infinite subsequence in nested closed intervals of lengths tending to zero. Repeating this for the finitely many coordinates gives a convergent subsequence. Its limit lies in the closed set, and the finite linear sum tends to its value there. This also proves the required compactness in finite dimension.

Taking absolute values of the τ-power identities and cancelling τ gives $\ell_v = B^{\mathsf{T}}(\ln|\alpha_j|_v)_j$. Consequently

$$u \cdot \ell_v = \sum_j (Bu)_j \ln|\alpha_j|_v \le \sum_j D_j |\ln|\alpha_j|_v| \qquad (u \in C_B).$$

Sum this inequality and use the inverse-height identity proved in Theorem 10.59. The set C_B is centrally symmetric because its inequalities use absolute values. Combining the resulting bound with (10.162) proves (10.163). □

The same height bound holds for a nonempty affine box constraint $|(Bu)_j - z_j| \le D_j$, with arbitrary real z_j. Indeed its local support maximum is at most $\sum_j z_j \ln|\alpha_j|_v + \sum_j D_j|\ln|\alpha_j|_v|$. The first sum disappears after summing over places, by the product formula, even if z is outside the image of B. Thus the bound uses the half-widths in the original coordinates and follows the original torus products through a change of basis. An enclosing box in the new coordinates is unnecessary.

The coefficient and nonzero-value budgets

Corollary 10.61 (initial coefficients with the support height). Retain every hypothesis and notation of Theorem 10.55, including $N > M_*$, and put $M = \Lambda - \lambda_0$, $H_+ = \mathfrak{H}_\theta(M)$, $H_- = \mathfrak{H}_\theta(-M)$. Define $b(s) = PsH_+$ for $s \ge 0$, and $b(s) = P|s|H_-$ for $s < 0$. Its nonzero integral kernel exists with

$$\begin{aligned} h_\infty(c) \le h_2(c) \le{}& \tfrac{1}{2}\ln N + \frac{\ln|\Delta_K|}{2d} \\ &+ \frac{1}{N - M_*}\sum_{s=-S}^{S}\sum_{\boldsymbol{t}\in\mathcal{J}_T}\left(\tfrac{1}{2}\ln N + \ln\mathcal{H}_0(|s|;\boldsymbol{t}) + b(s)\right), \end{aligned} \tag{10.164}$$

and hence with

$$h_2(c) \le \tfrac{1}{2}\ln N + \frac{\ln|\Delta_K|}{2d} + \frac{M_*}{N-M_*}\left(\tfrac{1}{2}\ln N + \ln \mathcal{B}_0^*(S,T) + \frac{PS(S+1)}{2(2S+1)}(H_+ + H_-)\right). \tag{10.165}$$

Both bounds are at least as strong as their respective bounds (10.152) and (10.153).

Proof. The normalized torus monomials in a row are $\alpha_0^{sw_\lambda}\theta^{Psm_\lambda}$. At an archimedean embedding, their maximum times $\sqrt{N}\mathcal{H}_0(|s|;\boldsymbol{t})$ bounds the row's Euclidean norm. At a finite place their maximum bounds its maximum norm, since every scalar is an integer. The phase has absolute value one. Summing the logarithms of these maxima gives exactly $b(s)$, by Theorem 10.59. The support contains zero, so each local torus maximum is at least one; the scalar majorant is also at least one, by Theorem 10.44. These are valid row factors in the completely proved weighted Siegel lemma (8.40), with a zero row allowed factor one. Its proof therefore gives (10.164) over K, exactly as in Theorem 10.55. The same row system, integral vector and polynomial-independence argument impose every original zero.

The positive and negative node sums are both $S(S+1)/2$. There are J_T rows at each node and $M_* = (2S+1)J_T$. Using the uniform scalar bound gives (10.165). Finally (10.161) bounds each of H_+, H_- by $2\sum_i A_i h(\theta_i)$; substitution gives the original two torus terms, without changing any other term. □

Corollary 10.62 (the support budget for a nonzero value). In the phase family of Theorem 10.55, suppose the γ_i are units at the chosen prime, and consider an integer-node prepared value $V = \Psi_I(s;\boldsymbol{t})$, with $t_0 \le D$. Put $C_c = \min\{h_\infty(c) + \ln N, h_2(c) + \frac{1}{2}\ln N\}$ and $\delta = \min_j v_p(c_j)$. If $V \ne 0$, then

$$v_p(V) - \delta \le \frac{d}{ef\ln p}\left(C_c + \ln \mathcal{H}_I(|s|;\boldsymbol{t}) + \Xi_{I,0,t_0}\ln q + b(s)\right). \tag{10.166}$$

In the first fractional family of Corollary 10.57, with its support field E_Λ, the nonzero value at $x = s/q$ satisfies instead

$$v_p(V) - \delta \le \frac{q^{h_\Lambda} d}{ef\ln p}\left(C_c + \ln \mathcal{H}_I(|s|/q;\boldsymbol{t}) + \Xi_{I,1,t_0}\ln q + \frac{b(s)}{q}\right). \tag{10.167}$$

Uniformly for $|s| \le X$ in the integer case and $|s| \le qX$ in the fractional case, the torus term may be replaced by

$$PX\max\{H_+, H_-\}. \tag{10.168}$$

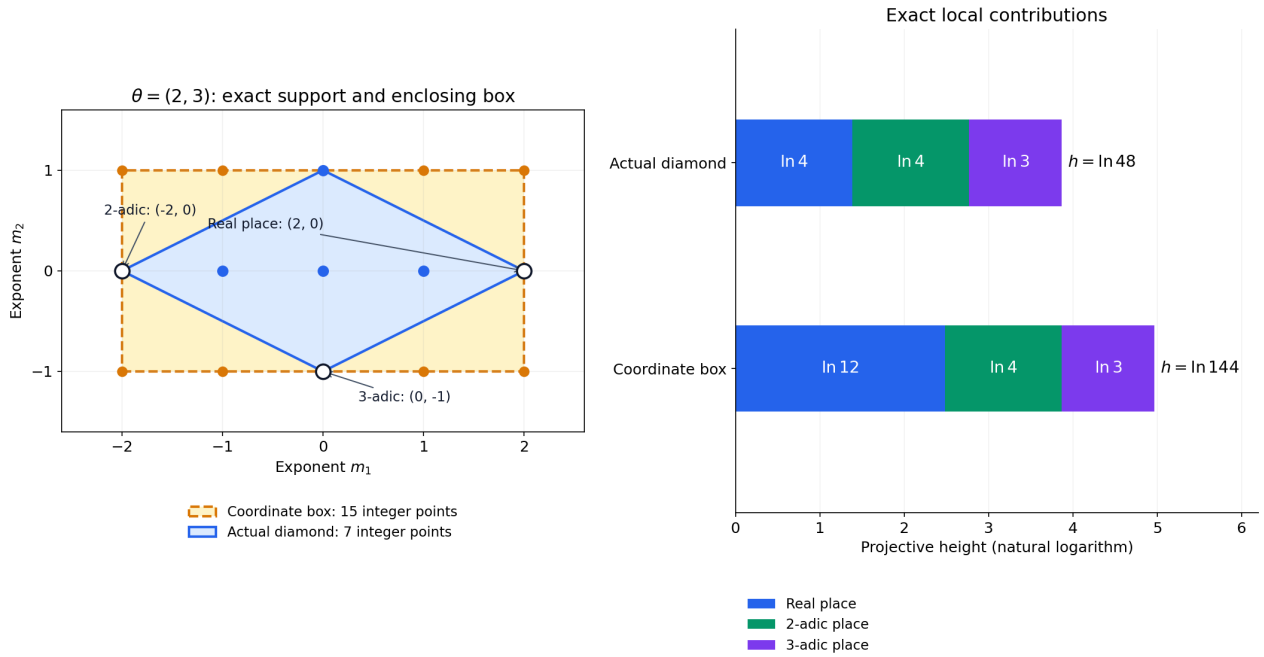
The corresponding strict zero-forcing comparisons of Theorem 10.43 and Corollaries 10.54 and 10.58 remain valid with these smaller budgets, under all their original analytic and field hypotheses.

Proof. For the integer case the phase congruence puts every torus monomial in K. Repeat Theorem 10.41's product formula after dividing by a coefficient of minimum local valuation. Its coefficient contribution is C_c; Theorem 10.44 supplies the individual scalar bound, and (10.115) supplies its clearing exponent. At each place the torus contribution is the maximum of the logarithms of θ^{Psm_λ}, with the phase removed. Its sum is $db(s)$. Its omitted contribution at the chosen prime is zero: the γ_i are units, and taking absolute values of $\gamma_i = \theta_i^P\zeta^{d_i}$ shows that the θ_i are units there. These local maxima are nonnegative because $0 \in M$. This proves (10.166).

In the fractional case, $U_\lambda^q = \alpha_0^{w_\lambda}\theta^{Pm_\lambda}$. Taking local logarithmic absolute values divides the preceding torus maximum by q, and its field-normalized sum is $b(s)/q$, by the field-invariance proof of Theorem

10.59. Theorem 10.56 still gives the global degree $q^{h_\Lambda} d$ and the original local e, f; none of these degree factors has been discarded. The same coefficient-normalized product formula gives (10.167). The bounds on s give (10.168).

The analytic conclusions in the cited zero-forcing proofs supply their unchanged lower bounds for the same valuations. A strict inequality between that lower bound and the new arithmetic upper bound again contradicts $V \neq 0$. Taking the maximum over the asserted output indices proves all the same zeros. This changes only the torus height estimate; every slope, depth, order and field condition remains a hypothesis. □



Theorems 10.59–10.60 and Corollaries 10.61–10.62; equations (10.159)–(10.168); Solution 31.

Figure 10.11. Over $\mathbb{Q}$, with $\theta = (2, 3)$, the diamond $C = \{(u_1, u_2) : |u_1| + 2|u_2| \leq 2\}$ has seven integer points; its enclosing box $[-2, 2] \times [-1, 1]$ has fifteen. At the real, 2-adic and 3-adic places the diamond's support contributions are $\ln 4, \ln 4, \ln 3$, attained respectively at $(2, 0), (-2, 0), (0, -1)$. The box contributions are $\ln 12, \ln 4, \ln 3$. Their exact projective heights are therefore $\ln 48$ and $\ln 144$. The bar lengths are numerical renderings of these labelled exact logarithms. The polygon, lattice points, active points and height identities are exact. Equations (10.159)–(10.168) prove the general support, contraction and budget mechanism; Solution 31 checks the primitive monomial vectors and a divided-support example. Human-source context: the original-coordinate torus identity and support containment (4.16), (5.1), and the product-formula steps (5.34)–(5.36), (7.13) in Yu's free 2013 paper, cited below. Figure program.

The enclosing-box estimate is valid, but this example loses exactly $\ln 3$ in its torus height. The support budget avoids that loss, and (10.163) retains bounds in the original bases after a multiplicative change of coordinates.

24. The field and the numerical scale of the initial kernel

The initial rows need not generate the entire ambient number field. We can construct their coefficients over the field they actually generate. Its discriminant is controlled by the heights of explicit normalized torus monomials, so an unrelated extension of the ambient field does not enter the coefficient budget.

The exact field of the row entries

Retain Theorem 10.55, and write $M = \Lambda - \lambda_0$, $H_+ = \mathfrak{H}_\theta(M)$, $H_- = \mathfrak{H}_\theta(-M)$. The normalized monomials and the field generated by all initial row entries are

$$W_\lambda = \alpha_0^{w_\lambda}\theta^{P(\lambda-\lambda_0)} \in K, \qquad W_{\lambda_0} = 1,$$
$$E_0 = \begin{cases} \mathbb{Q}, & S = 0, \\ \mathbb{Q}(W_\lambda : \lambda \in \Lambda), & S \geq 1, \end{cases} \qquad d_0 = [E_0 : \mathbb{Q}] \leq d. \tag{10.169}$$

Theorem 10.63 (initial coefficients over their row field). The field in (10.169) is exactly the smallest subfield of K containing every coefficient of the initial homogeneous system (10.150). If $N > M_*$, that system has a nonzero coefficient vector $c \in \mathcal{O}_{E_0}^N \subset \mathcal{O}_K^N$ giving the same nonzero Laurent polynomial and every originally specified zero.

For $S \geq 1$, choose normalized monomials $z_1, \ldots, z_a$ that generate E_0. Put $F_j = \mathbb{Q}(z_1, \ldots, z_j)$, $F_0 = \mathbb{Q}$, so $F_a = E_0$, and let $e_j = [F_j : F_{j-1}]$. Omit steps of degree one, and define $\Gamma = \sum_j (e_j - 1)h(z_j)$. For $S = 0$, use the empty tower and $\Gamma = 0$. Then

$$\frac{\ln|\Delta_{E_0}|}{2d_0} \leq \frac{1}{2}\ln d_0 + \Gamma, \qquad \Gamma \leq (d_0 - 1)P\min\{H_+, H_-\}. \tag{10.170}$$

With $b(s)$ as in Corollary 10.61, the coefficients can be chosen with

$$\begin{aligned} h_\infty(c) \leq h_2(c) \leq & \frac{1}{2}\ln(Nd_0) + \Gamma \\ & + \frac{1}{N - M_*}\sum_{s=-S}^{S}\sum_{\boldsymbol{t}\in\mathcal{J}_T}\left(\frac{1}{2}\ln N + \ln\mathcal{H}_0(|s|;\boldsymbol{t}) + b(s)\right), \end{aligned} \tag{10.171}$$

and therefore with

$$\begin{aligned} h_2(c) \leq & \frac{1}{2}\ln(Nd_0) + \Gamma \\ & + \frac{M_*}{N - M_*}\left(\frac{1}{2}\ln N + \ln\mathcal{B}_0^*(S,T) + \frac{PS(S+1)}{2(2S+1)}(H_+ + H_-)\right). \end{aligned} \tag{10.172}$$

The actual discriminant term $\ln|\Delta_{E_0}|/(2d_0)$ may instead be retained, replacing $\frac{1}{2}\ln d_0 + \Gamma$ in either formula.

Proof. Every entry of the row at s is an integer prepared scalar times W_λ^s. If $S = 0$, all entries are rational integers, so their field is $\mathbb{Q}$. If $S \geq 1$, these expressions first show that every entry belongs to the monomial field in (10.169). Conversely take the row at $s = 1$ with derivative index zero. Its column with $a = 0, \ell = 1$ and any λ is $\Delta(1;k)W_\lambda = (k+1)W_\lambda$. Every Euler factor there is $\Delta(\omega;0) = 1$. Division by the nonzero rational integer $k+1$ recovers W_λ from that row. Thus every field containing the entries contains each monomial, proving minimality.

The discriminant-and-generators proof, equation (8.33), gives the first bound in (10.170) for this explicit tower, including nonintegral monomials. Each selected z_j is a coordinate of the monomial vector containing 1. Its ordinary height is at most that vector's maximum projective height PH_+: locally, the vector maximum bounds $\max\{1, |z_j|_v\}$. The inverse vector also contains 1, so it bounds $h(z_j^{-1}) = h(z_j)$ by PH_-. Consequently $h(z_j) \leq P\min\{H_+, H_-\}$. The integer degrees have product d_0; induction using $(u-1)+(v-1) \leq uv - 1$ gives $\sum_j (e_j - 1) \leq d_0 - 1$. This proves the second bound. The empty-tower case has $d_0 = 1, \Delta_{E_0} = 1$.

Apply the completely proved weighted Siegel lemma (8.40) over E_0 to the actual representative rows. Their number M_*, their integer scalar bounds and all their dependencies are unchanged. Their normalized torus row height is $b(s)$, as in Corollary 10.61. This height can be computed over K, because the row vector lies in E_0 and projective heights are invariant under extension. The individual

θ_i need not lie in E_0. The Euclidean row bound adds $\frac{1}{2}\ln N + \ln \mathcal{H}_0$ as before. The finite-place conditions give $c \in \mathcal{O}_{E_0}^N$, and such elements are also algebraic integers in K.

That lemma supplies $\frac{1}{2}\ln N + \ln|\Delta_{E_0}|/(2d_0)$, followed by the displayed row sum divided by $N - M_*$. Substitution of (10.170) proves (10.171). Summing the positive and negative nodes separately proves (10.172), exactly as in (10.165). A nonzero coefficient vector still gives a nonzero polynomial by Lemma 10.29, and the same representative-row argument preserves all jets in (10.150). This proves every assertion. □

Full widths in the original coordinates

The original torus widths can now be inserted before any numerical constants are rounded. Let $\alpha_1, \dots, \alpha_r \in K^\times$, and suppose the multiplicative change of basis in Theorem 10.60 satisfies, for a positive integer τ, $\theta_i^\tau = \omega_i \prod_j \alpha_j^{\tau b_{ji}}$, where ω_i is a root of unity and every τb_{ji} is integral. The root-of-unity factors contribute zero to every local logarithm, so the proof of (10.163) still applies. Suppose the original support coordinates $B\Lambda$ lie in a translated box of full widths $D_1, \dots, D_r$, and put $L_\alpha = \sum_j D_j h(\alpha_j)$. After choosing the reference point λ_0, its original coordinates BM satisfy affine box constraints with half-widths $D_j/2$. The affine-centre proof following (10.163) gives

$$H_+, H_- \le L_\alpha, \qquad \Gamma \le (d_0 - 1) P L_\alpha. \tag{10.173}$$

Here the box condition is imposed on Bm; it does not require its centre to be in the image of B.

Corollary 10.64 (an initial budget in the original heights). Under these hypotheses and $N > M_*$, the initial coefficient vector may be chosen with

$$\begin{aligned} h_2(c) \le{}& \frac{1}{2}\ln(Nd_0) + (d_0 - 1) P L_\alpha \\ &+ \frac{M_*}{N - M_*}\left(\frac{1}{2}\ln N + \ln \mathcal{B}_0^*(S,T) + \frac{PS(S+1)}{2S+1} L_\alpha\right). \end{aligned} \tag{10.174}$$

Replacing d_0 everywhere by its upper bound d gives a valid further bound.

Proof. Substitute both parts of (10.173) into (10.172). The factor of two from $H_+ + H_-$ cancels the two in its denominator. The replacement by d increases $\ln d_0$ and $d_0 - 1$, while all other terms stay unchanged and $PL_\alpha \ge 0$. □

In particular a full width D_j is not a half-width D_j. This distinction accounts for a factor of two in the torus contribution.

Substitution at the numerical scale

Corollary 10.65 (the scaled coefficient inequality). Suppose $S \ge 1$, $N \ge c_* M_*$ with $c_* > 1$, and that positive numbers $\mathcal{D}, c_1, c_2, \sigma_1, \dots, \sigma_r$ satisfy

$$h(\alpha_j) \le \sigma_j, \qquad D_j = \frac{\mathcal{D}}{c_1 c_2 r P d \sigma_j}, \qquad Z = \frac{S\mathcal{D}}{d}. \tag{10.175}$$

Then a coefficient vector with every zero in Theorem 10.63 exists with

$$\begin{aligned} \frac{h_2(c)}{Z} \le{}& \frac{d\ln(Nd_0)}{2S\mathcal{D}} + \frac{d_0 - 1}{c_1 c_2 S} \\ &+ \frac{1}{c_* - 1}\left(\frac{d(\frac{1}{2}\ln N + \ln \mathcal{B}_0^*(S,T))}{S\mathcal{D}} + \frac{1}{2c_1c_2}\left(1 + \frac{1}{2S+1}\right)\right). \end{aligned} \tag{10.176}$$

An explicit generator tower may improve the second term to Γ/Z. If $S \geq g > 0$, the last factor $1 + 1/(2S+1)$ is at most $1 + 1/(2g+1)$.

Proof. Formula (10.175) gives

$$PL_\alpha = P\sum_j D_j h(\alpha_j) \leq P\sum_j D_j \sigma_j = \frac{\mathcal{D}}{c_1 c_2 d}.$$

Also $M_*/(N - M_*) \leq 1/(c_* - 1)$: divide $N \geq c_* M_*$ by $M_* > 0$ and subtract one. Every term in the parentheses of (10.174) is nonnegative. Divide that bound by Z, insert these inequalities, and use the exact identity

$$\frac{S+1}{2S+1} = \frac{1}{2}\left(1 + \frac{1}{2S+1}\right).$$

This proves (10.176), including its coefficient $\frac{1}{2c_1c_2}$. Retaining Γ in (10.172) proves the tower refinement. Finally $S \geq g$ increases the positive denominator $2S+1$, giving the asserted upper bound. □

Equation (10.176) supplies a direct numerical comparison once the column surplus, scalar majorant and coefficient-field tower have been determined. It retains the full-width torus term without an ambient-field discriminant. The later zero-forcing inequalities still require their own strict comparisons.

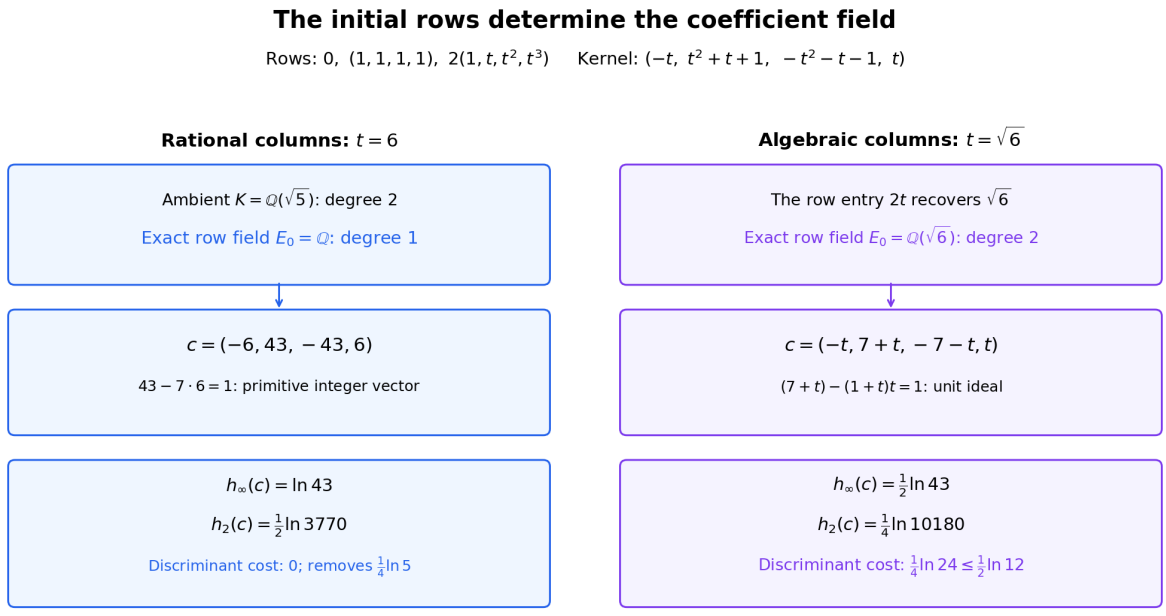


Figure 10.12. For $k = L = S = 1, T = 0, \Lambda = \{0,1,2,3\}, P = 1$, the initial rows are 0, $(1,1,1,1)$, $2(1,t,t^2,t^3)$. The vector $(-t, t^2+t+1, -t^2-t-1, t)$ is their exact integral kernel. Left: $t = 6$, with ambient $K = \mathbb{Q}(\sqrt{5})$, gives row field $\mathbb{Q}$, vector $(-6, 43, -43, 6)$, and no discriminant cost. The ambient-field cost would be $\frac{1}{4}\ln 5$. Right: $t = \sqrt{6}$ generates row field $\mathbb{Q}(\sqrt{6})$, with discriminant 24. Its cost $\frac{1}{4}\ln 24$ is retained; the generator proof bounds it by $\frac{1}{2}\ln 12$. The exact coefficient heights are $\ln 43$, $\frac{1}{2}\ln 3770$ on the left and $\frac{1}{2}\ln 43$, $\frac{1}{4}\ln 10180$ on the right. Theorem 10.63 and equations (10.169)–(10.176) prove the field and budget mechanism; Solution 32 verifies the rows, discriminants and heights. Human-source context: the initial coefficient-field reduction in Lemma 4.2 of Yu's free 2013 paper; the complete discriminant and Siegel inputs are proved in the earlier lesson linked above, from Matveev's free paper. Figure program.

25. Constructing enough columns in one phase class

The coefficient inequalities require a surplus of columns over equations. We now produce that surplus from the volume of a closed box, the covolume of the saturated exponent lattice and the number of possible phase classes. The boundary of the box matters when its normalized volume is an integer.

Points in a translated closed box

Theorem 10.66 (the closed-box lattice count). Let $\mathcal{L} = B\mathbb{Z}^r$, with B an invertible real matrix, and let $C = \prod_{j=1}^{r}[0, D_j]$, with every $D_j > 0$. Write

$$\Delta = |\det B|, \qquad V = \prod_j D_j, \qquad m = \lfloor V/\Delta \rfloor. \tag{10.177}$$

There are $m+1$ distinct points $y_0, \dots, y_m$ of C such that all $y_j - y_0$ lie in $\mathcal{L}$. Consequently a translate of C contains $m + 1$ distinct lattice points, including zero.

If $\mathcal{L} \supset \mathbb{Z}^r$ has finite index $J = |\mathcal{L}/\mathbb{Z}^r|$, then

$$\Delta = J^{-1}, \qquad m = \lfloor JV \rfloor. \tag{10.178}$$

Proof. Take the half-open fundamental cell $F = B[0, 1)^r$. Its translates by $\mathcal{L}$ partition $\mathbb{R}^r$, and its volume is Δ. For $x \in F$, let $n_C(x)$ count the points of $x + \mathcal{L}$ in C. This is a finite sum of indicator functions: boundedness of $C - F$ bounds the integer coordinates $B^{-1}\ell$ of any participating lattice vector. Translating each integral and using the partition gives

$$\int_F n_C(x)\, dx = \sum_{\ell \in \mathcal{L}} \int_{F+\ell} \mathbf{1}_C(y)\, dy = V.$$

The volume and translation rules used here are the same ones proved in the lattice argument of Small solutions over the coefficient field. If $V/\Delta > m$, this integral prevents $n_C(x) \le m$ everywhere. Some coset therefore supplies $m + 1$ points.

For the equality case $V/\Delta = m$, enlarge C about its centre by factors $1 + 1/n$. Each enlarged closed box has volume strictly greater than $m\Delta$, so it contains $m + 1$ points $x_n + \ell_{0n}, \dots, x_n + \ell_{mn}$, with $x_n \in F$. All participating lattice vectors lie in one fixed bounded set, because the enlarged boxes lie in the box enlarged by two and F is bounded. There are finitely many such lattice vectors. Passing to a subsequence fixes the distinct vectors $\ell_0, \dots, \ell_m$; a further subsequence makes x_n converge in the closed bounded cell $B[0, 1]^r$. Coordinatewise interval bisection gives this convergent subsequence, as in the proof of Theorem 10.60. The limits $x + \ell_j$ lie in C: its closed inequalities are the limits of the enlarged inequalities. Their differences are the distinct lattice differences $\ell_j - \ell_0$. This proves the count also at integer normalized volume.

For (10.178), since every coordinate unit vector belongs to $\mathcal{L}$, the matrix $U = B^{-1}$ is integral. Multiplication by B^{-1} identifies $\mathcal{L}/\mathbb{Z}^r$ with $\mathbb{Z}^r/U\mathbb{Z}^r$. Its index is $|\det U|$. Here is an integer reduction proof of that identity. Interchanging rows or columns, changing a sign, and adding an integer multiple of another row or column preserve index and absolute determinant. The Euclidean algorithm makes the first column have a positive pivot and zeros below. If an entry in its first row is not divisible by the pivot, column division and interchange produce a smaller positive pivot; repeat. This must stop because positive integers decrease. Column subtraction then clears the first row and leaves a nonsingular block of size $r - 1$. Induction diagonalizes the matrix by these operations. The quotient for a diagonal matrix has the product of the absolute diagonal entries as its index, by independent residue representatives. This is its absolute determinant. Thus $J = |\det U| = 1/\Delta$, proving (10.178). □

The enlargement argument uses a closed box. For the open interval $C = (0, 1)$ and lattice $\mathbb{Z}$, the normalized volume is one but no translate contains two lattice points.

Retaining a single phase

Lemma 10.67 (the phase-class count). Take the $m+1$ lattice differences in Theorem 10.66 and write them as $B\lambda_j$, with $\lambda_j \in \mathbb{Z}^r$. For integers $G_0 \geq 1$ and $d_1, \ldots, d_r$, put

$$g = \gcd(G_0, d_1, \ldots, d_r), \qquad H = G_0/g. \tag{10.179}$$

There is a set Λ of at least $\lceil (m+1)/H \rceil$ of these distinct integer vectors, all satisfying the same congruence $\boldsymbol{d} \cdot \lambda \equiv \varepsilon \pmod{G_0}$. Moreover $B\Lambda \subset C - y_0$.

Proof. Each phase value is a multiple of g modulo G_0, so there are at most H possible values. If every class contained fewer than $\lceil (m+1)/H \rceil$ of the $m+1$ points, their total would be smaller than $m+1$. Choose a class of at least that size. The map B is injective, so no points coalesce. The containment holds because every original difference was $y_j - y_0$. This includes the case $d_i = 0$ for every i, when $g = G_0, H = 1$. □

A certified surplus of prepared columns

Corollary 10.68 (box volume supplies the initial kernel). Suppose the multiplicative and phase hypotheses of Theorem 10.55 are available for the lattice basis B and the phase data of Lemma 10.67. For integers $k, L \geq 1, S, T \geq 0$, choose the retained support Λ there. Its number of coefficient columns satisfies

$$\begin{gathered} kL \left\lceil \frac{\lfloor V/\Delta \rfloor + 1}{H} \right\rceil \leq N = kL|\Lambda| \leq kL(\lfloor V/\Delta \rfloor + 1), \\ N > \frac{kLV}{\Delta H}. \end{gathered} \tag{10.180}$$

Define the support-independent row upper bound

$$\overline{J}_T = \sum_{u=0}^{\min(kL,T)} \binom{T-u+r-1}{r-1}, \qquad \overline{M} = (2S+1)\overline{J}_T. \tag{10.181}$$

If $kLV/(\Delta H) \geq c_* \overline{M}$, with $c_* > 1$, then $N > c_* M_*$. In particular the complete integral initial kernel of Theorem 10.63 exists and its scaled budget (10.176) applies whenever its original-width hypotheses hold. The selected support retains that full-width height bound.

For a saturated lattice of index J and widths from (10.175), an explicit sufficient criterion is

$$\frac{kLJ}{H} \frac{\mathcal{D}^r}{(c_1 c_2 r P d)^r \prod_j \sigma_j} \geq c_*(2S+1) \sum_{u=0}^{\min(kL,T)} \binom{T-u+r-1}{r-1}. \tag{10.182}$$

Proof. The first inequality in (10.180) is the retained-class count; the upper bound follows because the retained class is a subset of the chosen $m+1$ points. Since $\lfloor V/\Delta \rfloor + 1 > V/\Delta$, and a ceiling is at least its argument, the strict inequality follows.

At additive order $u \leq \min(kL, T)$, allowing all $r-1$ Euler directions gives $\binom{T-u+r-1}{r-1}$ indices of order at most $T-u$, by the divider count proved in Theorem 10.32. Every representative index in (10.126) is among these; zero Euler directions only remove duplicates. Hence $M_* \leq \overline{M}$. Combining this fact with (10.180) proves the strict surplus.

Pick any reference λ_0 in the retained class. Its normalized phase differences are multiples of G_0, and its original coordinates lie in $C - y_0 - B\lambda_0$. Thus all the hypotheses for the initial coefficient

construction and the full-width estimate are retained. Apply Theorem 10.63 and, when (10.175) holds, Corollary 10.65. Substitution of $\Delta = 1/J$ and $V = \mathcal{D}^r/[(c_1c_2rPd)^r \prod_j \sigma_j]$ gives (10.182). □

The count certifies the initial coefficient construction. The analytic slope, derivative precision and strict nonzero-value comparisons used to extend its zeros remain separate hypotheses.

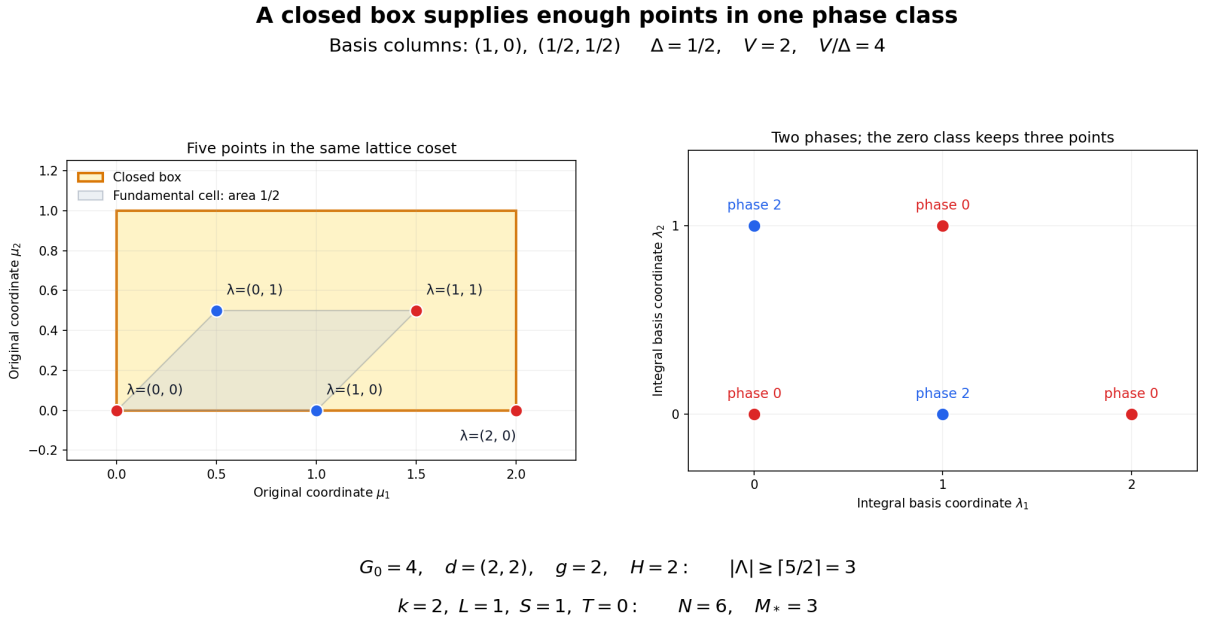


Figure 10.13. The exact lattice has basis columns $(1,0)$, $(1/2,1/2)$, *covolume* $1/2$ *and index two over* $\mathbb{Z}^2$. *The closed box is* $C = [0,2] \times [0,1]$, *so* $V/\Delta = 4$. *Five displayed points have integral basis coordinates* $(0,0), (0,1), (1,0), (1,1), (2,0)$. *For* $G_0 = 4, d = (2,2)$, *there are* $H = 2$ *phase classes. The class* 0 *retains three points with coordinates* $(0,0), (1,1), (2,0)$. *At* $k = 2, L = 1, S = 1, T = 0$, *they give* $N = 6$ *columns and* $M_* = 3$ *rows. The lattice, box, points and phase labels are exact. Theorem 10.66, Lemma 10.67 and Corollary 10.68 prove (10.177)–(10.182), including the integer-volume boundary case; Solution 33 verifies these data and the derivative-order limitation. Human-source context: the box and phase selection (4.18)–(4.21) in Yu's free 2013 paper, cited below. Figure program.*

26. Choosing parameters with a strict coefficient surplus

The lattice count becomes useful only when it exceeds the number of prescribed jets. We give a parameter choice for which that comparison holds with all integer roundings retained. The node and jet budgets are real numbers; their floors, rather than the budgets themselves, count the equations.

The real parameters

Let $r \geq 2$, and retain the independent $\mathfrak{p}$-adic units $\alpha_1, \ldots, \alpha_r$ in a number field of degree d. Write $e = e_\mathfrak{p}$, $f = f_\mathfrak{p}$, and $t = f \ln p$; thus $ef \leq d$. Let $q = 2$ for odd p, and $q = 3$ for $p = 2$, with $\zeta_3 \in K$ in the latter case. Let q^u be the order of the q-primary roots of unity, and $J = q^\nu$ the index of the q-primary saturation lattice. The roots reduce injectively: a nontrivial q-primary root in $1 + \mathfrak{p}$ would retain its positive finite valuation under its prime-to-p order by p-adic logarithms: power series, heights and the one-logarithm bound, Lemma 9.5, but its power is one. Their reductions form a subgroup of the finite field's group of $p^f - 1$ nonzero elements. Its distinct cosets partition that group into sets of size q^u, proving $q^u \mid p^f - 1$. Use its phase coefficients from Section 20, and put

$$G = p^f - 1, \qquad G_0 = G/q^u, \qquad \delta = \gcd(G_0, d_1, \dots, d_r), \quad H = G_0/\delta,$$
$$\ell_d = \max\{1, \ln d\}, \qquad \rho = \begin{cases} 58, & d \geq 2, \\ 17, & d = 1, \end{cases} \quad g_1 = 4 + \ln((r+1)d), \qquad A = \max\{g_1, e, t\}. \tag{10.183}$$

Take $\sigma_i \geq h(\alpha_i) > 0$, constants $c_0 > 1$ and $c_1, c_3, c_4 > 0$, and $g_0 > 39$. Choose $h \geq \max\{g_0, (r+1)t\}$. In this subsection h is this real auxiliary parameter, whereas $h(\alpha)$ denotes Weil height. Let $\kappa \geq 0$ be the integer with $p^{\kappa-1}(p-1) \leq 2e < p^{\kappa}(p-1)$, and set $P = p^{\kappa}$. For any $\tau \geq 0$ define

$$\vartheta = \begin{cases} (p-2)/(p-1), & p \geq 5, e = 1, \\ P/(2e), & \text{otherwise}, \end{cases} \qquad \theta = \vartheta/(1+\tau),$$
$$c_2 = \begin{cases} 7/4, & p > 2, \\ 13/9, & p = 2, \end{cases} \qquad a_* = \begin{cases} 7/2, & p \geq 5, e = 1, \\ 26/3, & p = 2, \\ 7, & \text{otherwise}. \end{cases} \tag{10.184}$$

Choose any $\widehat{\vartheta} \geq \vartheta$. The explicit choices $3/2$ at $p = 3$, $5/4$ at $p = 5, e \geq 2$, 1 at $p \geq 5, e = 1$, $7/6$ at $p \geq 7, e \geq 2$, and 2 at $p = 2$ work. Indeed $P \leq 2pe/(p-1)$, so $P/(2e) \leq p/(p-1)$. If $p \geq 5, e = 1$, then $\kappa = 0$, $P = 1$, and $\vartheta < 1$. In particular $c_2 qP/(e\theta) \geq a_*$ in every case.

Define the positive lower budgets

$$g_2 = \begin{cases} c_3 q(r+1)^2 d, & p \geq 5, \ e = 1, \text{ or } p \geq 7, \\ c_3 q(r+1) g_0 e / \ln p, & p = 2, 3, \text{ or } p = 5, \ e \geq 2, \end{cases}$$
$$g_3 = \frac{2c_0 c_1 c_4}{\rho} a_*^r \frac{r^r (r+1)^r}{(r!)^2} g_1 t,$$
$$g_4 = \frac{q(r+1)g_3}{c_1 \widehat{\vartheta} t} \begin{cases} 1, & e = 1 \text{ and } (p \geq 5 \text{ or } d = 1), \\ g_1^{-1}, & \text{otherwise}, \end{cases} \tag{10.185}$$
$$g_5 = \frac{c_3 q(r+1) g_3}{c_1 c_4 g_1 t}, \qquad 1 + \epsilon = \left(1 + \frac{r+1}{2g_4}\right)^r.$$

Here the $d = 1$ alternative in g_4 is compatible with $p = 3, e = 1$; $p = 2, d = 1$ cannot meet $\zeta_3 \in K$. Set

$$
\begin{aligned}
x &= \nu \ln q,\\
\gamma &= \frac{q^{\nu} hA}{(h+x)(A+x)},\\
S &= \frac{c_3 q(r+1)d(h+x)}{t},\\
\mathcal{D} &= \frac{\gamma}{q^{\nu+u}}(1+\epsilon)(2+g_2^{-1})c_0c_1c_4\left(\frac{c_2qPr(r+1)}{e\theta}\right)^r \frac{1}{r!}\\
&\quad \times \max\left\{\frac{p^f}{\delta t^r}, \frac{\mathrm{e}^r}{r^r}t\right\} d^{r+1}\ell_d \prod_i \sigma_i\,(A+x),\\
T &= \frac{q(r+1)\mathcal{D}}{c_1\theta e t},\\
k &= \lfloor h+x-1\rfloor + 1 = \lfloor h+x\rfloor,\\
\widetilde{D}_0 &= \frac{S\mathcal{D}}{c_1c_4kd(A+x)},\\
L &= \lfloor \widetilde{D}_0\rfloor + 1,\\
D_i &= \frac{\mathcal{D}}{c_1c_2rPd\sigma_i}.
\end{aligned}
\tag{10.186}
$$

The exponential constant $\mathrm{e} = \exp(1)$ in this formula is different from the ramification index e. The additive factors of the prepared family have precisely k choices of degree index and L choices of positive power, as in Lemma 10.29.

Lemma 10.69 (the saturation factor). The parameters above satisfy $1 \le \gamma \le q^{\nu}$.

Proof. With $x \ge 0$, $\gamma = \exp(x)hA/((h+x)(A+x))$. The logarithmic derivative with respect to x is $1-(h+x)^{-1}-(A+x)^{-1} > 0$, because $h > 39$ and $A \ge g_1 > 5$. At $x = 0$ the value is one. The upper bound follows from $(h+x)(A+x) \ge hA$. $\square$

Lower bounds and the rounded equation count

Lemma 10.70 (lower budgets without lost floors). We have

$$
\begin{gathered}
S \ge g_2, \qquad \frac{\mathcal{D}}{A+x} \ge \frac{g_3}{g_1}, \qquad \mathcal{D} \ge g_3, \qquad T \ge g_4,\\
\widetilde{D}_0 \ge g_5, \qquad \widetilde{D}_0 < L \le (1+g_5^{-1})\widetilde{D}_0,\\
\binom{\lfloor T\rfloor + r}{r} \le (1+\epsilon)\frac{T^r}{r!}.
\end{gathered}
\tag{10.187}
$$

Proof. In the first alternative for g_2, use $h+x \ge (r+1)t$ in S. In the second, use $h+x \ge g_0$ and $d/f \ge e$. These give both stated lower bounds, including $p = 5, e \ge 2$.

The q-primary lattice lies in the full saturation lattice of Theorem 10.18. Its index is q^{ν}, and $w_K \ge q^u$. Consequently (10.55) gives

$$
d^{r+1}\ell_d \prod_i \sigma_i \ge \frac{q^{\nu+u}}{\rho}\frac{r^r}{r!\mathrm{e}^r}.
\tag{10.188}
$$

For $d \ge 2$, replacing $\ln d$ by the larger ℓ_d preserves this bound; at $d = 1$ the degree and logarithmic factors equal one. The maximum in (10.186) is at least $(\mathrm{e}^r/r^r)t$. Combine this fact with (10.188), $\gamma \ge 1$, $(1+\epsilon)(2+g_2^{-1}) \ge 2$, and $c_2qP/(e\theta) \ge a_*$. This yields $\mathcal{D}/(A+x) \ge g_3/g_1$, including every displayed constant. Since $A+x \ge g_1$, it also yields $\mathcal{D} \ge g_3$.

If the first alternative in g_4 applies, $e = 1$; use $\mathcal{D} \ge g_3$ and $\theta \le \widehat{\vartheta}$. Otherwise use the stronger bound $\mathcal{D} \ge g_3(A + x)/g_1$ and $A + x \ge e$. Both give $T \ge g_4$.

We have $0 < k \le h + x$. Substitute the formula for S into $\widetilde{D}_0$, and then use the stronger bound for $\mathcal{D}$:

$$\widetilde{D}_0 = \frac{c_3 q(r+1)(h+x)\mathcal{D}}{c_1 c_4 k t (A+x)} \ge \frac{c_3 q(r+1) g_3}{c_1 c_4 g_1 t} = g_5.$$

For any positive real y, $y < \lfloor y \rfloor + 1 \le y + 1$. Taking $y = \widetilde{D}_0$ proves both assertions about L, including the strict inequality when y is an integer.

Finally,

$$\binom{\lfloor T \rfloor + r}{r} \le \frac{1}{r!} \prod_{j=1}^{r} (T + j) \le \frac{1}{r!} \left(T + \frac{r+1}{2} \right)^r .$$

The last inequality follows from the concavity of ln: its second derivative is $-z^{-2} < 0$, so the sum of the logarithms is at most r times the logarithm of their average. Divide by T^r, and use $T \ge g_4$ and (10.185). This proves the last line of (10.187). □

Theorem 10.71 (the parameter choice supplies a kernel). Use the closed box with widths D_i and choose its single phase class as in Lemma 10.67. In the prepared system impose every node $|s| \le S$ with integer s, and every jet of total order at most the real number T. Write $s_0 = \lfloor S \rfloor$, $t_0 = \lfloor T \rfloor$, let $N = kL|\Lambda|$, and let M_* be the actual representative row count. Put

$$c_{01} = c_0 \ell_d \frac{p^f}{p^f - 1} > 1. \tag{10.189}$$

Then

$$\begin{aligned} N &> \frac{kLq^\nu D_1 \cdots D_r}{H} > c_{01}(2S+1)\binom{t_0 + r}{r} \\ &\ge c_{01}(2s_0+1)\binom{t_0 + r}{r} \ge c_{01} M_*. \end{aligned} \tag{10.190}$$

In particular a nonzero global-integral coefficient vector and polynomial with all these zeros and jets exist, with the field and height conclusions of Lemma 10.63.

Proof. The first strict inequality is (10.180), with $J = q^\nu$, $\Delta = 1/J$, and $H = G/(q^u \delta)$. For the next comparison keep all widths in (10.186), and write $V = \prod_i D_i$. Since $L > \widetilde{D}_0$,

$$\frac{kLq^\nu V}{H} > \frac{Sq^{\nu+u}\delta \mathcal{D}^{r+1}}{c_1^{r+1} c_4 c_2^r r^r P^r d^{r+1} (A+x) G \prod_i \sigma_i}.$$

Divide by $(2S + 1)(1 + \epsilon)T^r/r!$. Substitute (10.186) for $\mathcal{D}$. The degree factors, widths and normalization factors cancel, leaving

$$\frac{S(2 + g_2^{-1})}{2S+1} \frac{\gamma c_0 \ell_d \delta t^r}{G} \max\left\{ \frac{p^f}{\delta t^r}, \frac{\mathrm{e}^r}{r^r} t \right\} \ge c_{01}.$$

Indeed $S \ge g_2$ makes the first factor at least one, $\gamma \ge 1$, and the first member of the maximum gives p^f. The binomial inequality (10.187) proves the second strict inequality in (10.190). A derivative order is an integer, so an order at most T is exactly an order at most t_0; similarly there are exactly $2s_0 + 1$ nodes. Counting all r derivative directions gives $\binom{t_0+r}{r}$ rows per node by the divider argument of Theorem 10.32. Removing zero additive orders or duplicate Euler directions only decreases the representative count. This proves the remaining inequalities.

Since $N > M_*$, the completely proved integral Siegel argument in Lemma 10.63 applies over the actual row field, with no unproved estimate used in place of an equation. Its nonzero vector gives a nonzero polynomial by Lemma 10.29. □

The height term with a real node budget

Corollary 10.72 (the rounded mean torus contribution). In this kernel let d_0 be the row-field degree, let $\mathcal{H}_0$ be the scalar row bound from Lemma 10.63 with node budget s_0 and jet budget t_0, and put $Z = S\mathcal{D}/d$ and $L_\alpha = \sum_i D_i h(\alpha_i)$. Then

$$\begin{aligned}\frac{h_2(\boldsymbol{c})}{Z} \le{}& \frac{\frac{1}{2}\ln(Nd_0)}{Z} + \frac{d_0 - 1}{c_1 c_2 S} \\ &+ \frac{1}{c_{01} - 1}\left[\frac{\frac{1}{2}\ln N + \ln \mathcal{H}_0}{Z} + \frac{1 + 1/(2g_2)}{2c_1 c_2}\right].\end{aligned} \tag{10.191}$$

Proof. Apply (10.174) at the integer budget s_0, and use $N > c_{01} M_*$ from (10.190). The full original width bound and $\sigma_i \ge h(\alpha_i)$ give $PL_\alpha \le \mathcal{D}/(c_1 c_2 d)$, so the generator term divided by Z is at most $(d_0 - 1)/(c_1 c_2 S)$. The exact mean of the absolute integer nodes is $s_0(s_0 + 1)/(2s_0 + 1)$. It satisfies

$$\frac{s_0(s_0+1)}{2s_0+1} = \frac{s_0}{2} + \frac{1}{4} - \frac{1}{4(2s_0+1)} \le \frac{S}{2} + \frac{1}{4}.$$

Thus its contribution divided by Z is at most $(1 + 1/(2S))/(2c_1 c_2) \le (1 + 1/(2g_2))/(2c_1 c_2)$. Substitution proves (10.191). This leaves the scalar row and row-field generator costs explicit; each must still be included in a numerical zero-forcing comparison. □

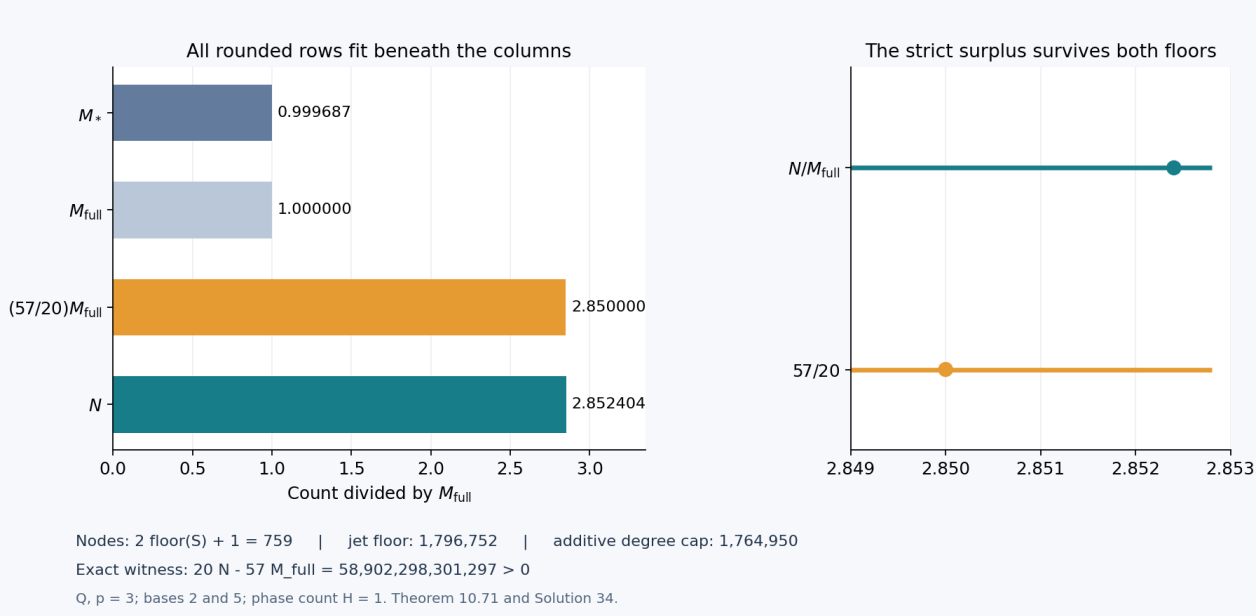


Figure 10.14. Take $K = \mathbb{Q}$, $p = 3$, $(\alpha_1, \alpha_2) = (2, 5)$, and the parameters in Solution 34. The prepared column count and every plotted comparison are exact integer or rational quantities. There are $2\lfloor S \rfloor + 1$ nodes and $\binom{\lfloor T \rfloor + 2}{2}$ possible jets per node. The surplus $c_{01} = 57/20$ is retained through both floors; the strict comparison follows from the full real parameters, without rounding either budget upwards. The phase count is one, so no support is lost at that step. See (10.187)–(10.191), Solution 34, and Yu's free paper for further reading.

27. The global size and scalar cost of the initial kernel

We now bound the number of coefficients and the size of each scalar row entry at the parameters of Section 26. Two factors need separate care. A prime-to-p root field is unramified, which strengthens the global degree bound by the ramification index. The additive prepared derivative also contains an lcm clearing factor; that factor contributes to its ordinary absolute value as well as to its integrality.

The degree available above an unramified root field

Lemma 10.73 (the torsion degree divided by ramification). With the notation of (10.183), we have

$$\frac{d}{e} \geq q^{u-1}(q-1). \tag{10.192}$$

Proof. Put $F = \mathbb{Q}(\zeta_{q^u}) \subseteq K$. The cyclotomic Eisenstein proof in p-adic logarithms: power series, heights and the one-logarithm bound, Proposition 9.7 at the prime q, gives $[F : \mathbb{Q}] = q^{u-1}(q-1)$: irreducibility over $\mathbb{Q}_q$ implies irreducibility over $\mathbb{Q}$. Since $q \neq p$, the completion $F_{\mathfrak{q}}$ at the prime below $\mathfrak{p}$ is unramified over $\mathbb{Q}_p$, by the complete prime-to-p proof in Cyclotomic, quadratic and Kummer extensions of local fields, Proposition 1.1.

Let $m = [K : F]$, and choose an F-basis $b_1, \dots, b_m$ of K. Inside $K_{\mathfrak{p}}$, the subspace $\sum_j F_{\mathfrak{q}} b_j$ has dimension at most m over the complete field $F_{\mathfrak{q}}$. It is complete, hence closed, by the finite-dimensional norm proof in Extensions of complete valued fields, Proposition 2.1. It contains K, whose closure is $K_{\mathfrak{p}}$, so it equals that completion. Therefore $[K_{\mathfrak{p}} : F_{\mathfrak{q}}] \leq m$.

Ramification indices multiply in a tower: restriction of a normalized integer valuation multiplies it by the ramification index at each step. The base root field has ramification index one, so $e = e(K_{\mathfrak{p}}/F_{\mathfrak{q}})$. The degree formula $ef = [L : K]$, fully proved by residue digits in Extensions of complete valued fields, Theorem 3.1, gives $e \leq [K_{\mathfrak{p}} : F_{\mathfrak{q}}] \leq m$. Therefore $d/e = [F : \mathbb{Q}]m/e \geq [F : \mathbb{Q}]$, as asserted. □

A lower volume and an upper coefficient count

Retain all real parameters in (10.183)–(10.186). Set

$$\begin{aligned} b &= \begin{cases} 1, & e = 1 \text{ and } (p \geq 5 \text{ or } d = 1), \\ g_1^{-1}, & \text{otherwise}, \end{cases} \\ g_{61} &= 2c_0 c_1^{1-r} c_4 \left(\frac{q}{\widehat{\vartheta}}\right)^r \frac{(r+1)^r \mathrm{e}^r}{r! r^r}\, t\, \frac{q-1}{q}\, g_1\, (b g_3)^{r-1}, \\ v &= q^{\nu} D_1 \cdots D_r. \end{aligned} \tag{10.193}$$

Lemma 10.74 (a uniform lower box volume). We have

$$v \geq g_{61} > 0. \tag{10.194}$$

Proof. In the maximum defining $\mathcal{D}$, use the second member $(\mathrm{e}^r/r^r)t$. Substituting one factor of $\mathcal{D}$ into v, and using $\gamma \geq 1$, $1 + \epsilon \geq 1$, and $2 + g_2^{-1} \geq 2$, gives

$$v \geq \frac{2c_0 c_1^{1-r} c_4}{q^u} \left(\frac{q}{\theta}\right)^r \frac{(r+1)^r \mathrm{e}^r}{r! r^r}\, t\, \frac{d}{e^r} \ell_d (A + x) \mathcal{D}^{r-1}.$$

Lemma 10.73 gives $d/q^u \geq e(q-1)/q$; also $\ell_d \geq 1$ and $\theta \leq \widehat{\vartheta}$. The remaining factor is $(A+x)(\mathcal{D}/e)^{r-1}$. In the first case for b, $e = 1$ and $\mathcal{D} \geq g_3$. In the other case, the stronger bound in (10.187) and $A + x \geq e$ give $\mathcal{D}/e \geq g_3/g_1$. Since $A + x \geq g_1$, both cases imply $(A + x)(\mathcal{D}/e)^{r-1} \geq g_1 (b g_3)^{r-1}$. This proves (10.194). □

Let w_K denote the full root-of-unity order, and put

$$
\begin{aligned}
g_6 &= \rho(1+g_5^{-1})(1+g_{61}^{-1})\frac{r!\mathrm{e}^r}{c_1^{r+1}c_4c_2^rP^rw_Kr^{2r}},\\
g_7 &= \frac{c_3q(r+1)g_0g_3}{t},\\
g_8 &= \frac{\ln g_7+g_1+\max\{\ln(g_6d/\mathrm{e}^{g_1}),0\}}{g_7}+\frac{r}{c_3q(r+1)^2}\frac{\ln g_3}{g_3}.
\end{aligned}
\tag{10.195}
$$

Theorem 10.75 (the logarithmic coefficient budget). Suppose $g_3 \ge \mathrm{e}$ and $g_7 \ge \mathrm{e}$. For the selected support of Theorem 10.71, $N = kL|\Lambda|$ and $Z = S\mathcal{D}/d$ satisfy

$$
N \le g_6 S\mathcal{D}^{r+1}, \qquad Z \ge g_7, \qquad \ln N \le g_8 Z. \tag{10.196}
$$

Proof. By (10.180), $|\Lambda| \le v+1$. Also

$$
kL \le (1+g_5^{-1})\frac{S\mathcal{D}}{c_1c_4d(A+x)}.
$$

Use $v \ge g_{61}$ to replace $v+1$ by at most $(1+g_{61}^{-1})v$. The full saturation bound (10.55) gives

$$
\frac{q^\nu}{d^{r+1}\prod_i\sigma_i} \le \frac{\rho\ell_d r!\mathrm{e}^r}{w_Kr^r}.
$$

For $d \ge 2$, use the same replacement of $\ln d$ by ℓ_d as in (10.188); for $d = 1$ the formula is exact. Substituting the widths D_i and $\ell_d \le g_1 \le A+x$ gives the first inequality of (10.196), with exactly g_6.

The formula for S, together with $h+x \ge g_0$ and $\mathcal{D} \ge g_3$, gives $Z \ge g_7$. Taking logarithms in the first inequality yields

$$
\ln N \le \ln(g_6d) + \ln Z + r\ln\mathcal{D}.
$$

Put $C = g_1 + \max\{\ln(g_6d/\mathrm{e}^{g_1}),0\} > 0$; then $\ln(g_6d) \le C$, so $C/Z \le C/g_7$. The function $\ln y/y$ decreases for $y \ge \mathrm{e}$, because its derivative is $(1-\ln y)/y^2$. Thus $\ln Z/Z \le \ln g_7/g_7$.

Finally $h+x \ge (r+1)t$ gives $S/d \ge c_3q(r+1)^2$, and therefore

$$
\frac{r\ln\mathcal{D}}{Z} \le \frac{r}{c_3q(r+1)^2}\frac{\ln\mathcal{D}}{\mathcal{D}} \le \frac{r}{c_3q(r+1)^2}\frac{\ln g_3}{g_3}.
$$

Combining these three bounds proves $\ln N \le g_8Z$. □

The lower hypotheses here have simple sufficient conditions. If $c_0 \ge 19/10, c_1 \ge 7/5, c_3 \ge 47/100, c_4 \ge 15/4$, then $g_3/t > 21$. Indeed $a_* \ge 7/2$, $\rho \le 58$, $g_1 > 5$, and $r^r(r+1)^r/(r!)^2 \ge 1$; hence

$$
\frac{g_3}{t} > \frac{2}{58}\frac{19}{10}\frac{7}{5}\frac{15}{4}\left(\frac{7}{2}\right)^2 5 > 21.
$$

The positive logarithm series in Solution 34 gives $\ln 2 > 2/3$, so $t \ge \ln 2 > 2/3$, and $g_3 > 14 > \mathrm{e}$. Also $g_7 > (47/100)\cdot 2\cdot 3\cdot 39\cdot 21 > 3 > \mathrm{e}$, where $\mathrm{e} < 3$ follows by summing its positive series and bounding the tail geometrically.

Retaining the clearing factor and the jet allocation

Write $s_0 = \lfloor S\rfloor, t_0 = \lfloor T\rfloor, D_{\mathrm{a}} = kL$, and $\mathfrak{v} = v(k) = \operatorname{lcm}(1,\dots,k)$. For the $r-1$ integer Euler forms of the actual prepared system, let Ω_j be the maximum of their absolute arguments on the support, and put

$$
m = \sum_{j=1}^{r-1}\Omega_j + r - 2, \qquad U = \min(D_{\mathrm{a}}, t_0). \tag{10.197}
$$

Lemma 10.76 (a complete scalar row bound). The scalar bound in Lemma 10.63 may be taken to be

$$\mathcal{H}_0 = \left[2\mathrm{e}\left(\frac{s_0}{k}+2\right)\right]^{D_{\mathrm{a}}} \max_{0\le u\le U}\left\{\mathfrak{v}^u\binom{m+t_0-u}{t_0-u}\right\}. \tag{10.198}$$

If $\mathfrak{v}=1$, a maximizing index is $u_*=0$. If $\mathfrak{v}>1$, one is

$$u_* = \min\left\{U,\ \max\left\{0, t_0-\left\lceil\frac{m}{\mathfrak{v}-1}\right\rceil+1\right\}\right\}. \tag{10.199}$$

More sharply, with $R=s_0+2k$, the scalar bound may be taken to be

$$\mathcal{H}_{\mathrm{fine}} = \frac{R^{D_{\mathrm{a}}}}{(k!)^L}\max_{0\le u\le U}\left\{\left(\frac{\mathfrak{v}}{R}\right)^u\binom{D_{\mathrm{a}}}{u}\binom{m+t_0-u}{t_0-u}\right\}. \tag{10.200}$$

A maximizing index $\widehat{u}$ is the first index after all the consecutive increasing steps. A step from u to $u+1$, for $u<U$, is increasing or constant precisely when

$$\mathfrak{v}(D_{\mathrm{a}}-u)(t_0-u) \ge R(u+1)(m+t_0-u). \tag{10.201}$$

These steps form an initial interval, so an integer binary search finds a maximizing index, with no numerical root approximation.

Proof. For $0\le a<k$, $1\le \ell\le L$, write

$$F_{a,\ell}(X) = \frac{1}{(k!)^\ell}\prod_{j=1}^{k}(X+a+j)^\ell.$$

It has degree $k\ell$, and every shift $a+j$ is at most $2k$. In the divided derivative of order u, choose the u differentiated linear factors. There are $\binom{k\ell}{u}$ choices. For $|s|\le s_0$, each remaining factor has absolute value at most s_0+2k. Thus

$$\left|\frac{F_{a,\ell}^{(u)}(s)}{u!}\right| \le \frac{\binom{k\ell}{u}(s_0+2k)^{k\ell-u}}{(k!)^\ell} \le \left[2\mathrm{e}\left(\frac{s_0}{k}+2\right)\right]^{D_{\mathrm{a}}}.$$

For the last inequality use $\binom{k\ell}{u}\le 2^{k\ell}$, the positive factor $s_0+2k\ge 1$, and $\ln(k!)\ge k\ln k-k$. The latter follows by comparing $\sum_{j=1}^{k}\ln j$ with the integral of the increasing function $\ln x$ on $[1,k]$. The resulting base is greater than one, and $k\ell\le D_{\mathrm{a}}$. If $u>k\ell$, the derivative is zero.

The prepared derivative multiplies this divided derivative by $\mathfrak{v}^u$, by (10.86); discarding that multiplier would give the wrong ordinary absolute-value bound. Each Euler factor satisfies

$$|\Delta(\omega_j;t_j)| \le \binom{\Omega_j+t_j}{t_j}.$$

Multiplying the positive generating series for these binomial coefficients, as in (10.124), shows that their product at total Euler order h is at most $\binom{m+h}{h}$. This increases with h, so for additive order u it is at most $\binom{m+t_0-u}{t_0-u}$. This proves (10.198).

For $F(u)=\mathfrak{v}^u\binom{m+t_0-u}{t_0-u}$, whenever $u<t_0$,

$$\frac{F(u+1)}{F(u)} = \mathfrak{v}\frac{t_0-u}{m+t_0-u}.$$

At $\mathfrak{v}=1$ this is at most one. Otherwise it is at least one exactly when $(\mathfrak{v}-1)(t_0-u)\ge m$. These ratios decrease as u increases. Hence F increases up to this integer threshold and decreases

afterwards; equality may give two adjacent maximizers. Clamping the last increasing step to $[0, U]$ gives (10.199), choosing the larger maximizer in the equality case.

For (10.200), retain the first divided-derivative estimate, rather than replacing its binomial coefficient by $2^{k\ell}$. Since $\binom{k\ell}{u} \le \binom{D_{\mathrm{a}}}{u}$ and $R^k/k! > 1$, it gives

$$\left|\frac{F_{a,\ell}^{(u)}(s)}{u!}\right| \le \binom{D_{\mathrm{a}}}{u} R^{-u} \left(\frac{R^k}{k!}\right)^{\ell} \le \frac{\binom{D_{\mathrm{a}}}{u} R^{D_{\mathrm{a}}-u}}{(k!)^L}.$$

Multiply by the clearing and Euler factors already proved. The ratio of the resulting expression at $u+1$ to that at u is

$$\frac{\mathfrak{v}}{R}\frac{D_{\mathrm{a}}-u}{u+1}\frac{t_0-u}{m+t_0-u}.$$

Every factor depending on u decreases on $0 \le u < U$; the first is strictly decreasing. Thus the ratios cross one at most once. Clearing their positive denominators gives exactly (10.201), which proves the asserted integer rule and the sharper bound. □

Corollary 10.77 (the normalized additive cost). If $c_3 q/t \le 8$, the first factor in (10.198) contributes at most

$$\frac{D_{\mathrm{a}}}{Z}\ln\left[2\mathrm{e}\left(\frac{s_0}{k}+2\right)\right] \le \frac{1+g_5^{-1}}{c_1 c_4}. \tag{10.202}$$

Proof. Since $h+x > 39$, $k = \lfloor h+x \rfloor \ge 39$ and $(h+x)/k < (k+1)/k \le 40/39$. Put $W = (r+1)d \ge 3$. Then

$$\frac{s_0}{k}+2 \le \frac{S}{k}+2 < \frac{320}{39}W+2 \le \frac{346}{39}W.$$

Since $\mathrm{e}^3 > 18 > 692/39$, it follows that $2\mathrm{e}(s_0/k+2) < \mathrm{e}^4 W$, and its logarithm is at most g_1. The inequality $\mathrm{e}^3 > 18$ follows already from the terms of degrees zero through five in its exponential series at three. Also $D_{\mathrm{a}}/Z \le (1+g_5^{-1})/[c_1 c_4(A+x)]$, by (10.187), and $g_1 \le A+x$. These prove (10.202). For odd p, the hypothesis holds whenever $c_3 \le 4$, since $\ln p \ge \ln 3 > 1$. At $p=2$, it holds whenever $c_3 \le 1/2$, since $\ln 2 > 2/3$. □

Theorem 10.78 (the initial height with all scalar costs). Assume the hypotheses of Theorems 10.71 and 10.75 and Corollary 10.77. Set $\Psi = u_* \ln \mathfrak{v} + \ln \binom{m+t_0-u_*}{t_0-u_*}$. Then the integral initial kernel can be chosen with

$$\begin{aligned}\frac{h_2(\boldsymbol{c})}{Z} \le{}& \frac{1}{2}g_8 + \frac{\ln d_0}{2g_7} + \frac{d_0-1}{c_1 c_2 S} \\ &+ \frac{1}{c_{01}-1}\left[\frac{1}{2}g_8 + \frac{1+g_5^{-1}}{c_1 c_4} + \frac{\Psi}{Z} + \frac{1+1/(2g_2)}{2c_1 c_2}\right].\end{aligned} \tag{10.203}$$

Proof. In (10.191), use $\ln N/Z \le g_8$, $Z \ge g_7$, and the exact scalar bound (10.198) at (10.199). Corollary 10.77 bounds its additive first factor; its remaining factor has logarithm exactly Ψ. Substitution proves every term of (10.203). The row-field term may further be bounded by $1/[c_1 c_2 c_3 q(r+1)^2]$, since $d_0 \le d$ and $S/d \ge c_3 q(r+1)^2$, but its exact vanishing at $d_0 = 1$ is preferable there.

More sharply, replace the sum $(1+g_5^{-1})/(c_1 c_4) + \Psi/Z$ in (10.203) by $\ln \mathcal{H}_{\mathrm{fine}}/Z$, evaluated at the exact integer maximizer $\widehat{u}$ in (10.201). This follows from the same substitution in (10.191), using (10.200).

This is a height bound for coefficients whose existence has already been proved. An extension of their zeros still requires the stated analytic slopes, derivative precision, arithmetic strict comparisons and final multiplicity contradiction. In particular the lcm and Euler term Ψ must be retained in each such comparison. □

Averaging the derivative orders

Corollary 10.79 (a mean scalar budget). Retain every Euler index, allowing duplicate rows, but omit the identically zero additive orders above D_{a}. Define

$$\begin{aligned} J_u &= \binom{t_0-u+r-1}{r-1} \quad (0\le u\le U), \quad \overline{J} = \sum_{u=0}^{U} J_u, \\ \overline{u} &= \frac{\sum_{u=0}^{U} uJ_u}{\overline{J}}, \qquad\qquad \overline{h} = \frac{r-1}{r}(t_0-\overline{u}), \quad \overline{M} = (2s_0+1)\overline{J}. \end{aligned} \tag{10.204}$$

Let $H(z) = -z\ln z - (1-z)\ln(1-z)$, with endpoint values zero, and put

$$\begin{aligned} A_{\mathrm{mean}} &= (D_{\mathrm{a}}-\overline{u})\ln R - L\ln(k!) + D_{\mathrm{a}}H(\overline{u}/D_{\mathrm{a}}), \\ E_{\mathrm{mean}} &= \begin{cases} \overline{h}\ln\!\left[\mathrm{e}(1+m/\overline{h})\right], & \overline{h}>0, \\ 0, & \overline{h}=0, \end{cases} \\ B_{\mathrm{mean}} &= A_{\mathrm{mean}} + \overline{u}\ln\mathfrak{v} + E_{\mathrm{mean}}. \end{aligned} \tag{10.205}$$

The mean logarithm of the scalar row bounds is at most B_{mean}, and an integral initial kernel exists with

$$\begin{aligned} \frac{h_2(\boldsymbol{c})}{Z} \le{}& \frac{\frac12\ln(Nd_0)}{Z} + \frac{d_0-1}{c_1c_2S} \\ &+ \frac{\overline{M}}{N-\overline{M}}\left[\frac{\frac12\ln N + B_{\mathrm{mean}}}{Z} + \frac{s_0(s_0+1)}{(2s_0+1)c_1c_2S}\right]. \end{aligned} \tag{10.206}$$

Moreover $\overline{u} \le t_0/(r+1)$.

Proof. At fixed additive order u, the $r-1$ Euler indices and their slack coordinate are all the nonnegative integer compositions of t_0-u into r parts. Permuting their coordinates is a bijection; the sum of their coordinate averages is t_0-u. Thus their mean total Euler order is $(r-1)(t_0-u)/r$. Weighting by J_u proves the formula for $\overline{h}$.

Without the additive cutoff, all r derivative coordinates and their slack are the compositions of t_0 into $r+1$ parts, so the mean additive order is $t_0/(r+1)$. Removing orders greater than U can only decrease that mean: each removed order exceeds every retained one. This proves the last assertion.

The sharper additive estimate in the proof of Lemma 10.76 has logarithm at most

$$\ln\binom{D_{\mathrm{a}}}{u} + (D_{\mathrm{a}}-u)\ln R - L\ln(k!).$$

For $0<u<D_{\mathrm{a}}$, the binomial theorem at $z=u/D_{\mathrm{a}}$ gives $\binom{D_{\mathrm{a}}}{u}z^u(1-z)^{D_{\mathrm{a}}-u} \le 1$. Hence $\ln\binom{D_{\mathrm{a}}}{u} \le D_{\mathrm{a}}H(u/D_{\mathrm{a}})$, including the endpoints by continuity. The second derivative $H''(z) = -1/z - 1/(1-z) < 0$ proves concavity, so averaging gives A_{mean}.

At total Euler order $h>0$, the factorial integral bound gives $\ln\binom{m+h}{h} \le h\ln[\mathrm{e}(1+m/h)]$. The right side extends continuously by zero at $h=0$. Its second derivative is $-m^2/[h(m+h)^2] \le 0$, so its average is at most E_{mean}; when $m=0$ it is the linear function h. The clearing multiplier has logarithm $u\ln\mathfrak{v}$, whose mean is exactly $\overline{u}\ln\mathfrak{v}$. This proves (10.205).

By (10.190), $N > c_{01}(2s_0+1)\binom{t_0+r}{r} \ge c_{01}\overline{M}$, so the completely proved weighted integral Siegel argument of Lemma 10.63 applies to these rows. Duplicate or further zero rows do not increase rank; assigning them the same positive upper scalar bounds preserves that argument. The row field remains E_0, and the torus-row mean remains that of (10.174). Use (10.173) for its full-width and field-generator terms, and substitute the mean scalar bound into the row sum in (10.171). This is (10.206), with the actual ratio $\overline{M}/(N-\overline{M})$. The nonzero polynomial and every required jet follow as before. □

A second way to clear the global rows

Corollary 10.80 (row scaling and an alternative mean bound). Define

$$B_{\mathrm{proj}} = (D_{\mathrm{a}} - \overline{u}) \ln R + D_{\mathrm{a}} H(\overline{u}/D_{\mathrm{a}}) + E_{\mathrm{mean}}. \tag{10.207}$$

In (10.206), B_{mean} may be replaced by

$$\min\{B_{\mathrm{mean}}, B_{\mathrm{proj}}\}. \tag{10.208}$$

One may also choose the smaller scalar bound separately at each row before averaging.

Proof. At the additive order u, every original prepared row contains the common multiplier $\mathfrak{v}^u$. Replace it by the common multiplier $(k!)^L$. This multiplies that entire row by the nonzero rational number $(k!)^L/\mathfrak{v}^u$, so it preserves its kernel and its coefficient field. Every nonzero row has unchanged projective height; zero rows remain zero. For a nonzero row, the logarithms of the absolute values of that common scalar sum to zero by the normalized product formula proved in Places of number fields in extensions and the product formula, Theorem 5.2.

The divided derivative of $F_{a,\ell}$ has denominator dividing $(k!)^\ell$. To verify this at every order, its polynomial coefficients have that denominator and divided differentiation multiplies a monomial coefficient by an integer binomial coefficient. At an integer node its denominator still divides $(k!)^\ell$, hence divides $(k!)^L$. The Euler binomials are integers at their integer arguments. Thus the replaced scalar entries are integers, as required for the same finite-place row estimate.

The sharper derivative bound in Lemma 10.76 now gives the archimedean scalar bound

$$\binom{D_{\mathrm{a}}}{u} R^{D_{\mathrm{a}}-u} \binom{m + t_0 - u}{t_0 - u}.$$

The factorial denominator cancels against the new common multiplier. Averaging its logarithm with the concavity arguments of Corollary 10.79 gives B_{proj}. Its coefficient matrix has exactly the same kernel as the original prepared matrix, so the identical weighted Siegel and full-width argument proves (10.206) with this bound. Both bounds therefore apply to the same row heights; choosing their smaller value, globally or row by row, is valid. The original lcm normalization remains available for every later local jet-precision computation. □

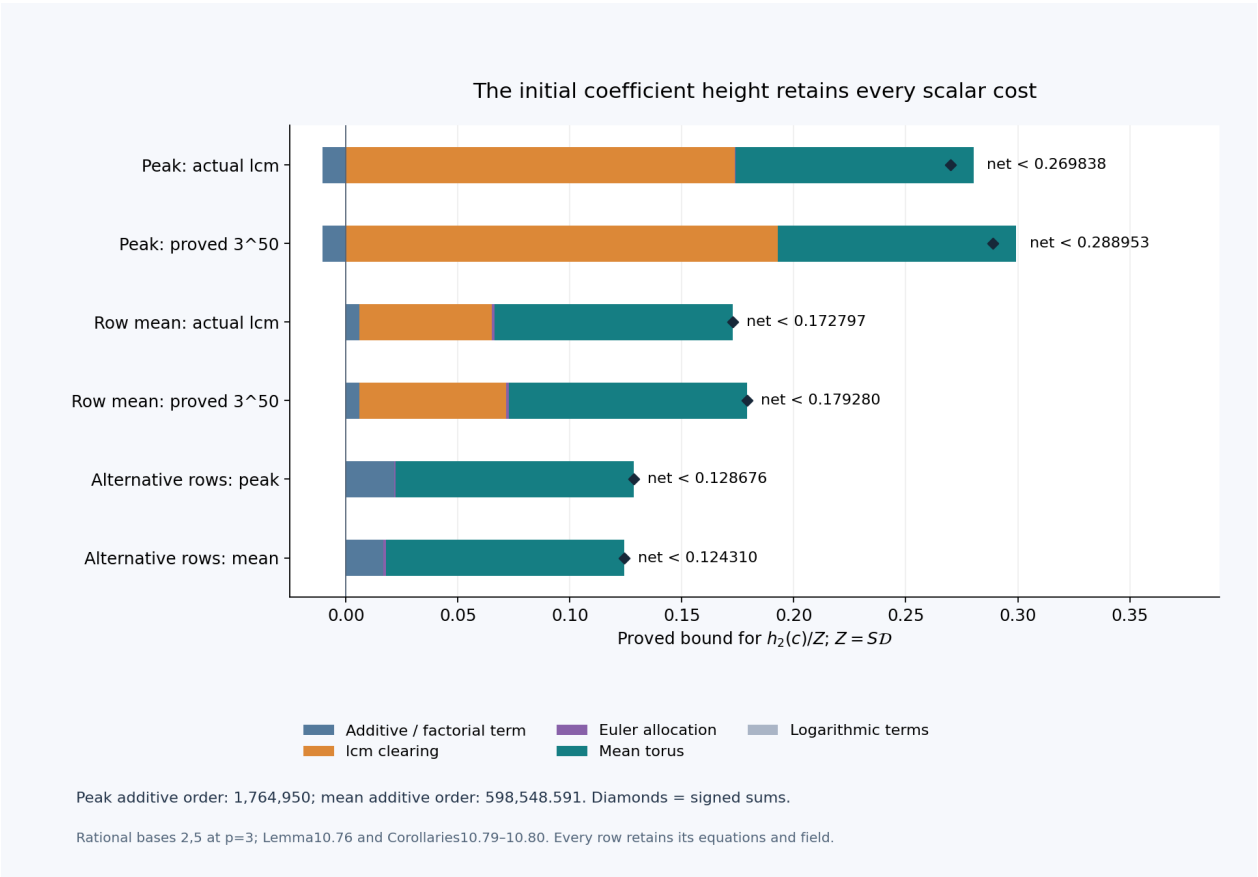


Figure 10.15. The parameters of Solution 34 give the proved coefficient-height bounds in Solution 35. The sharper allocation and row average retain the factorial denominator; lcm clearing, Euler orders, mean torus and logarithmic terms are all included. Left of zero is the factorial contribution to be subtracted when the highest additive order is used. The diamonds mark the net sums. Each pair uses either the actual integer $\operatorname{lcm}(1,\dots,50)$ *or the completely proved coarser* 3^{50} *bound in Two logarithms II: explicit lower bounds. The final pair uses the alternative global row clearing of Corollary 10.80. All bars give proved upper bounds on the height of some initial kernel. See (10.198)–(10.208), Solution 35, and Yu's free paper for further reading.*

28. Principal-unit depth and the analytic torus

The local torus coordinates must define convergent exponential functions on the enlarged disc used for interpolation. We prove the depth with the same power P as in Section 26. The ramification index enters this choice; the small strict reduction from ϑ to θ also has a mathematical purpose.

The power needed for an enlarged disc

Let $L/\mathbb{Q}_p$ be a finite extension with ramification index e, and normalize $v_p(p) = 1$. Write $a_p = 1/(p-1)$, and choose the nonnegative integer κ by

$$p^{\kappa-1}(p-1) \le 2e < p^{\kappa}(p-1), \qquad P = p^{\kappa}, \qquad \vartheta = \begin{cases} (p-2)/(p-1), & p \ge 5, e = 1, \\ P/(2e), & \text{otherwise.} \end{cases} \tag{10.209}$$

This integer exists: take the least nonnegative κ satisfying the strict upper inequality. At $\kappa = 0$, its lower inequality still holds because $(p-1)/p < 1 \le 2e$. Fix $0 < \theta < \vartheta$; in particular the choice $\theta = \vartheta/(1+\varepsilon)$ with $\varepsilon > 0$ is permitted.

Lemma 10.81 (the original principal-unit depth). If $\beta \in L$ and $v_p(\beta - 1) > 0$, then

$$v_p(\beta^P - 1) \ge \vartheta + a_p > \theta + a_p. \tag{10.210}$$

The statement includes $\beta^P = 1$, with valuation infinity.

Proof. The valuation group of L is $e^{-1}\mathbb{Z}$, by the complete degree-and-residue proof in Extensions of complete valued fields, Theorem 3.1. Thus $v_p(\beta - 1) \geq 1/e$. The elementary power step in p-adic logarithms: power series, heights and the one-logarithm bound, Lemma 9.5, gives $c_{j+1} \geq F(c_j)$ for $c_j = v_p(\beta^{p^j} - 1)$, where $F(c) = \min\{pc, c + 1\}$. Below a_p the step is exactly multiplication by p; at or above that threshold the lower bound still holds. The function F is increasing and $F(c) \geq c$ for $c \geq 0$. If any power is one, all subsequent assertions are immediate.

In the exceptional case $p \geq 5, e = 1$, we have $\kappa = 0$, so the initial bound is $1 = \vartheta + a_p$. Consider the other cases. The inequalities defining κ give $a_p < \vartheta \leq pa_p$. If $\kappa = 0$, then $1/e = 2\vartheta > \vartheta + a_p$, which proves the claim.

Suppose $\kappa \geq 1$. We first prove $c_{\kappa-1} \geq P/(pe)$. If none of $c_0, \ldots, c_{\kappa-2}$ reaches a_p, the exact steps below the threshold give $c_{\kappa-1} = p^{\kappa-1}c_0 \geq P/(pe)$; for $\kappa = 1$ this uses no step. Otherwise, after the first such crossing there is at least one further step before index $\kappa-1$. That step gives $c \geq a_p+1 \geq 2a_p$. Subsequent steps cannot decrease the valuation, and the lower defining inequality gives $P/(pe) \leq 2a_p$. This proves the same bound in that case.

Applying the increasing function F once more yields

$$c_\kappa \geq \min\{P/e, 1 + P/(pe)\} = \min\{2\vartheta, 1 + 2\vartheta/p\} \geq \vartheta + a_p. \tag{10.211}$$

Indeed $2\vartheta > \vartheta + a_p$. For the other member, use $(1 - 2/p)\vartheta \leq (p - 2)/(p - 1) = 1 - a_p$. This remains valid at $p = 2$, where both sides of that last inequality are zero. Finally $\theta < \vartheta$ supplies the strict inequality in (10.210). □

Coefficients and divided jets of the torus functions

Corollary 10.82 (normality with the actual depth). Let $w_1, \ldots, w_s \in L$ satisfy $v_p(w_i - 1) \geq \vartheta + a_p$, and let $\lambda_i \in L$ have $v_p(\lambda_i) \geq 0$. Put

$$\ell = \sum_i \lambda_i \log w_i, \qquad F(Z) = \exp(Z\ell) = \sum_{n\geq 0} A_n Z^n. \tag{10.212}$$

Then $F \in \mathcal{A}_{p^\theta}$, its Gauss norm is one, and for $n \geq 1$

$$v_p(A_n) \geq n\vartheta + a_p, \qquad v_p(A_n) - n\theta \geq n(\vartheta - \theta) + a_p. \tag{10.213}$$

If $v_p(z) \geq -\theta$, its divided jets satisfy

$$\frac{F^{(j)}(z)}{j!} = F(z)\frac{\ell^j}{j!}, \qquad v_p\left(\frac{F^{(j)}(z)}{j!}\right) \geq j\vartheta + a_p \quad (j \geq 1). \tag{10.214}$$

If $\ell = 0$, the positive-order coefficients and jets are zero. The constant coefficient and value have valuation zero.

Proof. The logarithm is an isometry at this depth, by Proposition 9.3 of the linked preceding lesson. Thus each logarithm has valuation at least $\vartheta + a_p$, and the integral multipliers and ultrametric inequality give the same bound for ℓ. If $\ell = 0$, the assertions follow from $F = 1$. Otherwise $A_n = \ell^n/n!$. The complete factorial digit formula (9.4) gives $v_p(n!) = (n - s_p(n))a_p \leq (n - 1)a_p$. This proves the first bound in (10.213); subtracting $n\theta$ proves the second. It tends to infinity, so the series belongs to the complete coefficient algebra $\mathcal{A}_{p^\theta}$. The constant coefficient is one and every other weighted coefficient has positive valuation, proving that the Gauss norm is exactly one.

For the asserted nodes, $v_p(z\ell) \geq \vartheta - \theta + a_p > a_p$. The exponential converges there and is a unit, by Proposition 9.3. The coefficient algebra and analytic identities of Lemma 9.2 and Proposition 9.4 justify differentiation and translation on this disc; alternatively the coefficient identity for each divided derivative is verified directly by the convergent exponential series. It gives (10.214), with the same factorial estimate. □

This applies to the integer exponent vectors of every torus monomial, including negative coordinates, and to exponents divided by any power of $q \neq p$. Such scalars are p-adically integral. Finite sums of these exponential series with locally integral coefficients have Gauss norm at most one. An additional additive polynomial still contributes its own coefficient norm; (10.213) does not discard that contribution.

The strict choice of θ cannot generally be removed. If $v_p(\ell) = \vartheta + a_p$, then the exact factorial formula gives $v_p(A_n) - n\vartheta = s_p(n)a_p$. At $n = p^j$ this is the constant a_p. The weighted coefficients therefore fail to tend to zero at radius p^{ϑ}. In this equality case the exponential does not converge at a point of valuation $-\vartheta$. Solution 36 supplies an actual example.

Residue phases and a coherent root choice

Corollary 10.83 (principal roots of the phased coordinates). Let $G = p^f - 1$, let $\zeta \in L$ be a primitive G-th root of unity, and suppose that the residue of the local unit α is the residue of ζ^a. Such residue lifts, including their primitivity, are proved in Lemma 10.53. Put

$$\beta = \alpha\zeta^{-a}, \qquad V = \beta^P = \alpha^P\zeta^{-aP}. \tag{10.215}$$

For a prime $q \neq p$, the element

$$\eta = \exp(q^{-1}\log V) \in L, \qquad \eta^q = V, \qquad v_p(\eta - 1) = v_p(V - 1) \geq \vartheta + a_p \tag{10.216}$$

is the unique q-th root of V with residue one. Fix $\xi \in \mathbb{C}_p$ with $\xi^q = \zeta$. A q-th root ρ of α can be chosen so that

$$\rho^P\xi^{-aP} = \eta. \tag{10.217}$$

The choice may be made separately for every member of a finite family, without changing any of their q-th power identities.

Proof. The residue condition gives $v_p(\beta - 1) > 0$. Lemma 10.81 therefore proves the depth of V. Since q is a local unit, $q^{-1}\log V$ has this same valuation; Proposition 9.4 gives both $\eta^q = V$ and the valuation identity. The series defining η lies in the complete field L. If two such roots had residue one, their ratio would be a q-th root of unity with positive valuation of its difference from one. The prime-to-p power identity in Lemma 9.5 forces that ratio to be one. This proves uniqueness, also when $V = 1$.

Start with any $\rho_0^q = \alpha$ in $\mathbb{C}_p$. Then $w = \rho_0^P\xi^{-aP}$ satisfies $w^q = V$, so η/w is a q-th root of unity. Fix a primitive one ω, and write $\eta/w = \omega^j$. Since $P = p^{\kappa}$ is coprime to q, integer division gives an integer b with $Pb \equiv j \pmod q$. Set $\rho = \rho_0\omega^b$. Its q-th power is still α, and its P-th power supplies the required correction, proving (10.217). No equation for another member of the family is affected. □

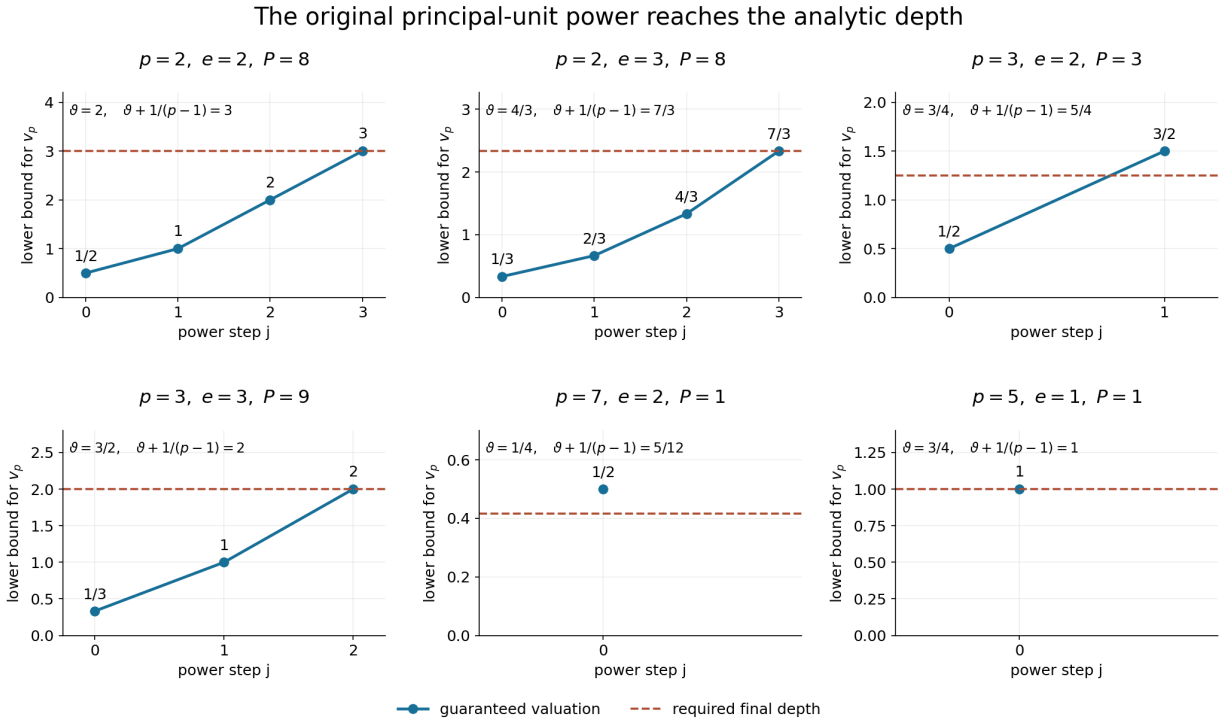


Figure 10.16. Start with the smallest possible positive valuation $1/e$ and iterate the guaranteed step $c \mapsto \min\{pc, c+1\}$. The points show lower bounds, not claims that every step is attained by a unit. The horizontal line is the exact final depth $\vartheta + 1/(p-1)$; its special unramified branch is retained at $p = 5, e = 1$. The zero-step case $p = 7, e = 2$ is included. Lemma 10.81 proves all six bounds and all other ramification cases. Corollary 10.82 then permits every closed radius p^{θ} with $\theta < \vartheta$. See Solution 36, the figure program, and Yu's free paper, equations (2.1), (4.7) and (4.11), for further reading.

29. The uniform clearing and Euler budget

The saturated coordinates explain the integer arguments of the Euler factors. Their full q^{ν} multiplier matters for ordinary absolute values. Averaging the derivative orders then controls that multiplier and the additive lcm cost together.

Euler arguments in a saturated basis

Let $\beta_1, \dots, \beta_n \in K^{\times}$, with $n \geq 2$ and positive heights $h_j = h(\beta_j)$. Let $A_0 = (a_{ij})$ be an r-by-n integer matrix, $2 \leq r \leq n$, and suppose that the original independent bases α_i of Section 26 have the form $\alpha_i = \zeta_i \prod_j \beta_j^{a_{ij}}$, with roots of unity ζ_i, and that

$$\sum_{j=1}^{n} |a_{ij}| h_j \leq \sigma_i, \qquad R_j = \frac{|b_n|}{h_j} + \frac{|b_j|}{h_n}, \qquad R_{\max} = \max_{1 \leq j < n} R_j. \tag{10.218}$$

Here $b_j \in \mathbb{Z}$ and $b_n \neq 0$. The product and inverse height inequalities proved earlier give $h(\alpha_i) \leq \sigma_i$, as required in Section 26.

Let B be a column-basis matrix of the q-primary saturation lattice $M \supseteq \mathbb{Z}^r$, of index $J = q^{\nu}$. Suppose $B\Lambda$ lies in a translated closed box of full widths D_i. Fix $\lambda_0 \in \Lambda$, and write $\mu = B\lambda, \mu_0 = B\lambda_0$. For any selected indices $j < n$, put

$$\omega_j(\lambda) = J \sum_{i=1}^{r} (b_n a_{ij} - b_j a_{in})(\mu_i - \mu_{0i}). \tag{10.219}$$

Lemma 10.84 (the original integer Euler arguments). These are integer linear forms in $\lambda-\lambda_0$. They are the eigenvalues of $\sum_i (b_n a_{ij} - b_j a_{in}) Y_i \partial/\partial Y_i$ on the Laurent monomials $Y^{J(\mu-\mu_0)}$. With the widths of (10.186),

$$\Omega_j = \max_{\lambda\in\Lambda} |\omega_j(\lambda)| \le J R_{\max} \sum_i D_i \sigma_i = \frac{q^\nu \mathcal{D} R_{\max}}{c_1 c_2 P d}. \tag{10.220}$$

Proof. The quotient $M/\mathbb{Z}^r$ has order J. The cyclic subgroup generated by any class has order dividing J, because its distinct cosets partition this finite group into equal-sized sets. Hence J annihilates every class, and JB has integer entries. This proves integrality in (10.219). Acting on a Laurent monomial multiplies it by its integer exponent in the differentiated coordinate, which proves the eigenvalue identity, including negative exponents.

Both μ and μ_0 lie in the same translated box, so $|\mu_i - \mu_{0i}| \le D_i$. No extra factor two is needed for its full widths. Also

$$|b_n a_{ij} - b_j a_{in}| \le |b_n||a_{ij}| + |b_j||a_{in}| \le R_j \sigma_i.$$

The last inequality uses separately $|a_{ij}|h_j \le \sigma_i$ and $|a_{in}|h_n \le \sigma_i$, which follow from (10.218). Sum over i, and substitute $D_i\sigma_i = \mathcal{D}/(c_1 c_2 r P d)$. This gives (10.220). □

The same proof divides the bound by q^I when the full original widths have contracted by q^{-I}. It keeps the original denominator-clearing multiplier J.

The combined mean cost

Use the full composition rows of Corollary 10.79, including duplicate Euler rows, and let $t_0 = \lfloor T \rfloor \ge 1$. Retain its $\overline{u}, \overline{h}$, and $m = \sum_{j=1}^{r-1} \Omega_j + r - 2$. Then $\overline{h} > 0$.

Lemma 10.85 (one budget for clearing and Euler factors). With $\mathfrak{v} = \operatorname{lcm}(1,\dots,k)$, put

$$\begin{aligned} \mathfrak{a}_{\mathrm{av}} &= \max\left\{ \ln\left[\mathrm{e}(1+m/\overline{h})\right], \frac{1}{2}\ln\mathfrak{v} \right\}, \\ E_{\mathrm{mean}} + \overline{u}\ln\mathfrak{v} &\le t_0 \mathfrak{a}_{\mathrm{av}}. \end{aligned} \tag{10.221}$$

Proof. The exact composition means in (10.204) give

$$\overline{h} + 2\overline{u} = \frac{r-1}{r} t_0 + \frac{r+1}{r}\overline{u} \le t_0,$$

since $\overline{u} \le t_0/(r+1)$. Formula (10.205) bounds the mean Euler logarithm by $\overline{h}\ln[\mathrm{e}(1+m/\overline{h})]$; the mean clearing logarithm is exactly $\overline{u}\ln\mathfrak{v}$. Both are bounded by their respective multipliers times $\mathfrak{a}_{\mathrm{av}}$. Adding proves (10.221). □

For the original parameters, we can retain more information than the maximum in this lemma. Define

$$W(d) = \begin{cases} \ln^3(3d), & d \ge 2, \\ \dfrac{\ln 6}{\ln 2\,\ln 3}, & d = 1, \end{cases} \qquad g_{91} = 1 + \frac{1+\ln W(d)}{g_0}. \tag{10.222}$$

Thus g_{91} is $1 + (1 + 3\ln\ln(3d))/g_0$ for $d \ge 2$. Assume in addition to Section 26 that

$$h \ge \ln\left(\frac{R_{\max}}{dW(d)}\right), \qquad c_0 \ge 19/10, \quad c_4 \ge 15/4, \quad \widehat{\vartheta} \le 2, \qquad H_1 = h + \nu\ln q. \tag{10.223}$$

The added height condition is exactly $R_{\max} \le dW(d)\mathrm{e}^h$. It is the coefficient-height condition associated with (10.218).

Theorem 10.86 (the original uniform mean Euler budget). Under these assumptions,

$$\begin{aligned} \frac{m}{\overline{h}} &\le \frac{4}{7}\,\mathrm{e}^{H_1} tW(d) + \frac{r+1}{t_0}, \\ \ln\bigl[\mathrm{e}(1+m/\overline{h})\bigr] &\le \left(g_{91} + \frac{1}{2(r-1)}\right) H_1, \end{aligned} \tag{10.224}$$

and

$$E_{\mathrm{mean}} + \overline{u}\ln\mathfrak{v} \le g_{91} t_0 H_1. \tag{10.225}$$

Proof. First $T \ge g_4 > 9$. To verify the latter lower bound directly, the factor defining g_4 is at least g_1^{-1}. Insert g_3 from (10.185), cancel c_1, g_1, t, and use $q \ge 2, r+1 \ge 3, \rho \le 58, a_* \ge 7/2$ and $r! \le r^r$. This gives

$$g_4 \ge \frac{6}{2}\frac{2(19/10)(15/4)}{58}\left(\frac{7}{2}\right)^2 = \frac{8379}{928} > 9.$$

Consequently $t_0 \ge 1$ and $T/t_0 < 2$.

We have $\overline{h} \ge (r-1)t_0/(r+1)$. Sum (10.220), use (10.223), and use $T = q(r+1)\mathcal{D}/(c_1\theta e t)$. Then

$$\frac{m}{\overline{h}} \le \mathrm{e}^{H_1}\frac{T}{t_0}\frac{\theta e}{P}\frac{tW(d)}{c_2 q} + \frac{(r+1)(r-2)}{(r-1)t_0}.$$

Indeed the factor d in $R_{\max} \le dW(d)\mathrm{e}^h$ cancels the global degree in (10.220). Formula (10.184) gives $\theta e/P \le 1$, including its special unramified case, and $c_2 q \ge 7/2$. These inequalities prove the first line of (10.224).

Both branches have $W(d) \ge 1$. For $d \ge 2$, $\ln(3d) \ge \ln 6 > 1$. For $d = 1$, use $\ln 6 = \ln 2 + \ln 3$ and $\ln 2 < 1$, so $W(d) = 1/\ln 2 + 1/\ln 3 > 1$. Since $t \ge \ln 2 > 2/3$ and $H_1 \ge (r+1)t$, we have $r + 1 < 3H_1/2$. Moreover $H_1 > 39$ implies

$$1 + \frac{3H_1}{2} < \frac{3}{14}H_1^2 \le \frac{3}{7}\mathrm{e}^{H_1}.$$

The last inequality follows from the quadratic term of the positive exponential series. This absorbs the $1 + (r+1)/t_0$ term in the first line of (10.224), yielding

$$1 + m/\overline{h} \le \mathrm{e}^{H_1} W(d)\max\{t, 1\}.$$

Take logarithms and add one. As $H_1 \ge g_0$, the constant $1 + \ln W(d)$ is at most $(g_{91} - 1)H_1$. The function $\ln z/z$ has maximum $1/\mathrm{e}$ for $z > 0$, by differentiating. Hence $\ln\max\{t, 1\} \le t/\mathrm{e} \le H_1/[\mathrm{e}(r+1)] \le H_1/[2(r-1)]$, since $\mathrm{e} > 2$. This proves the second line of (10.224).

The complete lcm bound in Two logarithms II: explicit lower bounds gives $\ln\mathfrak{v} < k\ln 3 < (3/2)H_1$. Here $k \le H_1$, and $\ln 3 < 3/2$ follows already from $1 + 3/2 + (3/2)^2/2 > 3$ in the positive exponential series.

Put $a = g_{91} + 1/[2(r-1)]$. We have bounded the combined mean by

$$H_1\left[\frac{r-1}{r}(t_0 - \overline{u})a + \frac{3}{2}\overline{u}\right].$$

This is affine in $\overline{u}$, so its maximum on $[0, t_0/(r+1)]$ occurs at an endpoint. At zero its coefficient of $t_0 H_1$ is $(r-1)g_{91}/r + 1/(2r) \le g_{91}$. At the other endpoint that coefficient is $((r-1)g_{91} + 2)/(r+1) \le g_{91}$. Both inequalities use $g_{91} \ge 1$. This proves (10.225). □

The full initial coefficient-height bound

Corollary 10.87 (a uniform bound for all initial scalar rows). Assume also the hypotheses of Theorem 10.75 and Corollary 10.77. An integral initial kernel with all required zeros can be chosen with mean scalar logarithm at most

$$\ln \mathcal{B}_{\mathrm{av}} = kL \ln\left[2\mathrm{e}\left(\frac{s_0}{k}+2\right)\right] + g_{91} T H_1. \tag{10.226}$$

In particular

$$\frac{\ln \mathcal{B}_{\mathrm{av}}}{Z} \le \frac{1+g_5^{-1}}{c_1 c_4} + \frac{g_{91}}{c_1 c_3 \theta e}. \tag{10.227}$$

With d_0 the actual row-field degree, its coefficients satisfy

$$\begin{aligned}\frac{h_2(\boldsymbol{c})}{Z} \le{}& \frac{1}{2}g_8 + \frac{\ln d_0}{2g_7} + \frac{d_0-1}{c_1 c_2 S} \\ &+ \frac{1}{c_{01}-1}\left[\frac{1}{2}g_8 + \frac{1+g_5^{-1}}{c_1 c_4} + \frac{g_{91}}{c_1 c_3 \theta e} + \frac{1+1/(2g_2)}{2c_1 c_2}\right].\end{aligned} \tag{10.228}$$

Proof. At every additive order, the first factor of (10.198) bounds its divided derivative before the lcm multiplier. Its logarithm is independent of the derivative allocation, so it also bounds its average. Theorem 10.86 bounds the mean lcm and Euler contributions together by $g_{91} t_0 H_1 \le g_{91} T H_1$. This proves (10.226). Corollary 10.77 bounds its first normalized term. The exact parameter formulas give

$$\frac{T H_1}{Z} = \frac{1}{c_1 c_3 \theta e},$$

proving (10.227).

Use the full composition rows and their actual ratio $\overline{M}/(N-\overline{M})$ in (10.206). The column surplus (10.190) bounds that ratio by $1/(c_{01}-1)$. Insert (10.227), $\ln N/Z \le g_8$, $Z \ge g_7$, and the exact mean-node bound from Corollary 10.72. This is (10.228), with every scalar, logarithmic, field-generator and torus cost retained. The row field and the integral nonzero coefficient conclusion are those of Lemma 10.63. □

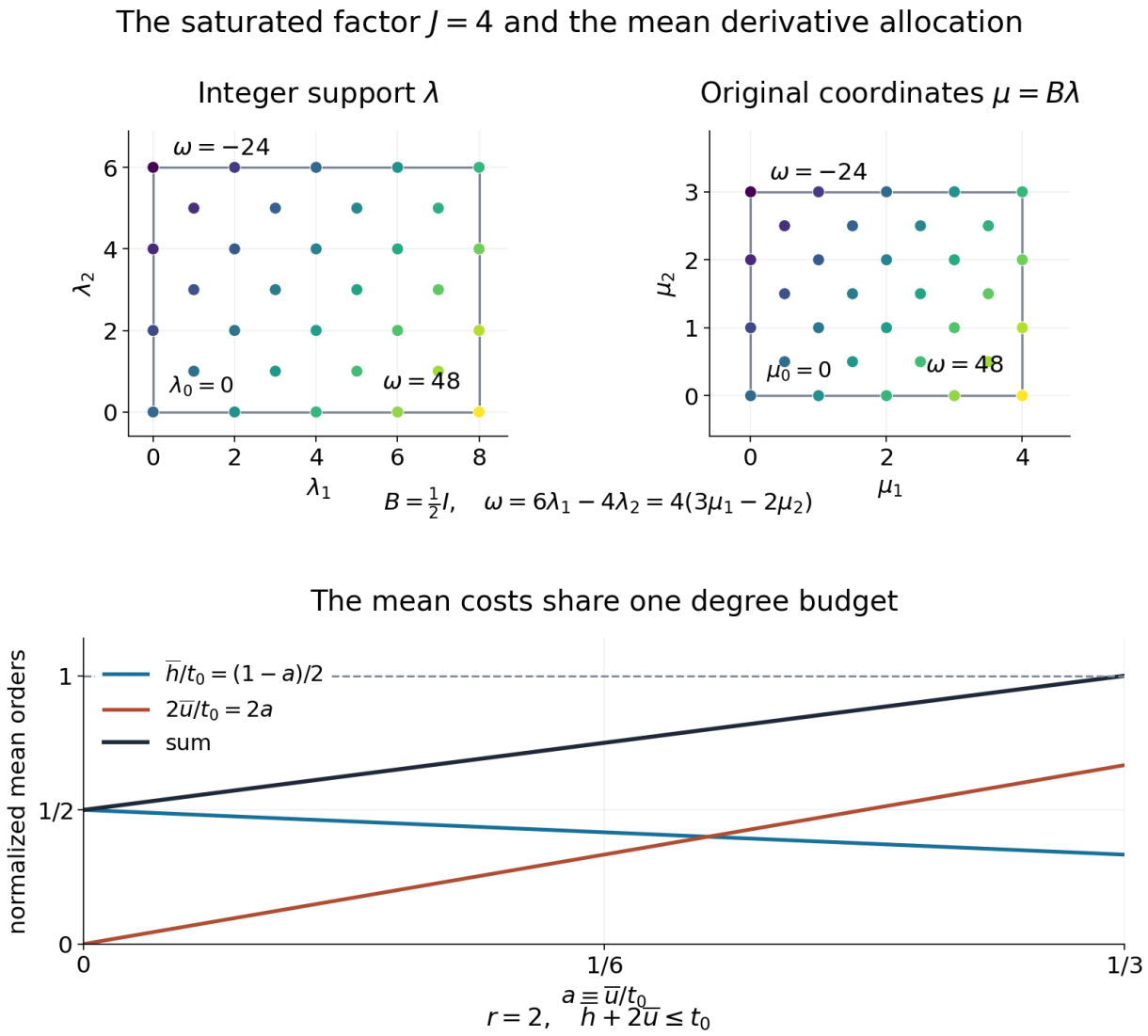


Figure 10.17. Take the rational bases $4, 9$, their q-primary saturated generators $2, 3$, and $J = 4$. The map $B = \frac{1}{2}I$ sends the integer support on the left to its original coordinates in the upper right panel. Retain the phase class $\lambda_1 + \lambda_2 \equiv 0 \pmod 2$. With $(b_1, b_2) = (2, 3)$, the prepared Euler argument is $6\lambda_1 - 4\lambda_2 = 4(3\mu_1 - 2\mu_2)$. Its extrema at the reference $\lambda_0 = 0$ are 48 and -24, marked at the corresponding vertices. The lower panel shows the exact mean relation at $r = 2$: for $a = \overline{u}/t_0 \le 1/3$, the normalized Euler mean is $(1-a)/2$ and their weighted sum $(1-a)/2 + 2a$ is at most one. Lemma 10.84 keeps the full clearing factor and original widths; Lemma 10.85 and Theorem 10.86 control the cost after averaging orders. The coordinates and values are reconstructed in Solution 37. Figure program. Human-source context: Yu's free paper, equations (2.8), (4.23) and (5.6).

30. The generator field and its uniform discriminant cost

The initial scalar bound is now uniform. We next control the field in which its coefficients are chosen. A large entry of a saturation basis can make its displayed generators have large heights while leaving the field unchanged. Centering the basis removes that loss. We then use the height-product theorem separately after each omission; the cancellation in $S\mathcal{D}/d$ keeps its bound independent of the saturation index.

A centered basis with small generator heights

Lemma 10.88 (a centered saturation basis). Let $r \ge 2$, and let $M \supseteq \mathbb{Z}^r$ be a rational lattice of finite index J. Suppose its basis B corresponds to generators $\theta_1, \dots, \theta_r \in K^\times$ in the independent original bases $\alpha_1, \dots, \alpha_r$, up to roots of unity. Put $\sigma_{\max} = \max_i \sigma_i$, where $h(\alpha_i) \le \sigma_i$.

There is a basis $B' = BU$, with $U \in \mathrm{GL}_r(\mathbb{Z})$, that is upper triangular and has the following alternatives for each column:

$$\begin{aligned} B'_{ii} = 1 \quad &\Longrightarrow \quad B'_i = e_i, \\ 0 < B'_{ii} < 1 \quad &\Longrightarrow \quad B'_{ii} \le \tfrac{1}{2}, \qquad |B'_{ji}| \le \tfrac{1}{2} \quad (j < i), \qquad B'_{ji} = 0 \quad (j > i). \end{aligned} \tag{10.229}$$

Choose $\theta'_j = \prod_i \theta_i^{U_{ij}}$. Then

$$\begin{aligned} h(\theta'_j) \le \sum_i |B'_{ij}| h(\alpha_i) \le \frac{r}{2}\sigma_{\max}, \\ E = \mathbb{Q}(\alpha_0, \theta_1, \ldots, \theta_r) = \mathbb{Q}(\alpha_0, \theta'_1, \ldots, \theta'_r), \end{aligned} \tag{10.230}$$

for any root of unity α_0. This change preserves every normalized monomial, phase class, original support width and integer Euler argument after relabelling $\lambda' = U^{-1}\lambda$.

Proof. The cyclic-subgroup coset argument in Lemma 10.84 shows that J kills $M/\mathbb{Z}^r$. Thus $L = JM$ is an integer lattice containing $J\mathbb{Z}^r$. Let $L_i = L \cap \mathrm{span}(e_1, \ldots, e_i)$. Its i-th coordinate image is a nonzero subgroup $h_i\mathbb{Z}$ containing $J\mathbb{Z}$. Hence h_i is a positive divisor of J.

Inductively choose $v_i \in L_i$ whose i-th coordinate is h_i. If $h_i = J$, choose $v_i = Je_i$, which belongs to L. Otherwise, subtract integer multiples of $v_{i-1}, \ldots, v_1$, in that descending order, to place coordinate $j < i$ in the interval $[-h_j/2, h_j/2]$. Ordinary integer division, with either consistent choice at a midpoint, makes this possible. Subtracting v_j does not change a coordinate larger than j.

The vectors $v_1, \ldots, v_i$ form a basis of L_i. Indeed, for $x \in L_i$, subtract the multiple of v_i specified by its i-th coordinate to obtain an element of L_{i-1}, and use induction. Reading the largest coordinate proves independence. Thus $B' = (v_1, \ldots, v_r)/J$ is a basis of M. Its diagonal lies in $(0, 1]$; a proper divisor $h_i < J$ is at most $J/2$. Every centered upper entry has absolute value at most $h_j/(2J) \le 1/2$. The special choice at $h_i = J$ gives the first alternative in (10.229).

Both B and B' are integer bases of the same lattice, so expressing one in the other gives inverse integer matrices U, U^{-1}. The chosen new generators and their inverse products therefore generate exactly the same field. Clear the denominators of B' and the finite root-of-unity factors by a common positive power. The height product inequality and exact power identity, already proved in the earlier height lesson, give the first inequality of (10.230). A proper-diagonal column has at most r entries of absolute value at most $1/2$. A unit column gives height at most $\sigma_{\max} \le r\sigma_{\max}/2$. This proves the second inequality.

Finally $B'\lambda' = B\lambda$ and $(\theta')^{\lambda'} = \theta^{\lambda}$. With the phase vector transformed to $d' = U^{\mathsf{T}}d$, we have $d' \cdot \lambda' = d \cdot \lambda$. Original-coordinate differences, hence their full-width bounds and the integer Euler forms in (10.219), are unchanged. Taking the same integer products of chosen q-th roots also preserves their powers and generated root field; the inverse products recover the old roots. The uniqueness assertion in Corollary 10.83 preserves the principal root choices. □

The unit-column alternative matters. Centering only the off-diagonal entries of a column with diagonal one would give the weaker $(r + 1)\sigma_{\max}/2$. Choosing the coordinate unit vector avoids this extra half-height.

Each omitted height product and the exact scale

Retain all parameters in (10.183)–(10.186). Write w_K for the full root-of-unity order, and define

$$
\begin{aligned}
g_{11} &= \frac{4q\mathrm{e}}{\rho}\, c_0c_1c_3c_4a_*^r \frac{(r+1)^{r+1}(r-1)^{r-1}}{(r!)^2} g_0g_1,\\
g_{12} &= \begin{cases} g_1/(2g_7) + 1/g_{11}, & d \geq 2,\\ 0, & d = 1. \end{cases}
\end{aligned}
\tag{10.231}
$$

These are the original constants in Yu's equation (3.16); $\mathrm{e} = \exp(1)$ is distinct from the ramification index.

Lemma 10.89 (the original omitted-product bound). For $Z = S\mathcal{D}/d$ and every $i = 1, \dots, r$,

$$
\begin{aligned}
d^r \ell_d \prod_{j \neq i} \sigma_j &\geq \frac{w_K}{\rho} \frac{(r-1)^{r-1}}{(r-1)!\,\mathrm{e}^{r-1}},\\
\frac{2Z}{rd\sigma_i} &\geq g_{11}.
\end{aligned}
\tag{10.232}
$$

Proof. Every subset of an independent multiplicative family is independent: a relation in that subset would extend by zero exponents to a relation in the full family. Apply the completely proved height-product inequality (10.57) to the $r-1$ original bases other than α_i. Since $\sigma_j \geq h(\alpha_j)$, replacing the heights preserves its lower bound. At $d \geq 2$, multiply by $d^r\ell_d$ and use $\ell_d \geq \ln d$. At $d = 1$, the proved constant is $17 = \rho$. This proves the first line of (10.232) for each omission separately.

For clarity the exact cancellation in the scale is

$$
\begin{aligned}
Z = \frac{c_3q(r+1)hA}{q^u t}(1+\epsilon)(2+g_2^{-1})c_0c_1c_4 \left(\frac{c_2qPr(r+1)}{e\theta}\right)^r \frac{1}{r!}\\
\times \max\left\{\frac{p^f}{\delta t^r}, \frac{\mathrm{e}^r}{r^r} t\right\} d^{r+1}\ell_d \prod_j \sigma_j.
\end{aligned}
\tag{10.233}
$$

It follows by substituting S and γ in (10.186): the factors q^ν, $h + x$ and $A + x$ cancel exactly. Use $h \geq g_0$, $A \geq g_1$, $(1+\epsilon)(2+g_2^{-1}) \geq 2$, $c_2qP/(e\theta) \geq a_*$, and the second member of the maximum. We obtain

$$
Z \geq \frac{2q\mathrm{e}^r}{q^u} c_0c_1c_3c_4a_*^r \frac{(r+1)^{r+1}}{r!} g_0g_1d^{r+1}\ell_d \prod_j \sigma_j.
$$

Divide by σ_i, insert the first line of (10.232), and use $w_K \geq q^u$. Multiplication by $2/(rd)$, together with $r(r-1)! = r!$, gives exactly g_{11} in (10.231). No bound on q^ν has been inferred from an omitted product. □

A uniform coefficient-field and initial height bound

Theorem 10.90 (the original field cost). Let $E = \mathbb{Q}(\alpha_0, \theta_1, \dots, \theta_r) \subseteq K$ be the generator field of the normalized initial rows, and let $d_E = [E : \mathbb{Q}]$. Then

$$
\frac{\ln|\Delta_E|}{2d_EZ} \leq g_{12}. \tag{10.234}
$$

Under the remaining hypotheses of Corollary 10.87, the initial rows have a nonzero integral kernel $c \in \mathcal{O}_E^N \subseteq \mathcal{O}_K^N$ giving every original zero and the same nonzero-polynomial conclusion, with

$$\frac{h_2(c)}{Z} \le g_{12} + \frac{g_8}{2} + \frac{1}{c_{01}-1}\left[\frac{g_8}{2} + \frac{1+g_5^{-1}}{c_1c_4} + \frac{g_{91}}{c_1c_3\theta e} + \frac{1+(2g_2)^{-1}}{2c_1c_2}\right]. \tag{10.235}$$

Proof. At $d = 1$, $E = \mathbb{Q}$ and its discriminant is one. Suppose $d \ge 2$, and use the centered generators of Lemma 10.88. Apply the complete discriminant-and-generators argument (8.33) in Linear forms in many logarithms to the tower obtained by first adjoining α_0, then the θ'_j. The first height is zero. The product of its nontrivial degrees is d_E; induction on $(a-1)+(b-1) \le ab-1$ bounds the sum of the degree decrements by $d_E - 1$. Consequently

$$\frac{\ln|\Delta_E|}{2d_E} \le \frac{\ln d_E}{2} + (d_E-1)\frac{r}{2}\sigma_{\max}. \tag{10.236}$$

Choose an index attaining $\sigma_{\max}$. Lemma 10.89 gives $r\sigma_{\max}/2 \le Z/(dg_{11})$. Since $d_E \le d$, the second term in (10.236), divided by Z, is at most $(d_E-1)/(dg_{11}) < 1/g_{11}$. Also $Z \ge g_7$ and $\ln d_E \le \ln d < g_1$, so its first term is at most $g_1/(2g_7)$. This proves (10.234), with the original g_{12}.

Every initial row entry is its integer prepared scalar times $\alpha_0^{sw_\lambda}\theta^{Ps(\lambda-\lambda_0)}$, by (10.151), hence belongs to E. Apply the complete weighted integral Siegel proof (8.40) over this field, allowing the full composition rows and retaining zero or duplicate rows as in Corollary 10.79. Their normalized heights do not change upon restriction to E, by the field-invariance argument in Theorem 10.59. The resulting bound is the original row-sum inequality with discriminant term $\ln|\Delta_E|/(2d_E)$.

Insert (10.234), $\ln N/Z \le g_8$, the exact row ratio bounded by $1/(c_{01}-1)$, the mean scalar bound (10.227), and the mean-node torus term in Corollary 10.72. This is (10.235). The vector is integral over E, hence over K; it is nonzero, and Lemma 10.29 gives the nonzero Laurent polynomial. The representative-row identities used in Theorem 10.55 supply every original jet zero. □

The field bound uses the same g_{12} as the original argument. Equation (10.235) retains the complete mean scalar budget proved with $\operatorname{lcm}(1,\dots,k) < 3^k$. A later individual prepared value still has its own scalar, denominator and field costs; averaging the initial rows does not change that value.

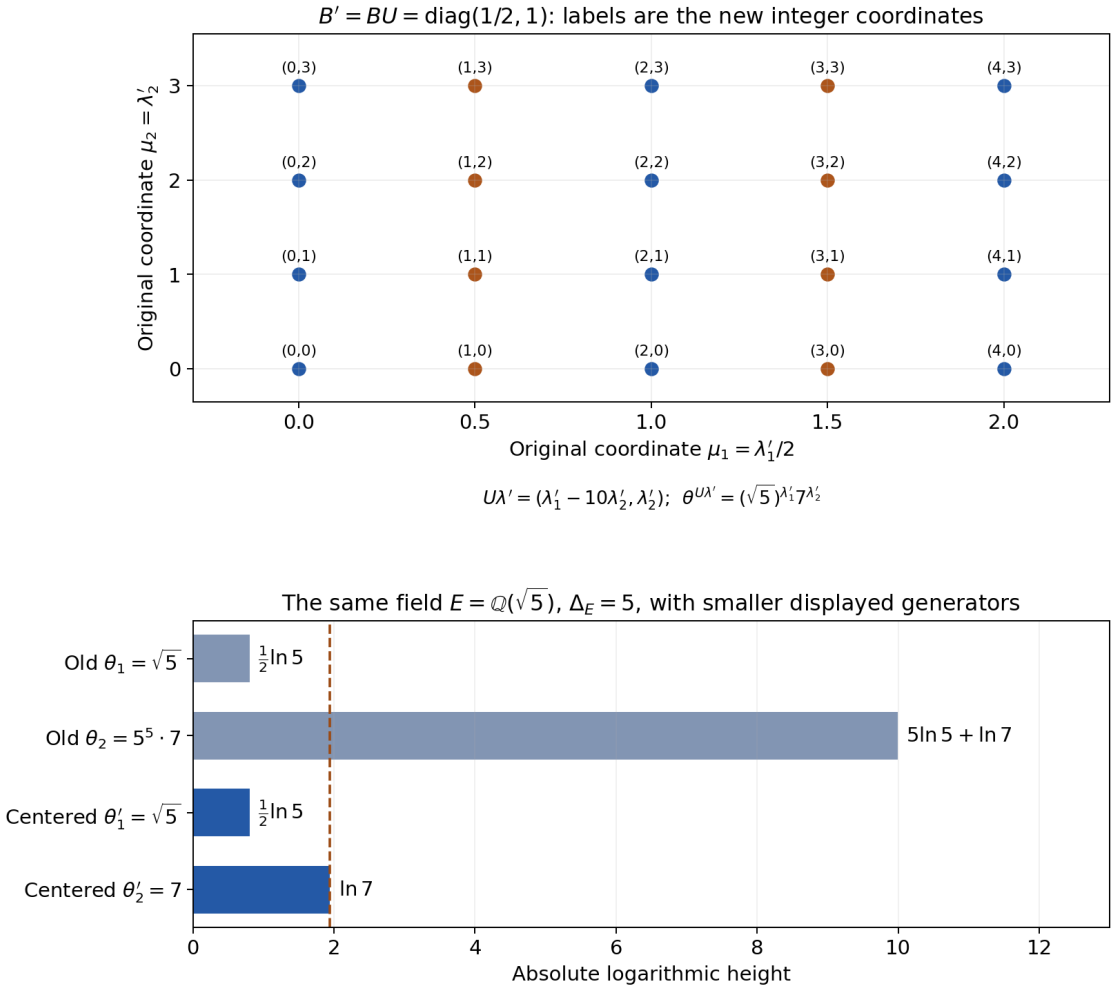


Figure 10.18. Over $K = \mathbb{Q}(\sqrt{5})$, the original bases $5, 7$ have saturation lattice $(\frac{1}{2}\mathbb{Z}) \times \mathbb{Z}$ and index two. The displayed basis $B = \left(\begin{smallmatrix} 1/2 & 5 \\ 0 & 1 \end{smallmatrix}\right)$ has generators $\sqrt{5}, 5^5 \cdot 7$. The integer matrix $U = \left(\begin{smallmatrix} 1 & -10 \\ 0 & 1 \end{smallmatrix}\right)$ gives $B' = BU = \mathrm{diag}(1/2, 1)$ and generators $\sqrt{5}, 7$. The sampled lattice points, blue for an even first coordinate and brown for an odd one, are labelled by λ', with original coordinates $B'\lambda' = (\lambda'_1/2, \lambda'_2)$. In the old basis they have labels $U\lambda'$, producing exactly the same points and monomials. The height bars are numerical renderings of the labelled exact logarithms; the dashed line is the centered bound $r\sigma_{\max}/2 = \ln 7$. Both lists generate the same quadratic field of discriminant five. Lemmas 10.88–10.89 and Theorem 10.90 prove the basis, omitted-product and field-budget mechanisms; Solution 38 checks this example and the normalized original constants. Figure program. Human-source context: Yu's free paper, equations (3.16), (3.23) and Lemma 4.2; Loher and Masser's free paper, Theorem 3, whose complete height-product proof is given in Section 9.

31. Individual scalar values and a complete integer extension

The initial coefficient estimate averages its rows. An individual value in the induction has its own additive order, ordinary absolute value and denominator. We retain these costs together with the actual derivative precision. At the end of this section, all three integer extensions of the original rational rank-two parameter example are verified uniformly over its permitted coefficient family.

The actual projected slopes and the additive order

Lemma 10.91 (normality at the original logarithmic slopes). Let $q \neq p$, $\nu \geq 0$, $A_0 = (a_{ij})$ an integer r-by-n matrix, and $b_1, \ldots, b_n \in \mathbb{Z}$ with $b_n \neq 0$ and $v_p(b_j) \geq v_p(b_n)$. Suppose $z_j \in \mathbb{C}_p$ satisfy $v_p(z_j) \geq \vartheta + 1/(p-1)$, and $0 < \theta < \vartheta$. Put

$$
\begin{aligned}
u_i &= q^{-\nu}\sum_{j=1}^{n} a_{ij}z_j, & \mathcal{L} &= \sum_j b_j z_j,\\
w_i &= u_i - \frac{q^{-\nu}a_{in}}{b_n}\mathcal{L} = q^{-\nu}\sum_{j<n}\left(a_{ij}-\frac{b_j}{b_n}a_{in}\right)z_j.
\end{aligned}
\tag{10.237}
$$

Every u_i, w_i has valuation at least $\vartheta + 1/(p-1) > \theta + 1/(p-1)$, with the convention $v_p(0) = \infty$. Thus the normality hypotheses of Proposition 10.37 hold for the original and projected curves.

More precisely, retain its prepared family, ρ, c_*, δ, k, L, and let $D_{\mathrm{a}} = kL$, $\mathfrak{v} = \operatorname{lcm}(1,\dots,k)$. For a fixed additive order $u = t_0 \le D_{\mathrm{a}}$, the following function is normal:

$$
\begin{aligned}
F_u(Z;\boldsymbol{t}) &= \frac{\rho^{D_{\mathrm{a}}-u}(k!)^L}{c_*\mathfrak{v}^u}\, f(Z/\rho;\boldsymbol{t}),\\
G_u &= (D_{\mathrm{a}}-u)\theta + Lv_p(k!) - uv_p(\mathfrak{v}),\\
v_p\!\left(\frac{\rho^{D_{\mathrm{a}}-u}(k!)^L}{c_*\mathfrak{v}^u}\right) &= G_u - \delta.
\end{aligned}
\tag{10.238}
$$

Its ordinary divided jets obey the chain-rule identity (10.107) with scale $G_u - \delta$. For $u > D_{\mathrm{a}}$, every prepared value is zero.

Proof. The fractions b_j/b_n are p-integral by their valuation hypothesis; $q^{-\nu}$ is a local unit. Every displayed linear combination therefore has the asserted lower valuation, by the ultrametric inequality. In particular the projected slope does not lose $v_p(b_n)$: the factors of b_n have already canceled in (10.237). The principal-unit depth proof in Lemma 10.81 and its logarithmic consequence in Corollary 10.82 supply these z_j when the original local units are used.

For the scale, the additive factor of a term is the coefficient of V^u in

$$
\frac{1}{(k!)^\ell}\prod_{a=1}^{k}(q^{-I}Z/\rho + \lambda_{-1} + a + \mathfrak{v}V)^\ell, \qquad 1 \le \ell \le L, \quad 0 \le \lambda_{-1} < k.
$$

Factoring $\mathfrak{v}^u$ from that coefficient leaves an integer-coefficient polynomial in $q^{-I}Z/\rho$, divided by $(k!)^\ell$, of degree at most $k\ell - u$. Multiplication by the factor in (10.238) makes every coefficient p-integral: q is a local unit, $k\ell - u \le D_{\mathrm{a}} - u$, and $\ell \le L$. The ratios c_j/c_* are p-integral and the integer Euler factors introduce no denominator.

Every exponential factor has argument $\beta Z/\rho$, where β is an integer linear combination of the w_i. Its positive-degree coefficients tend to zero and have positive valuation, by Corollary 10.82. Its Gauss norm is one. Multiplication by the normalized polynomial, followed by the finite sum over the support, proves normality through Lemma 9.2. The chain rule proves the jet identity. The degree bound proves the final zero assertion. □

The scale in (10.238) is allowed to have negative valuation; the prepared derivative already has the compensating local divisibility. The order-independent scale $G_0 - \delta$ remains valid, as proved in Proposition 10.37. We use that common scale in the numerical extension below.

A factorial-preserving pointwise majorant

Take a support at contraction depth I, and let Ω_j be nonnegative integer bounds for its $r-1$ integer Euler arguments. For $|x| \le X$, define

$$
R_I = q^{-I}X + 2k - 1, \qquad m_I = \sum_{j=1}^{r-1}\Omega_j + r - 2, \qquad D_{\mathrm{a}} = kL. \tag{10.239}
$$

Theorem 10.92 (the individual scalar and denominator budget). For a prepared index with additive order $u \le D_{\mathrm{a}}$ and Euler order $h = \sum_{j>0} t_j$, its rational scalar has ordinary absolute value at most

$$\mathcal{H}_{I,u,h}(X) = \frac{R_I^{D_{\mathrm{a}}-u}}{(k!)^L}\mathfrak{v}^u \binom{D_{\mathrm{a}}}{u}\binom{m_I+h}{h}. \tag{10.240}$$

Uniformly through total order T', its scalar bound is

$$\mathcal{H}_I^{\mathrm{fine}}(X,T') = \frac{R_I^{D_{\mathrm{a}}}}{(k!)^L}\max_{0\le u\le \min(D_{\mathrm{a}},T')}\left\{\left(\frac{\mathfrak{v}}{R_I}\right)^u\binom{D_{\mathrm{a}}}{u}\binom{m_I+T'-u}{T'-u}\right\}. \tag{10.241}$$

A maximizing index is the first one after all consecutive increasing or constant steps. For $u < \min(D_{\mathrm{a}}, T')$, their exact test is

$$\mathfrak{v}(D_{\mathrm{a}}-u)(T'-u) \ge R_I(u+1)(m_I+T'-u). \tag{10.242}$$

At $x = s/q^J$, $s \in \mathbb{Z}$, all scalar denominators divide $q^{\Xi_{I,J,u}}$, where

$$\Xi_{I,J,u} = \begin{cases} 0, & I+J=0,\\ \max\{0,(I+J)(D_{\mathrm{a}}-u)+Lv_q(k!)-uv_q(\mathfrak{v})\}, & I+J>0. \end{cases} \tag{10.243}$$

Suppose the normalized torus support has original full widths $q^{-I}D_i$, $h(\alpha_i) \le \sigma_i$, and the phase hypotheses of Theorem 10.55. Let the actual algebraic value V belong to a number field F, with degree d_F and chosen local degrees e_F, f_F. Its torus monomials are local units and arise from the consistent roots at $x = s/q^J$. If $V \ne 0$, then

$$\begin{aligned} v_p(V) - \delta \le \frac{d_F}{e_F f_F \ln p}\Big[&h_2(c) + \tfrac{1}{2}\ln N + \ln \mathcal{H}_{I,u,h}(X)\\ &+ \Xi_{I,J,u}\ln q + PXq^{-I}\sum_i D_i\sigma_i\Big]. \end{aligned} \tag{10.244}$$

The field factor is retained even when these roots lie in the old completion.

Proof. Differentiate the product of $k\ell$ linear factors at $q^{-I}x + \lambda_{-1}$, dividing by $u!$. Choosing the u differentiated factors gives $\binom{k\ell}{u}$ terms. Every remaining factor has ordinary absolute value at most R_I, since $\lambda_{-1} \le k-1$ and $a \le k$. Multiply by $\mathfrak{v}^u$, retaining the denominator $(k!)^\ell$.

If $u > k\ell$, the derivative is zero. Otherwise $\binom{k\ell}{u} \le \binom{D_{\mathrm{a}}}{u}$ and

$$\frac{R_I^{k\ell-u}}{(k!)^\ell} = R_I^{-u}\left(\frac{R_I^k}{k!}\right)^\ell \le \frac{R_I^{D_{\mathrm{a}}-u}}{(k!)^L},$$

because $R_I \ge k$ makes the parenthesized base at least one. The proved integer Euler generating-series estimate in Lemma 10.76 bounds their product at order h by $\binom{m_I+h}{h}$. This proves (10.240). It increases with h, so maximizing through $u+h \le T'$ gives (10.241).

The ratio of consecutive terms inside this maximum is

$$\frac{\mathfrak{v}}{R_I}\frac{D_{\mathrm{a}}-u}{u+1}\frac{T'-u}{m_I+T'-u}.$$

Both variable factors are nonincreasing, and the first strictly decreases. At $m_I = 0$, the last factor is one wherever a step is taken; the terminal value has Euler binomial one. Thus (10.242) describes an initial interval of steps, including this endpoint case. For rational R_I, integer division after clearing its denominator gives an exact binary search.

The scalar is integral at every prime other than q, by the fractional-node argument of Lemma 10.28. At q, the polynomial divided derivative has denominator dividing $(k!)^\ell$ and degree at most $k\ell - u$. Evaluation at $q^{-I-J}s + \lambda_{-1}$ costs at most $(I + J)(k\ell - u)$; its multiplier $\mathfrak{v}^u$ recovers $uv_q(\mathfrak{v})$. Taking $\ell \le L$ proves (10.243), with the globally integral $I + J = 0$ case stated separately.

Apply the completely proved nonzero-value product formula of Theorem 10.41 over F, dividing by a coefficient of minimum valuation δ. Cauchy–Schwarz gives $h_2(c) + \frac{1}{2}\ln N$. The scalar bounds just proved give its archimedean and q-denominator terms. The full-width support proof in Theorem 10.64, with contraction as in Theorem 10.60, bounds the torus contribution by $PXq^{-I}\sum_i D_i\sigma_i$. Taking consistent roots divides heights by q^J, while the node numerator multiplies them by $|s| \le q^J X$; these two factors cancel. The phase factors have height zero. Summing the place contributions and dividing by $e_F f_F \ln p$ gives (10.244). □

All three first integer extensions in the rational parameter family

Theorem 10.93 (a complete rank-two integer block). Use the original rational parameters and support of Solution 34, with bases $2, 5$ at three. Let $b_1, b_2 \ne 0$ be integers satisfying

$$\begin{gathered} v_3(b_2) \le v_3(b_1), \qquad R_{\max} = \frac{|b_2|}{\ln 2} + \frac{|b_1|}{\ln 5} \le W\exp(h), \qquad W = \frac{\ln 6}{\ln 2\,\ln 3}, \quad h = g_0, \\ \eta = 1 - \frac{0.538}{3} = \frac{1231}{1500}, \qquad U = \frac{8Z}{\ln 3}, \qquad Z = S\mathcal{D}. \end{gathered} \tag{10.245}$$

Choose the integral initial kernel of Theorem 10.90 for this support and these Euler arguments. Assume $v_3(2^{b_1}5^{b_2} - 1) \ge U$. Then the same prepared auxiliary family, without contraction of its support, satisfies

$$\Phi_0(s; \boldsymbol{t}) = 0 \quad \text{for } |s| \le \lfloor 2^j S\rfloor, \quad |\boldsymbol{t}| \le \lfloor \eta^j T\rfloor, \qquad j = 1, 2, 3. \tag{10.246}$$

The allowed coefficient family in (10.245), rather than one chosen pair b_1, b_2, is covered by these comparisons.

Proof. Put $\beta_1 = -2, \beta_2 = -5$, both in $1 + 3\mathbb{Z}_3$, and $u_i = \log(\beta_i^3)$. Since $\beta_1^3 - 1 = -9$ and $\beta_2^3 - 1 = -126$, the exact logarithm valuation identity in Proposition 9.3 gives $v_3(u_i) = 2$. Here $\nu = 0$, $P = 3$, and $\theta < 3/2$. The hypothesis on the multiplicative difference forces $b_1 + b_2$ even by reduction modulo three. Hence $\mathcal{L} = b_1u_1 + b_2u_2 = 3\log(2^{b_1}5^{b_2})$ has valuation at least U. The projected slopes are $w_1 = u_1$ and $w_2 = -(b_1/b_2)u_1$. Lemma 10.91 proves their enlarged-disc normality. The single selected Euler direction is $b_2E_1 - b_1E_2$, so Lemma 10.38 supplies its actual ordinary jet precision.

The coefficient height (10.235) is uniform over (10.245), since this is precisely the height condition of Theorem 10.86. It gives $h_2(c)/Z \le 0.310817025826$. Also $k = 50, L = 35299, D_{\mathrm{a}} = 1764950$, $v_3(50!) = 22$, and $v_3(\mathfrak{v}) = 3$. Set $G_0 = D_{\mathrm{a}}\theta + 22L$.

For the induction from j to $j + 1$, write

$$R_j = \lfloor 2^j S\rfloor, \quad T_j = \lfloor \eta^j T\rfloor, \quad n_j = 2R_j + 1, \quad \mu_j = T_j - T_{j+1} + 1.$$

Then $|\boldsymbol{t}| + \mu_j - 1 \le T_j$ for every output index. The initial case has every required zero by the kernel construction.

The height condition gives

$$v_3(b_2) \le \frac{h + \ln W + \ln(\ln 2)}{\ln 3} < 47.$$

This valuation is an integer, so it is at most46. Since every separation cost $B_j = \lfloor \log_3(2R_j) \rfloor$ below is at most7, we may use $M_0 = 46$ in (10.113). In particular $U - v_3(b_2) > \theta + 1/2$. The exact input budgets in Figure 10.19 satisfy

$$U + G_0 > n_j \mu_j \theta + \mu_j M_0, \qquad j = 0, 1, 2.$$

Corollary 10.40 therefore supplies the analytic valuation excess $(n_j \mu_j - D_{\mathrm{a}})\theta - 22L$ at every output index.

For the arithmetic bound, Lemma 10.84 and (10.245) give the common integer bound

$$\Omega \le \left\lceil \frac{\mathcal{D}}{c_1 c_2 P} W \exp(h) \right\rceil \le 131384680780655552448464590 51.$$

Use this bound, $\mathfrak{v} < 3^{50}$, and Theorem 10.92 at $I = J = 0$, $X = R_{j+1}$, $T' = T_{j+1}$. There is no scalar denominator. The normalized torus values at integer nodes belong to $\mathbb{Q}$, and their full-width height divided by Z is at most $R_{j+1}/(c_1 c_2 S)$. Retain the coefficient term $h_2(c) + \frac{1}{2} \ln N$, with $\ln N/Z \le g_8$.

The following are outward certified bounds, all valuation excesses divided by Z. Their computation and every integer rounding are detailed in Solution 39.

Extension	Analytic lower bound	Arithmetic upper bound
$j = 0$ to 1	1.339613527059	1.269329770046
$j = 1$ to 2	2.205396661115	1.946854665422
$j = 2$ to 3	3.629068791040	3.349870794944

Each comparison is strict. If a prepared output value were nonzero, (10.244) would contradict its analytic lower bound. Indices of additive order exceeding D_{a} are already zero. This proves the entire output range at each step, and induction proves (10.246). □

This proves the integer block for the stated original rank-two parameter family. Fractional-node extraction, contracted supports, the other ranks and primes, the second stopping branch and the final multiplicity contradiction require their own comparisons.

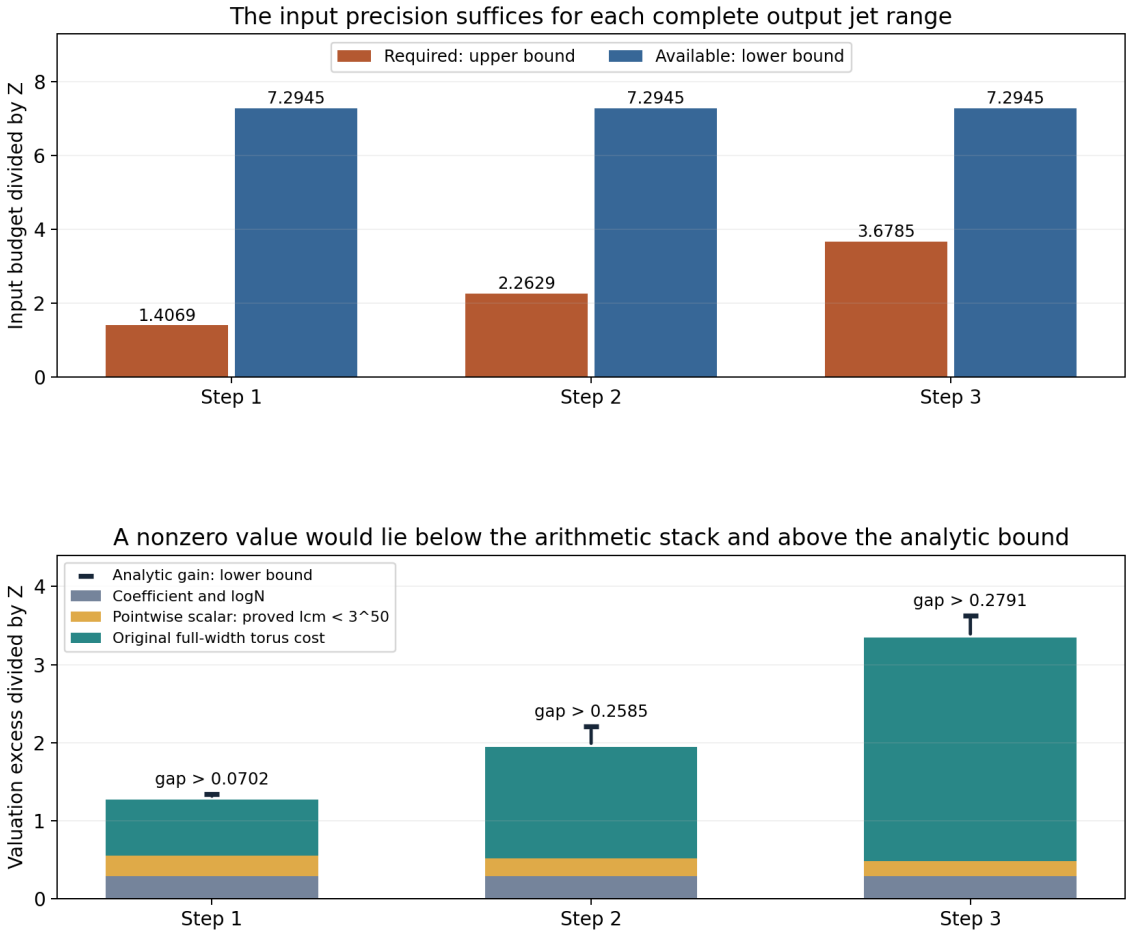


Figure 10.19. The constants, floors and permitted coefficient family are those of (10.245), with $k = 50, L = 35299$ and the proved clearing bound 3^{50}. Each upper panel pair compares the available lower input budget $(U + G_0)/Z$ with the required upper budget $(n_j\mu_j\theta + \mu_j M_0)/Z$. The lower stacks are certified upper bounds from the coefficient, individual scalar and full-width torus terms in (10.244); the black marks are the certified analytic lower bounds. Their strict gaps force zero values. These are bounds rather than sampled valuations of an auxiliary function. Lemma 10.91 proves actual normality, Theorem 10.92 proves the full pointwise budget, and Theorem 10.93 proves all three integer extensions. Solution 39 reconstructs every bound. Figure program. Human-source context: Yu's free paper, Lemma 5.2 and equations (5.25)–(5.38).

32. Nested integer jets and the first fractional step

Earlier integer extensions retain more derivatives at their inner nodes. These derivatives can be used together. The following interpolation argument keeps the full nested distribution rather than assigning its smallest multiplicity to every node.

Hermite interpolation on nested intervals

Lemma 10.94 (nested normal-series interpolation). Let $0 \le R_0 < \cdots < R_m$ be integers and let $\mu_0 \ge \cdots \ge \mu_m \ge 1$ be integers. Put

$$\begin{aligned} E_j &= [-R_j, R_j] \cap \mathbb{Z}, \quad \mu(s) = \mu_j \quad (s \in E_j \setminus E_{j-1}), \\ E_{-1} &= \varnothing, \qquad\qquad N_* = (2R_0 + 1)\mu_0 + 2\sum_{j=1}^{m} (R_j - R_{j-1})\mu_j. \end{aligned} \tag{10.247}$$

Choose $v_p(\rho) = \theta > 0$, a normal series F, and $C \geq 0$. Set $B = \lfloor \log_p(2R_m) \rfloor$ when $R_m \geq 1$, and $B = 0$ when $R_m = 0$. Suppose $v_p(F_h(\rho s)) + h\theta \geq \Lambda - hC$ for $s \in E_m$ and $0 \leq h < \mu(s)$. Then

$$v_p(F(\rho x)) \geq \min\{N_*\theta,\ \Lambda - (\mu_0 - 1)\max\{B, C\}\} \qquad (x \in \mathbb{Z}_p \subset \mathbb{Q}_p). \tag{10.248}$$

Proof. For $s \in E_m$, form the rational polynomial

$$L_s(X) = \prod_{\substack{t \in E_m \\ t \neq s}} \left(\frac{X - t}{s - t}\right)^{\mu(t)}.$$

It has value one at s and a zero of order $\mu(t)$ at every other node t. We claim

$$v_p(L_s(x)) \geq -(\mu_0 - \mu(s))B \qquad (x \in \mathbb{Z}_p).$$

The claim holds at a node by its zero or unit value, so consider a point that is not a node. Decompose the exponents as

$$\mu(t) = \mu_m + \sum_{j=0}^{m-1} (\mu_j - \mu_{j+1}) \mathbf{1}_{t \in E_j}.$$

For a full integer interval, the counts in residue classes modulo p^a differ by at most one. If $s \in E_j$, the contribution of that interval, omitting s, at level a is

$$N_a(x) - N_a(s) + 1 - \mathbf{1}_{x \equiv s \pmod{p^a}} \geq 0.$$

If $s \notin E_j$, no node is omitted and the contribution is $N_a(x) - N_a(s) \geq -1$. For $a > B$, every denominator difference $s - t$ has valuation less than a, since $0 < |s - t| \leq 2R_m < p^a$; those remaining contributions are nonnegative. The base interval E_m always contains s. Only increments belonging to intervals not containing s can lose valuation, and their sum is $\mu_0 - \mu(s)$. This proves the claim.

The formal Taylor inverse

$$A_s(U) = L_s(s + U)^{-1} = \prod_{t \neq s} (1 + U/(s - t))^{-\mu(t)}$$

has its degree-h coefficient of valuation at least $-hB$. Each factor has integer binomial coefficients, and each denominator difference has valuation at most B. Consequently the Hermite polynomial is

$$H(\rho X) = \sum_{s \in E_m} L_s(X) \sum_{h=0}^{\mu(s)-1} \rho^h F_h(\rho s)(X - s)^h [A_s(X - s)]_{\leq \mu(s) - 1 - h}.$$

Its degree is at most $N_* - 1$, and its divided jets at each node agree with those of F through order $\mu(s) - 1$. A summand with inverse coefficient of degree $a \leq \mu(s) - 1 - h$ has valuation at least

$$\Lambda - (\mu_0 - \mu(s))B - hC - aB \geq \Lambda - (\mu_0 - 1)\max\{B, C\}.$$

For completeness, the normal quotient argument of Lemma 10.33 applies to the nonuniform root polynomial $W(Z) = \prod_s (Z - \rho s)^{\mu(s)}$. Indeed $W = Z^{N_*} - A$, where every coefficient of A has positive valuation. Write Q and R for quotient and remainder on division by Z^{N_*}. Both have Gauss operator norm at most one. Let M_A mean multiplication by A. The convergent series

$$G = \sum_{j \geq 0} (Q \circ M_A)^j Q F$$

is normal, because multiplication by A strictly decreases Gauss norm and the normal-series algebra is complete by Lemma 9.2. With $H_0 = R(F + AG)$, one has $F = H_0 + WG$, $\deg H_0 < N_*$, and H_0 is

normal. Its jets are the required jets. Uniqueness of the polynomial with those jets follows because a polynomial of degree less than N_* divisible by every $(Z-\rho s)^{\mu(s)}$ is zero. Thus $H_0 = H$.

Finally $v_p(W(\rho x)) \geq N_*\theta$, and $G(\rho x)$ is integral. Combining this with the bound on H proves (10.248). □

In the prepared-family setting, suppose the values at a node in the j-th interval are zero through total order T_j. For a desired output order T', choose nonincreasing multiplicities $\mu_j \leq T_j - T' + 1$. Lemma 10.38 and the common normality scale of Proposition 10.37 give the hypotheses of Lemma 10.94 with $\Lambda = U - \beta + G_0$ and $C = C_0$. If $M_0 = \max\{B, C_0\}$, then $\beta \leq M_0$, so the sufficient input inequality is

$$U + G_0 \geq N_*\theta + \mu_0 M_0. \tag{10.249}$$

Under this inequality the prepared analytic valuation excess is at least $N_*\theta - G_0$ at every $x \in \mathbb{Z}_p$, for every total prepared order at most T'. The comparison with the unprojected family uses the same projection estimate as Corollary 10.40; no new precision hypothesis is introduced.

The original first fractional range

Theorem 10.95 (the first fractional step for the rational family). Under the hypotheses and with the initial family of Theorem 10.93, put $S_1 = \eta^{-3}S$, $T_1^* = \eta^3 T$. Then

$$\Phi_0(s/2; \boldsymbol{t}) = 0 \quad \text{for} \quad |s| \leq 2(\lfloor S_1 \rfloor + 1) = 1374, \qquad |\boldsymbol{t}| \leq \lfloor T_1^* \rfloor = 993090. \tag{10.250}$$

This range uses the original unfloored S, T, and the coefficient family is the whole family (10.245).

Proof. Theorem 10.93 and the initial construction give these nested integer data.

j	Radius R_j	Available order T_j
0	379	1796752
1	758	1474535
2	1517	1210101
3	3034	993090

For output order $T' = 993090$, take

$$(\mu_0, \mu_1, \mu_2, \mu_3) = (780000, 481446, 217012, 1).$$

Each multiplicity is at most the available order minus T' plus one. The ring counts are $759, 758, 1518, 3034$, hence $N_* = 1286383318$. Here $B = 7$, $C_0 \leq 46$, and $M_0 = 46$. The certified bounds are

$$\frac{N_*\theta + 780000 \cdot 46}{Z} \leq 7.244899709935 < 7.294535093395 \leq \frac{U + G_0}{Z}.$$

Thus (10.249) holds. The analytic valuation excess at every fractional output argument, divided by Z, is at least 7.100020502185. Division by two preserves membership in $\mathbb{Z}_3$.

We must retain the field of these values. Put $\gamma_1 = -8, \gamma_2 = -125$ and $\eta_i = \exp(\frac{1}{2}\log\gamma_i)$. Their logarithms have valuation two, so Lesson 9 proves that these are square roots in $\mathbb{Q}_3$, each congruent to one. Their global field is

$$F = \mathbb{Q}(\eta_1, \eta_2) = \mathbb{Q}(\sqrt{-2}, \sqrt{-5}), \qquad [F : \mathbb{Q}] = 4.$$

The two square classes are independent: $-2, -5, 10$ are all nonsquares, as their prime valuations or signs show. Equivalently, if $\sqrt{-5} = a + b\sqrt{-2}$ with rational a, b, squaring forces $ab = 0$; $b = 0$ would

make -5 a rational square, and $a = 0$ would make $5/2$ a rational square. Both are impossible. This proves the degree. The chosen completion of F is $\mathbb{Q}_3$, since $F \subset \mathbb{Q}_3$ and $\mathbb{Q}$ is dense there. Hence its chosen e, f are one. Every normalized monomial at $s/2$ is a product of integer powers of η_1^s, η_2^s, so every prepared value belongs to this ambient field. Individual values may belong to a proper subfield; the bound over F remains valid.

The outward parameter intervals give

$$686.229279178168 \leq S_1 \leq 686.229279178169.$$

Thus $X = 687$ covers the entire output range. In Theorem 10.92 take $I = 0, J = 1, R = X + 99 = 786$, the same common Euler bound of Theorem 10.93, and $T' = 993090$. Its scalar maximum, using the proved upper clearing bound 3^{50}, occurs at additive order 470396, leaving Euler order 522694. The ratio tests are exact integer products. The entropy estimates proved in Solution 39, with the negative factorial logarithm retained, give $\ln \mathcal{H}_0^{\mathrm{fine}}/Z \leq 0.208379911543$.

Also $v_2(50!) = 25 + 12 + 6 + 3 + 1 = 47$. Equation (10.243) therefore gives the common denominator exponent

$$\Xi_{0,1,u} \leq D_{\mathrm{a}} + 47L = 3424003$$

for every additive order. This deliberately retains a valid common upper bound; the order recovery can only improve it.

The four terms inside the arithmetic bracket of (10.244), divided by Z, have the following certified upper bounds.

Term	Upper bound
Coefficient height and $\frac{1}{2}\ln N$	0.310842088027
Individual scalar logarithm	0.208379911543
q-denominator logarithm	0.008748404912
Original full-width torus height	0.714102914710

Multiplying their sum by the actual ambient global-to-local factor $4/\ln 3$, using the unrounded rational intervals before the final rounding, gives an arithmetic valuation excess at most $4.522335429896Z$. It is strictly smaller than the analytic lower bound. Their gap exceeds $2.577685072290Z$. Thus every nonzero prepared value would contradict (10.244); orders above D_{a} are already zero. This proves (10.250). $\square$

This establishes the first fractional step for the stated original rank-two family. The following section completes its finite contraction chain and terminal contradiction; other parameter cases require separate arguments.

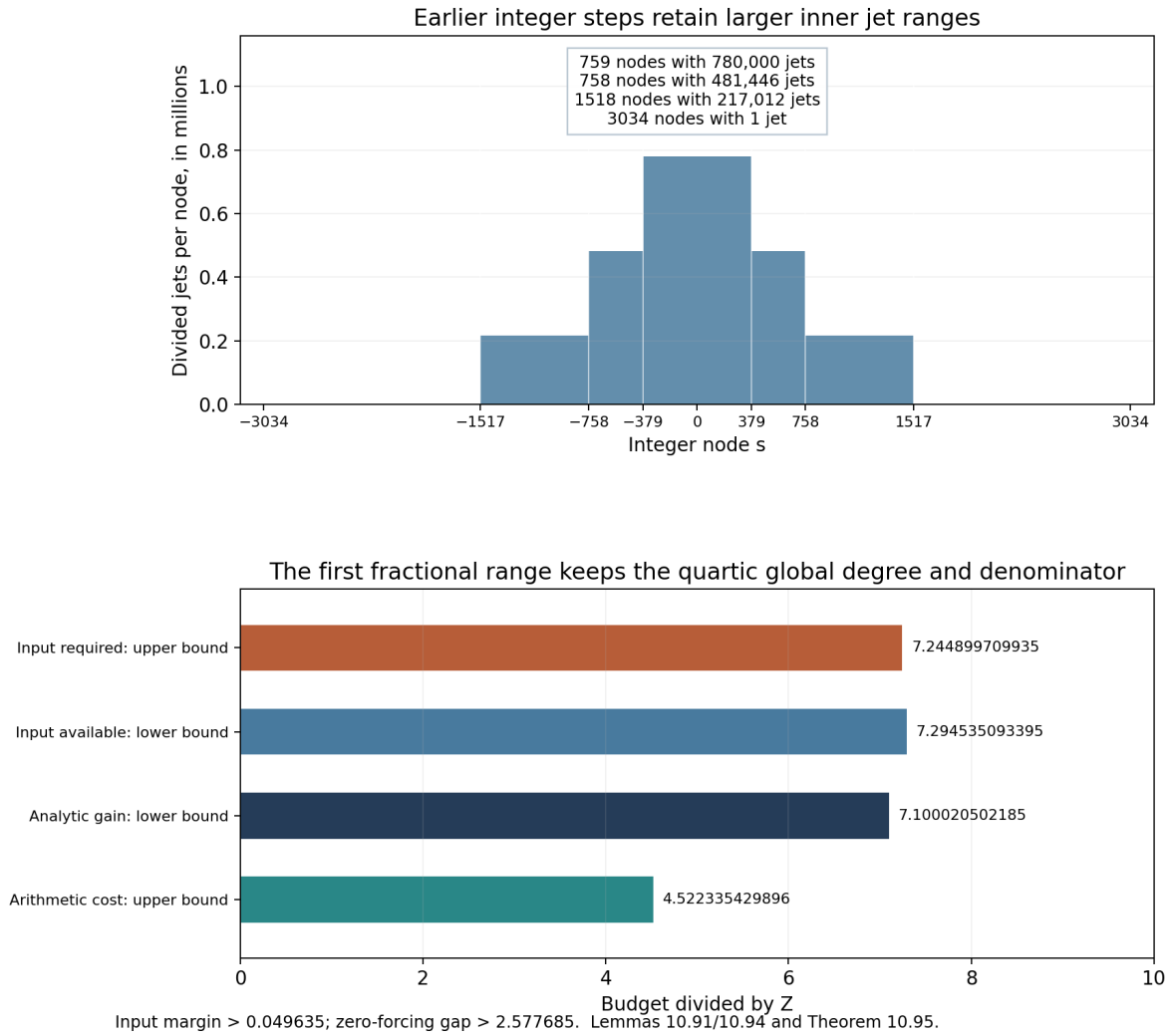


Figure 10.20. Each unit cell in the upper panel represents one integer node, with constant height equal to its available multiplicity $\mu(s)$; cell edges are half-integers. The exact interval radii are $379, 758, 1517, 3034$, and the four ring counts and multiplicities give $N_* = 1286383318$. The lower panel shows certified bounds divided by Z: the input required is below the input available, and the arithmetic upper bound is below the analytic lower bound. The arithmetic cost includes the quartic ambient field and the full common q-denominator exponent 3424003. These are proved bounds, not sampled values of an auxiliary function. Lemma 10.94 proves the nested interpolation; Theorem 10.95 proves the exact original first fractional range; Solution 40 checks the Hermite basis and calculations. Figure program. Human-source context: Yu's free paper, Lemma 5.3.

33. Seventeen contractions and a polynomial contradiction

The first fractional step can be repeated after restriction to an exponent coset. Three features make a complete finite descent possible for the family (10.245): the coefficients remain a subvector of the initial vector, the actual Euler arguments shrink with the support, and integer values return to the rational field. We first prove the interpolation estimate that uses the odd nodes together with two full intervals.

Odd nodes inside nested full intervals

Lemma 10.96 (mixed nested interpolation). Let $p \neq q$ be primes and $1 \leq A \leq R$ integers. Put

$$\begin{aligned} E_0 &= \{s \in \mathbb{Z} : |s| \leq A,\ q \nmid s\}, \qquad E_1 = [-A, A] \cap \mathbb{Z}, \qquad E_2 = [-R, R] \cap \mathbb{Z}, \\ \mu(s) &= \mu_0 \quad (s \in E_0), \qquad \mu(s) = \mu_1 \quad (s \in E_1 \setminus E_0), \quad \mu(s) = \mu_2 \quad (s \in E_2 \setminus E_1), \end{aligned} \tag{10.251}$$

where $\mu_0 \ge \mu_1 \ge \mu_2 \ge 1$. Define

$$N_* = (2R+1)\mu_2 + (2A+1)(\mu_1-\mu_2) + |E_0|(\mu_0-\mu_1), \qquad B = \lfloor \log_p(2R) \rfloor, \quad M = \max\{B, C\}.$$

Suppose F is normal, $v_p(\rho) = \theta > 0$, $C \ge 0$, and $v_p(F_h(\rho s)) + h\theta \ge \Lambda - hC$ for $0 \le h < \mu(s)$. Then

$$v_p(F(\rho x)) \ge \min\{N_*\theta,\ \Lambda - B(\mu_0-\mu_1) - (\mu_0-1)M\} \quad (x \in \mathbb{Z}_p \subset \mathbb{Q}_p). \tag{10.252}$$

Proof. Write the multiplicity as $\mu_2 + (\mu_1-\mu_2)\mathbf{1}_{E_1} + (\mu_0-\mu_1)\mathbf{1}_{E_0}$. For a full integer interval, residue-class counts modulo p^a differ by at most one. For E_0, they differ by at most two: subtract the counts of multiples of q from those of the full interval, and use that multiplication by q permutes residue classes modulo p^a.

Use the cardinal polynomial

$$L_s(X) = \prod_{\substack{t \in E_2 \\ t \ne s}} \left(\frac{X-t}{s-t} \right)^{\mu(t)}$$

from the proof of Lemma 10.94. At a residue level $a \le B$, an interval containing s contributes $N_a(x) - N_a(s) + 1 - \mathbf{1}_{x \equiv s \ (p^a)}$. This is nonnegative for a full interval and at least -1 for the deleted interval E_0. If $x \equiv s$, it is zero; otherwise the asserted bounds follow from the respective count differences. A level not containing s contributes at least -1 for a full interval and -2 for E_0. For $a > B$, no denominator difference contributes and all remaining contributions are nonnegative. Therefore

$$v_p(L_s(x)) \ge -B(\mu_0 - \mu(s) + \mu_0 - \mu_1).$$

At a node this inequality also follows from its zero or unit value.

The coefficient of U^a in $L_s(s+U)^{-1}$ has valuation at least $-aB$, by the integer negative-binomial expansion proved in Lemma 10.94. In its Hermite formula, a term using the divided jet $h < \mu(s)$ and inverse coefficient $a \le \mu(s) - 1 - h$ consequently has valuation at least

$$\Lambda - B(\mu_0 - \mu(s) + \mu_0 - \mu_1) - hC - aB \ge \Lambda - B(\mu_0-\mu_1) - (\mu_0-1)M.$$

The polynomial has degree less than N_* and the prescribed jets. The monic normal quotient proof of Lemma 10.94 applies unchanged to $W(Z) = \prod_{s \in E_2}(Z - \rho s)^{\mu(s)}$; it gives $F = H + WG$ with G normal. Thus $v_p(W(\rho x)G(\rho x)) \ge N_*\theta$. This proves (10.252). □

For the prepared family, $\Lambda = U - \beta + G_0$, $C = C_0$, and $\beta \le M$. Hence a sufficient input condition is

$$U + G_0 \ge N_*\theta + \mu_0 M + B(\mu_0 - \mu_1). \tag{10.253}$$

It gives analytic valuation excess at least $N_*\theta - G_0$, with the same projection comparison as Corollary 10.40. The last term in (10.253) is the cost of the odd-only data. It cannot be omitted by treating those nodes as a full interval.

What each contraction preserves

Proposition 10.97 (rational contraction invariants). Start with the support and integral coefficients of Theorem 10.93, and use $\gamma_1 = -8, \gamma_2 = -125$. Whenever the stage I family has exact zeros at $s/2$, for odd s, through total prepared order O, one can choose a nonzero stage $I+1$ family with exact integer zeros at those same odd s through order O. At every stage choose a reference exponent carrying a coefficient of minimum three-adic valuation. Then

$$\begin{gathered}\delta_I = \delta_0, \qquad h_2(c_I) \le h_2(c_0), \qquad N_I \le N_0,\\ \operatorname{width}_i(\mathcal{M}_I) \le D_i/2^I, \qquad 0 \in \mathcal{M}_I, \qquad |\omega(\boldsymbol{n})| \le \Omega_I := \left\lfloor \frac{\Omega_0}{2^I} \right\rfloor,\\ \Omega_0 = 131384680780655244846459051, \qquad \omega(\boldsymbol{n}) = b_2 n_1 - b_1 n_2.\end{gathered} \tag{10.254}$$

Each family is nonzero, has additive degree at most $D_{\mathrm{a}} = 1764950$, and has integral coefficients. Its normality scale remains $G_0 - \delta_0$, with $C_0 \le 46$. Integer prepared values belong to $\mathbb{Q}$; fractional values at $s/2$ belong to the fixed field $F = \mathbb{Q}(\sqrt{-2}, \sqrt{-5})$, whose global-to-local factor at the chosen three-adic place is four.

Proof. Theorem 10.95 proves that $1, \eta_1, \eta_2, \eta_1\eta_2$ are independent over $\mathbb{Q}$, where $\eta_i^2 = \gamma_i$. For odd s, reducing powers of η_i^s modulo two multiplies these four basis elements by nonzero rational factors. Thus an exact zero splits into four exact exponent-coset zeros. Choose a coset containing a coefficient of valuation δ_0, with exponent $\boldsymbol{m}_*$, and set $\boldsymbol{n} = (\boldsymbol{m} - \boldsymbol{m}_*)/2$. Theorem 10.48 proves that the new coefficients are precisely the selected old coefficients, with no mixing.

The old Euler argument is $2\omega(\boldsymbol{n}) + \omega(\boldsymbol{m}_*)$. Lemma 10.30 expands its binomial polynomials in the new binomial basis. The triangular matrix and its inverse lie in $\mathbb{Z}[1/2]$, are three-integral, and preserve total order. Their diagonal entries are powers of two. They carry all extracted zero jets to all new zero jets without lowering O. The additive factor at the old argument $s/2$ is exactly the stage $I+1$ factor at s: $2^I \mathfrak{v}\, \partial_X$ cancels the scaling 2^{-I} in each divided derivative. The factored root monomial is a local unit.

The chosen subvector retains δ_0, integrality and the projective-height bounds of Theorem 10.48. It contains a nonzero coefficient, and the additive-basis independence of Lemma 10.29 proves the new polynomial nonzero. Its support contains $\boldsymbol{n} = 0$, and every coordinate width is divided by two. Induction gives (10.254). Since zero is a reference inside the support,

$$|b_2 n_1 - b_1 n_2| \le 2^{-I}(|b_2| D_1 + |b_1| D_2) \le 2^{-I}\Omega_0;$$

the left side is an integer.

The ordinary analytic torus uses the same $\log \gamma_i$, the same projected slopes and integer exponents $\boldsymbol{n}$. The proof of Lemma 10.91 therefore supplies the same normality scale. Powers of two in additive differentiation are three-adic units, so the ordinary-jet loss in Lemma 10.38 still uses $C_0 \le 46$. At integers the torus is $\gamma_1^{sn_1}\gamma_2^{sn_2}$, hence rational. At $s/2$ it is $\eta_1^{sn_1}\eta_2^{sn_2}$. No further roots are introduced when a support is contracted: the torus bases remain γ_i. The global field is always the same quartic field proved in Theorem 10.95, with chosen completion $\mathbb{Q}_3$. This proves all the assertions. □

The inverse Euler change is three-integral, but need not be integral at two. It does not change the coefficient vector of the new polynomial. The height estimate below is applied directly to its new integer Euler arguments; it does not charge that jet-change matrix as an extra polynomial coefficient.

All stages with their original radii and orders

Theorem 10.98 (complete finite contraction chain). Under the contradiction hypothesis of Theorem 10.93, there are nonzero contracted families satisfying Proposition 10.97 for $I = 1, \dots, 17$. Define from the original real parameters

$$\begin{aligned} &S_I = \eta^{-3I} S, && T_I = \eta^{3I} T, && O_j = \lfloor \eta^j T_I \rfloor \quad (0 \le j \le 3),\\ &A_I = 2(\lfloor S_I \rfloor + 1), && R_I = \lfloor 4 S_I \rfloor, && X_I = \lfloor S_I/\eta^3 \rfloor + 1. \end{aligned} \tag{10.255}$$

The stage I family vanishes at every odd integer $|s| \le A_I$ through order O_0, at every integer $|s| \le A_I$ through order O_1, at every integer $|s| \le R_I$ through order O_2, and at every $s/2$ with $|s| \le 2X_I$ through order O_3.

Proof. Theorem 10.95 and Proposition 10.97 give the first odd-node input. We give all estimates for the induction.

For closure of the odd-node set, let $n = A_I$, $\mu = O_0 - O_1 + 1$, $B = \lfloor \log_3(2A_I) \rfloor$, and $M = \max\{46, B\}$. The radius A_I is even, so it contains exactly A_I odd nodes. The deleted-node interpolation of Lemma 10.39, with its proved residue-count loss, has sufficient input and analytic gain

$$\mathcal{B}_{I,0} = n\mu\theta + \mu(B + M), \quad \mathcal{G}_{I,0} = n\mu\theta - G_0. \tag{10.256}$$

Indeed its Hermite loss is $\mu B + (\mu - 1)M$; adding $\beta \le M$ proves the stated sufficient bound. This fills the even nodes as well. For extension from the full interval A_I to R_I, take $n = 2A_I + 1$, $\mu = O_1 - O_2 + 1$, and the same B, M. Full-interval interpolation gives

$$\mathcal{B}_{I,1} = n\mu\theta + \mu M, \qquad \mathcal{G}_{I,1} = n\mu\theta - G_0. \tag{10.257}$$

For the next fractional step use Lemma 10.96 with $q = 2$, $A = A_I$, $R = R_I$, and

$$\begin{aligned} &\mu_0 = \lfloor 780000\eta^{3I} \rfloor, && \mu_1 = O_1 - O_3, \quad \mu_2 = O_2 - O_3, \\ &N_* = (2R_I + 1)\mu_2 + (2A_I + 1)(\mu_1 - \mu_2) + A_I(\mu_0 - \mu_1), \\ &\mathcal{B}_{I,2} = N_*\theta + \mu_0 M + B(\mu_0 - \mu_1), && \mathcal{G}_{I,2} = N_*\theta - G_0, \end{aligned} \tag{10.258}$$

where now $B = \lfloor \log_3(2R_I) \rfloor$. For all seventeen stages, the exact floors give $\mu_0 \ge \mu_1 \ge \mu_2 \ge 1$ and $\mu_0 \le O_0 - O_3 + 1$. The smaller choices $\mu_1 = O_1 - O_3$, $\mu_2 = O_2 - O_3$ are valid: each is below the available order difference plus one. They do not lower the output order O_3. All $s/2$ in the asserted range belong to $\mathbb{Z}_3$.

Here is the arithmetic estimate used for each output comparison. Let its total order be O, its absolute argument bound be X, and put $J = 0$ for integers, $J = 1$ for half-integers. Take respectively

$$(O, X, J) = (O_1, A_I, 0),\ (O_2, R_I, 0),\ (O_3, X_I, 1).$$

Set $r = X/2^I + 99$, $D = D_{\mathrm{a}}$, and $V = 3^{50}$. The pointwise scalar bound in Theorem 10.92 is

$$\mathcal{H}_I(O, X) = \max_{0 \le u \le \min\{D, O\}} \frac{r^{D-u}}{(50!)^L} \binom{D}{u} V^u \binom{\Omega_I + O - u}{O - u}. \tag{10.259}$$

Keep the uniform denominator exponent $\Xi_{I,J} = (I + J)D + 47L$, which bounds (10.243) at every order. Proposition 10.97 and Theorem 10.92 give the nonzero-value upper bound

$$\begin{aligned} \mathcal{A}_{I,J}(O, X) = \frac{d_J}{\ln 3}\Big(&0.310817025826Z + \tfrac{1}{2}g_8 Z + \ln \mathcal{H}_I(O, X) \\ &+ \Xi_{I,J} \ln 2 + \frac{Z(X/2^I)}{1.4494 \cdot 1.75S}\Big), \\ d_0 = 1, \qquad d_1 = 4. \end{aligned} \tag{10.260}$$

The coefficient term is the uniform bound of Theorem 10.93; shrinking the vector can only improve it. The last term is the original full-width torus bound after I contractions. The common clearing term retains $v_2(50!) = 47$. Integer values use the actual rational field; every fractional value is bounded over the fixed quartic field. This explains every term in (10.260).

The next table gives an upper bound for $\max_j \mathcal{B}_{I,j}/Z$ and a lower bound for the minimum of the three gaps $(\mathcal{G}_{I,j} - \mathcal{A}_{I,j})/Z$, with the respective output arguments above. The common available input is $(U + G_0)/Z \geq 7.294535093395$.

I	Maximum required input, upper bound	Minimum zero-forcing gap, lower bound
1	7.192833169904	0.240138949661
2	7.156267657061	0.358632162785
3	7.136548280236	0.444858960780
4	7.125498620658	0.510791197596
5	7.119172043534	0.563520072068
6	7.115527327639	0.607407734375
7	7.113058026427	0.644775387448
8	7.112111071831	0.677281716283
9	7.112117675512	0.705361309159
10	7.109803325816	0.730342112918
11	7.110367794024	0.754317954142
12	7.105538214081	0.775918682856
13	7.095336861199	0.798823134852
14	7.111140057941	0.818899480846
15	7.120519275568	0.839066755948
16	7.021962525897	0.912928544835
17	7.002483730960	0.985921533843

These are outward rational bounds. Solution 41 gives their calculation from the convergent series and exact integer ratio tests already proved in Solutions 34 and 39. In particular the maximum in (10.259) is bounded without floating-point exponentiation or constructing the enormous Euler binomial. Both input and zero-forcing comparisons hold at every one of the fifty-one entries.

Interpolation now gives excess at least $\mathcal{G}_{I,j}$. If a respective prepared value were nonzero, (10.260) would bound its excess by a strictly smaller quantity. Thus each closure, extension and fractional output is zero through its stated order. For odd output numerators, Proposition 10.97 creates the next contracted family. Since $S_{I+1} = S_I/\eta^3$, $T_{I+1} = \eta^3 T_I$, its odd radius and order are exactly $2X_I = A_{I+1}$ and $O_3 = \lfloor T_{I+1} \rfloor$. This proves the induction and all asserted ranges. □

The last support has one exponent

Corollary 10.99 (a complete valuation bound for the stated family). For every pair of nonzero integers b_1, b_2 satisfying (10.245),

$$v_3(2^{b_1}5^{b_2} - 1) < \frac{8Z}{\ln 3}. \tag{10.261}$$

Proof. Suppose the reverse inequality holds. Theorem 10.98 gives a nonzero stage-seventeen family. The original parameter bounds of Solution 34 are $D_1 < 67805$, $D_2 < 29202$. Both are smaller than $2^{17} = 131072$, so (10.254) gives integer coordinate widths zero. The support is nonempty and contains its reference zero, hence is exactly $\{0\}$. The polynomial is therefore

$$Q_{17}(X) = \sum_{a,\ell} c_{17,a,\ell} \Delta(2^{-17}X + a; 50)^{\ell}, \qquad 0 \neq Q_{17}, \qquad \deg Q_{17} \leq 1764950. \tag{10.262}$$

At this stage $A_{17} = 18092274$, $O_1 = 61$. The full integer closure in Theorem 10.98 includes the prepared order $(0, 0)$, so $Q_{17}(s) = 0$ at all 36184549 distinct integers in that interval. A nonzero

degree-d polynomial has at most d distinct roots: at each root polynomial division gives a factor $X - s$, and division by the preceding distinct factors leaves the next factor nonzero at the new root. Induction proves the root bound. It contradicts (10.262). Also $2^{b_1}5^{b_2} \neq 1$, since its two-adic valuation is the nonzero integer b_1. Thus no infinite-valuation exception occurs, and (10.261) follows. □

This proves a complete finite descent and contradiction for the entire rational coefficient family (10.245). Other ranks, fields, primes and parameter families require their own comparisons; this corollary does not assert the general Yu estimate (2.9) or its variants.

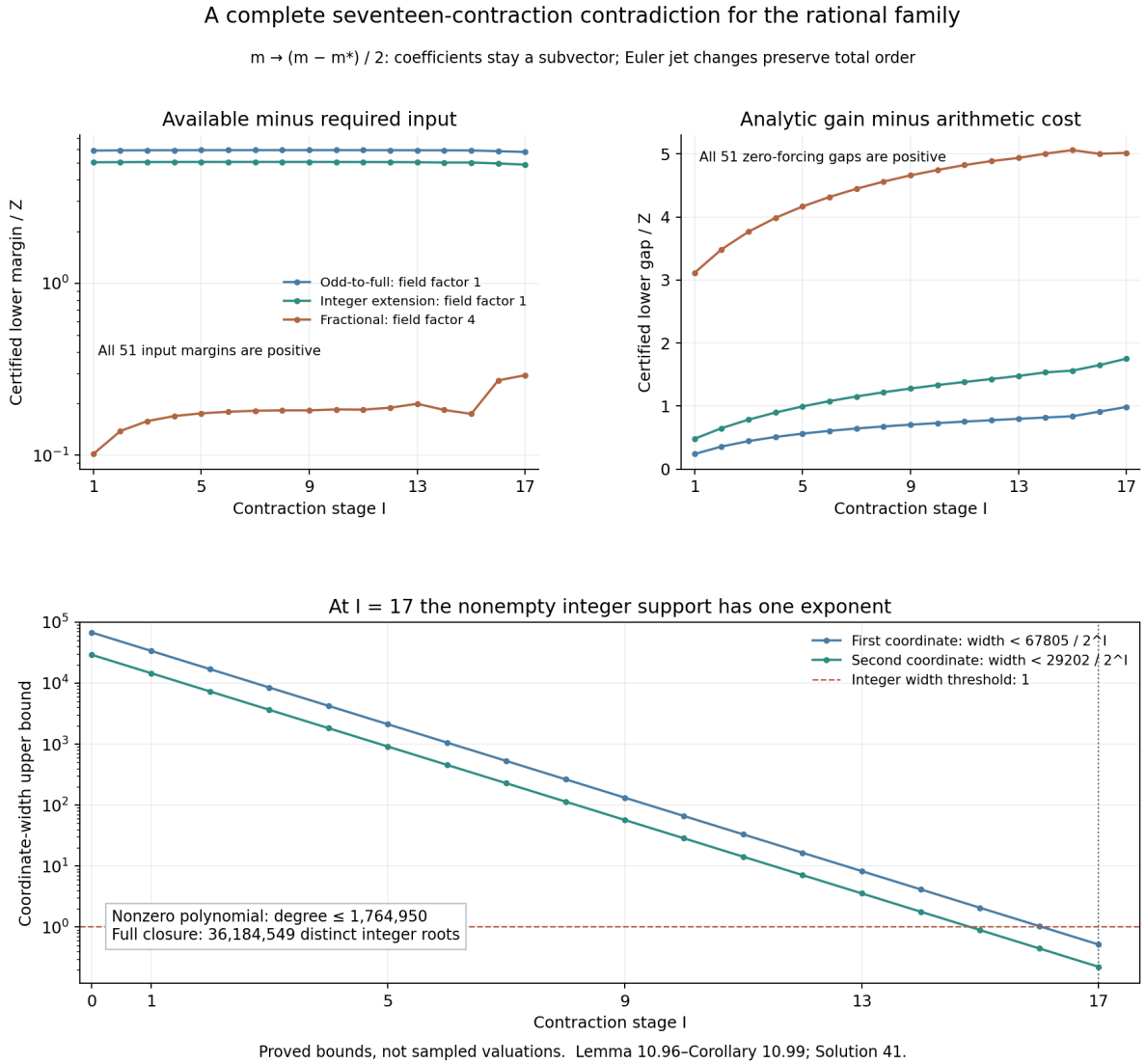


Figure 10.21. At stage I, restriction to a minimum-coefficient exponent coset sends $\boldsymbol{m}$ to $(\boldsymbol{m} - \boldsymbol{m}_*)/2$, with the forward and inverse Euler changes preserving total jet order. The upper panels give certified lower margins for all three comparisons at each stage; they are bounds, not sampled valuations. Integer arithmetic uses global-to-local factor one, fractional arithmetic factor four at every stage. The lower panel shows the proved coordinate-width upper bounds $D_i/2^I$; the horizontal line is width one. At seventeen both integer widths vanish, leaving the nonzero polynomial of degree at most1764950 with36184549 integer roots. Lemma10.96, Proposition10.97, Theorem10.98 and Corollary10.99 prove these mechanisms; Solution41 reconstructs the bounds. Figure program. Human-source context: Yu's free paper, Lemmas5.3–5.4 and the support passage in Section5.

34. Uniform degree comparisons at the geometric stop

The general descent has a geometric stopping depth, where the multiplicity theorem produces fewer independent algebraic bases. Its degree comparisons can be proved uniformly from the height-product bound of Theorem 10.18. In particular, they require no additional individual-height theorem.

We use the parameters (10.183)–(10.186), writing $a = r + 1$, $x = \nu \ln q$, and $\ell_d = \max\{1, \ln d\}$. The independent bases α_i belong to the same degree-d field, and $\sigma_i \geq h(\alpha_i) > 0$. Use the original choices of $\widehat{\vartheta}$ in (10.184), and $0 \leq \tau \leq 1/100$. The fixed case constants satisfy the following weaker rational bounds:

$$\begin{gathered} \frac{19}{10} \leq c_0 \leq 3, \qquad c_2 \geq 1, \qquad c_3 > \frac{47}{100}, \qquad \frac{15}{4} < c_4 < 21, \\ a_* \geq \frac{7}{2}, \qquad \rho \leq 58, \qquad \theta \leq \widehat{\vartheta} \leq 2, \qquad e\theta > \frac{49}{100}, \\ \eta = 1 - \frac{c_5}{r+1} > \frac{18}{25}, \qquad 1 < q\eta^{r+1}, \qquad q\eta^{2(r+1)} < 1. \end{gathered} \tag{10.263}$$

The bound $a_* \geq 7/2$, rather than seven, includes the special unramified case $p \geq 5, e = 1$. For the geometric comparisons the relevant ranges of c_5 are $0.5267 \leq c_5 \leq 0.56$ at $q = 2$, and $0.753 \leq c_5 \leq 0.827$ at $q = 3$. The other rational bounds are direct comparisons with the fixed case constants. The last two inequalities in (10.263), and all estimates used below, are proved explicitly.

Let I be any nonnegative integer with

$$\begin{gathered} E_I = (q\eta^{r+1})^I > \exp(3(A + x)), \\ \overline{S} = q\eta^{-(r+1)I} S, \qquad \overline{T} = \eta^{(r+1)I+1} T, \qquad \mathsf{D}_0 = kL, \qquad \mathsf{D}_j = 2q^{\nu - I} D_j \quad (1 \leq j \leq r). \end{gathered} \tag{10.264}$$

The original geometric stopping choice is $I = \lfloor 3(A + x)/\ln(q\eta^{r+1}) \rfloor + 1$. The widths D_j and their normalization are exactly those in (10.186). The positive real D_j are upper coordinate-degree bounds; the actual degrees are integers. For the ordinary jet budget set $S^\circ = \lfloor \overline{S} \rfloor$, $T^\circ = \lfloor \overline{T} \rfloor$, and allocate

$$S_0 = \left\lfloor \frac{\overline{S}}{3} \right\rfloor, \qquad S_j = \left\lfloor \frac{2\overline{S}}{3r} \right\rfloor \ (1 \leq j \leq r), \qquad T_j = \left\lfloor \frac{\overline{T}}{r+1} \right\rfloor \ (0 \leq j \leq r). \tag{10.265}$$

Theorem 10.100 (all geometric-stop degree inequalities). Under these hypotheses, the allocations satisfy (10.67), and

$$\mathsf{D}_0 > 20\mathsf{D}_j \quad (1 \leq j \leq r), \qquad \mathsf{D}_0 > 4\overline{T}, \qquad \mathsf{D}_0 > 4r. \tag{10.266}$$

Consequently $\mathsf{D}_0 > \max\{\mathsf{D}_1, \dots, \mathsf{D}_r, T_r + r\}$. For every $1 \leq m < r$ and every m-element subset J, define the ratios

$$\begin{aligned} \mathcal{R}_0 &= \frac{(S_0 + 1)(T_0 + 1)}{\mathsf{D}_0}, \\ \mathcal{R}_{m,J} &= \frac{(S_m + 1)\binom{T_m + m + 1}{m+1}}{(m+1)!\mathsf{D}_0 \prod_{j \in J} \mathsf{D}_j}, \\ \mathcal{R}_r &= \frac{(S_r + 1)\binom{T_r + r}{r}}{(r+1)!\mathsf{D}_0 \prod_{j=1}^{r} \mathsf{D}_j}. \end{aligned} \tag{10.267}$$

They satisfy the uniform strict bounds

$$\mathcal{R}_0 > \frac{36}{25}, \qquad \mathcal{R}_{m,J} > 8 \quad (1 \le m < r), \qquad \mathcal{R}_r > 31. \tag{10.268}$$

Thus every allocation and degree hypothesis of Theorem 10.25 holds, for every original rank, field and prime parameter case.

Proof. We keep the ramification index e distinct from $\mathrm{e} = \exp(1)$. The power-series proof of the logarithm gives $\ln 2 > 2/3$ and $\ln 3 > 1$. Hence $t = f \ln p > 2/3$, and $A \ge g_1 > 5$. Also $2 < \mathrm{e} < 3$: the exponential series gives the lower bound, while $n! \ge 2 \cdot 3^{n-2}$ for $n \ge 2$ bounds its tail by a geometric series. For $d \ge 1$ one has $\ell_d \le d$, since $\ln d < d$.

First check (10.263). The inequalities for c_5 give $\eta > 18/25$. For fixed $0 < c < 3$, the function $(1 - c/y)^y$ increases for $y \ge 3$. Its logarithmic derivative is $\ln(1 - c/y) + (c/y)/(1 - c/y) > 0$: writing $z = c/y$, the derivative of $\ln(1 - z) + z/(1 - z)$ is $z/(1 - z)^2 > 0$, and its value at zero is zero. Therefore $q\eta^{r+1}$ is bounded below by $2(61/75)^3 > 1$ at $q = 2$, or by $3(2173/3000)^3 > 1$ at $q = 3$. On the other hand, $\ln(1 - z) < -z$ gives $q\eta^{2(r+1)} < q \exp(-2c_5) < 1$. For the last comparison it suffices that $\ln 2 < 7/10$, $\ln 3 < 11/10$; the finite positive exponential sums at $7/10$ through degree four and at $11/10$ through degree six already exceed two and three, respectively. Finally $e\vartheta \ge 1/2$ by (10.184), so $e\theta \ge 50/101 > 49/100$. The stated upper bounds for $\widehat{\vartheta}$ were proved there.

Small uniform parameter bounds

The two alternatives in g_2 give $g_2 > 1$. For the first use $c_3 > 47/100, q \ge 2, r + 1 \ge 3, d \ge 1$. For the second, $g_0 > 39, e \ge 1$, and $p \in \{2, 3, 5\}$, with $\ln 5 < 2$. The latter follows from the finite exponential sum at two through degree three.

Let

$$F_r = \frac{r^r (r + 1)^r}{(r!)^2}.$$

Since $r! \le r^r$, one has $F_r \ge (1 + 1/r)^r > 2$. The definitions (10.185) consequently imply

$$\begin{gathered} g_5 > \frac{893}{7250}(r + 1)\left(\frac{7}{2}\right)^r > 2r, \qquad g_4 > \frac{57}{116}(r + 1)\left(\frac{7}{2}\right)^r > r(r + 1), \\ 1 + \epsilon < \exp(1/2) < 2, \qquad 1 + g_5^{-1} < \frac{5}{4}. \end{gathered} \tag{10.269}$$

Indeed $g_5 = (2c_0c_3q/\rho)(r + 1)a_*^r F_r$. For its second inequality the ratio $(r + 1)(7/2)^r/r$ increases: its consecutive ratio is $(7/2)r(r + 2)/(r + 1)^2 \ge 28/9$. At $r = 2$ the displayed lower bound is $131271/29000 > 4$. For g_4, its extra factor in (10.185) is at least g_1^{-1}, and $\widehat{\vartheta} \le 2$. Thus $g_4 > (qc_0c_4/\rho)(r + 1)a_*^r F_r$, giving its displayed lower bound. The function $(7/2)^r/r$ increases, and at two its comparison with $r(r + 1)$ reduces to $2793/928 > 1$. The bound on ϵ follows from its defining power and $\ln(1 + z) < z$. Since $\mathrm{e} < 3$, also $\exp(1/2) < 2$.

Lemma 10.70 and $k \ge 39$ now give $\mathsf{D}_0 = kL > kg_5 > 78r$. They also give the two useful bounds

$$\frac{S\mathcal{D}}{c_1c_4d(A + x)} < \mathsf{D}_0 < \frac{5}{4}\frac{S\mathcal{D}}{c_1c_4d(A + x)}.$$

Dominance of the additive degree

For a non-torsion α_i, Theorem 10.18 with one base, its saturation index at least one, and $w_K \ge 2$, gives

$$\sigma_i \ge \frac{2}{58\mathrm{e}\, d^2\ell_d}, \qquad d\sigma_i > \frac{1}{87d^2}.$$

At $d = 1$, weakening its proved constant17 to58 preserves the bound. Thus no sharper individual-height result is needed.

Because $q\eta^{2(r+1)} < 1$, we have $\eta^{-(r+1)I} > E_I > \exp(3(A+x))$. Also $q^{I-\nu} > \exp(3A)$. From the definitions of the two degree bounds,

$$\frac{\mathsf{D}_0}{\mathsf{D}_j} > \frac{c_3 c_2 q r (r+1) P}{2c_4}\, d\sigma_j\, \frac{h+x}{A+x}\, \frac{q^{I-\nu}}{t}.$$

Here $c_2 P \geq 1$, $(h+x)/(A+x) \geq 39/(A+39)$, and $t \leq A$. Use $\exp(3A) \geq \exp(2g_1)\exp A = \mathrm{e}^8 (r+1)^2 d^2 \exp A$. The function $\exp A/[A(A+39)]$ increases for $A \geq 5$, by its logarithmic derivative. The ratio is therefore larger than

$$\frac{47}{350}\,\frac{9\cdot 256}{87}\,\frac{39\cdot 32}{5\cdot 44} = \frac{135143424}{6699000} > 20.$$

Similarly the additive degree divided by the real output order satisfies

$$\frac{\mathsf{D}_0}{\overline{T}} > \frac{c_3 e\theta}{c_4}\,\frac{h+x}{A+x}\,\eta^{-1}\eta^{-(r+1)I} > \frac{47}{100}\,\frac{49}{100}\,\frac{39}{21\cdot 44}\,2^{15} > 4.$$

The function $\exp(3A)/(A+39)$ is increasing for $A \geq 5$; this justifies its endpoint bound in the second line. Together with $\mathsf{D}_0 > 4r$, these inequalities imply $\mathsf{D}_0 > 2(\overline{T}+r) > T_r + r$.

Every allocation inequality

The floor definitions in (10.265) have integer sums at most $\overline{S}, \overline{T}$, hence at most S°, T°. They are nonincreasing because $r \geq 2$. Moreover

$$S_0 + 1 > \overline{S}/3, \qquad S_j + 1 > 2\overline{S}/(3r), \qquad T_j + 1 > \overline{T}/(r+1).$$

The binomial product gives $\binom{T_j+b}{b} \geq (T_j+1)^b/b!$ for every positive integer b. These strict floor bounds retain all integer losses.

Put $\lambda = 1 + g_5^{-1} < 5/4$. For $m = 0$, substitution and cancellation give

$$\mathcal{R}_0 > B_0 := \frac{q^2 \eta c_4 d(A+x)}{3\theta e t\, \lambda} > \frac{36}{25}.$$

The last inequality uses $d \geq e$, $A + x \geq t$, $q \geq 2$, $\eta > 18/25$, $c_4 > 15/4$, and $\theta \leq 2$.

For $1 \leq m < r$, retain every subset J. The same substitutions give

$$\mathcal{R}_{m,J} > \frac{2B_0}{r}\,\frac{\prod_{j\in J}\sigma_j}{((m+1)!)^2}\left(\frac{\eta c_2 q P r d}{2e\theta t}\,\frac{E_I}{q^\nu}\right)^m.$$

All these subfamilies are independent, so Theorem 10.18 applies separately to each one:

$$\prod_{j\in J}\sigma_j \geq \frac{m^m}{29 m! \mathrm{e}^m d^{m+1}\ell_d}.$$

Furthermore $E_I/q^\nu > \exp(3A)$, and

$$\frac{\exp(3A)}{t} > \frac{8192}{5}(r+1)^2 d^2.$$

Use $c_2 qP/(e\theta) \geq a_* \geq 7/2$. The expression in parentheses is larger than $2048 r(r+1)^2 d^3$. Since $\mathrm{e} < 3$, $m^m \geq m!$, and $d^{2m-1} \geq \ell_d$, dropping only positive factors gives

$$\mathcal{R}_{m,J} > \frac{72}{725}\left(\frac{2048}{3}\right)^m \frac{(r+1)^{2m}}{r((m+1)!)^2} \geq \frac{72}{725}\,\frac{(2048/3)^m}{(m+1)^3} > 8.$$

For the middle inequality use $(m+1)! \le (m+1)^{m+1}$, and the increase of $(r+1)^{2m}/r$ for $r \ge m+1$. The last expression increases with $m \ge 1$: its consecutive ratio exceeds $(2048/3)(2/3)^3 > 1$. At $m = 1$ it is $147456/17400 > 8$.

For $m = r$, use the complete formula for $\mathcal{D}$ in (10.186). Write

$$M = \max\{p^f/(\delta t^r),\, \mathrm{e}^r t/r^r\}.$$

Since $\delta \ge 1$, $t > 2/3$, $p^f = \exp t \le \exp A$, and $r \ge 2$, $M < \exp A(3/2)^r$. The identities

$$q^{rI}\eta^{(r-1)(r+1)I} = q^I E_I^{r-1} > E_I^r, \qquad q^{\nu+u}/\gamma \ge 1$$

retain the saturation factors before they are bounded. Cancellation of $\mathcal{D}$ yields

$$\mathcal{R}_r > \frac{8\eta^r \exp((3r-1)A)}{135r(r+1)!\,[3(r+1)t]^r \ell_d}.$$

Indeed the remaining factor $3\lambda(1+\epsilon)(2+g_2^{-1})c_0$ is less than $135/2$. Now $t \le A$, $\exp A/A > 32/5$, and $\exp((2r-1)A) \ge [\mathrm{e}^4(r+1)d]^{2r-1}$. Using $(r+1)! \le (r+1)^{r+1}$ gives

$$\mathcal{R}_r > \frac{8(192/125)^r 16^{2r-1}}{135r(r+1)^2} > 31.$$

At $r = 2$ the last expression exceeds31. Its consecutive ratio is $(192/125)16^2 r(r+1)/(r+2)^2 > 1$, because $r(r+1)/(r+2)^2 \ge 3/8$. This proves all three parts of (10.268), hence all the hypotheses asserted. □

The resulting bases remain in the original field

Corollary 10.101 (weighted rank reduction at the geometric stop). Suppose the contracted prepared family at the depth in (10.264) vanishes at every integer $|s| \le \overline{S}$ through total order $\overline{T}$. Suppose its independent torus bases and selected Euler directions give the hyperplane W of Theorem 10.25, with nonzero vector $\boldsymbol{B}$. Then there are $1 \le m < r$ and independent integral vectors $u_1, \dots, u_m$, whose rational span contains $\boldsymbol{B}$. Put $\beta_i = \prod_j \alpha_j^{u_{ij}}$ and $R_i = \sum_j \sigma_j |u_{ij}|$. These are independent units in the original field, $h(\beta_i) \le R_i$, and

$$(S_m+1)\binom{T_m+m}{m}\prod_{i=1}^m R_i \le (m+1)! r^m \mathsf{D}_0 \left(\frac{2q^{\nu-I}\mathcal{D}}{c_1c_2rPd}\right)^m. \tag{10.270}$$

If the original multiplicative form satisfies $\Xi^{N_0} = \zeta \prod_j \alpha_j^{N_0 B_j}$, with integral exponents and a root of unity $\zeta \in K$, then for some positive integers N_1, w and integers c_i,

$$\Xi^{N_0 N_1 w} = \prod_{i=1}^m \beta_i^{N_0 w c_i}. \tag{10.271}$$

Proof. Proposition 10.31 converts all prepared zero jets into ordinary W-jets, and clears the Laurent monomial without changing the cutoff or the additive degree. Its coordinate bounds are precisely $\mathsf{D}_0, \mathsf{D}_j$ in (10.264). Theorem 10.100 verifies every degree and allocation hypothesis of Theorem 10.25. Apply that theorem with weights $A_j = \sigma_j$. If its first alternative (10.69) holds, multiply by $(T_m+m)/m \le \mathsf{D}_0$; it implies the second alternative (10.70). For that alternative use $C_{r,m} \le m! r^m$ and $\sigma_j \mathsf{D}_j = 2q^{\nu-I}\mathcal{D}/(c_1c_2rPd)$, independent of j. This proves (10.270).

The integer products β_i lie in K and are local units. Their independence and height inequalities are exactly the elementary argument of Corollary 10.26, applied to the original α_j. Clear denominators

in the rational-span identity to obtain $N_1 \boldsymbol{B} = \sum_i c_i u_i$. Raise the assumed identity for Ξ to N_1, then to the order w of ζ. This proves (10.271), without adjoining a new root field. $\square$

These degree comparisons apply uniformly to the geometric stopping depth. Reaching that depth requires the integer and fractional zero-production estimates with their actual global field and derivative-precision costs. Once those zeros are available, (10.270) gives the quantitative lower-rank output used in the general estimate.

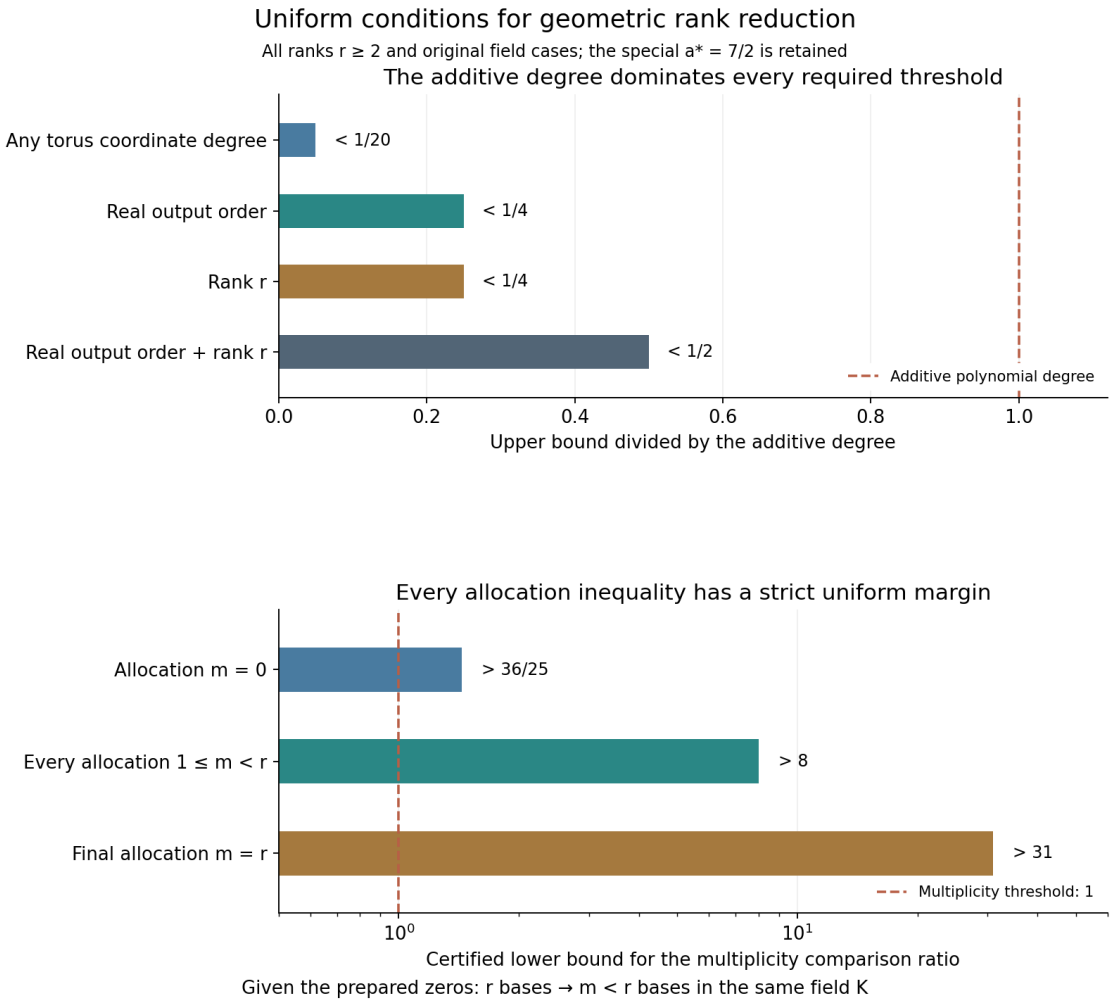


Figure 10.22. The coordinate-degree, output-order and rank bounds in (10.266) put the additive degree above every required degree threshold. The allocation ratios in (10.268) have proved lower bounds 36/25, 8, 31, uniformly over every original rank and field case, including $a_* = 7/2$. The plotted numbers are certified universal bounds, not samples of parameter cases. Theorem 10.100 proves the cancellations, floors and height-product comparisons; Corollary 10.101 keeps the new bases in the original field and gives their quantitative height product. Solution 42 checks the limiting rank constants and allocation algebra. Figure program. Human-source context: Yu's free paper, Section 6.

35. The original height profile survives rank reduction

The new bases from Corollary 10.101 must satisfy the original height-product allowance at their smaller rank. A small height product by itself does not prove this: the allowance changes with both the rank and the residue index. We retain that change exactly.

Keep all hypotheses and parameters of Theorem 10.100. For any independent family $\boldsymbol{a}$ of $j \geq 1$ local units in the same field, let $\delta(\boldsymbol{a})$ be its residue index. At $j \geq 2$ use the q-primary saturated residue index in (10.49). At $j = 1$ use the actual residue order as in (10.76), without adjoining the torsion

generator to that order. Both are positive integers. Set

$$\begin{aligned} M_j(\boldsymbol{a}) &= \max\left\{\frac{p^f}{\delta(\boldsymbol{a})t^j}, \frac{\mathrm{e}^j}{j^j}t\right\}, \\ H_* &= \frac{c_2 q P}{e\theta}, \qquad \Gamma_n = \mathrm{e}H_*(n+1)d \quad (n \geq r). \end{aligned} \tag{10.272}$$

Here $\mathrm{e} = \exp(1)$, whereas e is the ramification index. The field, θ, c_2, P, and n stay fixed when the rank changes. In particular, the maximum $M_r(\boldsymbol{\alpha})$ is exactly the one in $\mathcal{D}$ in (10.186).

Theorem 10.102 (uniform preservation of the original rank profile). Under the actual zero hypotheses of Corollary 10.101, let m, $\boldsymbol{\beta}$, and R_i be its output. For every $n \geq r$,

$$\prod_{i=1}^{m} R_i < \frac{1}{700}\,\Gamma_n^{r-m}\,\frac{M_r(\boldsymbol{\alpha})}{M_m(\boldsymbol{\beta})}\prod_{j=1}^{r}\sigma_j. \tag{10.273}$$

This preserves the original residue-index-dependent allowance, including the different definition at rank one.

Proof. First, $t > 1$ in every original parameter case. At odd primes this follows from $t \geq \ln 3 > 1$. At two, $\zeta_3 \in K$, and the injectivity of the prime-to-two roots under reduction, proved before (10.183), gives $3 \mid 2^f - 1$. Thus f is even and at least two, and $t \geq \ln 4 > 1$.

For any $j \geq 1$ and any positive residue index,

$$t^j M_j(\boldsymbol{a}) \leq p^f t.$$

Indeed $\delta(\boldsymbol{a})t \geq 1$ gives the first part of the maximum. For the second use $\ln y \leq y - 1$ at $y = t/j > 0$: it gives $(\mathrm{e}t/j)^j \leq \exp t = p^f$. This proves the bound at every rank, with its actual residue index.

Put $\lambda = 1 + g_5^{-1}$, $E = (q\eta^{r+1})^I$, and $C = \lambda(1+\epsilon)(2+g_2^{-1})c_0$. Theorem 10.100 gives $C < 45/2$, $\lambda < 5/4$, and $\mathsf{D}_0 < \lambda S\mathcal{D}/(c_1 c_4 d(A+x))$. In (10.270), bound its denominator from below by

$$\frac{2\overline{S}}{3r}\,\frac{(\overline{T}/(r+1))^m}{m!}.$$

Both lower bounds are strict because $\lfloor y \rfloor + 1 > y$. Divide the resulting upper bound for $\prod_i R_i$ by $\Gamma_n^{r-m}(M_r/M_m)\prod_j \sigma_j$. Substitute the complete $\mathcal{D}, S, T, \overline{S}, \overline{T}$ formulas. Writing this normalized upper bound as $\mathcal{Q}$, the exact cancellation gives

$$\begin{aligned} \mathcal{Q} = \frac{3r}{2q}\,\frac{(m+1)!m!}{r!}\,C\,\frac{\gamma}{q^{\nu+u}}\,\frac{r^r(r+1)^r}{\mathrm{e}^{r-m}(n+1)^{r-m}} \\ \times\left(\frac{2t}{\eta}\right)^m \ell_d M_m(\boldsymbol{\beta}) q^{m\nu}\frac{\eta^{(r+1)I}}{E^m}. \end{aligned} \tag{10.274}$$

The factors H_* and d cancel completely. The original M_r cancels against the profile ratio; the new M_m remains. Neither $(m+1)!$ nor $m!$ has disappeared.

Monotonicity of ln gives $\ln(r!) \geq \int_1^r \ln y\,dy = r\ln r - r + 1$, hence $r^r/r! < \mathrm{e}^r$. Use $n \geq r$, $q \geq 2$, $\gamma/q^{\nu+u} \leq 1$, and $C < 45/2$ in (10.274). The stopping inequalities give

$$q^{m\nu}\frac{\eta^{(r+1)I}}{E^m} < \exp\bigl(-3(m+1)A - (2m+3)x\bigr) \leq \exp\bigl(-3(m+1)A\bigr).$$

Combine this with $t^m M_m \le p^f t \le \exp(A)A$, $\mathrm{e} < 3$, and $\eta > 18/25$. It follows that

$$\mathcal{Q} < \frac{135}{8}\, r(m+1)!m! \left(\frac{25}{3}\right)^m (r+1)^m \ell_d A \exp\bigl(-(3m+2)A\bigr). \tag{10.275}$$

This remaining inequality is uniform in the rank, field, prime, ramification and residue degree. Set $V = (r+1)d$, so $g_1 = 4 + \ln V$ and $A \ge g_1 > 5$. The function $y\exp(-(3m+2)y)$ decreases for $y \ge 5$. Also $g_1 \le 5V$, since $\ln V \le V$ and $V \ge 1$, and $\ell_d \le d$. Thus (10.275) is at most

$$\frac{675}{8}\left(\frac{25}{3}\right)^m \exp(-4(3m+2)) \frac{r(m+1)!m!}{(r+1)^{2m+1} d^{3m}}.$$

For fixed $m \ge 1$, $r/(r+1)^{2m+1}$ decreases for $r \ge 1$, by its logarithmic derivative. Since $r \ge m+1$, and $(m+1)!m! \le (m+1)^{2m+1}$, its last factorial-rank factor is smaller than $m+1 \le 2^m$. The latter inequality follows by induction from its equality at one. Finally $\mathrm{e} > 2$ gives

$$\mathcal{Q} < \frac{675}{2048}\left(\frac{25}{6144}\right)^m \le \frac{5625}{4194304} < \frac{1}{700}. \tag{10.276}$$

The final rational comparison is $5625 \cdot 700 < 4194304$. All these bounds hold for every $1 \le m < r$; they are not deductions from a finite list of ranks. This proves (10.273). □

A genuine minimum-rank contradiction

Let $a_1, \dots, a_n$ be independent local units in K, and let a_0 generate its q-primary roots of unity. Fix $\mathcal{L} = \sum_j b_j z_j \ne 0$. A rank-$j$ representation consists of integral forms $L_0 = z_0, L_1, \dots, L_j$ independent over $\mathbb{Q}$, rationals $B_0, \dots, B_j$ with $\mathcal{L} = \sum_i B_i L_i$, and positive weights s_i. If $L_i = \sum_{k=0}^n a_{ik} z_k$, require

$$\begin{aligned} \alpha_i &= a_0^{a_{i0}} \prod_{k=1}^n a_k^{a_{ik}}, \qquad s_i \ge \sum_{k=1}^n |a_{ik}| h(a_k), \\ \prod_{i=1}^j s_i &\le \Psi_j(\boldsymbol{\alpha}) \prod_{k=1}^n h(a_k), \qquad \Psi_j(\boldsymbol{\alpha}) = \Gamma_n^{n-j} \frac{M_n(\boldsymbol{a})}{M_j(\boldsymbol{\alpha})}. \end{aligned} \tag{10.277}$$

The weighted condition implies $h(\alpha_i) \le s_i$ by the height product and power laws; the torsion factor has height zero. The bases α_i are independent: a torsion multiplicative relation, after killing the finite torsion factor, would give a linear relation among the rows of L_i modulo z_0, contradicting independence with L_0. At $j = n$, take $L_i = z_i, s_i = h(a_i)$. Its profile equals one, so a representation exists. There is therefore a least positive admissible rank. After renumbering its forms one may assume $B_j \ne 0$.

Corollary 10.103 (the geometric stop contradicts the original minimum rank). Suppose $r \ge 2$ is this least admissible rank, and the parameters and actual prepared zeros satisfy Corollary 10.101 for its bases and weights. Then those zeros cannot exist.

Proof. Take the integral vectors u_i of Corollary 10.101 and put $L_i' = \sum_{j=1}^r u_{ij} L_j$. Together with L_0, these forms are independent: their projections modulo L_0 are the independent combinations of the independent projections of $L_1, \dots, L_r$. Because $\boldsymbol{B}$ belongs to the rational span of the u_i, $\mathcal{L} = B_0 L_0 + \sum_i c_i L_i'$ for rationals c_i, not all zero. The bases associated with L_i' are precisely $\beta_i = \prod_j \alpha_j^{u_{ij}}$, in the same field and units at the same prime. The triangle inequality for each coefficient gives

$$\sum_{k=1}^n \left| \sum_{j=1}^r u_{ij} a_{jk} \right| h(a_k) \le \sum_{j=1}^r |u_{ij}| s_j = R_i.$$

Thus every required independence, representation and weighted-height condition survives. Finally Theorem 10.102 and (10.277) give

$$\prod_i R_i < \frac{1}{700}\,\Gamma_n^{r-m}\,\frac{M_r}{M_m}\,\Gamma_n^{n-r}\,\frac{M_n}{M_r}\prod_k h(a_k) = \frac{1}{700}\,\Psi_m(\boldsymbol{\beta})\prod_k h(a_k).$$

So $m < r$ is admissible with the original allowance, contrary to the choice of r. No enlargement of the original field occurs in this argument. □

A rank-two example with a proved gap

Take $K = \mathbb{Q}, p = 3, (a_1, a_2) = (2, 5)$, and $\mathcal{L} = 101z_1 + 103z_2$. Here $d = e = f = 1$, $P = 3$, $H_* = 7(1 + \tau)$, and $0 \le \tau \le 1/100$. The rational valuations at two and five prove independence and show that the q-primary saturation is already $\mathbb{Z}^2$. The residue group generated by the torsion root and the two bases has order two, giving $\delta(\boldsymbol{a}) = 1$. The proved logarithm bounds $1 < t = \ln 3 < 11/10$, $2/3 < \ln 2 < 7/10$, $1 < \ln 5 < 2$, and $2 < \mathrm{e} < 3$ imply

$$\Gamma_2 < 64, \qquad M_2(\boldsymbol{a}) < 3, \qquad M_1(\beta) \ge \mathrm{e}t > 2, \qquad \Psi_1(\beta)h(2)h(5) < \frac{672}{5}. \tag{10.278}$$

Every possible rank-one form has non-torsion coefficients $(a_{11}, a_{12}) = k(101, 103)$, where k is a nonzero integer. Indeed rational proportionality follows from the representation; $\gcd(101, 103) = 1$ makes its proportionality factor integral. Its required weight is larger than $101(2/3) + 103 = 511/3 > 672/5$. Thus rank one is impossible, whereas the standard rank-two representation is admissible. This is a concrete instance of the minimum-rank contradiction once the geometric-stop zeros are produced. It does not assert those zero hypotheses without their precision proof.

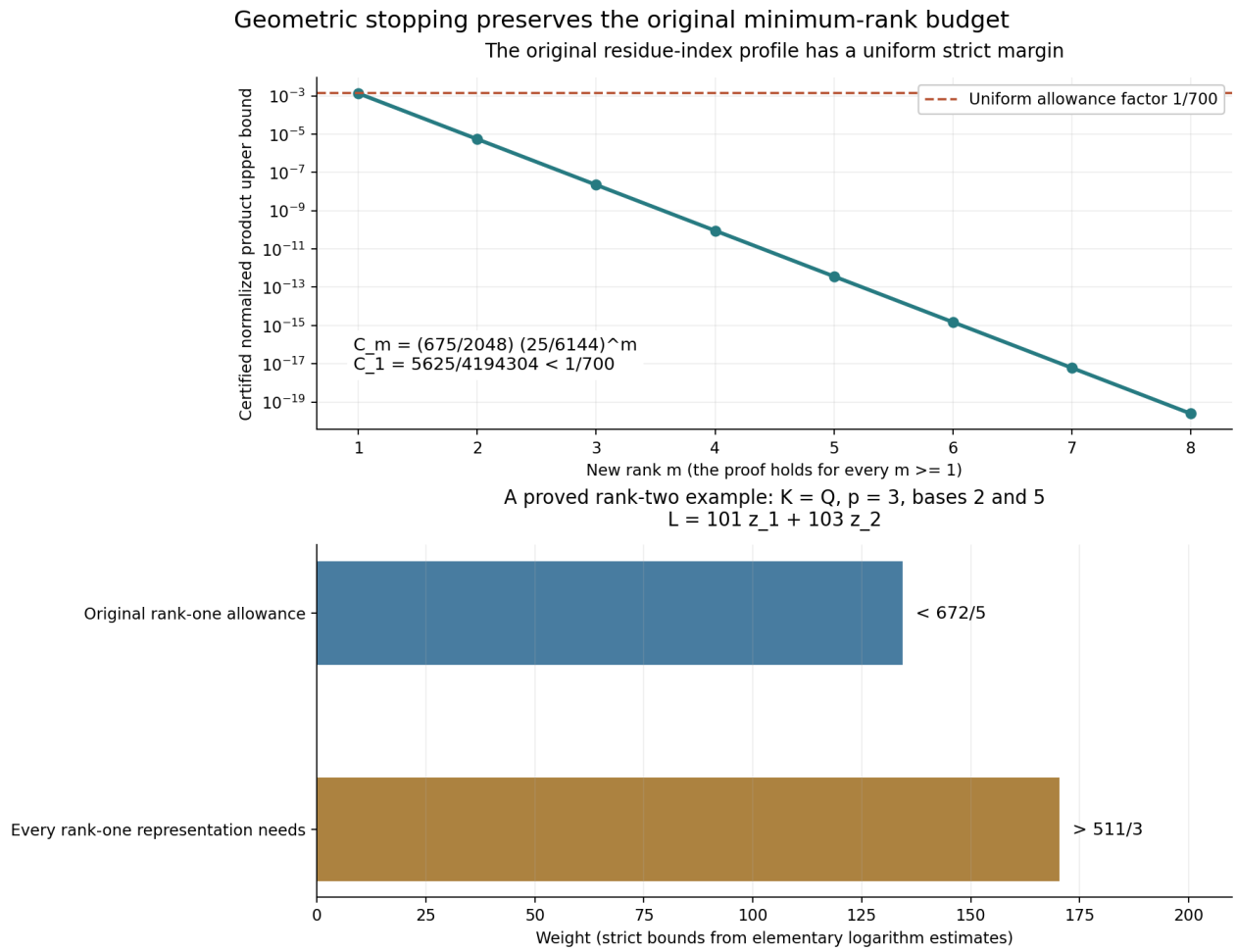


Figure 10.23. The universal envelope in (10.276) decreases geometrically for every smaller rank $m \geq 1$. The plotted first eight values illustrate the proved formula; its validity for all ranks comes from Theorem 10.102. The rank-two example has an admissible rank-one weight below $672/5$, while every rank-one representation needs a weight above $511/3$. Corollary 10.103 proves the general minimum-rank contradiction, retaining the original residue-index profile and original field. Solution 43 checks the complete cancellation and the example. Figure program. Human-source context: Yu's free paper, (2.10) and (6.18)–(6.19).

The geometric branch now has its full degree, multiplicity and minimum-rank closure under the actual prepared-zero hypotheses. The general descent still requires those hypotheses to be proved with their original integer and fractional precision budgets, and requires the separate stopping branch when its first phase range ends earlier.

36. Complete composition averages at the original general parameters

We keep the parameters of Section 26, including their full coordinate widths, the original coefficient field and every field and prime case. The derivative coordinates together with their slack form a complete simplex. Retaining its zero rows makes the average of each coordinate exact. This gives a smaller initial height budget while keeping the original kernel and all its zeros.

Throughout this section use the following rational constants. In the last column, $\widehat{\vartheta}$ is the upper bound used in (10.185).

Case	Conditions	c_0	c_1	c_3	c_4
I.1	$p = 3, d \geq 2$	2.66	1.449	1.4647	20.74
I.2	$p = 3, d = 1$	1.9	1.4494	1.3852	20.8
II	$p = 5, e \geq 2$	2.74	1.4372	0.8412	19
III.1	$p \geq 5, e = 1, d \geq 2$	2.78	1.4341	2.992	18.7

Case	Conditions	c_0	c_1	c_3	c_4
III.2	$p \geq 5, e = 1, d = 1$	2.6	1.432	3.26	18.2757
IV	$p \geq 7, e \geq 2$	3	1.4441	3.849	20
V	$p = 2, \zeta_3 \in K$	2.5	2.5347	0.4757	3.765

Case	a_*	q	$\widehat{\vartheta}$
I.1, I.2	7	2	3/2
II	7	2	5/4
III.1, III.2	7/2	2	1
IV	7	2	7/6
V	26/3	3	2

Every terminating decimal in this table denotes an exact rational number. Put

$$F_r = \frac{r^r(r+1)^r}{(r!)^2}, \qquad B_r = \frac{2c_0c_1c_4}{\rho}a_*^r F_r, \qquad b_r = \frac{7200}{29}\,14^{r-2}, \qquad Z = \frac{S\mathcal{D}}{d}. \tag{10.279}$$

Coupled rounding and coefficient-count bounds

Lemma 10.104 (uniform lower budgets in every original case). For every integer $r \geq 2$, the table and (10.183)–(10.186) give

$$\begin{gathered} t > 1, \qquad B_r > b_r, \qquad \mathcal{D} > B_r(A+x)t, \\ S > \frac{7}{5}(r+1)^2 d, \qquad g_2 > 58, \qquad g_5 > 188, \\ \lambda := 1 + g_5^{-1} < \frac{189}{188}, \qquad 1 + \epsilon < \frac{251}{250}. \end{gathered} \tag{10.280}$$

Proof. For odd p, $t = f \ln p \geq \ln 3 > 1$. At $p = 2$, the primitive cube root reduces to an element of order three, by the injectivity proof preceding (10.183). Thus $3 \mid 2^f - 1$, so f is even and at least two. Consequently $t \geq \ln 4 > 1$.

Direct multiplication of the seven rational rows gives

$$c_0c_1c_4a_*^2 > 800, \quad c_0c_3qa_*^2 > 202, \quad \frac{c_0c_4qa_*^2}{\widehat{\vartheta}} > 1000, \quad c_3q > \frac{7}{5}. \tag{10.281}$$

For explicit checks at the possible minima, the first product is smallest in III.2 and equals 833.54005644. The second is smallest in III.1 and equals 203.78512. The third is smallest in V and equals 1060.475; the fourth is also smallest in V and equals 1.4271. These are exact products of the displayed rationals.

We have $F_2 = 9$ and

$$\frac{F_{r+1}}{F_r} = \frac{r+2}{r+1}\left(1 + \frac{2}{r}\right)^r > 4. \tag{10.282}$$

To verify the last inequality for every real $r \geq 2$, the derivative of $r\ln(1+2/r)$ is $\ln(1+z) - z/(1+z) > 0$, where $z = 2/r > 0$. The latter function vanishes at zero and has derivative $z/(1+z)^2 > 0$. Its value at $r = 2$ gives $(1+2/r)^r \geq 4$. Since $a_* \geq 7/2$ and $\rho \leq 58$, (10.281)–(10.282) give

$$B_r > \frac{800}{29}\,9\,14^{r-2} = b_r. \tag{10.283}$$

The stronger lower bound in Lemma 10.70 is $\mathcal{D} \geq B_r(A+x)t$; its proof uses the strict inequality $2 + g_2^{-1} > 2$, so this inequality is strict. The formula for S, with $h \geq (r+1)t$, proves its bound in (10.280).

In cases III and IV, the smallest possible value of $g_2 = c_3 q(r+1)^2 d$ is $3.26 \cdot 2 \cdot 9 = 58.68$, in III.2. In I, II and V use the other formula for g_2, $g_0 > 39$, and respectively $\ln 3 < 11/10$, $\ln 5 < 2$, and $\ln 2 < 1$. In II also $e \geq 2$. Each lower bound is greater than 58.

The formulas for g_5 and g_4, including the smaller alternative g_1^{-1} for the latter, give

$$g_5 > \frac{404}{58}(r+1)F_r(7/2)^{r-2}, \qquad g_4 > \frac{2000}{58}(r+1)F_r(7/2)^{r-2}. \tag{10.284}$$

The first bound is at least $5454/29 > 188$. In the second, (10.282) implies $(r+1)F_r(7/2)^{r-2} > (87/20)r(r+1)$: at $r = 2$, $9 > 87/10$, and each next ratio of $F_r(7/2)^{r-2}$ is greater than 14 while $(r+1)/r \leq 3/2$. Hence $r(r+1)/(2g_4) < 1/300$. Since $(1+z)^r \leq \exp(rz)$ for $z \geq 0$,

$$1 + \epsilon < \exp(1/300) \leq \sum_{j \geq 0} (1/300)^j = \frac{300}{299} < \frac{251}{250}.$$

This also proves the remaining rounding bounds. The elementary logarithm inequalities used above follow from the positive exponential series: its terms through degree five at $11/10$ already exceed three; its terms through degree four at two exceed five. □

Lemma 10.105 (a general logarithmic coefficient budget). Let Λ be the selected initial support of Theorem 10.71, $N = kL|\Lambda|$, and $1 \leq d_E \leq d$. Then

$$\frac{\ln(Nd_E)}{Z} < \varepsilon_r := \frac{3 + 14/(r+1)}{7b_r} \leq \frac{667}{151200}\, 14^{2-r} < \frac{1}{225}\, 14^{2-r}. \tag{10.285}$$

Proof. The height-product bound of Theorem 10.18, applied separately to each singleton, gives $d\sigma_i > 1/(87d^2)$. Indeed for $d \geq 2$, its lower bound is $d^2 \ell_d \sigma_i \geq w_K/(58\mathrm{e}) > 1/87$, using $w_K \geq 2$, $\mathrm{e} < 3$, and $\ell_d \leq d$. At $d = 1$ its denominator is 17 and the same weaker bound follows.

Every coordinate of the lattice lies in $q^{-\nu}\mathbb{Z}$, by the annihilator argument in Lemma 10.84. A translated interval of full width D_i therefore contains at most $\lfloor q^\nu D_i \rfloor + 1$ possible coordinates. The phase restriction only decreases the count. As $c_1 c_2 r P > 1$, (10.186) gives $q^\nu D_i < 87 q^\nu d^2 \mathcal{D}$. Moreover $kL \leq \lambda S\mathcal{D}/[c_1 c_4 d(A+x)] < S\mathcal{D}$, and $\mathcal{D} > 1$. Consequently

$$N < S\mathcal{D}(88 q^\nu d^2 \mathcal{D})^r, \qquad \ln(Nd_E) < \ln S + (r+1)\ln \mathcal{D} + (2r+1)\ln d + rx + r\ln 88. \tag{10.286}$$

Set $S_* = \frac{7}{5}(r+1)^2 d$. Both S_* and $5b_r$ exceed e; the function $\ln z/z$ decreases there. Also $\ln S_* < 2A$, since $\ln(7/5) < 1$ and $A \geq 4 + \ln((r+1)d)$. Thus the contribution of $\ln S$, after division by Z, is less than $10/[7(r+1)^2 b_r]$.

Since $\mathcal{D} > b_r(A+x)t > 5b_r$,

$$\frac{\ln \mathcal{D}}{\mathcal{D}} < \frac{\ln(5b_r)}{5b_r} < \frac{3r+2}{5b_r}.$$

Here $\ln(36000/29) < 8$ and $\ln 14 < 3$; the positive exponential series proves both, using respectively terms through degree eight at eight and through degree four at three. The contribution of $(r+1)\ln \mathcal{D}$ is therefore less than $(3r+2)/[7(r+1)b_r]$.

Finally $\ln 88 < 5 < A$, by the positive exponential series through degree six at five, and $\ln d < A$. The remaining terms in (10.286) are at most $(3r+1)(A+x)$. Their contribution is less than $5(3r+1)/[7(r+1)^2 b_r]$. Adding these three bounds gives

$$\frac{3r^2+20r+17}{7(r+1)^2 b_r} = \frac{3+14/(r+1)}{7b_r}.$$

This factor decreases apart from the displayed 14^{2-r}. At $r=2$ it is $667/151200 < 1/225$. All divisions used the original scale $Z = S\mathcal{D}/d$. □

The original height parameter

For the next average take the original

$$g_0 = (r+1)\big(a_0 r + a_1 + \ln(a_0 r + a_2) + \ln d\big), \tag{10.287}$$

with the following choices. In III, $a_0 = 2 + \ln 7$; in I, II and IV, $a_0 = 2 + \ln 14$; in V, $a_0 = 2 + \ln 26$. The values of a_1, in the order of the seven rows above, are $4.03, 4.79, 3.44, 4.71, 5.84, 5.12, 2.52$. Set $a_2 = a_1$ in I.2 and III.2, and $a_2 = a_1 + \ln 2$ otherwise. These choices are compatible with the larger h in Section 26.

Lemma 10.106 (the general Euler logarithm constant). For these original choices, g_{91} of (10.222) satisfies

$$g_0 > (r+1)(39r/10 + 36/5 + \ln d), \qquad g_{91} < 1 + \frac{1}{3(r+1)}. \tag{10.288}$$

Proof. The elementary positive logarithm series $\ln z = 2\sum_{j\ge 0} y^{2j+1}/(2j+1)$, $y = (z-1)/(z+1)$, proves $\ln 7 > 19/10$, $\ln 14 > 13/5$, and $\ln 13 > 5/2$ by retaining respectively six, twelve and twelve terms. This series and its explicit geometric tail were proved in Solution 34. In every case $a_0 > 39/10$. Directly at $r = 2$, each of $2a_0 + a_1 + \ln(2a_0 + a_2)$ exceeds 15. For the smallest checks in II and III.1 use $2a_0 + a_1 > 12.64$ and > 12.51, respectively, together with $2a_0 + a_2 > 13$ and $\ln 13 > 5/2$. In V, $a_0 > 2 + 2/3 + 5/2 = 31/6$, and $2a_0 + a_2 > 13$. The other four checks have larger lower bounds. Increasing r increases the first linear term by at least $(39/10)(r-2)$ and the logarithm cannot decrease. This proves the first bound.

For $d \ge 2$, set $y = \ln d$. Concavity gives the tangent inequality $\ln z \le \ln 9 + (z-9)/9$. At $z = y + \ln 3$,

$$1 + 3\ln(y + \ln 3) \le \frac{19}{3}\ln 3 - 2 + \frac{y}{3} < 5 + \frac{y}{3}.$$

The last inequality uses $\ln 3 < 11/10$. The first bound in (10.288) gives $g_0 > (r+1)(15+y)$. Substituting in $g_{91} - 1 = [1 + 3\ln\ln(3d)]/g_0$ proves the second bound. At $d = 1$, use $\ln 6 < 9/5$, $\ln 2 > 2/3$, and $\ln 3 > 1$, so $W(1) < 27/10 < \mathrm{e}$. The last comparison follows from the exponential series through degree five at one. Hence $1 + \ln W(1) < 2$ and $g_0 > 15(r+1)$, which is stronger than required. The exponential series through degree six at $9/5$ proves $\ln 6 < 9/5$. □

Keeping the zero rows in the simplex

Put $t_0 = \lfloor T \rfloor$, and now retain **all** $\binom{t_0+r}{r}$ composition indices at every node, including those whose additive order exceeds D_{a}. Such rows are zero. Assign them the same positive scalar majorants as below. Let u be the additive coordinate and h_{E} the sum of the $r-1$ Euler coordinates. The slack completes the $r+1$ coordinates to total t_0.

Theorem 10.107 (the complete-simplex mean bound). Assume the coefficient-height condition (10.223) and the original choices (10.287). Retaining these full rows preserves every initial zero and the coefficient field. Their mean scalar logarithm is at most

$$D_{\mathrm{a}} \ln\!\left[2\mathrm{e}\left(\frac{s_0}{k}+2\right)\right] + C_r T H_1, \qquad C_r = \frac{r+1/10}{r+1} + \frac{17(r-1)}{24(r+1)^2} < 1, \tag{10.289}$$

where $s_0 = \lfloor S \rfloor$, $H_1 = h + x$. In fact

$$1 - C_r = \frac{23r+193}{120(r+1)^2} > 0. \tag{10.290}$$

Proof. Permutation of the $r+1$ composition coordinates is a bijection. Their means are therefore exactly $t_0/(r+1)$. Consequently

$$\overline{u} = \frac{t_0}{r+1}, \qquad \overline{h}_{\mathrm{E}} = \frac{(r-1)t_0}{r+1}. \tag{10.291}$$

For $u \le D_{\mathrm{a}}$, Lemma 10.76 gives its additive majorant in (10.198). For $u > D_{\mathrm{a}}$ the derivative is zero, so the same positive expression is still a valid majorant; no division by a zero entry is used. At Euler total h_{E}, retain the majorant $\binom{m+h_{\mathrm{E}}}{h_{\mathrm{E}}}$. Its logarithm is bounded by $h_{\mathrm{E}} \ln[\mathrm{e}(1 + m/h_{\mathrm{E}})]$; concavity, as proved in Corollary 10.79, bounds its mean by its value at $\overline{h}_{\mathrm{E}}$.

The first part of the proof of Theorem 10.86 used only $\overline{h} \ge (r-1)t_0/(r+1)$. Repeating its substitution with equality (10.291) gives

$$\frac{m}{\overline{h}_{\mathrm{E}}} \le \frac{4}{7}\mathrm{e}^{H_1} t W(d) + \frac{r+1}{t_0}, \qquad 1 + \frac{m}{\overline{h}_{\mathrm{E}}} \le \mathrm{e}^{H_1} W(d) t. \tag{10.292}$$

For the second inequality, use exactly its quadratic-series absorption of $1 + (r+1)/t_0$, with $H_1 > 39$, $H_1 \ge (r+1)t$, $W(d) \ge 1$ and now $t > 1$. Taking logarithms and adding one bounds the mean Euler contribution by $\overline{h}_{\mathrm{E}}(g_{91} H_1 + \ln t)$. Differentiating $\ln t / t$ gives its maximum $1/\mathrm{e}$; hence $\ln t \le H_1/[\mathrm{e}(r+1)]$.

The complete lcm proof in lesson seven gives $\ln \mathfrak{v} < k \ln 3 \le H_1 \ln 3$. Combining this with (10.291), the sum of the mean clearing and Euler logarithms is less than

$$\frac{t_0 H_1}{r+1}\left[(r-1)g_{91} + \ln 3 + \frac{r-1}{\mathrm{e}(r+1)}\right]. \tag{10.293}$$

Lemma 10.106, $\ln 3 < 11/10$, and $1/\mathrm{e} < 3/8$ bound its coefficient by C_r: the latter exponential inequality follows from the first four terms at one, whose sum is $8/3$, and the remaining terms are positive. The identity (10.290) follows by subtraction. Adding the allocation-independent additive majorant proves (10.289), since $t_0 \le T$.

Finally Theorem 10.71 supplies $N > c_{01}(2s_0+1)\binom{t_0+r}{r}$, so the complete weighted integral Siegel argument applies to this full row matrix. The additional zero rows change no kernel equation, and repeated Euler rows preserve the same equations. Every entry still lies in $E = \mathbb{Q}(\alpha_0, \theta_1, \ldots, \theta_r)$. The row field, nonzero vector, independent polynomial basis and all prescribed initial jets are therefore retained. □

Corollary 10.108 (a sharper general initial coefficient bound). In the original generator field $E \subseteq K$, an integral initial coefficient vector with every prescribed zero and a nonzero polynomial may be chosen with

$$\frac{h_2(c)}{Z} < g_{12} + \frac{\varepsilon_r}{2} + \frac{1}{c_{01}-1}\left[\frac{\varepsilon_r}{2} + \frac{\lambda}{c_1 c_4} + \frac{C_r}{c_1 c_3 \theta e} + \frac{1+(2g_2)^{-1}}{2c_1 c_2}\right]. \tag{10.294}$$

Here $g_{12} = 0$ when $d = 1$; in every case one may further use

$$g_{12} \le \frac{1 + 5/(r+1)}{546 b_r}. \tag{10.295}$$

Proof. Apply the weighted field argument of Theorem 10.90 to the full rows just proved, retaining its actual discriminant bound. Lemma 10.105 bounds $\ln N/Z$. Corollary 10.77 bounds the allocation-independent additive term; its hypothesis holds for every displayed case. The identity $TH_1/Z = 1/(c_1 c_3 \theta e)$, with Theorem 10.107, bounds the combined mean term. The exact row ratio is less than $1/(c_{01}-1)$, and Corollary 10.72 bounds the mean torus contribution. These substitutions give (10.294).

For (10.295), write $g_3 = B_r g_1 t$. Then $g_1/(2g_7) = 1/[2c_3 q(r+1)g_0 B_r] < 5/[546(r+1)b_r]$, by $c_3 q > 7/5$ and $g_0 > 39$. The formula (10.231) for g_{11} can be written as

$$g_{11} = 2\mathrm{e}\, c_3 q B_r g_0 g_1 \frac{(r+1)(r-1)^{r-1}}{r^r}.$$

The last ratio exceeds $1/\mathrm{e}$, because $(1 + 1/(r-1))^{r-1} < \mathrm{e}$. Thus $g_{11} > 2c_3 q B_r g_0 g_1 > 546 b_r$, using $g_1 > 5$. Adding its reciprocal proves (10.295); at $d = 1$ its exact discriminant contribution remains zero. □

An individual value in its normal coordinates

The exact normalizing factor in Lemma 10.91 also gives a second arithmetic estimate. Here its factorial and lcm valuations cancel against a common rational rescaling of the actual algebraic value. The global field degree is still required.

Theorem 10.109 (a second arithmetic bound for the normalized value). Retain all hypotheses and notation of Theorem 10.92. Let $I, J \ge 0$, $x = s/q^J$, $u \le D_{\mathrm{a}}$, and h_{E} the total Euler order. Put

$$G_u = (D_{\mathrm{a}} - u)\theta + L v_p(k!) - u v_p(\mathfrak{v}), \qquad \widehat{\Xi} = (I+J)(D_{\mathrm{a}} - u). \tag{10.296}$$

If the actual value $V \in F$ is nonzero, then

$$v_p(V) + G_u - \delta \le \frac{d_F}{e_F f_F \ln p}\Bigg[h_2(c) + \tfrac{1}{2}\ln N + (D_{\mathrm{a}} - u)\ln R_I + \ln \binom{D_{\mathrm{a}}}{u} + \ln \binom{m_I + h_{\mathrm{E}}}{h_{\mathrm{E}}} + \widehat{\Xi}\ln q + PXq^{-I}\sum_i D_i \sigma_i\Bigg] + (D_{\mathrm{a}} - u)\theta. \tag{10.297}$$

For this same normal quantity one may take the smaller of (10.297) and the right side of (10.244) plus G_u. Both estimates use the same actual F, d_F, e_F, f_F.

Proof. Multiply the whole algebraic value, at this one fixed additive order, by the common nonzero rational scalar

$$W = q^{\widehat{\Xi}} \frac{(k!)^L}{\mathfrak{v}^u} V \in F. \tag{10.298}$$

It is essential that this multiplier is independent of the support index and the binomial-basis column. For a column of power $\ell \le L$, divided differentiation of $F_{a,\ell}$ has denominator dividing $(k!)^\ell$. Multiplication by $(k!)^L$ leaves integer polynomial coefficients. At $q^{-I}x$, its degree is at most $k\ell - u \le D_{\mathrm{a}} - u$, so $q^{\widehat{\Xi}}$ clears every remaining rational denominator. The Euler binomials are integers at their original integer arguments. Thus the rational scalar multiplying each torus monomial in W is an integer.

Its ordinary absolute value is at most

$$q^{\widehat{\Xi}} R_I^{D_{\mathrm{a}}-u} \binom{D_{\mathrm{a}}}{u} \binom{m_I + h_{\mathrm{E}}}{h_{\mathrm{E}}}.$$

Indeed the column estimate before taking the largest power is $(k!)^{L-\ell}\binom{k\ell}{u} R_I^{k\ell-u}$. Since $R_I \ge k$, we have $R_I^k \ge k!$; increasing ℓ to L and the binomial to $\binom{D_{\mathrm{a}}}{u}$ therefore gives the stated bound, exactly as in Theorem 10.92.

Apply its proved product-formula and row-height argument to this nonzero W, using the integer scalar bound just obtained. The coefficient vector and full-width torus term are unchanged. At the chosen prime, q is a unit, so $v_p(W) = v_p(V) + Lv_p(k!) - uv_p(\mathfrak{v})$. Adding $(D_{\mathrm{a}} - u)\theta$ proves (10.297). Applying (10.244) to the original V and adding its same G_u proves the other bound. Their minimum is valid because both bound the same number. Restriction of the completion, or the fact that roots lie in an old completion, has not replaced the actual global field degree. □

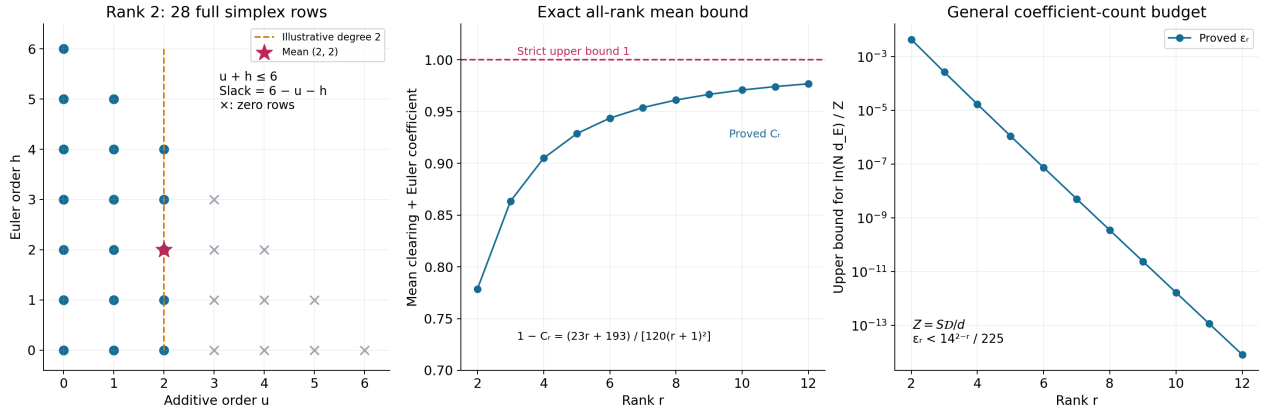


Figure 10.24. Left: every pair of nonnegative additive and Euler orders with sum at most six, with the remaining slack, at rank two. The 28 equally weighted rows have mean additive and Euler order two; rows above the illustrative additive degree two are zero and remain in the average. Middle: the proved constants C_r for ranks two through twelve. Equation (10.290) proves their strict upper bound one for every rank. Right: the logarithmic coefficient-count bounds ε_r of (10.285), with logarithmic vertical scale; the displayed points illustrate the analytical all-rank formula. The actual scale is $Z = S\mathcal{D}/d$. Lemmas 10.104–10.106 and Theorem 10.107 keep every original field and prime case, and Corollary 10.108 retains all scalar, discriminant and mean torus costs. Figure program. Human-source background: Yu's free paper, equations (1.11), (3.12)–(3.18) and the initial row system in Section 4.

37. A fully proved elementary bound for least common multiples

The individual prepared derivatives contain $v(k)^u$, where $v(k) = \mathrm{lcm}(1, \dots, k)$. We will prove

$$\ln v(k) < \frac{107}{103} k \qquad (k \ge 1). \tag{10.299}$$

This strengthens the complete elementary bound in lesson seven. It also keeps the lcm constant used in the general logarithmic-form parameters. The proof follows the finite-recurrence method of Costa Pereira's freely available paper, pages 321–323. We give the recurrence, its induction, and the complete exact finite computation here. The computation regenerates its primes and encloses each logarithm by rational numbers; no numerical table is an assumed input.

A factorial combination

For $x > 0$, put $\psi(x) = \sum_{p^a \le x} \ln p$, summing over primes p and integers $a \ge 1$. It is zero below one and constant on each interval $[n, n+1)$. Define the integer weights

r	1	2	3	5	6	7	10	11	13	14	15	30	42	110	182
v_r	1	−1	−1	−1	1	−1	1	−1	−1	1	1	−1	−1	−1	1

The symbols v_r are weights; they are distinct from the integer $v(k)$. Put

$$F(y) = \sum_r v_r \lfloor y/r \rfloor, \qquad \sigma(x) = \sum_r v_r \ln(\lfloor x/r \rfloor !), \qquad \omega = -\sum_r \frac{v_r \ln r}{r}. \tag{10.300}$$

As usual, $0! = 1$.

Lemma 10.110 (the factorial combination and its error). These weights satisfy $\sum v_r/r = 0$, $\sum v_r = -3$ and $\sum |v_r| = 15$. For every $y \ge 0$, $-1 \le F(y) \le 4$. Moreover

$$\ln v(k) = \psi(k), \qquad \sigma(x) = \sum_{p^a \le x} (\ln p) F(x/p^a), \qquad |\sigma(x) - \omega x| \le 15(\ln x + 1) \quad (x \ge 182). \tag{10.301}$$

Proof. The exponent of p in the least common multiple is the largest integer a for which $p^a \le k$; summing its logarithm proves the first identity. In $n!$, the exponent of p is $\sum_{a \ge 1} \lfloor n/p^a \rfloor$, by counting multiples of each power. Thus $\ln(\lfloor x \rfloor !) = \sum_{p^a \le x} (\ln p) \lfloor x/p^a \rfloor$. Substituting this finite sum in (10.300) proves the second identity.

For the weight sum, add the four rational identities $1 + 1/15 - 1/2 - 1/3 - 1/5 - 1/30 = 0$, $1/6 - 1/7 - 1/42 = 0$, $1/10 - 1/11 - 1/110 = 0$, and $1/14 + 1/182 - 1/13 = 0$. The other two sums follow directly from the table.

Write $F = F_0 + F_1 + F_2 + F_3$, using respectively the weights at $\{1, 2, 3, 5, 15, 30\}$, $\{6, 7, 42\}$, $\{10, 11, 110\}$, and $\{13, 14, 182\}$. The first rational identity makes F_0 periodic with period 30. Its values at the residues $0, 1, \dots, 29$, obtained by substituting in its six floors, are $0, 1, 1, 1, 1, 1, 0, 1, 1, 1, 0, 1, 0, 1, 1, 1, 1, 2, 1, 2, 1, 1, 1, 2, 1, 1, 1, 1, 1, 2$. It is constant between consecutive integers, so $F_0 \le 2$ at every real argument. The identities for $1/6$ and $1/10$, together with $\lfloor a + b \rfloor - \lfloor a \rfloor - \lfloor b \rfloor \in \{0, 1\}$, show that $F_1, F_2 \in \{0, 1\}$. The identity for $1/13$ gives $F_3 \in \{-1, 0\}$. Consequently $F \le 4$.

For the lower bound, all denominators divide $\tau = 30030$; the zero inverse-weight sum makes F periodic with this period. At integral n, $\lfloor n/r \rfloor + \lfloor (\tau - 1 - n)/r \rfloor = \tau/r - 1$. Therefore $F(n) + F(\tau - 1 - n) = 3$. Since $F \le 4$, it follows that $F(n) \ge -1$. Each floor in F is constant between consecutive integers, so the same bounds hold at real $y \ge 0$.

Finally let $t \ge 1$, $n = \lfloor t \rfloor$. Monotonicity of $\ln z$ gives $n \ln n - n + 1 \le \ln(n!) \le n \ln n - n + 1 + \ln n$. The change in $t \ln t - t$ between n and t is between zero and $\ln t$. Hence $|\ln(n!) - (t \ln t - t)| \le \ln t + 1$. Apply this at every $t = x/r \ge 1$. The $x \ln x - x$ terms cancel because $\sum v_r/r = 0$; the remaining linear term is ωx. The fifteen absolute weights give the stated error. □

The finite certificate

The selected half-open intervals in the next table have integer endpoints. Each pair (b, c) denotes $[b, c)$; pairs in a given row are disjoint.

Level s	Intervals with $F \geq s$, for the upper recurrence	Intervals with $F < s$, for the lower recurrence
1	(67,126), (157,176), (179,220), (223,275), (277,330), (359,429)	none
2	(17,22), (23,26), (29,35), (47,52), (59,65), (71,78), (79,88), (191,210)	(26,29), (65,71), (117,139)
3	(19,21)	(21,31), (33,61), (63,73), (84,103), (110,193), (208,229), (242,271), (294,323), (325,373), (440,493)
4	none	(440,877)

Let $\mathcal{U}, \mathcal{L}$ be the respective lists of pairs, retaining their level. The initial cutoffs are $a_0 = 877, a_1 = 66, a_2 = 17, a_3 = 19, a_4 = 439$. Here $F(y) \geq s$ for $1 \leq y < a_s$, $s = 0, 1$, and $F(y) < s$ for $1 \leq y < a_s$, $s = 2, 3, 4$.

Lemma 10.111 (exact finite certificate). The cutoffs and every interval in the table have the asserted property. Set

$$\begin{aligned} A &= 1 - 1/a_0 - 1/a_1 - \sum_{(b,c)\in\mathcal{U}} 1/c, \\ B &= \sum_{(b,c)\in\mathcal{U}} 1/b, \\ C &= 1/a_2 + 1/a_3 + 1/a_4 + \sum_{(b,c)\in\mathcal{L}} 1/c, \\ D &= 1 - \sum_{(b,c)\in\mathcal{L}} 1/b. \end{aligned} \tag{10.302}$$

With $Q = 10^{12}$, $N = 10^6$, $U = 27/26$, $L = 25/26$, and $\varepsilon = 9/40000$, the following rational inequalities hold:

$$\begin{gathered} A \geq .732878, \quad B \geq .297381, \quad C \leq .263738, \quad D \leq .802853, \\ 1.046524 < \omega < 1.046525, \\ AU + BL - \omega - \varepsilon > \frac{6731}{26000000}, \qquad \omega - \varepsilon - CU - DL > \frac{11523}{26000000}. \end{gathered} \tag{10.303}$$

Also $\psi(x) < Ux$ for $114 \leq x < N$, and $\psi(x) > Lx$ for $227 \leq x < N$. For integers $1 \leq n < 113$, $\psi(n)/n < \psi(113)/113$, and $U < \psi(113)/113 < 107/103$.

Proof and reproducible finite computation. The full standard-library program chebyshev_certificate.py is part of this lesson's editable sources. We describe every finite check it performs, including its rational logarithm bounds.

For the floor conditions it substitutes $\sum v_r(n \,//\, r)$ at every integer in each initial range and every listed half-open interval. This covers real arguments as well, because F is constant on $[n, n+1)$. It checks the within-level disjointness and that each interval begins at or beyond its level's initial

cutoff. The fractions in (10.302) are then summed with exact integer numerators and denominators. Their comparison with the displayed terminating decimals is an exact rational comparison.

To enclose a logarithm, first write an integer $m = 2^a t$, $1 \le t < 2$, and put $z = (t-1)/(t+1)$, so $0 \le z < 1/3$. For $\ln 2$, take $z = 1/3$. Integrating the finite geometric identity for $(1-z^2)^{-1}$ proves

$$0 \le \ln \frac{1+z}{1-z} - 2\sum_{j=0}^{15} \frac{z^{2j+1}}{2j+1} \le \frac{2z^{33}}{33(1-z^2)} \le \frac{9}{132\,3^{33}} < \frac{1}{Q}. \tag{10.304}$$

Round each of the sixteen positive terms down to an integer multiple of $1/Q$. Their sum is a lower bound; adding $17/Q$ gives an upper bound, since there are sixteen rounding errors and the tail is less than $1/Q$. At $z = 0$ use the exact value zero. Adding a copies of the interval for $\ln 2$ gives integer endpoints l_m, u_m with $l_m/Q \le \ln m \le u_m/Q$. This uses integer division and powers only. Apply these enclosures, with the appropriate signs, to the fifteen terms defining ω. They give the displayed strict bracket. Substituting its two endpoints and the four rational coefficient bounds proves (10.303); the residual fractions are exactly those shown there.

The sieve begins with unmarked integers from two through N. Each unmarked p is prime: a composite would already have been marked by its smaller prime divisor. It marks multiples starting at p^2; smaller multiples were marked earlier. For every discovered prime, it assigns the interval for $\ln p$ to each of $p, p^2, \ldots \le N$. No two different primes assign the same integer. Successive cumulative sums give integer endpoints L_n, U_n for $Q\psi(n)$. The program checks all the following inequalities by integer arithmetic:

Integer range	Checked positive integer expression	Smallest value
$114 \le n < N$	$27Qn - 26U_n$	13207722459676, at $n = 199$
$227 \le n < N$	$26L_n - 25Q(n+1)$	55077005461136, at $n = 346$
$1 \le n \le 113$	$107Qn - 103U_n$	167297005424
$1 \le n < 113$	$nL_{113} - 113U_n$	33103591701014

The same calculation gives $117386725268647 \le Q\psi(113) \le 117386725271792$ and verifies $26L_{113} > 27Q \cdot 113$. The sieve finds 78498 primes and 78734 prime powers at most N; these counts are outputs, not inputs. Every prime-power assignment and cumulative sum is integral, and the maximum cumulative upper endpoint is checked to be below 2^{63}, the storage limit. The finite sums for (10.302) use arbitrary-size integers.

For a real $x \in [n, n+1)$, the upper comparison follows from $\psi(x) = \psi(n) < Un \le Ux$. The lower comparison follows from $\psi(x) = \psi(n) > L(n+1) > Lx$; the extra $n+1$ in its certificate is essential. These arguments prove all assertions of the lemma. □

Continuing the bounds to every real argument

Theorem 10.112 (a sharp elementary lcm bound). For every real $x > 0$,

$$\psi(x) \le \frac{\psi(113)}{113}x < \frac{107}{103}x. \tag{10.305}$$

Consequently (10.299) holds at every positive integer k.

Proof. We first derive the two recurrences from the actual factorial combination. For an integer $j \in [-1, 4]$, $j - 1 = -\sum_{s=0}^{1} \mathbf{1}_{j<s} + \sum_{s=2}^{4} \mathbf{1}_{j \ge s}$. Apply this at $j = F(x/p^a)$ in (10.301). All weights $\ln p$ are positive. The condition $x/p^a \in [b, c)$ contributes exactly $\psi(x/b) - \psi(x/c)$, with the indicated half-open convention.

For $s = 0, 1$, occurrences of $F < s$ begin only at a_s. Their total mass is at most $\psi(x/a_s)$, minus the masses of any disjoint selected intervals where $F \geq s$. For $s = 2, 3, 4$, the mass where $F \geq s$ is at least that of its selected intervals. This gives the upper recurrence. Reverse the two roles to obtain the lower recurrence:

$$\begin{aligned}\psi(x) &\leq \sigma(x) + \psi(x/a_0) + \psi(x/a_1) - \sum_{(b,c)\in\mathcal{U}} [\psi(x/b) - \psi(x/c)],\\ \psi(x) &\geq \sigma(x) - \sum_{s=2}^{4} \psi(x/a_s) + \sum_{(b,c)\in\mathcal{L}} [\psi(x/b) - \psi(x/c)].\end{aligned} \tag{10.306}$$

There is no subtraction of intersecting intervals within a level: Lemma 10.111 checked disjointness. Intervals at different levels represent different indicator terms and must retain their multiplicity.

For $x \geq N$, (10.301) gives $|\sigma(x)/x - \omega| < \varepsilon$. Indeed $(\ln x + 1)/x$ decreases for $x > 1$, and $\ln N < 14$. The latter follows from $\ln 10 < 7/3$: the exponential series at $7/3$ through degree six is already greater than ten.

Now prove simultaneously $Lx < \psi(x) < Ux$, starting at $x = N$, by induction on the interval $[n, n + 1)$. The finite base is Lemma 10.111. Every argument x/d on the right of (10.306) has $d \geq 17$, so it is less than n. Every argument requiring an upper estimate is at least $N/877 > 114$. Every argument requiring a lower estimate is at least $N/440 > 227$. Thus either the finite base or the earlier induction intervals apply to each term. Division of the upper recurrence by x gives $\psi(x)/x < \omega+\varepsilon+(1-A)U-BL < U$. The lower recurrence gives $\psi(x)/x > \omega-\varepsilon-CU+(1-D)L > L$. The two strict final comparisons are precisely (10.303). This proves the bounds at every $x \geq N$, and hence the upper bound $\psi(x) < Ux$ at every $x \geq 114$.

At $0 < x < 1$ the assertion is immediate. At $1 \leq x < 114$, let $n = \lfloor x \rfloor \leq 113$. Lemma 10.111 gives $\psi(x)/x \leq \psi(n)/n \leq \psi(113)/113$, with equality permitted at $x = 113$. At $x \geq 114$, use $\psi(x)/x < U < \psi(113)/113$. The last strict bound in (10.305) is the certificate at 113. Finally use $\ln v(k) = \psi(k)$. □

Corollary 10.113 (the sharper full-simplex coefficient budget). Every field, prime and rank case of Theorem 10.107 and Corollary 10.108 remains valid with C_r replaced by

$$C_r^{\sharp} = \frac{r + 4/103}{r + 1} + \frac{17(r - 1)}{24(r + 1)^2} = C_r - \frac{63}{1030(r + 1)} < C_r < 1. \tag{10.307}$$

The individual clearing factor also satisfies $u \ln v(k) < (107/103)uk$ whenever $u > 0$, and is zero when $u = 0$.

Proof. In (10.293), substitute $\ln v(k) < (107/103)k \leq (107/103)H_1$, using Theorem 10.112. All Euler, additive and field arguments are unchanged. The rational calculation of its mean coefficient now gives (10.307). The full row composition, its zero rows and its actual coefficient field are the same. In (10.294), only the scalar mean term containing C_r changes, so the same substitution gives the asserted coefficient-height bound. Multiplication of (10.299) by u proves the last statement. □

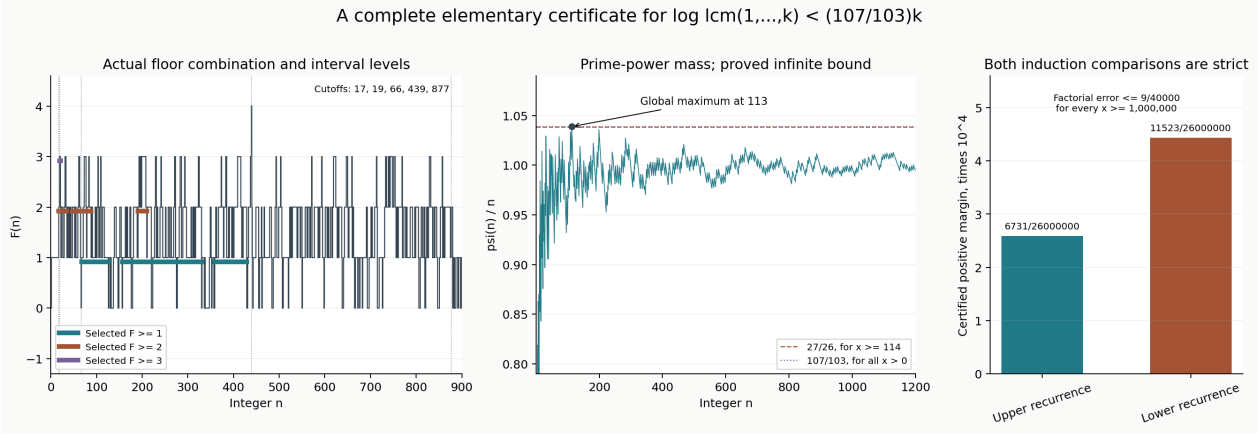


Figure 10.25. Left: the actual integer floor combination $F(n)$ and the selected intervals in the upper recurrence, colored by their level; the vertical marks are its initial cutoffs. Middle: $\psi(n)/n$ for integers through 1200, with the maximum at 113 and the proved constants $27/26$ *and* $107/103$. *These discrete points illustrate (10.305); the simultaneous induction (10.306) proves the infinite statement. Right: the exact positive lower bounds for the two induction margins in (10.303). The factorial combination is estimated at $x \geq 10^6$; the finite base is generated by the integer certificate of Lemma 10.111. Figure program. Human method and interval choices: Costa Pereira's free paper, pages 321–323; all required identities, finite checks, error bounds and induction are given above.*

38. An individual clearing and Euler bound at every original parameter

The full-simplex average controls the initial coefficient height. A later nonzero-value argument needs a bound for each individual row. We obtain one from a positive generating series. It preserves the factorial denominator in the additive polynomial and makes the remainder uniform in the field, prime and rank.

Retain Sections 26 and 29, with $H_1 = h + x$, $Z = S\mathcal{D}/d$, the coefficient-height condition (10.223), and the seven original constant cases in (10.281). Define $g_9 = \max\{g_{91}, 107/103\}$.

The exact sharp regime for the Euler constant

Lemma 10.114. For every original case and every $r \geq 2$,

$$g_9 \leq \max\left\{1 + \frac{1}{3(r+1)}, \frac{107}{103}\right\}. \qquad g_9 = \frac{107}{103} \quad \text{if } d = 1 \text{ or } r \geq 8. \tag{10.308}$$

Proof. The first statement follows from Lemma 10.106. At $r \geq 8$, its bound gives $g_{91} - 1 < 1/27 < 4/103$; the difference of the last two fractions is $5/2781$.

At $d = 1$, the only cases are I.2 and III.2. Four terms of the positive logarithm series give $\ln 2 > 693/1000$, and five terms give $\ln 3 > 1098/1000$. The already proved $\ln 6 < 9/5$ therefore gives $W(1) < (9/5)/[(693/1000)(1098/1000)] < 19/8$. The exponential series through degree five at $7/8$ exceeds $19/8$, so $\ln W(1) < 7/8$. All these comparisons are between explicit rational numbers; the positive tails have the required direction.

At rank two in I.2, (10.287) and $\ln 14 > 13/5, \ln 13 > 5/2$ give $g_0 > 3(9.2 + 4.79 + 2.5) = 49.47$. In III.2, use $\ln 7 > 19/10$ to obtain $g_0 > 3(7.8 + 5.84 + 2.5) = 2421/50$. Every term defining g_0 increases with r, so the latter lower bound holds throughout both cases. By (10.222), $g_{91} - 1 < (15/8)/(2421/50) = 125/3228 < 4/103$. The last difference is $37/332484 > 0$. Taking the maximum with $107/103$ proves (10.308). □

A positive generating-series majorant

For a support contracted by q^{-I}, $I \ge 0$, take its actual integer Euler maxima Ω_j, and set $m_I = \sum_{j=1}^{r-1} \Omega_j + r - 2$, as in (10.239). Lemma 10.84 permits the same bound with the integer part of its upper estimate. Put $a = g_9 H_1$, $z = \exp(-a)$, and $E_I = (m_I + 1)[-\ln(1 - z)]$. This remainder depends on the support, not on the allocation of derivative orders.

Theorem 10.115 (an individual clearing/Euler bound). For every pair of nonnegative integers u, h_{E},

$$u \ln v(k) + \ln \binom{m_I + h_{\mathrm{E}}}{h_{\mathrm{E}}} \le a(u + h_{\mathrm{E}}) + E_I. \tag{10.309}$$

In the original parameters the remainder satisfies

$$\frac{E_I}{Z} < \frac{1000}{999} \frac{q^{-I}(r-1)}{\mathrm{e}\, c_1 c_2 c_3 q P (r+1)^2} + \frac{r-1}{999Z} < \frac{q^{-I}}{300} + \frac{r-1}{999Z}. \tag{10.310}$$

The symbol e in this formula is $\exp(1)$, and $P = p^{\kappa}$ is the original prime power.

Proof. The positive geometric series and its $m_I + 1$-fold product give $(1-z)^{-m_I-1} = \sum_{j \ge 0} \binom{m_I+j}{j} z^j$. Indeed its coefficient counts the $m_I + 1$ nonnegative parts summing to j; placing m_I separators among $m_I + j$ positions gives the displayed binomial. Positivity bounds any one term by the entire sum. By Theorem 10.112, $\ln v(k) < (107/103)k \le a$, because $k \le H_1$ and $g_9 \ge 107/103$. Thus $v(k)^u \binom{m_I+h_{\mathrm{E}}}{h_{\mathrm{E}}} \le z^{-u-h_{\mathrm{E}}}(1-z)^{-m_I-1}$. Taking logarithms proves (10.309), including $u = 0$ or $h_{\mathrm{E}} = 0$.

We now bound its actual remainder. The full-width argument (10.220), with contraction, gives $m_I + 1 \le (r-1)q^{\nu-I}\mathcal{D}R_{\max}/(c_1 c_2 P d) + r - 1$. There is no extra factor $1/r$: summing the r widths cancels that factor in (10.186). Substituting $R_{\max} \le dW(d)\exp(h)$, $q^{\nu} = \exp(x)$, and the exact formula for S gives

$$\frac{m_I + 1}{Z} \le \frac{q^{-I}(r-1)tW(d)\exp(H_1)}{c_1 c_2 c_3 q P (r+1) H_1} + \frac{r-1}{Z}. \tag{10.311}$$

For $0 < z < 1$, integration gives $-\ln(1-z) \le z/(1-z)$. Since $(g_9 - 1)H_1 \ge (g_{91} - 1)g_0 = 1 + \ln W(d)$, we have $W(d)\exp(H_1)z \le 1/\mathrm{e}$. Also $H_1 > 39$ and $g_9 > 1$. The first four exponential terms at 39 exceed 1000, so $z < 1/1000$ and $(1-z)^{-1} < 1000/999$. Finally $t/H_1 \le 1/(r+1)$. These substitutions in (10.311) prove the first inequality of (10.310).

For its uniform last inequality, the definition $2e < P(p-1)$ gives $P \ge 3$ in I, $P \ge 5$ in II, and $P \ge 4$ in V; in III and IV use $P \ge 1$. Direct substitution of the seven rational rows gives the following lower bounds for $c_1 c_2 c_3 q P$, all greater than 15:

Case	Rational lower bound
I.1	445693563/20000000
I.2	527023581/25000000
II	52892553/2500000
III.1	18772369/1250000
III.2	204239/12500
IV	389083863/20000000
V	522494609/25000000

For all $r \ge 2$, $(r-1)/(r+1)^2 \le 1/8$, since $(r-3)^2 \ge 0$. The positive exponential series proves $1/\mathrm{e} < 3/8$, as in Theorem 10.107. The coefficient of q^{-I} is therefore less than $(1000/999)(3/8)/(8 \cdot$

15) = 25/7992 < 1/300. This proves the last inequality. It holds at every rank and degree; the proof did not test a finite list of ranks or degrees. □

Retaining the factorial and the actual field

Corollary 10.116 (a factorial-preserving general point budget). Retain every hypothesis and notation of Theorem 10.92, with its actual $R_I = q^{-I}X + 2k - 1$, number field F, q-denominator exponent $\Xi_{I,J,u}$, and $u \le D_{\mathrm{a}}$. For an integer total-order cutoff $O \ge u + h_{\mathrm{E}}$, put

$$\begin{aligned} B_{I,u}(X,O) &= (D_{\mathrm{a}} - u)\ln R_I + \ln \binom{D_{\mathrm{a}}}{u} - L\ln(k!) + aO + E_I, \\ B_I(X,O) &= D_{\mathrm{a}}\ln(R_I + 1) - L\ln(k!) + aO + E_I. \end{aligned} \tag{10.312}$$

Then $\ln \mathcal{H}_{I,u,h_{\mathrm{E}}}(X) \le B_{I,u}(X,O) \le B_I(X,O)$. The exact maximization (10.241) remains available, so its logarithm and B_I may be replaced by their smaller value when the whole cutoff is used. For a nonzero value, its same actual normal quantity satisfies

$$\begin{aligned} v_p(V) + G_u - \delta \le \frac{d_F}{e_F f_F \ln p}\Big[& h_2(c) + \tfrac{1}{2}\ln N + B_{I,u}(X,O) \\ & + \Xi_{I,J,u}\ln q + PXq^{-I}\sum_i D_i\sigma_i \Big] + G_u. \end{aligned} \tag{10.313}$$

Here $G_u = (D_{\mathrm{a}} - u)\theta + Lv_p(k!) - uv_p(v(k))$, exactly as in Lemma 10.91. One may take the minimum with the previously proved normal-coordinate bound (10.297).

Proof. Insert (10.309) into the individual scalar expression (10.240), and use $u + h_{\mathrm{E}} \le O$. This gives its bound by $B_{I,u}$. The ordinary binomial expansion gives $\binom{D_{\mathrm{a}}}{u} R_I^{D_{\mathrm{a}}-u} \le (R_I + 1)^{D_{\mathrm{a}}}$, proving $B_{I,u} \le B_I$. Both bounds keep the negative term $-L\ln(k!)$; it has not been canceled by changing the rational clearing convention.

Apply (10.244) with this scalar bound and then add the same exact G_u to both sides. No global field has been restricted, no q-denominator has been removed, and no derivative order has been averaged. Equation (10.297) bounds this identical normal quantity in the identical field. Their minimum is therefore legitimate. At $u > D_{\mathrm{a}}$, the value is zero by Lemma 10.91 and no nonzero-value argument is needed. □

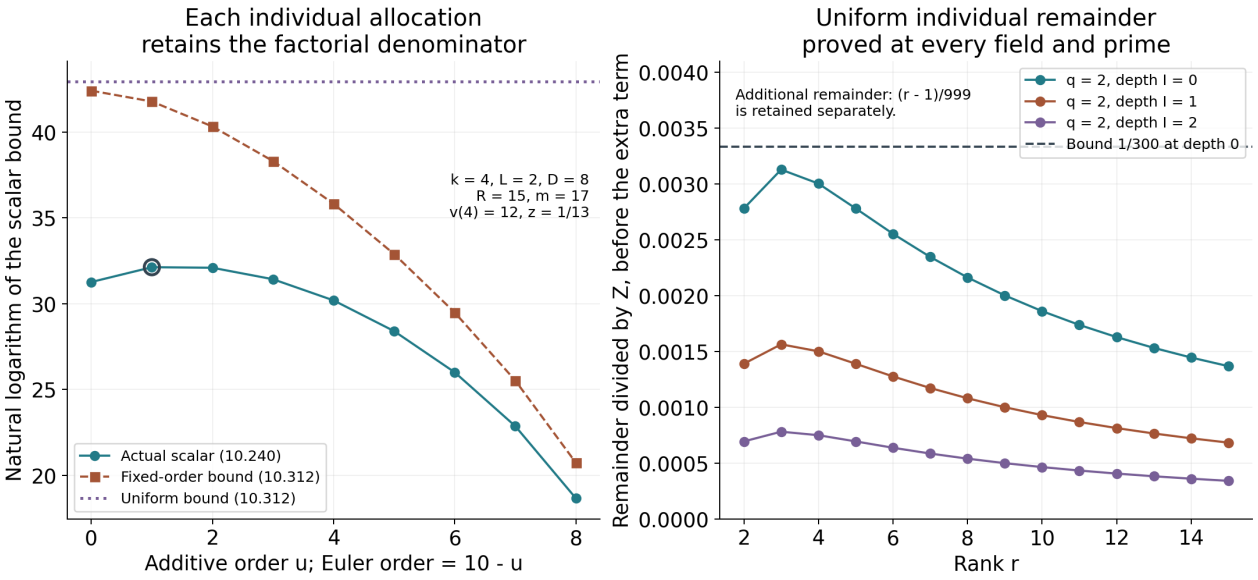


Figure 10.26. Left: all additive allocations $0 \le u \le 8$ with Euler order $10 - u$, at $k = 4, L = 2, D_{\mathrm{a}} = 8, R_I = 15, m_I = 17$. The lcm is exactly 12. The plotted scalar is (10.240), including its denominator $(4!)^2$; its largest value occurs at $u = 1$. The generating comparison uses $z = 1/13$, so $a = \ln 13$, and displays both the fixed-allocation and uniform bounds (10.312). This is an illustrative instance of the positive-series proof, not a substitution for the original general parameters. Right: the coefficient of Z in the first term of the analytical bound (10.310), using $1/\mathrm{e} < 3/8$ and $c_1c_2c_3qP > 15$, for $q = 2$ and depths zero, one and two. Its bound by $1/(300q^I)$ is proved for every rank; the separate additive remainder $(r-1)/999$ is retained in (10.310). Proof locators: Theorem 10.115, Corollary 10.116 and Solution 46. Figure program. Human background: Yu's free paper, its definitions (3.16) and the individual scalar estimate (5.38); the positive-series bound and its uniform remainder are proved here.

39. Keeping additive divisibility in the ordinary jets

The prepared equations vanish at their input nodes. Their projection error already contains a useful factor from the additive polynomial. Keeping that factor through differentiation gives a more precise input to interpolation, at each fixed additive order. The argument applies to the original fields and to every contraction depth; it does not average derivative orders.

Divisibility of a prepared scalar

Use the prepared family (10.92), its original and projected curves (10.104), and the logarithmic slopes of Lemma 10.91. Write

$$D_{\mathrm{a}} = kL, \qquad \ell_p = v_p(\mathfrak{v}), \qquad \chi_p = Lv_p(k!), \qquad \gamma_u = \max\{0, u\ell_p - \chi_p\}, \qquad \mathfrak{v} = \operatorname{lcm}(1, \dots, k). \tag{10.314}$$

The symbol ℓ_p is a valuation, not the exponent of an additive basis column. The selected Euler arguments are integers, as in Lemma 10.38.

Lemma 10.117 (order-dependent projection precision). Suppose the hypotheses of Lemma 10.36 hold, with $b = b_n$, $\beta = v_p(b_n)$, and coefficient minimum δ. At any $y \in \mathbb{Z}_p$, every prepared scalar of additive order $u \le D_{\mathrm{a}}$ has valuation at least γ_u. Consequently

$$v_p\big(f(y; \boldsymbol{t}) - \Phi(y; \boldsymbol{t})\big) \ge \delta + U - \beta + \gamma_u, \qquad u = t_0. \tag{10.315}$$

If $u > D_{\mathrm{a}}$, both prepared functions vanish identically.

Proof. Consider a column with exponent $1 \le l \le L$. Its additive prepared scalar is $\mathfrak{v}^u$ times the coefficient of V^u in

$$\frac{1}{(k!)^l} \prod_{a=1}^{k} (q^{-I} y + \lambda_{-1} + a + V)^l.$$

Every coefficient of the numerator is p-integral, since $q \neq p$ and $y \in \mathbb{Z}_p$. Thus its valuation is at least $u\ell_p - l v_p(k!) \ge u\ell_p - \chi_p$. Independently, prepared Taylor integrality, extended to $\mathbb{Z}_p$ in the proof of Lemma 10.36, gives valuation at least zero. These are simultaneous bounds, giving their maximum γ_u. Each integer Euler binomial is p-integral, so multiplication by the Euler factors preserves that maximum.

In the proof of Lemma 10.36, the ratio of the two exponential monomials minus one has valuation at least $U - \beta$; the original monomial is a local unit. Multiply this by the scalar bound just proved and by a coefficient of valuation at least δ. The ultrametric inequality proves (10.315). Beyond degree D_{a}, every additive derivative is zero. □

This proof also applies at $y = s/q^J$. A q-denominator has p-adic valuation zero. It still incurs its full arithmetic cost at primes above q in Theorem 10.92.

The precision at one additive order

Retain the selected derivations and coefficients λ_{aj} of Lemma 10.38. Assume the full original depth condition $v_p(z_j) \ge \vartheta + 1/(p-1)$, the p-integrality of b_j/b_n, and $0 < \theta < \vartheta$. For an integer $j \ge 0$, define

$$\begin{aligned} H_{u,j} &= \min\{j, D_{\mathrm{a}} - u\}, \\ M_u(j) &= \min_{0 \le a \le H_{u,j}} \{\max\{-a\ell_p, u\ell_p - \chi_p\} + (j-a)(\vartheta - \beta)\}. \end{aligned} \tag{10.316}$$

The minimum is over integers. It computes a lower-bound envelope for the jets; it does not assert that a jet attains that valuation.

Theorem 10.118 (ordinary jets with the additive divisibility retained). Fix a prepared index $\boldsymbol{t}$ of additive order $u \le D_{\mathrm{a}}$. Suppose the original prepared values at a node $s \in \mathbb{Z}_p$ vanish through total order T. For $|\boldsymbol{t}| + j \le T$,

$$\begin{aligned} v_p(f_j(s;\boldsymbol{t})) &\ge \delta + U - \beta + M_u(j), \\ v_p((F_u)_j(\rho s;\boldsymbol{t})) + j\theta &\ge U - \beta + G_u + M_u(j). \end{aligned} \tag{10.317}$$

Here F_u, G_u are exactly (10.238), and the subscript j is ordinary divided differentiation. In particular, with

$$C_{\mathrm{sharp}} = \max\{\ell_p, \beta - \vartheta, 0\}, \qquad M_u(j) \ge -jC_{\mathrm{sharp}}, \tag{10.318}$$

one may use the linear loss C_{sharp} instead of C_0. Formula (10.316) can be stronger than this linear consequence.

Proof. Differentiate along the projected curve with the commuting operator $\partial_X + \sum_{a \in J} \xi_a \delta_a$ of Lemma 10.38. Its coefficients satisfy $v_p(\xi_a) \ge \vartheta + 1/(p-1) - \beta$, unless they are zero. For every positive integer h, the factorial bound (9.4) gives

$$v_p(\xi_a^h/h!) \ge h(\vartheta - \beta).$$

The bound is also valid at $h = 0$. Multiplication of an Euler binomial by a power of its argument has integer expansion coefficients in the same binomial basis, by the identity used in Lemma 10.38. Thus all Euler operations of combined order $j - a$ have scalar valuation at least $(j - a)(\vartheta - \beta)$, without an additional factorial loss.

The additive part of order a has multiplier $(q^I \mathfrak{v})^{-a}\binom{u+a}{a}$, whose valuation is at least $-a\ell_p$. Its resulting prepared additive order is $u + a$. The resulting total prepared order is at most $|\boldsymbol{t}| + j$, so its original value is zero at s. Lemma 10.117 bounds its projected value by $\delta + U - \beta + \gamma_{u+a}$. If $u + a > D_{\mathrm{a}}$, the term is identically zero and can be omitted. Every remaining term therefore has valuation at least

$$\delta + U - \beta + \gamma_{u+a} - a\ell_p + (j - a)(\vartheta - \beta).$$

Since $\gamma_{u+a} - a\ell_p = \max\{-a\ell_p, u\ell_p - \chi_p\}$, minimization and the ultrametric inequality prove the first line of (10.317). The exact chain rule (10.238) proves its second line in the same normal coordinates.

For the simpler bound, $\gamma_{u+a} \geq 0$ and $\vartheta - \beta \geq -C_{\mathrm{sharp}}$, while $\ell_p \leq C_{\mathrm{sharp}}$. Each term is at least $-jC_{\mathrm{sharp}}$, proving (10.318). All q-powers used here are local units; this assertion does not remove a q-cost from the separate height estimate. □

Full, nested and q-deleted interpolation

For an integer maximum multiplicity $M \geq 1$ and separation bound $B \geq 0$, put

$$\begin{aligned} K_u(M, B) &= \min_{0 \leq j < M} \{M_u(j) + jB\} \\ &= \min_{0 \leq a \leq H} \{\max\{-a\ell_p, u\ell_p - \chi_p\} + aB \\ &\qquad + (M - 1 - a)\min\{0, \vartheta - \beta + B\}\}, \\ H &= \min\{M - 1, D_{\mathrm{a}} - u\}. \end{aligned} \tag{10.319}$$

These are finite integer minima. To evaluate the second one when $\ell_p > 0$, it suffices to check $0, H$ and the floor and ceiling of $(\chi_p - u\ell_p)/\ell_p$, clipped to $[0, H]$. Indeed its expression is affine on either side of that one breakpoint, and a linear function on an integer interval takes its minimum at an endpoint. When $\ell_p = 0$, it equals $(M - 1)\min\{0, \vartheta - \beta + B\}$.

To prove the equality in (10.319), write $j = a + h$. At each fixed a, the admissible Euler order is $0 \leq h \leq M - 1 - a$. Its coefficient in the expression to minimize is $\vartheta - \beta + B$. Choosing the appropriate endpoint gives exactly the second line. No real relaxation of an integer order is being made.

Theorem 10.119 (interpolation at the retained jet precision). Use any one of the following proved node systems:

- The full nested intervals of Lemma 10.94. Set N_* as in (10.247), $M = \mu_0$, and $E = 0$.
- The mixed q-deleted and full intervals of Lemma 10.96. Set N_* as in (10.251), $M = \mu_0$, and $E = \mu_0 - \mu_1$.
- A q-deleted interval $|s| \leq R$, $q \nmid s$, with $q \mid R$ and constant multiplicity M. Set $N_* = 2(1 - 1/q)RM$ and $E = M$.

In the first two cases use their proved separation bound B; in the last use $B = \lfloor \log_p(2R) \rfloor$. At every node s, suppose the original prepared values vanish through order $|\boldsymbol{t}| + \mu(s) - 1$, where $\boldsymbol{t}$ has fixed additive order $u \leq D_{\mathrm{a}}$. Put $\Lambda_u = U - \beta + G_u$. Then, for every $x \in \mathbb{Z}_p \subset \mathbb{Q}_p$,

$$
\begin{aligned}
v_p(f(x;\boldsymbol{t})) + G_u - \delta &\geq \mathcal{P}_u, \\
v_p(\Phi(x;\boldsymbol{t})) + G_u - \delta &\geq \mathcal{P}_u, \\
\mathcal{P}_u &= \min\{N_*\theta, \Lambda_u - EB - (M-1)B + K_u(M,B)\}.
\end{aligned}
\tag{10.320}
$$

Proof. Lemma 10.91 makes F_u normal. Theorem 10.118 gives its weighted jets with baseline $\Lambda_u = U - \beta + G_u$ and increment $M_u(j)$. Use the complete cardinal polynomial, Taylor inverse and normal quotient constructions of Lemmas 10.94 and 10.96. At a node of multiplicity $\mu(s)$, its cardinal factor has valuation at least $-B(M - \mu(s) + E)$. The inverse coefficient of degree $a \leq \mu(s) - 1 - j$ has valuation at least $-aB$. These assertions also hold for the constant q-deleted system, with $E = M$, by the cardinal and inverse proof of Theorem 10.39.

A Hermite summand with divided jet j thus has valuation at least

$$
\Lambda_u + M_u(j) - B(M - \mu(s) + E) - aB \geq \Lambda_u - EB - (M-1)B + M_u(j) + jB.
$$

Every such j is less than M, so (10.319) gives the second entry of (10.320). The same monic normal quotient has valuation at least $N_*\theta$ at ρx, exactly as in those proved interpolation lemmas. Their sum proves the asserted lower bound for f.

At $j = 0$, (10.316) gives $M_u(0) = \gamma_u$. Hence $K_u(M,B) \leq \gamma_u$, and the second entry of (10.320) is at most $\Lambda_u + \gamma_u$. Lemma 10.117 gives at least this last precision for the normalized projection error at x. Adding that error to f proves the same assertion for Φ. □

The q-deleted extra term EB survives. The lower bound also retains the exact factorial/lcm terms in G_u. From (10.318), its second entry is at least $U - \beta + G_u - EB - (M-1)\max\{B, C_{\text{sharp}}\}$. The finite envelope (10.319) can give additional precision.

The comparison with an actual algebraic value

Corollary 10.120 (a fixed-order zero criterion). Retain the hypotheses and precision $\mathcal{P}_u$ of Theorem 10.119. At $x = s/q^J$, suppose the phase and field hypotheses of Theorem 10.92 identify the analytic original value $\Phi(x;\boldsymbol{t})$ with its actual algebraic value $V \in F$. Take its same coefficient minimum and fixed additive/Euler orders, full global degree and local degrees. Let $\mathcal{A}_u(x)$ be the smaller of the right sides of (10.312) and (10.297), with these exact data. Then

$$
\mathcal{P}_u > \mathcal{A}_u(x) \quad \Longrightarrow \quad V = 0. \tag{10.321}
$$

In particular the sufficient input budget

$$
U - \beta + G_u \geq N_*\theta + EB + (M-1)B - K_u(M,B) \tag{10.322}
$$

and the strict comparison $N_*\theta > \mathcal{A}_u(x)$ imply that zero.

Proof. For a nonzero V, Corollary 10.116 and Theorem 10.109 give $v_p(V) + G_u - \delta \leq \mathcal{A}_u(x)$ in the same actual field. Theorem 10.119 gives the opposite strict inequality under (10.321), a contradiction. Substitution of (10.322) proves the last assertion. □

The criterion keeps the separate remainder in (10.311), the original root field, q-denominators and derivative order. Its two inequalities must still be verified at the particular original radii, multiplicities and arithmetic data.

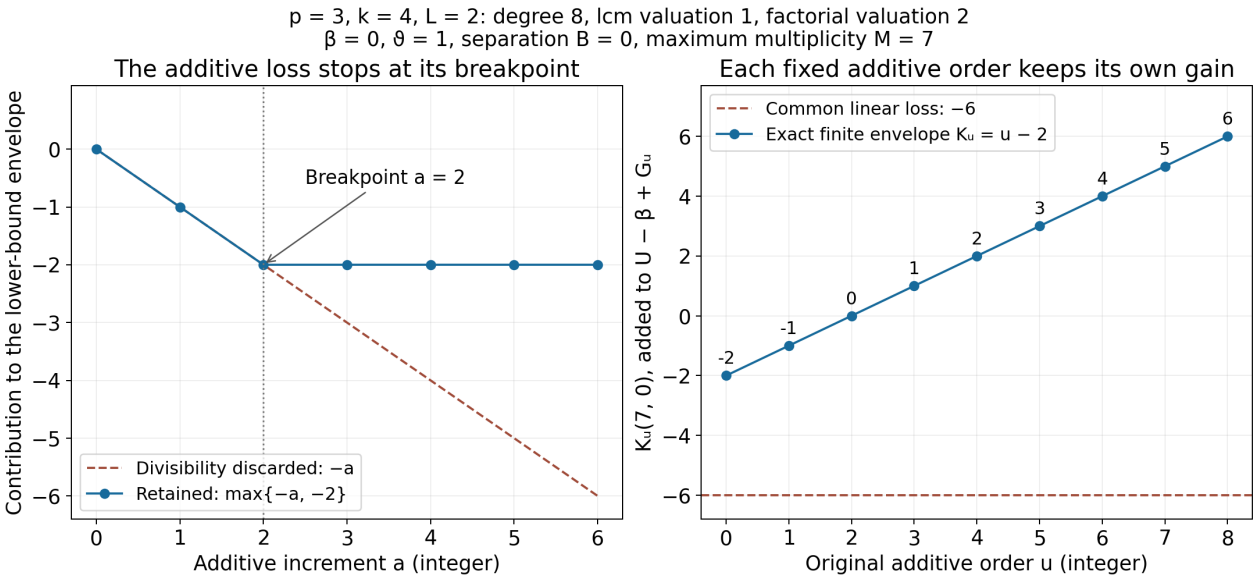


Figure 10.27. An exact illustrative scalar calculation at $p = 3, k = 4, L = 2, D_{\mathrm{a}} = 8$, so $\ell_p = 1, \chi_p = 2$, with $\beta = 0, \vartheta = 1, B = 0, M = 7$. Left: at $u = 0$, the expression minimized in (10.319) is $\max\{-a, -2\}$, for integer $0 \le a \le 6$; its minimum is -2, attained from $a = 2$ onward. The dashed line displays the bound $-a$ obtained after discarding additive divisibility. Right: the exact lower-bound envelope is $K_u(7, 0) = u - 2$ for every $0 \le u \le 8$, compared with the common linear-loss contribution -6 from $C_0 = 1$. These are contributions to $U - \beta + G_u$, not valuations of actual nonzero auxiliary values. The displayed integers are checked in Solution 47; Theorems 10.118–10.119 prove the general mechanism. Figure program. Human-source context: the original ordinary-jet extrapolation in Yu's freely accessible paper, Section 5, equations (5.25)–(5.32); the order-dependent refinement is proved above.

40. General integer zero production at the original parameters

We now perform the strict analytic and arithmetic comparisons for every original field, prime and rank case. The integer radii and jet orders are formed from the original real parameters. The q-denominator at a contracted stage is charged in full; its depth-dependent cost is controlled by the corresponding decrease in the torus height.

The original height and descent ranges

Retain Sections 26, 29 and 36–39. Choose the original $\theta = \vartheta/(1 + \varepsilon_n)$, where $\varepsilon_n = 10^{-26}/(2n)$ and n is the number of original logarithms, as in (10.74). The estimates below are also valid for $\theta = \vartheta/(1 + \tau)$ with $0 < \tau \le 10^{-6}$. Let $b_n \ne 0$ have minimum p-adic valuation among the nonzero coefficients. Put

$$\begin{aligned} &B_\dagger = \min_{b_j \ne 0} |b_j|, \qquad h \ge \max\{g_0, (r+1)t, \ln B_\dagger, \ln(R_{\max}/(dW(d)))\}, \\ &U = \frac{q^{r+1} S\mathcal{D}}{et}, \quad W_* = \frac{et}{dZ}, \qquad W_* U = q^{r+1}, \\ &S_I = S\eta^{-(r+1)I}, \quad T_I = T\eta^{(r+1)I}, \qquad \eta = 1 - \frac{c_5}{r+1}. \end{aligned} \tag{10.323}$$

Here $t = f \ln p$, $Z = S\mathcal{D}/d$, $k = \lfloor h + \nu \ln q \rfloor$, and every other parameter is exactly (10.186). The following values of c_5 complete the original constant table.

Case	$r = 2$	$3 \le r \le 7$	$r \ge 8$
I.1	0.5377	0.55	0.56

Case	$r = 2$	$3 \le r \le 7$	$r \ge 8$
I.2	0.538	0.551	0.56
II	0.53	0.54	0.55
III.1	0.528	0.536	0.55
III.2	0.5267	0.534	0.55
IV	0.5345	0.543	0.56
V	0.753	0.78	0.827

Use a nonnegative integer stage I satisfying $I \ln(q\eta^{r+1}) \le 3(A + \nu \ln q)$ and $c_5 T_{I+1}/(r+1) \ge 1$. These are the original geometric and multiplicity restrictions before either stopping point. They include $I = 0$: Lemma 10.104 gives $T \ge g_4 > 6(r+1)$, while $c_5 > 1/2$ and $\eta^{r+1} > 1/3$.

An admissible stage family has the same selected Euler derivations, an integral coefficient subvector of the initial kernel, its reference coefficient of minimum valuation, the phase condition of Theorem 10.55, full widths at most $q^{-I} D_i$, and additive factors evaluated at $q^{-I} x$. The original torus slopes and their projection are those of Lemma 10.91. The phase identity (10.151) puts every integer value in K; the height and coefficient-minimum conclusions of Theorems 10.48 and 10.60 keep the original coefficient bound. Thus these data describe the full original induction, without a special rational or fixed-rank family.

Coupled field bounds

Lemma 10.121 (surplus and depth in the same field). Write

$$C_{01} = c_0 \ell_d \frac{p^f}{p^f - 1}, \qquad Y = e\theta, \qquad \Omega_{\mathrm{n}} = \frac{e}{d}\left(\theta + \frac{1}{p-1}\right). \tag{10.324}$$

The following bounds hold simultaneously. In the third column, the lower bound for Y is the displayed number divided by $1 + 10^{-6}$.

Case	Lower bound for C_{01}	Numerator of the lower bound for Y	Upper bound for Ω_{n}
I.1	$9c_0/8$	3/2	2
I.2	57/20	3/2	2
II	$5c_0/4$	5/2	3/2
III.1	c_0	3/4	1/2
III.2	c_0	3/4	1
IV, $P = 1$	c_0	1/2	1/2
IV, $P \ge 7$	c_0	7/2	4/3
V	10/3	2	3/2

Proof. The local degree identity gives $ef \le d$. In I.1, if $d = 2$, then $f \le 2$ and $3^f/(3^f - 1) \ge 9/8$. At $d = 3$, use $f \le 3$ and $\ln 3 > 13/12$, giving $\ell_d 3^f/(3^f - 1) > (13/12)(27/26) = 9/8$. At $d \ge 4$, $\ell_d \ge \ln 4 > 4/3$ suffices. In I.2, $e = f = d = 1$, so $C_{01} = (19/10)(3/2) = 57/20$. In II, $e \ge 2$. At $d = 2, 3$, we have $f = 1$, giving the factor $5/4$. At larger d, $\ell_d > 4/3 > 5/4$. The other odd-prime surplus bounds follow directly from $\ell_d \ge 1$ and $p^f/(p^f - 1) > 1$.

In V, K contains a primitive cube root. Its quadratic field has degree two, so d is even. Its reduction has order three, hence f is even and at least two, as proved in Lemma 10.104. At $d = 2$, $f = 2$ and the surplus is $(5/2)(4/3) = 10/3$. At $d \ge 4$, use $\ell_d > 4/3$. The logarithm bounds follow from the positive series in Solution 34; in particular $\ln 2 > 2/3$ and $\ln 3 > 1098/1000 > 13/12$.

Outside III, $Y = P/[2(1+\tau)]$. The defining strict inequality $2e < P(p-1)$ gives $P \geq 3, 5, 4$ respectively in I, II and V. In IV, P is either one or at least $p \geq 7$. In III, $e = 1$ and $\vartheta = (p-2)/(p-1) \geq 3/4$. These prove the depth column.

For I, use $\theta \leq 3/2$, $e/d \leq 1$, giving $\Omega_{\rm n} \leq 2$; for II use $\theta \leq 5/4$. In III, $\theta + 1/(p-1) \leq 1$, and $d \geq 2$ in III.1. If IV has $P = 1$, then $2e < p - 1$, so $\Omega_{\rm n} = [1/(2(1+\tau)) + e/(p-1)]/d < 1/d \leq 1/2$. For its other branch, $\theta \leq p/(p-1)$ gives $\Omega_{\rm n} \leq (p+1)/(p-1) \leq 4/3$. Finally Lemma 10.73 gives $e/d \leq 1/2$ in V, and $\theta \leq 2$, giving $\Omega_{\rm n} \leq 3/2$. □

The additive polynomial and the contracted denominator

Lemma 10.122 (integer-stage scalar costs). At the target radius $X = q^{j+1}S_I$, $j \geq 0$, the additive part of the bound (10.312) satisfies

$$\frac{D_{\rm a}\ln(R_I+1) - L\ln(k!)}{Z} < \frac{\lambda}{c_1c_4}\left(1 + \frac{j\ln q}{A + \nu\ln q}\right), \qquad \lambda < \frac{189}{188}. \tag{10.325}$$

At $I = 0$ there is no q-denominator. At $I \geq 1, j \geq 1$, its full denominator cost and the torus height obey

$$\frac{\Xi_{I,0,u}\ln q}{Z} + \frac{PXq^{-I}\sum_i D_i\sigma_i}{Z} \leq \frac{q^{j+1}}{c_1c_2} + \frac{\lambda\ln q}{c_1c_4(q-1)(A+\nu\ln q)}. \tag{10.326}$$

Proof. Monotonicity of $\ln x$ gives $\ln(k!) \geq \int_1^k \ln x\,dx = k\ln k - k + 1$. Thus the additive numerator in (10.325) is at most $D_{\rm a}\ln[{\rm e}(R_I+1)/k]$. Put $H_1 = h + \nu\ln q$. Since $k \geq 39$, $H_1/k < 40/39$. We have $R_I + 1 = 2k + q^{j+1}S(q\eta^{r+1})^{-I}$. The table gives $c_3q^2 < 77/5$, and $(r+1)d \geq 3, t > 1$. Therefore

$$\frac{R_I+1}{k} < q^j(r+1)d\left(\frac{2}{3} + \frac{77}{5}\frac{40}{39}\right) = \frac{214}{13}q^j(r+1)d.$$

The exponential series through degree five at three exceeds $214/13$, so $\ln[{\rm e}(R_I+1)/k] < g_1 + j\ln q \leq A + j\ln q$. Insert $D_{\rm a}/Z \leq \lambda/[c_1c_4(A+\nu\ln q)]$ from Lemma 10.70. This proves (10.325), retaining the factorial contribution.

Let $Q = q\eta^{r+1} > 1$, $D = A + \nu\ln q > 5$. The exact full-width term is $q^{j+1}Q^{-I}/(c_1c_2)$. Formula (10.243) and the proved factorial bound give $\Xi_{I,0,u} \leq D_{\rm a}(I + 1/(q-1))$. Only the part proportional to I needs absorption. Set $a = q^{j+1}/(c_1c_2)$, $b = \lambda\ln q/(c_1c_4D)$. The function $aQ^{-z} + bz$ is convex for $z \geq 0$. On $0 \leq z \leq 3D/\ln Q$, its maximum is bounded by its two endpoints. At zero its value is a. At the other endpoint its ratio to a is $\exp(-3D) + 3\lambda c_2\ln q/(c_4q^{j+1}\ln Q)$.

For odd primes, $c_5 \leq 14/25$ and the increasing function $(1-c/y)^y$, $y \geq 3$, gives $Q \geq 2(61/75)^3 > 43/40$. Hence $\ln Q > 6/83$, by the first term of the positive logarithm series. With $c_4 > 18$, $j \geq 1$, and $\ln 2 < 347/500$, the second ratio term is less than $4233747/6016000 < 3/4$. In V, apply the same increasing function separately in the three rank bands: their endpoints give $3(749/1000)^3, 3(161/200)^4, 3(8173/9000)^9 > 5/4$. Thus $\ln Q > 2/9$. Using $c_4 > 15/4$ and $\ln 3 < 1099/1000$, the ratio term is less than $300027/470000 < 2/3$. The first ratio term is less than $1/1000$: $D > 5$, and the exponential series through degree six at 15 exceeds 1000. Both endpoint ratios are less than one. Convexity therefore absorbs the entire I-dependent cost. The remaining $1/(q-1)$ term is exactly the second term in (10.326).

For the stated monotonicity, the derivative of $y\ln(1-c/y)$ is $\ln(1-z) + z/(1-z) > 0$, $z = c/y$; its latter expression has derivative $z/(1-z)^2 > 0$ and value zero at zero. No denominator at q has been treated as globally harmless. □

The analytic input at every order

Lemma 10.123 (all integer input budgets). Put $\beta = v_p(b_n)$. For a fixed additive order $u \le D_{\mathrm{a}}$, the exact envelopes of Section 39 satisfy

$$G_u + K_u(M,B) \ge (D_{\mathrm{a}} - u)\theta - (M-1)\max\{0, \beta - \vartheta - B\}. \tag{10.327}$$

Consequently full-node interpolation has precision at least

$$\min\{N_*\theta,\ U - \beta + (D_{\mathrm{a}} - u)\theta - (M-1)\max\{B, \beta - \vartheta, 0\}\} \tag{10.328}$$

in the unchanged normal quantity $v_p(\Phi) + G_u - \delta$.

At any allowed stage I, and for $0 \le j \le r$, define $R = \lfloor q^j S_I \rfloor$, $\mu = \lfloor c_5 \eta^j T_I/(r+1) \rfloor + 1$, $N_* = (2R+1)\mu$, and $B = \lfloor \log_p(2R) \rfloor$. Then, simultaneously for all $u \le D_{\mathrm{a}}$,

$$U - \beta + (D_{\mathrm{a}} - u)\theta > N_*\theta + (\mu - 1)\max\{B, \beta - \vartheta, 0\}. \tag{10.329}$$

The analytic gain obeys

$$W_* N_* \theta > \frac{c_5 q \eta^j}{c_1}\left(2q^j - \frac{1}{S_I}\right). \tag{10.330}$$

Proof. Write $\ell = v_p(\mathfrak{v}), \chi = L v_p(k!)$. In the expression defining $M_u(a)$, the identity $\chi - u\ell + \max\{-b\ell, u\ell - \chi\} = \max\{\chi - (u+b)\ell, 0\}$ is nonnegative. The remaining Euler contribution is at least $a\min\{\vartheta - \beta, 0\}$. Thus $G_u + M_u(a) \ge (D_{\mathrm{a}} - u)\theta + a\min\{\vartheta - \beta, 0\}$. Minimizing after adding aB proves (10.327), and substitution in Theorem 10.119 proves (10.328).

For each nonzero b_j, its integer factorization gives $v_p(b_j)\ln p \le \ln |b_j|$. The minimum-valuation choice of b_n therefore gives $\beta \ln p \le \ln B_\dagger \le h$. Next, $\ln(3q^r S) < g_1 + (11/10)r + \ln H_1$, since $3c_3 q < 24 < \mathrm{e}^4$ and $t > 1$. We have $g_1 \le 4 + r + \ln d$, while (10.288) implies $H_1 > 2(4 + 21r/10 + \ln d)$. Also $\ln H_1 < H_1/2$, by the maximum $\ln z/z = 1/\mathrm{e} < 1/2$. Hence $\ln(3q^r S) < H_1$. With $Y_I = \eta^{-(r+1)I} \ge 1$, the inequality $\ln Y_I \le Y_I - 1$ gives $\ln(2R+1) \le \ln(3q^r S_I) < Y_I H_1$. It follows that $\max\{B, \beta - \vartheta, 0\} \le Y_I H_1/\ln p$.

The identities $S_I T_I = ST$ and $W_* ST\theta = q(r+1)/c_1$, with $\mu > c_5\eta^j T_I/(r+1)$ and $2R+1 > 2q^j S_I - 1 > 0$, prove (10.330). For the upper input bound, the stage restriction gives $\mu \le c_5(\eta^j + \eta^{r+1})T_I/(r+1)$. Moreover $W_*\beta < 1/[7(r+1)b_r]$, using $ef \le d$, $\mathcal{D} > b_r(A + \nu \ln q)t$, and $c_3 q > 7/5$. The remaining jet loss satisfies $W_*(\mu - 1)\max\{B, \beta - \vartheta, 0\} \le c_5\eta^j/[c_1 c_3 Y(r+1)]$.

For completeness, $2c_5(\eta^r + \eta^{r+1}) < 27/20$ at odd primes and is less than $3/2$ in V. To prove these uniform bounds, set $c = c_5, y = r+1, z = c/y$. The logarithmic derivative of $(1 - c/y)^{y-1}(2 - c/y)$ is $\ln(1-z) + z/(1-z) - z/[y(1-z)(2-z)]$. Its first two terms equal $\int_0^z t/(1-t)^2\,dt \le z^2/[2(1-z)^2]$. The full derivative is nonpositive because $c(2-z) \le 2(1-z)$: at $z \le c/3$ this follows from $6 - 8c + c^2 > 0$ for $c \le 827/1000$. Thus the expression is largest at $r = 2$. The product $2c(1 - c/3)^2(2 - c/3)$ increases up to $c = 827/1000$, since its derivative has the sign of $18 - 24c + 4c^2 > 0$ there. At the odd endpoint $c = 14/25$, and at the V endpoint $c = 827/1000$, direct rational substitution gives the asserted bounds.

Divide the upper input sum by q^{r+1}/c_1. Since $q\eta > 1$, both $(q\eta)^j$ and q^j increase with j; their endpoint $j = r$ bounds the principal part by these last constants. The floor-radius remainder is less than $1/200$, because $S_I > 58$ and $q^r \ge 4$. The jet-loss remainder is less than $1/75$, since $c_3 Y > 19/10$ at odd primes and $c_3 Y > 19/20$ in V, by Lemma 10.121. The coefficient-minimum remainder is less than $1/10000$, using $c_1 < 3$ and $b_r \ge 7200/29$. Their sum is less than $1/50$. Hence the input sum is

less than $(137/100)q^{r+1}/c_1$ at odd primes and $(38/25)q^{r+1}/c_1$ in V. These are less than $W_*U = q^{r+1}$, since $c_1 > 7/5$ or $c_1 > 5/2$, respectively. Dropping the nonnegative $(D_{\mathrm{a}} - u)\theta$ proves (10.329).

The projection is defined with the required depth: from (10.186), $U/H_1 > c_3 q^{r+2}(r+1)b_r > 5000$, while $\beta < 3H_1/2$. Thus $U - \beta > 1 \geq 1/(p-1)$. □

The strict arithmetic comparison

Use the simultaneous field bounds of Lemma 10.121, denoting its last three columns by C, Y_-, Ω_+; thus $Y_- = Y_0/(1+10^{-6})$. Set $\bar{\lambda} = 189/188$, $\ell_2 = 347/500$, $\ell_3 = 1099/1000$, and

$$b_r = \frac{7200}{29}14^{r-2}, \qquad \epsilon_r = \frac{3 + 14/(r+1)}{7b_r},$$
$$\bar{g}_{12} = \begin{cases} 0 & d = 1, \\ (1 + 5/(r+1))/(546b_r) & d > 1, \end{cases} \qquad \bar{g}_9 = \begin{cases} 107/103 & d = 1 \text{ or } r \geq 8, \\ \max\{107/103, 1 + 1/(3(r+1))\} & \text{otherwise.} \end{cases} \tag{10.331}$$

The following lower bounds $S_- \leq S$ will retain the floor-radius loss.

Case	S_-
I.1 and I.2	$(10/11)c_3q(r+1)^2(39r/10 + 36/5)$
II	$c_3q(r+1)^2(39r/10 + 36/5)$
III.1 and both IV branches	$2c_3q(r+1)^2$
III.2	$c_3q(r+1)^2$
V	$(500/347)c_3q(r+1)^2(39r/10 + 36/5)$

Indeed $S = c_3q(r+1)(d/f)H_1/\ln p$, and $d/f \geq e$. Use $h \geq g_0$ with $\ln 3 < 11/10$ in I, $e \geq 2$ and $\ln 5 < 2$ in II, and $\ln 2 < 347/500$ in V. These are the first three bounds. For III and IV use $H_1 \geq (r+1)t$ and their minimum degree, two except in III.2. The resulting S_- increase with r; direct substitution at $r = 2$ gives $S_- > 58$ in every row. Thus the same lower bound applies to S_I.

Define

$$\begin{aligned} A_r &= c_1(\bar{g}_{12} + \epsilon_r) \\ &\quad + \frac{c_1\epsilon_r/2 + \bar{\lambda}/c_4 + C_r^\sharp/(c_3Y_-) + 117/(232c_2)}{C-1}, \\ \mathcal{R}_r(j) &= A_r + c_1\left(\frac{1}{300} + \frac{1}{273\cdot 999 b_r}\right) + \frac{\bar{g}_9\eta^{j+1}}{c_3Y_-} \\ &\quad + \frac{\bar{\lambda}}{c_4}\left(1 + \Omega_+ + \frac{j\ell_q}{5} + \frac{\chi_j\ell_q}{5(q-1)}\right) + \frac{q^{j+1}}{c_2}, \\ \mathcal{L}_r(j) &= c_5q\eta^j\left(2q^j - \frac{1}{S_-}\right) - \mathcal{R}_r(j), \qquad \chi_0 = 0, \quad \chi_j = 1 \ (j \geq 1). \end{aligned} \tag{10.332}$$

The number 117/232 bounds $(1 + (2g_2)^{-1})/2$, since $g_2 > 58$. Corollary 10.113 and Lemma 10.105 give $c_1[h_2(c) + \frac{1}{2}\ln N_0]/Z < A_r$. These bounds persist for every contracted coefficient subvector: each local vector norm decreases under deletion, hence so does the projective height, and $N_I \leq N_0$. Its chosen minimum reference coefficient remains present.

Lemma 10.124 (the comparison margin increases along the integer steps). For every original rank and field case, $\mathcal{L}_r(j) > \mathcal{L}_r(0)$ at every integer $1 \leq j \leq r-1$.

Proof. Write $c = c_5$, $a = \bar{\lambda}\ell_q/(5c_4)$. First omit the indicator term $\chi_j a/(q-1)$ in $\mathcal{R}_r(j)$, and allow a real variable $s \in [0, r-1]$. Call the resulting difference $\widetilde{\mathcal{L}}(s)$. Differentiation and deletion of its positive terms give

$$\widetilde{\mathcal{L}}'(s) \geq q^{s+1}\left[2c\eta^s \ln(q\eta) - \frac{\ln q}{c_2}\right] - a. \tag{10.333}$$

The terms deleted here are positive because $\ln \eta < 0$: they come from $-cq\eta^s/S_-$ and $-\bar{g}_9\eta^{s+1}/(c_3 Y_-)$.

We establish a uniform positive lower bound for the bracket. Put $b = c/(r+1-c)$. The inequality $-\ln(1-z) \leq z/(1-z)$ follows by integration of $1/(1-z)$. Hence $\eta^{r-1} \geq \exp[-c+(2-c)b]$ and $\ln(q\eta) \geq \ln q - b > 0$. For $s \leq r-1$, the bracket is therefore at least $2c\exp(-c)\exp((2-c)b)(\ln q - b) - \ln q/c_2$. The factor $2c\exp(-c)$ increases for $0 < c < 1$. For fixed positive a_0, l, the derivative of $\exp(a_0 b)(l-b)$ changes sign at most once, from positive to negative. Its minimum on a compact interval consequently occurs at an endpoint.

At odd primes, $5267/10000 \leq c \leq 14/25$, $b \leq 14/61$, $2 - c \geq 36/25$, and $\ln 2 > 693/1000$. The exponential at $5267/10000$ is less than $17/10$. At the two endpoints, the resulting product has the following lower bounds:

$$\frac{693}{1000}, \qquad \left(\frac{693}{1000} - \frac{14}{61}\right)\sum_{m=0}^{5}\frac{(504/1525)^m}{m!} > \frac{129}{200}.$$

It follows that the bracket is greater than $2(5267/10000)(10/17)(129/200) - (347/500)(4/7) = 36901/11900000 > 1/400$. In V, use $753/1000 \leq c \leq 827/1000$, $b \leq 827/2173$, $2 - c \geq 1173/1000$, $\ln 3 > 1098/1000$, and $\exp(753/1000) < 213/100$. The endpoint at zero is $1098/1000$; the other endpoint, bounded with the exponential series through degree five, exceeds that number. The bracket is therefore greater than $2(753/1000)(100/213)(1098/1000) - (1099/1000)(9/13) = 71469/4615000 > 1/100$. All exponential upper bounds just used follow by summing through degree ten and bounding the remaining positive series by $x^{11}/[11!(1-x/12)]$, for $0 \leq x < 1$; the ratio of each following term to its predecessor is at most $x/12$. The logarithm intervals follow from the positive series of Solution 34, with four terms for $\ln 2$ and five for $\ln 3$, and its geometric tail estimate. Thus none is an unproved numerical decimal.

For $s \geq 1$, (10.333) is positive: at odd primes, $q^{s+1}/400 \geq 1/100 > a$, since $a < 1/125$; in V, $q^{s+1}/100 \geq 9/100 > a$, since $a < 3/50$. We must also pay the jump of the indicator at the first integer. On $0 \leq s \leq 1$, use $\eta^s \geq \eta$. At odd primes, $\eta \geq 61/75$, $\ln(2\eta) > 12/25$, and therefore the bracket exceeds $2(5267/10000)(61/75)(12/25) - (347/500)(4/7) > 7/500$. In V, $\eta \geq 2173/3000$, $\ln(3\eta) > 3/4$, giving a bracket greater than $2(753/1000)(2173/3000)(3/4) - (1099/1000)(9/13) > 1/20$. The two logarithm lower bounds follow from the first two positive terms, at $2\eta \geq 122/75$ and $3\eta \geq 2173/1000$. Integrating (10.333) on this interval yields $\widetilde{\mathcal{L}}(1) - \widetilde{\mathcal{L}}(0) > qd_0 - a > a/(q-1)$, using $d_0 = 7/500$ or $1/20$. Indeed $2(7/500) - 1/125 > 1/125$, and $3/20 - 3/50 > 3/100$. After the full indicator cost is subtracted, $\mathcal{L}_r(1) > \mathcal{L}_r(0)$. At later integers that cost is constant, and the positive derivative proves the assertion. □

Lemma 10.125 (strict margins in all original cases). For every $r \geq 2$ and $0 \leq j \leq r-1$,

$$\mathcal{L}_r(j) > \frac{1}{200}. \tag{10.334}$$

Proof. By Lemma 10.124 it suffices to check $j = 0$. For $r = 2, 3, \ldots, 7$, substitute their six integer values into (10.331)–(10.332), using the original constants of (10.186), the displayed c_5 table and the coupled field bounds. Every quantity is rational. The following table gives strict lower bounds for the minimum of those six results.

For every $r \geq 8$, use the single last-band value of c_5, $C_r^\sharp < 1$, $\eta \leq 1$, $\bar{g}_9 = 107/103$, and the values of $\epsilon_r, \bar{g}_{12}, b_r, S_-$ at $r = 8$. This is a uniform bound: b_r increases, ϵ_r and $\bar{g}_{12}$ decrease, and S_- increases. Thus every positive arithmetic cost is enlarged and every gain is diminished by these substitutions. The inequality $C_r^\sharp < 1$ follows directly from (10.307), or by subtracting it from one and using $99/103 > 17/24$. The last column is the resulting strict lower bound for every integer $r \geq 8$.

Case	Minimum over $2 \leq r \leq 7$, greater than	Uniformly for $r \geq 8$, greater than
I.1	80/1000	76/1000
I.2	17/1000	5/1000
II	91/1000	89/1000
III.1	52/1000	65/1000
III.2	22/1000	45/1000
IV, $P = 1$	28/1000	47/1000
IV, $P \geq 7$	592/1000	690/1000
V	248/1000	402/1000

These are finite rational inequalities, obtainable by multiplying positive denominators. The complete rational inputs and all 56 exact differences are generated by the editable arithmetic certificate. Its calculation uses precisely (10.331)–(10.332), including S_-, the normality contribution and the separate remainder. Solution 48 spells out the narrowest uniform row and its comparison with 1/200. The eight uniform rows use proved inequalities for all larger ranks; they are not an inference from sampled ranks. Lemma 10.124 then supplies every integer step, including the first q-denominator cost. □

The integer extension theorem

Theorem 10.126 (general integer zero production). Assume the original contradiction $v_p(\mathcal{L}) \geq U$, the integral initial kernel of Corollary 10.113, and an admissible original stage I as above. Let $j \in \{0, \dots, r-1\}$ if $I = 0$, or $j \in \{1, \dots, r-1\}$ if $I \geq 1$. Suppose all its prepared values vanish at the integers $|s| \leq \lfloor q^j S_I \rfloor$ and selected derivative orders $|\boldsymbol{t}| \leq \lfloor \eta^j T_I \rfloor$. Then all its prepared values vanish at

$$|s| \leq \lfloor q^{j+1} S_I \rfloor, \qquad |\boldsymbol{t}| \leq \lfloor \eta^{j+1} T_I \rfloor. \tag{10.335}$$

Proof. Fix one target derivative, with additive order u and total selected order at most $O = \lfloor \eta^{j+1} T_I \rfloor$. If $u > D_{\mathrm{a}}$, it is identically zero by polynomial degree. Otherwise take R, μ, N_*, B from Lemma 10.123. The integer inequality $\lfloor A \rfloor - \lfloor B \rfloor \geq \lfloor A - B \rfloor$, for $A \geq B$, follows by writing $A - B = m + a$ with integer m and $0 \leq a < 1$. Apply it to $A = \eta^j T_I$, $B = \eta^{j+1} T_I$. It gives $O + \mu - 1 \leq \lfloor \eta^j T_I \rfloor$. The ordinary-jet transfer in Theorem 10.118 consequently provides all μ input jets at each of the $2R + 1$ integer nodes. The projected function is normal on the same enlarged disc by Lemma 10.91 and Corollary 10.82; multiplication of the arguments by powers of q does not change their p-adic depths. Formula (10.329) and Theorem 10.119 give analytic precision at least $N_* \theta$ in the original quantity $v_p(\Phi) + G_u - \delta$.

Suppose the integer value V were nonzero. The phase identity (10.151) puts it in the original field K. Apply Corollary 10.116 in that field, with exactly its degree d and local degrees e, f. The coefficient term is bounded by A_r. Lemma 10.122 bounds the additive polynomial and the entire torus/q-denominator contribution. Its indicator remainder is needed at $I \geq 1$, where $j \geq 1$; at $I = 0, j \geq 1$ the same term is a harmless upper bound, and at $I = j = 0$ it is absent. The normal scalar satisfies $W_* G_u \leq \lambda \Omega_{\mathrm{n}} / (c_1 c_4)$, by $u \leq D_{\mathrm{a}}$ and Lemma 10.70. The individual Euler/lcm term is at most $\bar{g}_9 \eta^{j+1} / (c_1 c_3 Y_-)$, by Lemma 10.114 and the target order. Finally its separate scalar

remainder is bounded using (10.311) and $Z > 273(r+1)b_r$, giving $\mathcal{E}_u/Z < 1/300 + 1/(273 \cdot 999 b_r)$. Together these give $c_1 W_* [v_p(V) + G_u - \delta] \le \mathcal{R}_r(j)$. But (10.330), $S_I \ge S_-$, and (10.334) give the strict opposite inequality $c_1 W_* N_* \theta > \mathcal{R}_r(j) + 1/200$. Corollary 10.120 now forces $V = 0$, a contradiction. This holds for every target order and integer, proving (10.335). □

Corollary 10.127 (the full initial integer block). Under the same original contradiction, the initial auxiliary polynomial of Corollary 10.113 has every prepared zero at $|s| \le \lfloor q^j S \rfloor$, $|\boldsymbol{t}| \le \lfloor \eta^j T \rfloor$, simultaneously for $j = 0, 1, \dots, r$, for every original field, prime and rank case.

Proof. Its prescribed initial kernel equations give $j = 0$. The initial stage satisfies both descent restrictions. Apply Theorem 10.126 successively at $j = 0, 1, \dots, r-1$, using each concluded zero block as the next input. The coefficient vector, field, logarithmic slopes, radii and real multiplicities stay the original ones throughout. □

Theorem 10.126 also proves the integer part of each subsequent admissible contraction, once that stage's first q-expanded zero block has been obtained. Producing that input uses fractional-node interpolation and coset extraction, separately from the integer induction proved here.

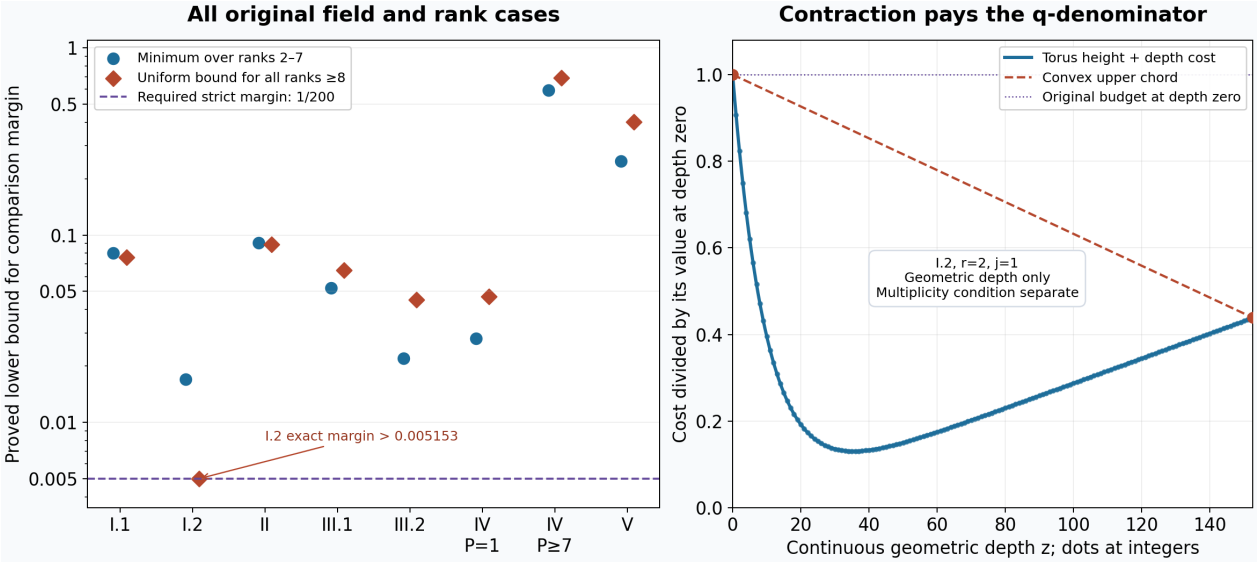


Figure 10.28. Left: the strict lower bounds in Lemma 10.125, on a logarithmic vertical scale. Each first marker bounds all six ranks 2–7; each second marker bounds every rank at least eight by a single proved uniform estimate. The horizontal line is 1/200; the I.2 uniform marker equals that line as a displayed lower bound, and its exact margin is strictly larger, as Solution 48 verifies. Right: the convex denominator-absorption function of Lemma 10.122, divided by its value at depth zero, for the illustrative case I.2, $r = 2, j = 1, D = 4 + \ln 3$, $c_5 = 269/500$. The continuous curve covers $0 \le z \le 3D/\ln Q$; dots select integer depths in that geometric range. This plot does not assert that all these depths satisfy the separate multiplicity stopping condition. Both endpoint values lie at most one, so the upper chord proves the entire q-denominator budget. Figure program and exact comparison certificate. Human-source context: the original constant cases and integer extrapolation in Yu's freely accessible paper, Table 3.1 and Section 5, equations (5.23)–(5.41); the coupled bounds, floor estimates and all strict comparisons are proved here.

41. The full first fractional block and contraction

The preceding integer induction gives a hierarchy of zero blocks. Its earlier blocks have more derivatives than its last block. Retaining all those derivatives supplies the first fractional extrapolation in every original field and rank case, with the full degree of the normalized root field. We then pass to a coefficient class and fill its missing multiples of q.

Retain the original parameters and contradiction of Section 40. Write $c = c_5$, $a = r + 1$, $H = \eta^{r+1}$, $S_1 = H^{-1} S$, and $T_1 = HT$. Here the subscript 1 denotes the first stage; it does not replace any real parameter by its floor. Use the original phase and saturated Kummer data of Theorems 10.51 and 10.56.

The mass of the complete nested input

Lemma 10.128 (all initial integer blocks as one interpolation system). For every $0 \le j \le r$, set

$$\begin{aligned} &R_j = \lfloor q^j S \rfloor, \qquad T_j = \lfloor \eta^j T \rfloor, \qquad O = \lfloor HT \rfloor, \\ &\mu_j = T_j - O + 1, \quad N_* = (2R_0+1)\mu_0 + 2\sum_{j=1}^{r}(R_j - R_{j-1})\mu_j. \end{aligned} \tag{10.336}$$

These are valid nested multiplicities for every output derivative of order at most O. They are positive and nonincreasing, and

$$\begin{aligned} N_* &= (2R_r+1)\mu_r + \sum_{j=0}^{r-1}(2R_j+1)(\mu_j - \mu_{j+1}), \\ \frac{c_1 W_* N_* \theta}{q^r} > \Gamma_r, &\qquad \Gamma_r = \frac{2cq}{q^r}\sum_{j=0}^{r}(q\eta)^j - F_r, \\ F_r &= \frac{q(r+1)}{q^r S_-}\left[1 - H + 2\sum_{j=1}^{r}(\eta^j - H)\right]. \end{aligned} \tag{10.337}$$

Here S_- is the original-case lower bound in Section 40. For a fixed additive order $u \le D_{\mathrm{a}}$, the resulting normalized analytic precision at every $x \in \mathbb{Z}_p$ is at least

$$\min\{N_*\theta,\ U - \beta + (D_{\mathrm{a}} - u)\theta - (\mu_0 - 1)\max\{B, \beta - \vartheta, 0\}\}, \quad B = \lfloor \log_p(2R_r) \rfloor. \tag{10.338}$$

Proof. Corollary 10.127 provides the zeros through order T_j in each interval. For an output index of order at most O, its next $\mu_j - 1 = T_j - O$ ordinary jets are consequently available there, by Theorem 10.118. The inequality $\lfloor A \rfloor - \lfloor B \rfloor + 1 > A - B$ gives $\mu_j > (\eta^j - H)T$. The smallest real difference, at $j = r$, is $c\eta^r T/(r+1)$. The original first-stage restriction $cHT/(r+1) \ge 1$ makes this at least $1/\eta > 1$; hence all multiplicities are positive. Monotonicity follows from $T_j \ge T_{j+1}$.

Telescoping proves the first equality in (10.337). For its strict lower bound, count the annuli instead of rounding the multiplicity differences. The inner interval has more than $2S - 1$ nodes. The j-th annulus has more than $2(q^j - q^{j-1})S - 2$ nodes, all positive. Multiply these lower counts by $(\eta^j - H)T$. The principal real sum is

$$2ST\left[(1 - H) + \sum_{j=1}^{r}(q^j - q^{j-1})(\eta^j - H)\right] = \frac{2cST}{r+1}\sum_{j=0}^{r}(q\eta)^j.$$

The removed radius-floor mass is at most $T[1 - H + 2\sum_{j=1}^{r}(\eta^j - H)]$. Use $c_1 W_* ST\theta = q(r+1)$ and $S \ge S_-$. This proves the second inequality of (10.337), without replacing a difference of floors by its real value.

Finally apply Theorem 10.119 to these full nested intervals, with $M = \mu_0$ and $E = 0$. The cancellation (10.327) applies to its same fixed additive order. It gives (10.338), retaining the complete maximum jet order and all factorial/lcm contributions in the original normal quantity $v_p(\Phi) + G_u - \delta$. □

A comparison in the actual root field

Let $C, Y_-, \Omega_+, \bar{g}_9, \bar{g}_{12}, b_r, \epsilon_r, A_r$ have the meanings and simultaneous field bounds of (10.331)–(10.332). Write $d_r = 5$ for $2 \le r \le 7$, and $d_r = 6$ for $r \ge 8$; this number is a lower bound for $A + \nu \ln q$, not a field degree. Define

$$\begin{aligned} \mathcal{A}_r^{\mathrm{f}} &= A_r + c_1 \left(\frac{1}{300} + \frac{1}{273 \cdot 999 b_r} \right) + \frac{\bar{g}_9 H}{c_3 Y_-} + \frac{\bar{\lambda}}{c_4} \\ &\quad + \frac{\bar{\lambda} \ell_q}{d_r c_4} \left(1 + \frac{1}{q-1} \right) + \frac{H^{-1} + S_-^{-1}}{c_2} + \frac{\bar{\lambda} \Omega_+}{c_4 q^r}, \\ \mathcal{I}_r &= c_1 q - \frac{c_1}{7(r+1) b_r q^r} - \frac{1 - H + 1/(6(r+1))}{q^r c_3 Y_-}. \end{aligned} \tag{10.339}$$

Lemma 10.129 (both original first fractional comparisons). For every original field, prime and rank case, the arithmetic upper bound for a nonzero value at $x = s/q$, $|x| \le \lfloor S_1 \rfloor + 1$, satisfies

$$\frac{c_1 W_*}{q^r} [v_p(V) + G_u - \delta] \le \mathcal{A}_r^{\mathrm{f}}. \tag{10.340}$$

The second analytic entry of (10.338), in that same normalization, is greater than $\mathcal{I}_r$. Moreover

$$\Gamma_r - \mathcal{A}_r^{\mathrm{f}} > \frac{1}{50}, \qquad \mathcal{I}_r - \mathcal{A}_r^{\mathrm{f}} > 1. \tag{10.341}$$

Proof. Theorem 10.56 gives the actual normalized support field E_Λ, with $[E_\Lambda : K] = q^{h_\Lambda}$, $h_\Lambda \le r$, and unchanged chosen local degrees e, f. Its nonzero hyperplane relation requires cancellation of $\gcd(G_0, d_1, \dots, d_r)$; those degree conclusions are therefore valid also when q divides the original phase modulus. They concern the full field, not just its selected completion.

Use the individual scalar bound of Corollary 10.116 over E_Λ, as in Corollary 10.57. Its positive coefficient/scalar/torus envelope is multiplied by $q^{h_\Lambda} \le q^r$. The normal term G_u is added after that multiplication; it is not multiplied by the degree factor. The coefficient envelope is A_r. The separate remainder is the second term of $\mathcal{A}_r^{\mathrm{f}}$. Since the output order is at most HT, its individual Euler/lcm contribution is at most $\bar{g}_9 H/(c_3 Y_-)$.

The target radius obeys $X/S \le H^{-1} + 1/S_- < q$. To check the last inequality uniformly, at odd primes Lemma 10.122 gives $H \ge (61/75)^3$, hence $H^{-1} < 15/8$; also $1/S_- < 1/58 < 1/8$. In V its three rank bands give $H > 5/12$, so $H^{-1} < 12/5$, again leaving the required room. The additive term is consequently bounded by Lemma 10.122 at radius qS, namely $\bar{\lambda}/c_4$ after multiplication by c_1. Its factorial contribution stays in that estimate. Formula (10.243) gives the full rational clearing bound $\Xi_{0,1,u} \le D_{\mathrm{a}}(1 + 1/(q-1))$. Lemma 10.70 converts its cost to the fifth term of $\mathcal{A}_r^{\mathrm{f}}$. The torus height gives the sixth term, including the rounded radius's +1. Finally $c_1 W_* G_u \le \bar{\lambda} \Omega_+ / c_4$, proving the last term and (10.340). For $r \ge 8$, $A + \nu \ln q \ge g_1 = 4 + \ln((r+1)d) > 6$, since $\ln 9 > 2$. Thus the stated d_r bounds are valid.

For the analytic input, the coefficient-minimum loss satisfies $c_1 W_* \beta < c_1/[7(r+1)b_r]$, by Lemma 10.123. Its same node estimate gives $\max\{B, \beta - \vartheta, 0\} \le H_1/\ln p$. Here $\mu_0 - 1 = \lfloor T \rfloor - \lfloor HT \rfloor \le (1-H)T + 1$, and

$$c_1 W_* T \frac{H_1}{\ln p} = \frac{f}{d c_3 \theta} \le \frac{1}{c_3 Y_-}.$$

Since $T > 6(r+1)$, the input loss is bounded by the last term of $\mathcal{I}_r$. Divide by q^r, use $W_* U = q^{r+1}$, and discard only the nonnegative $(D_{\mathrm{a}} - u)\theta$. This proves the asserted input lower bound.

For ranks 2–7, every expression in (10.337) and (10.339) is rational. Substitute each of the six integer ranks in each of the eight coupled field rows. For all larger ranks use the last constant $c = c_5$, and set

$$\begin{aligned} H_- &= (1 - c/9)^9, & H_+ &= \left(\sum_{j=0}^{10} \frac{c^j}{j!} \right)^{-1}, \\ E_+(c) &= \sum_{j=0}^{10} \frac{c^j}{j!} + \frac{c^{11}}{11!(1 - c/12)}, \\ \Gamma^-_{\geq 8} &= \frac{2cq}{E_+(c)} \sum_{j=0}^{3} q^{-j} - \frac{153q}{q^8 S_-(8)}. \end{aligned} \tag{10.342}$$

The positive exponential series and its geometric tail prove $\exp(c) < E_+(c)$ and $\exp(c) > 1/H_+$. The increasing function $(1 - c/y)^y$ from Lemma 10.122 gives $H_- \leq H < \exp(-c) < H_+$. Also $r \ln(1 - c/(r+1)) \geq -rc/(r+1-c) > -c$, so $\eta^r > 1/E_+(c)$. In the reversed geometric sum of (10.337), retain its last four terms and use $q\eta \leq q$. This gives the principal part of $\Gamma^-_{\geq 8}$. For its floor loss, $F_r \leq q(r+1)(2r+1)/(q^r S_-(r))$. Each original $S_-(r)/(r+1)^2$ is nondecreasing. The factor $(2r+1)/((r+1)q^r)$ decreases: its consecutive ratio is less than $2/q \leq 1$. Hence its maximum on $r \geq 8$ is at eight, giving the last term of (10.342).

In the arithmetic upper bound replace $C_r^\sharp$ by one, H by H_+ in the Euler term and by H_- in its inverse, and every $\epsilon_r, \bar{g}_{12}, b_r, S_-, q^r$ by its conservative rank-eight value. In the input lower bound replace H by H_- and the decreasing losses by their rank-eight values. Every substitution has the correct direction. There are therefore eight rational uniform comparisons, in addition to the 48 finite-rank comparisons.

The strict lower bounds for the first gap are as follows. Each displayed integer is divided by 1000.

Case	Minimum at ranks 2–7	Uniformly at every rank at least 8
I.1	1108	647
I.2	1075	587
II	1121	679
III.1	1041	589
III.2	1023	580
IV, $P = 1$	1040	574
IV, $P \geq 7$	1473	1061
V	508	25

Every corresponding input gap is strictly greater than one. These are exact rational substitutions in the formulas just proved; the editable comparison certificate generates all 56 differences and checks them using integer arithmetic. In particular the smallest uniform fractional comparison, in V, exceeds $25/1000 > 1/50$. The infinite rank range follows from the displayed monotonicity and exponential bounds, not from numerical rank sampling. This proves (10.341). □

First fractional zero production

Theorem 10.130 (the full original first fractional block). Under the original contradiction and phase data above, the integral initial auxiliary family has all its prepared zeros at

$$\Phi(s/q; \boldsymbol{t}) = 0, \qquad |s| \leq q(\lfloor S_1 \rfloor + 1), \qquad |\boldsymbol{t}| \leq \lfloor T_1 \rfloor. \tag{10.343}$$

This holds for every original field, prime and rank case.

Proof. Fix a target derivative. An additive order greater than D_{a} gives an identically zero derivative. Otherwise use the complete nested integer input of Lemma 10.128. The projected function and its unprojected prepared value have the same lower precision (10.338), by the order-specific projection theorem. Since q differs from p, $s/q \in \mathbb{Z}_p$; the principal exponential laws of Lesson 9 identify this analytic value with the normalized algebraic support value in E_Λ. Its global degree factor is exactly q^{h_Λ}, and its selected local degrees are the original ones.

If the value were nonzero, Lemma 10.129 would bound its normalized valuation by $\mathcal{A}_r^{\mathrm{f}}$. The first analytic entry is strictly larger, by (10.337) and (10.341). The second entry is also strictly larger, by the input part of (10.341). Their minimum is therefore strictly larger than the arithmetic upper bound. Corollary 10.120 forces zero. Both inequalities are proved at the original rounded radius, including its +1, for all derivative indices. □

Completing the first contraction

Theorem 10.131 (the original first full contracted block). A coefficient class containing an initial coefficient of minimum valuation may be chosen, with a nonzero contracted polynomial, original coefficient-height bound and original minimum. Its phase family is defined over K, has torus widths at most $q^{-1}D_i$ and additive argument $q^{-1}x$. Initially it has the stronger zeros

$$\Phi_1(s;\boldsymbol{t}) = 0, \qquad |s| \le q(\lfloor S_1 \rfloor + 1), \quad q \nmid s, \quad |\boldsymbol{t}| \le \lfloor T_1 \rfloor. \tag{10.344}$$

It also has all the zeros, including multiples of q, at

$$\Phi_1(s;\boldsymbol{t}) = 0, \qquad |s| \le q(\lfloor S_1 \rfloor + 1), \quad |\boldsymbol{t}| \le \lfloor \eta T_1 \rfloor. \tag{10.345}$$

Proof. Apply the full phase and coset passage of Theorem 10.51 to (10.343), using its second phase partition when q divides G_0. The saturated root-monomial basis separates the coefficient classes at $q \nmid s$. Choose a class containing an initial coefficient attaining δ. Its coefficient subvector preserves that minimum, remains nonzero, and has no larger projective height. The affine prepared-jet transfer of Theorem 10.48 proves (10.344) at every selected order; its triangular leading entries are powers of q, hence are units at p. The widths and the additive argument are exactly (10.141)–(10.142); Theorem 10.60 preserves the original height envelope. The phase condition of Theorem 10.55 puts every integer prepared value of this new family in K. No global root-degree factor is needed for those integer values.

Set $R = q(\lfloor S_1 \rfloor + 1)$, $M = \lfloor cT_1/(r+1) \rfloor + 1$, $N_* = 2(q-1)(\lfloor S_1 \rfloor + 1)M$, and $B = \lfloor \log_p(2R) \rfloor$. The nodes are exactly the q-deleted interval of (10.344), whose radius is divisible by q. The floor-difference inequality gives $\lfloor \eta T_1 \rfloor + M - 1 \le \lfloor T_1 \rfloor$, so all the required ordinary jets are present. Apply the q-deleted case of Theorem 10.119. Its extra term is $EB = MB$, in addition to the inverse-coefficient and derivative loss.

Define the following arithmetic bound in the original field:

$$\begin{aligned} \mathcal{A}_r^{\mathrm{c}} &= A_r + c_1\left(\frac{1}{300} + \frac{1}{273 \cdot 999 b_r}\right) + \frac{\bar{g}_9 \eta H}{c_3 Y_-} + \frac{\bar{\lambda}}{c_4} \\ &\quad + \frac{\bar{\lambda}\ell_q}{d_r c_4}\left(1 + \frac{1}{q-1}\right) + \frac{H^{-1} + S_-^{-1}}{c_2} + \frac{\bar{\lambda}\Omega_+}{c_4}, \\ \mathcal{J}_r &= c_1 q^{r+1} - \frac{c_1}{7(r+1)b_r} - \frac{3c}{(r+1)c_3 Y_-}. \end{aligned} \tag{10.346}$$

For a nonzero target integer value, $c_1 W_*[v_p(V) + G_u - \delta] \le \mathcal{A}_r^{\mathrm{c}}$. Indeed its coefficient/scalar envelope is unchanged, its order is at most ηHT, its torus width and argument give $Rq^{-1}/(c_2 S) \le$

$(H^{-1}+S_-^{-1})/c_2$, including the rounded radius's $+1$, and $\Xi_{1,0,u}\le D_{\mathrm{a}}(1+1/(q-1))$ gives the full q-denominator term. Its additive scalar is evaluated at $q^{-1}x$, of absolute size at most $\lfloor S_1\rfloor+1<qS$, by Lemma 10.129; Lemma 10.122 therefore bounds it by the same additive term. The normal contribution now has no q^{-r} factor, because the value is in K.

For the two analytic entries, we have

$$\begin{aligned} c_1W_*N_*\theta &> 2cq(q-1),\\ c_1W_*\mathcal{P}_{u,\mathrm{input}} &> \mathcal{J}_r. \end{aligned} \tag{10.347}$$

The first uses $\lfloor S_1\rfloor+1>S_1$, $M>cT_1/(r+1)$, and $S_1T_1=ST$. For the second, the node bound of Lemma 10.123 gives $\max\{B,\beta-\vartheta,0\}\le H^{-1}H_1/\ln p$. After the exact cancellation (10.327), the input loss is at most $MB+(M-1)\max\{B,\beta-\vartheta,0\}$. Since $M-1\le cT_1/(r+1)$ and $M\le cT_1/(r+1)+1$, it is at most $[2cT_1/(r+1)+1]H^{-1}H_1/\ln p$. The identity used in Lemma 10.129 and the restriction $cT_1/(r+1)\ge 1$ bound its normalized value by $3c/((r+1)c_3Y_-)$. The same coefficient-minimum estimate bounds β. This proves (10.347), retaining both q-deleted losses.

The two strict comparisons are

$$2cq(q-1)-\mathcal{A}_r^{\mathrm{c}}>\frac{1}{50},\qquad \mathcal{J}_r-\mathcal{A}_r^{\mathrm{c}}>1. \tag{10.348}$$

They follow from the same 48 finite and eight uniform rational substitutions used in Lemma 10.129. For $r\ge 8$ bound ηH by H_+ and H^{-1} by H_-^{-1}; the other monotone terms have their same rank-eight bounds. The gain has its constant last-band value. In $\mathcal{J}_r$, use $q^{r+1}\ge q^9$ and the decreasing losses at rank eight. Strict lower bounds for the first gap, again divided by 1000, are

Case	Minimum at ranks 2–7	Uniformly at every rank at least 8
I.1	372	392
I.2	338	332
II	402	420
III.1	366	384
III.2	326	348
IV, $P=1$	371	391
IV, $P\ge 7$	725	836
V	5635	6334

Every corresponding input gap exceeds one; the exact certificate checks all four comparisons in each of its 56 rows. Thus both analytic entries exceed the arithmetic bound for every target integer. Corollary 10.120 forces those values to vanish. This proves (10.345), and completes the first contraction in all original cases. □

The higher-order q-deleted zeros (10.344) remain available alongside the full block (10.345). They belong to the input of later mixed extrapolations, rather than being discarded after the missing multiples of q have been filled.

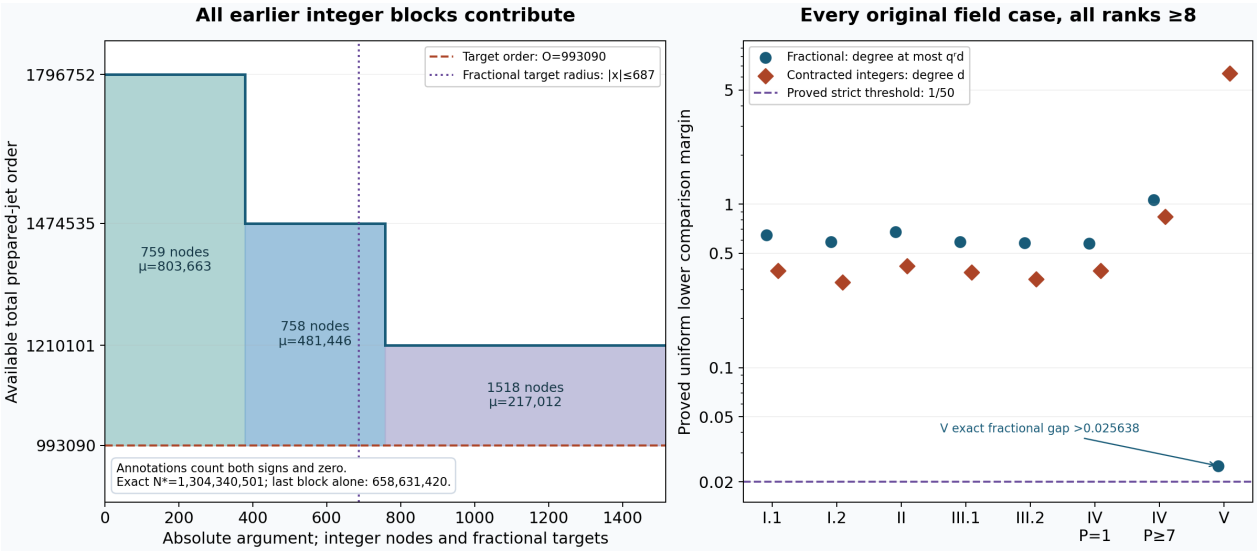


Figure 10.29. Left: the actual certified integer radii and derivative floors of Solutions 39 and 49, with output order993090. Each annotation counts the full symmetric annulus, including zero in the first interval. The shaded excess derivative orders give the variable multiplicities of (10.336), with cardinal degree1304340501; the fractional target radius687 is the original rounded radius from (10.343). This is a precision profile, not a graph of auxiliary-function values. Right: the strict rational lower bounds for every original field case at every rank at least eight, under the proved uniform substitutions of (10.342). Fractional values retain their actual normalized root-field factor at mostqˆr; after the exact phase passage, integer values belong toK. The horizontal threshold is1/50; the exact V fractional margin exceeds0.025638. Both input comparisons also pass, as proved in Lemma10.129 and Theorem10.131. Figure program and exact rational certificate. Human-source context: the original fractional range and first phase passage in Yu's free paper, Section5, equations(5.42)–(5.60); all field, nested-jet, floor and strict comparison arguments used here are proved in this lesson.

42. Retaining the complete mixed profile

The full contracted interval and the higher-order q-deleted interval provide different information. Both remain available when the integer block is extended to larger radii. We now put an arbitrary number of these full blocks into the same interpolation system. The cardinal-polynomial loss must account separately for the deleted nodes.

Mixed interpolation with any number of full intervals

Theorem 10.132 (the complete mixed normal interpolation bound). Let $p \neq q$ be primes. Let $1 \leq A \leq R_1 < \cdots < R_m$ be integers, where $m \geq 1$, and put

$$\begin{aligned} E_j &= [-R_j, R_j] \cap \mathbb{Z}, & D &= \{s \in \mathbb{Z} : |s| \leq A,\ q \nmid s\}, \\ \nu_1 &\geq \cdots \geq \nu_m \geq 1, & E &\geq 0, & M &= \nu_1 + E, \\ \mu(s) &= \nu_m + \sum_{j=1}^{m-1} (\nu_j - \nu_{j+1}) \mathbf{1}_{E_j}(s) + E\mathbf{1}_D(s). \end{aligned} \tag{10.349}$$

All the multiplicities and E are integers. Their total degree is

$$\begin{aligned} N_* &= (2R_m + 1)\nu_m + \sum_{j=1}^{m-1} (2R_j + 1)(\nu_j - \nu_{j+1}) + 2(A - \lfloor A/q \rfloor)E, \\ B &= \lfloor \log_p(2R_m) \rfloor. \end{aligned} \tag{10.350}$$

Choose ρ with $v_p(\rho) = \theta > 0$ and a normal series F. Suppose, for real numbers Λ, C with $C \geq 0$, that its divided jets satisfy $v_p(F_h(\rho s)) + h\theta \geq \Lambda - hC$ at every $s \in E_m$ and $0 \leq h < \mu(s)$. Then

$$v_p(F(\rho x)) \geq \min\{N_*\theta,\ \Lambda - EB - (M-1)\max\{B, C\}\} \qquad (x \in \mathbb{Z}_p \subset \mathbb{Q}_p). \tag{10.351}$$

For the prepared family, suppose the original values vanish through order $|\boldsymbol{t}| + \mu(s) - 1$ at those nodes, with fixed additive order $u \leq D_{\mathrm{a}}$. Use the exact jet quantities $G_u, M_u(h), K_u(M, B)$ of Theorems 10.117–10.119. Then the projected and unprojected values have the common lower bound

$$\begin{aligned} &v_p(f(x;\boldsymbol{t})) + G_u - \delta \geq \mathcal{P}_u, \\ &v_p(\Phi(x;\boldsymbol{t})) + G_u - \delta \geq \mathcal{P}_u, \\ &\qquad\qquad \mathcal{P}_u = \min\{N_*\theta,\ U - \beta + G_u - EB - (M-1)B + K_u(M,B)\}. \end{aligned} \tag{10.352}$$

In particular,

$$\mathcal{P}_u \geq \min\{N_*\theta,\ U - \beta + (D_{\mathrm{a}} - u)\theta - EB - (M-1)\max\{B, \beta - \vartheta, 0\}\}. \tag{10.353}$$

Proof. The intervals are nested, and $D \subset E_1$. The representation of μ in (10.349) is therefore its exact decomposition into a full outer layer, nonnegative full-interval increments and one nonnegative deleted increment. Counting each layer gives (10.350): there are $2(A - \lfloor A/q \rfloor)$ nonmultiples of q in the symmetric interval, because zero is a multiple.

At every residue level p^a, counts in the residue classes of a full consecutive integer interval differ by at most one. To prove this, write its length as $kp^a + b$, $0 \leq b < p^a$. Each class has either k or $k+1$ representatives. For the deleted interval, subtract the representatives that are multiples of q. Their indices form the full interval $[-\lfloor A/q \rfloor, \lfloor A/q \rfloor]$, and multiplication by q permutes the classes modulo p^a. Each of the two full-interval count differences is at most one in absolute value. Deleted-interval counts consequently differ by at most two. A bound of one does not hold in general.

For $s \in E_m$, take the cardinal factor

$$L_s(X) = \prod_{\substack{t \in E_m \\ t \neq s}} \left(\frac{X - t}{s - t} \right)^{\mu(t)}.$$

If x is another node, it is zero; if $x = s$, it is one. Consider a nonnode $x \in \mathbb{Z}_p$. At a residue level $a \leq B$, an increment on a set containing s contributes $N_a(x) - N_a(s) + 1 - \mathbf{1}_{x \equiv s\ (p^a)}$. For a full interval this is nonnegative: if the residues agree it is zero, and otherwise its first difference is at least -1. For the deleted interval it is at least -1. An increment on a set not containing s contributes $N_a(x) - N_a(s)$, at least -1 for a full interval and at least -2 for the deleted interval.

The full outer layer always contains s. If $s \in D$, all the full increments contain s, and only the deleted increment can lose valuation, by at most E at each level. If $s \in E_1 \setminus D$, all full increments still contain s, but the missing deleted increment can lose at most $2E$. If s belongs to a later annulus, the missing full increments have total weight $\nu_1 - \mu(s)$, and the deleted increment can again lose at most $2E$. All three cases give the same bound

$$v_p(L_s(x)) \geq -B\big(M - \mu(s) + E\big). \tag{10.354}$$

For $a > B$, every nonzero denominator difference has valuation less than a, since $0 < |s - t| \leq 2R_m < p^a$. The remaining contributions are nonnegative. Thus the sum over the residue levels proves (10.354), also at the nodes by their zero or unit values.

The coefficient of degree a in

$$L_s(s+Z)^{-1}=\prod_{t\neq s}(1+Z/(s-t))^{-\mu(t)}$$

has valuation at least $-aB$. Indeed each negative-binomial coefficient is an integer, and each denominator difference has valuation at most B. Use the Hermite polynomial constructed in Lemma 10.94, with this exact multiplicity function. A term with divided jet h and Taylor-inverse degree $a\le\mu(s)-1-h$ has valuation at least

$$\Lambda-hC-B(M-\mu(s)+E)-aB\ge\Lambda-EB-(M-1)\max\{B,C\}.$$

Its degree is less than N_*, and it has the required jets at every node.

The full monic normal-quotient construction in Lemma 10.94 applies to $W(Z)=\prod_{s\in E_m}(Z-\rho s)^{\mu(s)}$. It produces $F=H+WG$, with G normal and H the unique polynomial of degree less than N_* having those jets. At ρx, each factor of W has valuation at least θ, and G is integral. The quotient contribution is therefore at least $N_*\theta$. The ultrametric inequality proves (10.351).

For the prepared family, Theorem 10.118 gives the weighted jet baseline $U-\beta+G_u$ and increment $M_u(h)$. The same Hermite term now has valuation at least

$$U-\beta+G_u-EB-(M-1)B+M_u(h)+hB.$$

Every h is less than M. Taking the minimum in the definition of $K_u(M,B)$ proves the asserted lower bound for f. The projection error has precision at least $U-\beta+G_u+\gamma_u$, by Lemma 10.117. Since $K_u(M,B)\le M_u(0)=\gamma_u$, it has at least the second entry of (10.352). Adding it gives the same bound for Φ. Finally the exact fixed-order cancellation in Lemma 10.123 gives (10.353). This proves the theorem with both the deleted-cardinal cost and the complete inverse/jet cost retained. □

An exact small profile

Take $p=2,q=3,A=R_1=9,R_2=18,\nu_1=3,\nu_2=1,E=2$. The twelve deleted nodes have multiplicity five, the remaining seven inner nodes have multiplicity three, and the eighteen outer nodes have multiplicity one. Thus

$$N_*=12\cdot5+7\cdot3+18\cdot1=37+19(3-1)+12\cdot2=99,\qquad M=5,\quad B=5.\tag{10.355}$$

The deleted nodes have residue counts $(4,3,2,3)$ modulo four. The discrepancy of two occurs here, so it is necessary to distinguish the deleted increment from a full interval. The uniform cardinal bounds in (10.354) are respectively $-10,-20,-30$ at a deleted inner node, a nondeleted inner node and an outer node. With $C=0$, (10.351) gives precision at least $\min\{99\theta,\Lambda-30\}$. These are bounds on the interpolation calculation; they assert no value or zero for a particular auxiliary function.

For later course stages, the available full-interval orders and higher q-deleted orders must be inserted into this theorem without discarding either. The resulting cardinal count and input precision are exact. A zero-production conclusion additionally requires the corresponding strict arithmetic comparisons in the actual coefficient field.

The exact inherited profile at a later depth

Lemma 10.133 (the complete inherited mixed mass and input budget). Retain every original rank, field and prime parameter of Section 40, and let $I\ge1$ be an admissible depth with $cT_{I+1}/(r+1)\ge1$. Write $c=c_5,a=r+1,H=\eta^{r+1}$, $S_I=H^{-I}S,T_I=H^IT$, and $O=\lfloor HT_I\rfloor$. Suppose the stage-I family has all its q-deleted zeros through order $\lfloor T_I\rfloor$ at radius $q(\lfloor S_I\rfloor+1)$, all its full integer zeros through order $\lfloor\eta T_I\rfloor$ at that radius, and all its full integer zeros through order $\lfloor\eta^jT_I\rfloor$ at radius $\lfloor q^jS_I\rfloor$, $2\le j\le r$. Put

$$
\begin{aligned}
&R_1 = q(\lfloor S_I \rfloor + 1), && R_j = \lfloor q^j S_I \rfloor \quad (2 \le j \le r),\\
&\mu_j = \lfloor \eta^j T_I \rfloor - O + 1 \quad (0 \le j \le r), && E = \mu_0 - \mu_1,\\
&N_I = (2R_r + 1)\mu_r + \sum_{j=1}^{r-1} (2R_j + 1)(\mu_j - \mu_{j+1}) + 2(q-1)(\lfloor S_I \rfloor + 1)E.
\end{aligned}
\tag{10.356}
$$

These are valid multiplicities for every output derivative of order at most O. For a fixed additive order $u \le D_{\mathrm{a}}$, Theorem 10.132 applies with $M = \mu_0$, and its first analytic entry satisfies

$$
\begin{aligned}
&\frac{c_1 W_* N_I \theta}{q^r} > \Gamma_{r,I}^{\mathrm{mix}},\\
&\Gamma_{r,I}^{\mathrm{mix}} = \frac{2cq}{q^r}\left[\sum_{j=0}^{r} (q\eta)^j + q - 2\right]\\
&\qquad - \frac{q(r+1)}{q^r S_I}\left[2(q+1)(\eta^2 - H) + 2\sum_{j=3}^{r}(\eta^j - H)\right].
\end{aligned}
\tag{10.357}
$$

An empty sum is zero. Its second analytic entry, in the same normalization, is strictly greater than

$$
\mathcal{I}_r^{\mathrm{mix}} = c_1 q - \frac{c_1}{7(r+1)b_r q^r} - \frac{1 - H + c(1+2H)/(r+1)}{q^r c_3 Y_-}.
\tag{10.358}
$$

Here b_r, Y_- are the original coupled bounds in Lemma 10.121.

Proof. The original bound $S_I \ge S > 58$ gives $\lfloor q^2 S_I \rfloor > q(\lfloor S_I \rfloor + 1)$, since $q(q-1)S_I - q - 1 > 0$. All the outer radii are strictly increasing. The floor-difference inequality gives $\mu_j > (\eta^j - H)T_I$. At the last layer, $(\eta^r - H)T_I = c\eta^r T_I/(r+1) \ge 1/\eta > 1$. Thus the multiplicities are positive and nonincreasing. Each node has exactly the available additional ordinary jets required by Theorem 10.118. The top increment on deleted nodes is $E = \lfloor T_I \rfloor - \lfloor \eta T_I \rfloor \ge 0$. Formula (10.350) is therefore precisely (10.356).

For the lower mass, count the nodes by disjoint types, with their full available multiplicities. There are exactly $2(q-1)(\lfloor S_I \rfloor + 1) > 2(q-1)S_I$ deleted inner nodes, with multiplicity μ_0. There are $2(\lfloor S_I \rfloor + 1) + 1 > 2S_I$ inner multiples of q, with multiplicity μ_1. The first outer annulus has more than $2(q^2 - q)S_I - 2(q+1)$ nodes, with multiplicity μ_2. Every subsequent annulus j has more than $2(q^j - q^{j-1})S_I - 2$ nodes, with multiplicity μ_j. Each lower node count is positive.

Multiply these counts by their respective strict lower multiplicities $(\eta^j - H)T_I$. The principal mass is

$$
\begin{aligned}
&2S_I T_I\left[(q-1)(1-H) + (\eta - H) + \sum_{j=2}^{r}(q^j - q^{j-1})(\eta^j - H)\right]\\
&\qquad = \frac{2cS_I T_I}{r+1}\left[\sum_{j=0}^{r}(q\eta)^j + q - 2\right].
\end{aligned}
\tag{10.359}
$$

For the equality, subtract the full initial real annular sum of Lemma 10.128. The difference is $(q-2)(1-\eta) = (q-2)c/(r+1)$. The remaining removed floor mass is at most $T_I[2(q+1)(\eta^2 - H) + 2\sum_{j=3}^{r}(\eta^j - H)]$. Use $S_I T_I = ST$ and $c_1 W_* ST\theta = q(r+1)$. This proves (10.357), including the larger rounded inner radius. It does not round the differences of derivative floors independently.

For the input entry of (10.353), Lemma 10.123 gives

$$B,\ \max\{B,\beta-\vartheta,0\}\le H^{-I}H_1/\ln p,\qquad H_1=h+\nu\ln q.$$

Also $E\le cT_I/(r+1)+1$ and $\mu_0-1\le(1-H)T_I+1$. The two separate losses are consequently bounded by

$$[(1-H+c/(r+1))T_I+2]\,H^{-I}H_1/\ln p.$$

Since $cHT_I/(r+1)\ge 1$, we have $2/T_I\le 2cH/(r+1)$. The identity $c_1W_*T_IH^{-I}H_1/\ln p=f/(dc_3\theta)\le 1/(c_3Y_-)$ therefore gives the last term of (10.358). Finally $W_*U=q^{r+1}$ and the strict coefficient-minimum bound of Lemma 10.123 give the first two terms. Discard only the nonnegative $(D_{\mathrm{a}}-u)\theta$. This proves the claimed strict input bound, retaining both EB and the maximum inverse/jet loss. $\square$

Thus the complete inherited profile supplies two explicitly proved analytic entries at every admissible later depth. To produce fractional zeros one must compare both of them with the nonzero-value upper bound in the actual normalized support field. These analytic bounds alone do not remove its global degree factor or the q-denominator cost.

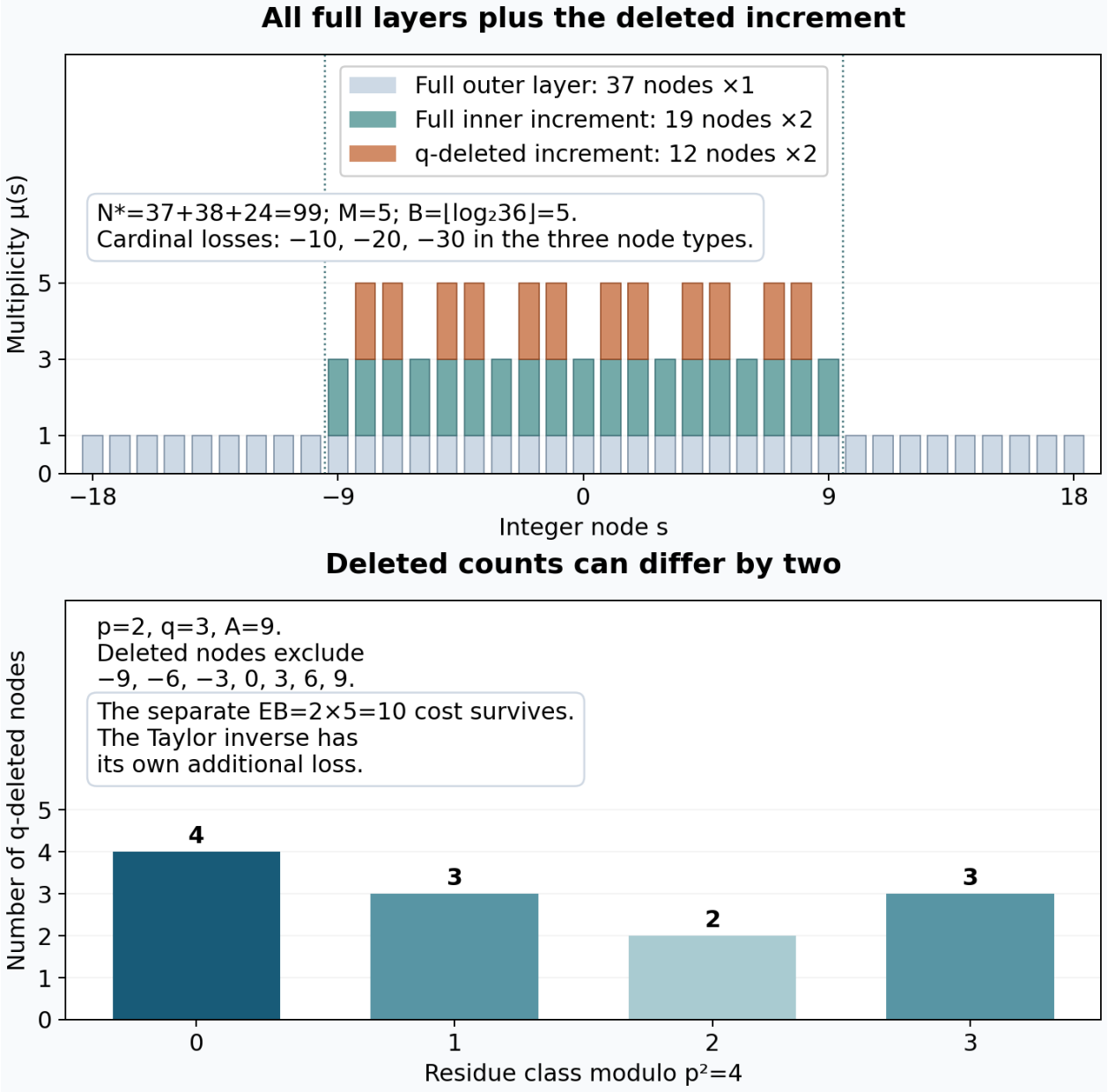


Figure 10.30. The exact profile of (10.355) and Solution51, withp=2,q=3,A=R1=9,R2=18. Each bar represents one integer node and shows its complete multiplicity, decomposed into the full outer layer, full inner increment and q-deleted increment. The total degree is99. The lower panel counts the deleted nodes in the four residue classes modulo4; the counts4,3,2,3 show why discrepancy two is needed. Theorem10.132 proves the cardinal losses and the separateEB cost; Lemma10.133 supplies the later original floor profile without discarding any layer. These diagrams describe the interpolation system, not auxiliary-function values or a later zero-production comparison. Figure program. Human-source context: the original integer and fractional phase ranges in Yu's freely accessible paper, Section5, equations(5.23) and(5.42)–(5.60); the arbitrary-layer mixed interpolation, floor-mass and fixed-order input proofs are given here.

43. The complete prime-two phase induction

In the original case V, retaining more integer derivatives supplies the fractional interpolation with enough precision throughout the full descent. We choose each derivative loss from its actual arithmetic envelope. The subsequent comparison keeps the complete global root-field factor. It also treats the last contracted block, which can occur one step beyond the interval used to produce fractional zeros.

Throughout this section take the original case V, so $p = 2, q = 3$. Retain all the original field, phase, coefficient, height and projection hypotheses of Section 40. Put $a = r + 1, c = c_5, \eta = 1 - c/a, H = \eta^a, Q = qH$, and $D = A + \nu \ln q$. The additive degree remains $D_{\mathrm{a}} = kL$; it is distinct from D. Let

$S_I = H^{-I}S, T_I = H^I T$. The exact original stopping indices are

$$L_D = \frac{3D}{\ln Q}, \qquad I^* = \lfloor L_D \rfloor + 1, \qquad I_1 = \max\{I \in \mathbb{Z}_{\geq 0} : cT_I/a \geq 1\}, \qquad I_0 = \min\{I^*, I_1\}. \tag{10.360}$$

These indices exist: $0 < H < 1, Q > 1$, and the initial multiplicity bounds give $cT_1/a > 1$, as in Section 41. For $1 \leq I < I_0$, both $I \leq L_D$ and $cHT_I/a \geq 1$ hold. At $J = I_0$, only $J \leq L_D + 1$ and $cT_J/a \geq 1$ are needed for closing the deleted interval.

Arithmetic costs and chosen ordinary-jet losses

Use the exact rational coupled bounds $Y_- = 2/(1 + 10^{-6}), \Omega_+ = 3/2, C = 10/3$ from Lemma 10.121, and $b_r, \epsilon_r, \bar{g}_{12}, \bar{g}_9, S_-$ from (10.331) and its table. The coefficient bound A_r of (10.339) is

$$A_r = c_1(\bar{g}_{12} + \epsilon_r) + \frac{c_1\epsilon_r/2 + \bar{\lambda}/c_4 + C_r^\sharp/(c_3 Y_-) + 117/(232c_2)}{C - 1}. \tag{10.361}$$

Here $c_1 = 2.5347, c_2 = 13/9, c_3 = 0.4757, c_4 = 3.765$, $\bar{\lambda} = 189/188$, and $C_r^\sharp$ is (10.307). Define

$$\begin{aligned} b &= A_r + c_1\left(\frac{1}{300} + \frac{1}{273 \cdot 999 b_r}\right) + \frac{\bar{\lambda}}{c_4}, \quad & n &= \frac{\bar{\lambda}\Omega_+}{c_4}, \\ e_0 &= \frac{\bar{g}_9}{c_3 Y_-}, & \ell &= \frac{\bar{\lambda}(1099/1000)}{D c_4}, \\ \mathcal{C}(I) &= b + n + e_0 H^I + \ell(I + 1/2), \\ B_0(I) &= \mathcal{C}(I) + \frac{q(1 + S_-^{-1})}{c_2} Q^{-I}, \\ B_j(I) &= \mathcal{C}(I) + \ell j + \frac{q^{j+1}}{c_2} Q^{-I} \quad (1 \leq j < r). \end{aligned} \tag{10.362}$$

The symbol e_0 is a numerical envelope, not the ramification index. Choose

$$\begin{aligned} \varepsilon &= 1/100, & \lambda_{\text{node}} &= 232/231, \\ \Delta_0(I) &= \frac{B_0(I) + \varepsilon}{2q(q-1)a}, & \Delta_j(I) &= \frac{\lambda_{\text{node}}(B_j(I) + \varepsilon)}{2q^{j+1}a} \quad (1 \leq j < r), \\ \tau_0(I) &= 1, & \tau_j(I) &= 1 - \sum_{k=0}^{j-1} \Delta_k(I) \quad (1 \leq j \leq r). \end{aligned} \tag{10.363}$$

All quantities preceding the depth variable are those of the original construction. These choices change the retained derivative profile; they do not change its stopping indices, initial coefficients or height constants.

Lemma 10.134 (the adaptive comparison envelopes). Suppose $1 \leq I < I_0$. A nonzero integer target at radius $q(\lfloor S_I \rfloor + 1)$, with total order at most T_I, has normalized valuation $c_1 W_*[v_p(V) + G_u - \delta]$ at most $B_0(I)$. A nonzero integer target at radius $\lfloor q^{j+1} S_I \rfloor$, $1 \leq j < r$, has the same normalized valuation at most $B_j(I)$. The additive order u is fixed in each comparison.

For the fractional targets $x = s/q$, $|s| \leq q(\lfloor S_I/H \rfloor + 1)$, of total order at most $\lfloor HT_I \rfloor$, the arithmetic envelope in normalization $c_1 W_*/q^r$ is

$$A_{\text{f}}(I) = b + e_0 H^{I+1} + \ell(I + 3/2) + \frac{H^{-1} + S_-^{-1}}{c_2} Q^{-I} + \frac{n}{q^r}. \tag{10.364}$$

Assuming the necessary integer input zeros, the two analytic entries for deleted closure are bounded below by $B_0(I)+\varepsilon$ and

$$J_0(I)=c_1q^{r+1}-\frac{c_1}{7ab_r}-\frac{2\Delta_0(I)+cH/a}{c_3Y_-}. \tag{10.365}$$

For the full extension from the jth to the $(j+1)$st radius, with cutoffs $\lfloor\tau_jT_I\rfloor,\lfloor\tau_{j+1}T_I\rfloor$, they are bounded below by $B_j(I)+\varepsilon$ and

$$J_j(I)=c_1q^{r+1}-\frac{c_1}{7ab_r}-\frac{\Delta_j(I)}{c_3Y_-}. \tag{10.366}$$

If $1\ge\tau_1\ge\cdots\ge\tau_r>H$, the complete mixed fractional profile has its two analytic entries strictly greater than

$$\begin{aligned}\Gamma(I)&=\frac{2qa}{q^r}\left[(q-1)(1-H)+(\tau_1-H)+\sum_{j=2}^{r}(q^j-q^{j-1})(\tau_j-H)\right]\\&\quad-\frac{qa}{q^rS_-}\left[2(q+1)(\tau_2-H)+2\sum_{j=3}^{r}(\tau_j-H)\right],\\J_{\mathrm f}(I)&=c_1q-\frac{c_1}{7ab_rq^r}-\frac{1-H+\Delta_0(I)+2cH/a}{q^rc_3Y_-}.\end{aligned} \tag{10.367}$$

Proof. At an integer target, Theorem 10.55 places the actual value in K. Corollary 10.116 therefore uses its full global degree d and local degrees e,f. The coefficient term is at most A_r, and the scalar remainder is the separate term already included in b, by Theorem 10.115. An order at most T_I has Euler/lcm cost at most e_0H^I, by Lemma 10.114. The normal contribution is at most n.

At the full extension radius, Lemma 10.122 bounds the additive part by $\bar\lambda/c_4+\ell j$. Retain the actual torus term $q^{j+1}Q^{-I}/c_2$ and clearing term $\ell(I+1/2)$; no chord absorption is used here. These give B_j.

For deleted closure, the additive argument has size at most $q^{-I}q(\lfloor S_I\rfloor+1)\le qSQ^{-I}+q^{1-I}\le S/H+1<qS$. The last inequality is Lemma 10.129. The scalar monotonicity thus gives the j=0 additive cost. Its torus cost is at most $q(1+S_-^{-1})Q^{-I}/c_2$, since $q^{-I}\le Q^{-I}$. This proves B_0.

At a fractional target the corresponding additive argument is at most $q^{-I}(\lfloor S_I/H\rfloor+1)\le(S/H)Q^{-I}+q^{-I}<qS$. The torus cost is at most $(H^{-1}+S_-^{-1})Q^{-I}/c_2$, and the clearing exponent obeys $\Xi_{I,1,u}\le D_{\mathrm a}(I+1+1/(q-1))$. Its order is at most HT_I, giving e_0H^{I+1}. Theorem 10.56 gives global degree dq^{h_Λ}, $h_\Lambda\le r$, with the original selected local degrees. Dividing the product-formula bound by q^r bounds every positive algebraic-height term by the terms displayed in (10.364). The normal contribution is added after that product-formula comparison; it is consequently n/q^r. This proves (10.364), retaining the worst global factor even if the selected completion has local degree one.

For deleted closure take $M_0=\lfloor\Delta_0T_I\rfloor+1$. The inequality $\lfloor A\rfloor-\lfloor B\rfloor\ge\lfloor A-B\rfloor$ ensures that an output through $\lfloor\tau_1T_I\rfloor$ has all M ordinary input jets at every deleted node. Their number is $2(q-1)(\lfloor S_I\rfloor+1)>2(q-1)S_I$, and $M_0>\Delta_0T_I$. The identity $c_1W_*ST\theta=qa$ gives first entry strictly greater than $2q(q-1)a\Delta_0=B_0+\varepsilon$.

Theorem 10.119 has both the deleted loss M_0B and the inverse/jet loss. Its exact cancellation (10.327) bounds their sum by $[2\Delta_0T_I+1]H^{-I}H_1/\ln p$. Here and below, the rounded inner radius also satisfies $2R_1<3q^rS_I$; hence the separation bound of Lemma 10.123 applies to it. Since $1/T_I\le cH/a$ and $c_1W_*T_IH^{-I}H_1/\ln p\le1/(c_3Y_-)$, this gives (10.365). The coefficient-minimum loss is at most $c_1/(7ab_r)$. Only the nonnegative term $(D_{\mathrm a}-u)\theta$ is discarded.

For a full extension take $M_j = \lfloor \Delta_j T_I \rfloor + 1$. The same floor-difference inequality provides all those ordinary jets between successive τ cutoffs. The full node count exceeds $2q^j S_I - 1$. Since $S_I \geq S_- > 58$ and $q^j \geq 2$, $1 - 1/(2q^j S_I) > 231/232$. The factor λ_{node} in (10.363) thus gives first entry strictly greater than $B_j + \varepsilon$. There is no deleted increment in this full system; $M_j - 1 \leq \Delta_j T_I$ gives (10.366) by the same exact fixed-order cancellation. This proves both integer input estimates without presuming a zero.

For fractional interpolation take $O = \lfloor HT_I \rfloor$, full radii $R_1 = q(\lfloor S_I \rfloor + 1), R_j = \lfloor q^j S_I \rfloor$ for $2 \leq j \leq r$, and multiplicities $\mu_j = \lfloor \tau_j T_I \rfloor - O + 1$, with $\mu_0 = \lfloor T_I \rfloor - O + 1$. They are positive and nonincreasing. Every available input order covers the required additional jets, including when two derivative floors coincide. Apply Theorem 10.132 with deleted increment $E = \mu_0 - \mu_1$. The disjoint annular counts used in Lemma 10.133 now multiply $(\tau_j - H)T_I$, giving (10.367). Replacing S_I by its lower bound S_- enlarges the floor loss. No differences of floors are rounded separately.

For its second entry, $E \leq \Delta_0 T_I + 1$ and $\mu_0 - 1 \leq (1 - H)T_I + 1$. Thus both deleted and inverse/jet losses together are at most $[(1 - H + \Delta_0)T_I + 2]H^{-I}H_1/\ln p$. Use $2/T_I \leq 2cH/a$, then divide by q^r. The identities $W_* U = q^{r+1}$ and the same minimum-valuation bound give J_{f}. The comparison throughout is in the unchanged normal quantity with the same u and δ. □

Continuous depths and every rank

Lemma 10.135 (strict adaptive bounds at every depth). For every original V rank $r \geq 2$ and every $1 \leq I \leq L_D$,

let $\mathcal{G}(I) = \Gamma(I)$ at ranks 2–7, and let $\mathcal{G}(I) = qa(2 - 1/(q^r S_-))(\tau_r(I) - H)$ at ranks at least eight. The latter is the lower mass from the last full block alone. Then

$$\begin{aligned} \Delta_0(I) < c/a, \qquad \tau_r(I) > H, \qquad J_j(I) - B_j(I) > 1 \ (0 \leq j < r), \\ \mathcal{G}(I) - A_{\mathrm{f}}(I) > 19/10, \qquad J_{\mathrm{f}}(I) - A_{\mathrm{f}}(I) > 12/5. \end{aligned} \tag{10.368}$$

For $2 \leq r \leq 7$, each $\tau_j(I) > \eta^j$. For every $r \geq 8$, $\tau_r(I) > 2/3$; the original η^j blocks are retained independently by Theorem 10.126.

There is also a strict closure comparison on the larger interval $1 \leq J \leq L_D + 1$:

$$2cq(q-1) - B_0(J) > 3, \qquad c_1 q^{r+1} - \frac{c_1}{7ab_r} - \frac{3c}{ac_3 Y_-} - B_0(J) > 1. \tag{10.369}$$

Proof. Fix r and the original D. Each $B_j(I)$ is convex: it is a sum of positive multiples of H^I, Q^{-I}, a constant and an affine function of I. Consequently each Δ_j is convex and each τ_j is concave. Each coefficient of τ_j in Γ is positive. For j=2 this follows from $2(q^2 - q) > 2(q+1)/S_-$, and for j at least three from $2(q^j - q^{j-1}) > 2/S_-$; the first coefficient is positive directly. The bounds $S_- > 58, q = 3$ imply both inequalities. The last-block mass is also a positive affine function of τ_r. Thus $\mathcal{G} - A_{\mathrm{f}}$, $J_{\mathrm{f}} - A_{\mathrm{f}}$, and every $J_j - B_j$ are concave. It suffices to bound their two endpoints; likewise, an upper bound on Δ_0 and lower bounds on τ_j follow from the endpoints. This is a proof for a continuous interval, not a check of sampled depths.

At I=1 every exponential is rational. At $I = L_D$, use $Q^{-L_D} = \exp(-3D) < 1/1000$, $H^{L_D} \leq H$, and $\ell L_D = 3\bar{\lambda}(1099/1000)/(c_4 \ln Q)$. The exponential inequality follows from the positive series through degree six at 15 and $D > 5$, as proved in Lemma 10.122. For a rational x greater than one put

$$\mathcal{L}_{20}(x) = 2\sum_{k=0}^{19} \frac{((x-1)/(x+1))^{2k+1}}{2k+1} < \ln x. \tag{10.370}$$

The identity follows by integrating the finite geometric expansion of $2/(1-z^2)$ from zero to $(x-1)/(x+1)$; its omitted integral is positive. This also proves the inequality without a numerical logarithm oracle. Replace the remaining $1/D$ terms by $1/5$ at ranks 2–7. In (10.362), the two upper endpoint common costs are then

$$\begin{aligned} C_{\mathrm{E}} &= b+n+e_0H+3\ell_-/2, \\ C_{\mathrm{L}} &= b+n+e_0H+\frac{3\bar{\lambda}(1099/1000)}{c_4\mathcal{L}_{20}(Q)}+\ell_-/2, \qquad \ell_- = \frac{\bar{\lambda}(1099/1000)}{5c_4}. \end{aligned} \tag{10.371}$$

Use Q^{-1} and $1/1000$, respectively, for the torus terms; use e_0H^2 for the fractional Euler cost. All resulting $B_j, \Delta_j, A_{\mathrm{f}}$ are upper bounds. The resulting τ_j are lower bounds, and the positive coefficients just proved make their resulting Γ a lower bound. Formula (10.367) similarly gives a lower input entry.

The following are strict rational lower bounds, each divided by 1000, on the minima of the two endpoint differences. They are obtained from (10.361)–(10.367), (10.370)–(10.371) by multiplying positive denominators. The editable rational certificate supplies every integer numerator and denominator, and performs precisely these finite operations.

Rank	Fractional mass gap	Fractional input gap	Terminal closure gap
2	1930	2456	3358
3	4129	2531	3657
4	7334	2838	3934
5	9512	3000	4088
6	11191	3100	4186
7	12895	3168	4254
Every rank at least 8	7299	2520	4171

In the twelve finite endpoint substitutions, every integer input gap exceeds one, every $\Delta_0 < c/a$, and every $\tau_j > \eta^j$. These are also finite rational inequalities with their original c bands, not extrapolations in r. Concavity proves their full depth ranges.

Here is the analytic bound that justifies the last row for all ranks. Take $c = 827/1000$ and define

$$H_- = \left(1-\frac{c}{9}\right)^9, \qquad H_+ = \left(\sum_{k=0}^{10}\frac{c^k}{k!}\right)^{-1}, \qquad Q_- = 3H_-. \tag{10.372}$$

Lemma 10.122 proves $H \geq H_-$. The inequality $\ln(1-z) < -z$, obtained by integrating $-1/(1-z) < -1$, gives $H < \exp(-c) < H_+$. Thus $Q \geq Q_- > 1$. The increasing b_r, S_-, decreasing $\epsilon_r, \bar{g}_{12}, C_r^\sharp < 1$, and $\bar{g}_9 = 107/103$ bound b and e_0 from above by their rank-eight substitutions with $C_r^\sharp$ replaced by one. Write these bounds as b_+, e_+, let $S_8 = S_-(8)$, and use $\ell_+ = \bar{\lambda}(1099/1000)/(6c_4)$. The original D exceeds six in this rank range, by the parameter bounds in Lemma 10.129.

The upper common endpoints are (10.371) with b_+, e_+, H_+, ℓ_+, and $\mathcal{L}_{20}(Q_-)$. Write them as $C_{\mathrm{E}}^+, C_{\mathrm{L}}^+$, and put $C_{\max} = \max\{C_{\mathrm{E}}^+, C_{\mathrm{L}}^+\}$. To bound the sum of the adaptive losses, the finite positive sums are at most $\sum_{j\geq 1} q^{-j-1} = 1/[q(q-1)] = 1/6$ and $\sum_{j\geq 1} jq^{-j-1} = 1/(q-1)^2 = 1/4$. The first follows by multiplying a finite geometric sum by $1-1/q$ and passing to its vanishing remainder. For the second, write $j = \sum_{k=1}^{j} 1$ and sum the nonnegative double series in either order, obtaining $(1/q^2)/(1-1/q)^2$. Hence

$$\sum_{j=0}^{r-1}\Delta_j(I)\le\frac{1}{a}\left[\frac{(1+\lambda_{\mathrm{node}})(\mathcal{C}(I)+\varepsilon)}{12}+\frac{\lambda_{\mathrm{node}}\ell}{8}+\left(\frac{1+S_-^{-1}}{4c_2}+\frac{\lambda_{\mathrm{node}}(r-1)}{2c_2}\right)Q^{-I}\right].\tag{10.373}$$

At the early endpoint replace Q^{-I} by Q_-^{-1}; at the late endpoint replace it by $1/1000$. Use $(r-1)/a\le 1$ and $1/a\le 1/9$. If these two torus bounds are v, the resulting loss bound is

$$\frac{\lambda_{\mathrm{node}}v}{2c_2}+\frac{1}{9}\left[\frac{(1+\lambda_{\mathrm{node}})(C^+ +\varepsilon)}{12}+\frac{\lambda_{\mathrm{node}}\ell_+}{8}+\frac{(1+S_8^{-1})v}{4c_2}\right].\tag{10.374}$$

The exact rational substitutions give $1-\mathrm{loss}>671/1000>2/3$ at the early endpoint and $1-\mathrm{loss}>892/1000>2/3$ at the late endpoint. Concavity then gives $\tau_r>2/3$ throughout. Both comparisons are included in the two uniform certificate rows.

The last full block alone now has normalized fractional mass strictly greater than

$$q\cdot 9\left(2-\frac{1}{q^8S_8}\right)(2/3-H_+).\tag{10.375}$$

Indeed it has more than $2q^rS_I-1$ nodes and multiplicity greater than $(\tau_r-H)T_I$. Use $a\ge 9, q^rS_I\ge q^8S_8$, and $c_1W_*ST\theta=qa$. All the other increments in the mixed system are nonnegative. This is a lower bound for its actual total mass, without any requirement that every adaptive τ_j exceed η^j.

At both endpoints, enlarge (10.364) with $b_+, e_+, H_+^2, H_-^{-1}, S_8^{-1}, \ell_+$, $q^{-r}\le q^{-8}$, and the respective common depth terms. Subtracting this upper bound from (10.375) gives differences greater than $9753/1000$ and $7299/1000$. Since $\Delta_0<c/a$, the input formula (10.367) is bounded below by (10.358). In that formula use $1-H\le 1-H_-$, $2cH/a\le 2cH_+/9$ and $q^r\ge q^8$. Its two gaps exceed $4974/1000$ and $2520/1000$. These bounds prove the asserted uniform fractional comparisons.

For the remaining uniform integer assertions, the two endpoint bounds give $B_0<8$. Consequently $\Delta_0<(801/1200)/a<c/a$. For $j\ge 1$, $(C_{\max}+\ell_+ j)/q^{j+1}$ decreases with j: cross multiplication reduces its next-step comparison to $(q-1)C_{\max}+((q-1)j-1)\ell_+>0$. Thus its largest value is at one, and exact rational substitution gives

$$\frac{C_{\max}+\ell_+}{q^2}+\frac{1}{c_2Q_-}<3/2,\qquad\frac{\lambda_{\mathrm{node}}(3/2+\varepsilon/q^2)}{2}<4/5.\tag{10.376}$$

It follows that $B_j<(3/2)q^{j+1}$, $\Delta_j<4/(5a)$. The full input gap is bounded below by $q^8(qc_1-3/2)-c_1/(7\cdot 9b_8)-(4/5)/(9c_3Y_-)$, which exceeds one. The closure gap is bounded below by $c_1q^9-c_1/(7\cdot 9b_8)-[2(801/1200)+cH_+]/(9c_3Y_-)-8$, also greater than one. This proves every integer comparison at every rank, rather than testing finitely many larger ranks.

Finally, (10.369) needs the interval $[1,L_D+1]$. The function $B_0(J)$ remains convex. Its early endpoint has the same upper bound as before. At the new late endpoint, $Q^{-J}<1/1000, H^J\le H$, and $\ell J\le 3\bar{\lambda}(1099/1000)/(c_4\mathcal{L}_{20}(Q))+\ell_-$, using ℓ_+ in the uniform case. Its upper bound is therefore the previous late B_0 bound plus that one extra ℓ_- or ℓ_+. The third column of the table uses exactly this extra cost. The associated input bound in (10.369) exceeds its arithmetic bound by more than one in each of the twelve finite rows and the two uniform rows. Convexity proves (10.369) on the whole larger interval. All strict assertions of the lemma follow. □

Producing the actual zeros and passing the phase

Theorem 10.136 (the complete original V first induction). Under the original contradiction and initial integral auxiliary kernel of Section 40, there are nonzero stage families Φ_I for every $0 \le I \le I_0$. Each has the original coefficient-height bound, minimum valuation δ, selected derivations and phase conditions, torus widths at most $q^{-I}D_i$, and additive argument $q^{-I}x$. Each has every original zero at

$$\Phi_I(s;\boldsymbol{t}) = 0, \qquad |s| \le \lfloor qS_I \rfloor, \quad |\boldsymbol{t}| \le \lfloor \eta T_I \rfloor. \tag{10.377}$$

For each $1 \le I \le I_0$, the stronger radius $q(\lfloor S_I \rfloor + 1)$ is available at that same full cutoff, together with all q-deleted zeros at that radius through $\lfloor T_I \rfloor$. For every $1 \le I < I_0$, all full original outer blocks at radii $\lfloor q^j S_I \rfloor$, cutoffs $\lfloor \eta^j T_I \rfloor$, $2 \le j \le r$, remain available. The same family has the additional full adaptive blocks at those radii through $\lfloor \tau_j(I) T_I \rfloor$, with the rounded first radius and cutoff τ_1. Its original fractional output is

$$\Phi_I(s/q;\boldsymbol{t}) = 0, \qquad |s| \le q(\lfloor S_{I+1} \rfloor + 1), \quad |\boldsymbol{t}| \le \lfloor T_{I+1} \rfloor. \tag{10.378}$$

Proof. Corollary 10.127 and Theorems 10.130–10.131 supply the initial family, full first fractional block and stage-one family, including its q-deleted zeros. The original multiplicity bound ensures $I_0 \ge 1$.

Suppose the stage-I q-deleted zeros have been obtained and $1 \le I < I_0$. First close them at cutoff $\lfloor \tau_1 T_I \rfloor$, using M_0 from Lemma 10.134. Both analytic entries exceed B_0, by (10.368); a nonzero value in K has the opposite arithmetic bound. Corollary 10.120 therefore forces every missing integer value to be zero. Since $\Delta_0 < c/a$, this includes the original full ηT_I cutoff. Theorem 10.126 independently extends those original integer blocks through all $\eta^j T_I$ cutoffs. In addition, apply Lemma 10.134 successively at $j = 1, \dots, r-1$, with the just-proved adaptive block as the next input. Both entries exceed the corresponding B_j. This gives every stated adaptive block for the very same coefficient family. No selection or coefficient change occurs during these integer extensions.

Use their complete mixed multiplicities for a target in (10.378). Both normalized analytic entries exceed A_{f}, by Lemma 10.135. The projected and unprojected values have the same proved precision, and the principal exponential laws identify the analytic original value with the actual algebraic value in the normalized support field. Corollary 10.120, with its full global degree factor and original selected local degrees, forces that value to vanish. This proves every fractional zero in (10.378) at its exact rounded radius and every derivative order.

Apply Theorem 10.51 to its nodes prime to q. Retain a class containing an original coefficient of valuation δ; when q divides G_0, perform the required second phase partition too. The entire class vanishes at every prescribed node and order. The invertible affine binomial transfer preserves the total-order cutoff; q is a unit at p. The new coefficient subvector is integral, nonzero, of the same minimum and no larger height. Its widths shrink by q and its additive argument becomes $q^{-(I+1)}x$. The actual phase identity puts its integer prepared values in K. This constructs Φ_{I+1} with all q-deleted zeros at the rounded radius and cutoff $\lfloor T_{I+1} \rfloor$.

For clarity, the full closure at the successor $J = I + 1$ is valid even when $J = I_0$. Set $M = \lfloor cT_J/a \rfloor + 1$ and take the q-deleted radius $q(\lfloor S_J \rfloor + 1)$. The floor inequality supplies every ordinary jet for output $\lfloor \eta T_J \rfloor$. Its first analytic entry exceeds $2cq(q-1)$. The two separate deleted and inverse/jet losses are at most $[2cT_J/a + 1]H^{-J}H_1/\ln p$. Since $cT_J/a \ge 1$, its second entry exceeds $c_1 q^{r+1} - c_1/(7ab_r) - 3c/(ac_3Y_-)$. Its nonzero integer arithmetic envelope is $B_0(J)$, by the same scalar, clearing, width and field argument as Lemma 10.134. Formula (10.369) forces every remaining integer value to vanish on $[1, L_D + 1]$. This completes the full original final block without applying the fractional-depth comparison beyond its proved interval.

Induction now constructs the families through I_0, with (10.377) and all the asserted stronger blocks. Every passage retains a coefficient attaining the original minimum, so none can vanish identically. The result proves the original V first induction at both possible stopping indices; the subsequent stopping contradictions and the other prime cases are separate arguments. □

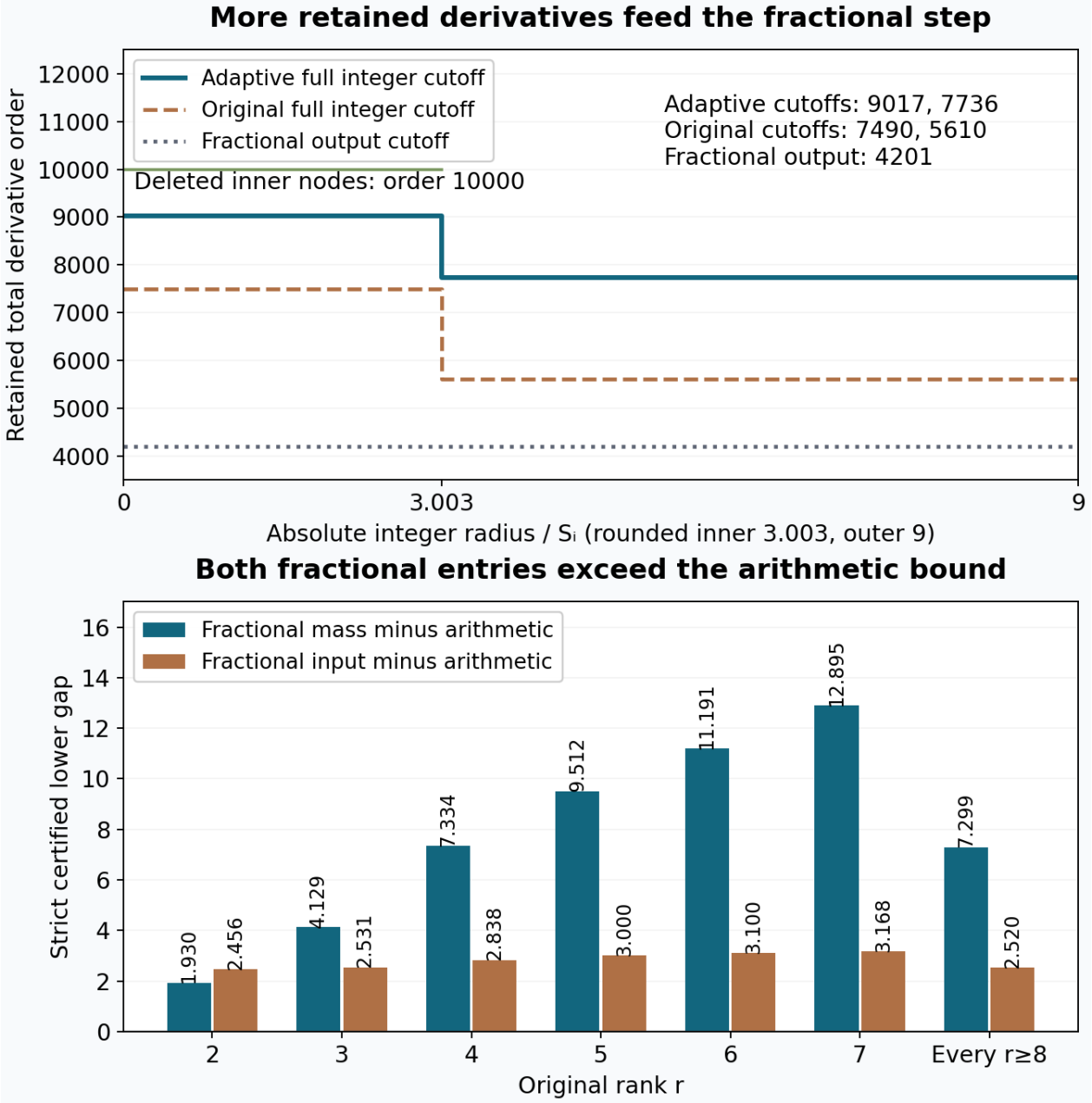


Figure 10.31. Left: a conservative rank-two early-endpoint envelope, with D=5, evaluated by the exact rational formulas (10.361)–(10.363), at illustrative $S_I = 1000, T_I = 10000$. The full cutoffs are $\lfloor \tau_1 T_I \rfloor, \lfloor \tau_2 T_I \rfloor$; the original cutoffs are $\lfloor \eta T_I \rfloor, \lfloor \eta^2 T_I \rfloor$, and the fractional output is $\lfloor H T_I \rfloor$. Deleted inner nodes additionally retain order10000. The panel describes available derivative cutoffs, not auxiliary-function values. Right: strict rational lower bounds for the mass and input gaps in Lemma10.135, minimized over both continuous-depth endpoints. The last column represents the proved uniform bound for every rank at least eight. The full factor q^r has already been charged in these fractional comparisons. The extra terminal closure is proved separately in (10.369). Figure program, rational certificate. Human-source context: the original first induction and stopping indices in Yu's freely accessible paper, Section5, equations(5.14)–(5.16),(5.20),(5.42)–(5.62). The adaptive bounds, complete interpolation comparisons and terminal closure are proved above.

44. The complete odd-prime phase induction

For odd p the original integer derivative cutoffs already give enough fractional precision. We prove this for all seven coupled field rows, every rank and the entire original depth interval. The proof uses the full mixed profile of Section 42 and the global degree of the actual fractional support field.

Retain every original hypothesis of Section 40, now with odd p and q=2. The rows are I.1, I.2, II, III.1, III.2 and the two branches of IV. Put $a = r+1, c = c_5, \eta = 1 - c/a, H = \eta^a, Q = qH, D = A + \nu \ln q$, and retain the stopping indices (10.360). Thus $S_I = H^{-I}S, T_I = H^I T$. The additive degree $D_{\mathrm{a}} = kL$ is unchanged. The original depth restriction gives $1 \le I < I_0 \Rightarrow I \le L_D$ and $cHT_I/a \ge 1$. At the final stage only $cT_{I_0}/a \ge 1$ is asserted.

Arithmetic envelopes and the complete input profile

Use the constants c_1, c_2, c_3, c_4, their original rank bands for c, and the simultaneous bounds C, Y_-, Ω_+ in Lemma 10.121. In particular $c_2 = 7/4$. Let $b_r, \epsilon_r, \bar{g}_{12}, \bar{g}_9, S_-$ be exactly the row-dependent bounds in (10.331). The number $\bar{g}_{12}$ is zero in the degree-one rows I.2 and III.2; $\bar{g}_9 = 107/103$ in those rows and at every rank at least eight. Define

$$\begin{aligned} A_r &= c_1(\bar{g}_{12} + \epsilon_r) + \frac{c_1\epsilon_r/2 + \bar{\lambda}/c_4 + C_r^\sharp/(c_3Y_-) + 117/(232c_2)}{C-1}, \\ b &= A_r + c_1\left(\frac{1}{300} + \frac{1}{273\cdot 999 b_r}\right) + \frac{\bar{\lambda}}{c_4}, \\ e_0 &= \frac{\bar{g}_9}{c_3Y_-}, \qquad n = \frac{\bar{\lambda}\Omega_+}{c_4}, \qquad \ell = \frac{\bar{\lambda}(347/500)}{Dc_4}, \qquad \bar{\lambda} = 189/188. \end{aligned} \tag{10.379}$$

The coefficient bound is precisely (10.339); $C_r^\sharp$ is (10.307). The symbol e_0 again denotes a numerical cost rather than a local degree. Our two arithmetic envelopes are

$$\begin{aligned} A_{\mathrm{f}}(I) &= b + e_0H^{I+1} + \ell(I+2) + \frac{H^{-1} + S_-^{-1}}{c_2}Q^{-I} + \frac{n}{q^r}, \\ B_0(J) &= b + n + e_0H^J + \ell(J+1) + \frac{q(1+S_-^{-1})}{c_2}Q^{-J}. \end{aligned} \tag{10.380}$$

Lemma 10.137 (odd-prime fractional and terminal envelopes). For an admissible stage $1 \le I < I_0$, a nonzero fractional target $x = s/q$, with $|s| \le q(\lfloor S_{I+1}\rfloor + 1)$ and total derivative order at most $\lfloor T_{I+1}\rfloor$, has $(c_1W_*/q^r)[v_p(V) + G_u - \delta] \le A_{\mathrm{f}}(I)$. Assuming all the original stage-I integer blocks, its two analytic entries in that same normalization are strictly greater than

$$\begin{aligned} \Gamma_r &= \frac{2cq}{q^r}\sum_{j=0}^{r}(q\eta)^j - \frac{qa}{q^rS_-}\left[2(q+1)(\eta^2 - H) + 2\sum_{j=3}^{r}(\eta^j - H)\right], \\ J_{\mathrm{f},r} &= c_1q - \frac{c_1}{7ab_rq^r} - \frac{1 - H + c(1+2H)/a}{q^rc_3Y_-}. \end{aligned} \tag{10.381}$$

For any integer stage $J \ge 1$ with $cT_J/a \ge 1$, a nonzero integer target at the rounded radius $q(\lfloor S_J\rfloor + 1)$, of total order at most $\lfloor \eta T_J\rfloor$, has normalized valuation at most $B_0(J)$. If all q-deleted zeros are available at this radius through $\lfloor T_J\rfloor$, the two analytic closure entries are strictly greater than

$$2cq(q-1), \qquad J_{\mathrm{c},r} = c_1q^{r+1} - \frac{c_1}{7ab_r} - \frac{3c}{ac_3Y_-}. \tag{10.382}$$

All comparisons fix the same additive order $u \le D_{\mathrm{a}}$ and use the original coefficient minimum δ.

Proof. Theorem 10.115 gives the coefficient bound A_r and the separate scalar remainder in b. At the fractional radius, the additive argument has size at most

$$q^{-I}(\lfloor S_I/H \rfloor + 1) \le (S/H)Q^{-I} + q^{-I} < qS.$$

The last strict inequality is the original scalar bound of Lemma 10.129. It gives the additive cost $\bar{\lambda}/c_4$. The actual torus cost is at most $(H^{-1} + S_-^{-1})Q^{-I}/c_2$, since $S \ge S_-$ and $q^{-I} \le Q^{-I}$. The fractional clearing exponent is $\Xi_{I,1,u} \le D_{\mathrm{a}}(I + 1 + 1/(q-1)) = D_{\mathrm{a}}(I+2)$. The upper logarithm bound $\ln 2 < 347/500$, proved in Lemma 10.122, therefore gives $\ell(I+2)$. The total output order is at most HT_I, so Lemma 10.114 gives the Euler/lcm cost $e_0 H^{I+1}$.

The actual normalized support field in Theorem 10.56 has global degree dq^{h_Λ}, $h_\Lambda \le r$, and the original selected local degrees e,f. The product-formula comparison charges this entire factor. Divide its positive algebraic-height terms by q^r; because $q^{h_\Lambda - r} \le 1$, each is bounded above by its displayed cost. The normal contribution is added after that comparison and is at most n/q^r. This proves the first line of (10.380). It remains valid when the chosen completion has local degree one.

The original full and deleted blocks have the exact nested multiplicities (10.356). Lemma 10.133 proves their complete mass and fixed-order input estimates, including the separate deleted-cardinal and inverse/jet losses. Since q=2 and $S_I \ge S_-$, (10.357)–(10.358) imply (10.381). All terms $\eta^j - H$ in the subtracted floor loss are positive. Replacing S_I by S_- thus decreases that lower mass. No derivative floor is changed.

At the integer closure radius,

$$q^{-J}q(\lfloor S_J \rfloor + 1) \le qSQ^{-J} + q^{1-J} \le S/H + 1 < qS.$$

The additive bound is the same. The integer clearing exponent obeys $\Xi_{J,0,u} \le D_{\mathrm{a}}(J+1)$, since the original binomial denominator contribution is at most $D_{\mathrm{a}}/(q-1)$. The torus cost is at most $q(1 + S_-^{-1})Q^{-J}/c_2$, the Euler/lcm cost at most $e_0 H^J$, and the normal term at most n. The actual prepared integer value lies in K by Theorem 10.55. Corollary 10.116 therefore uses d, e and f without a fractional root-field factor. These facts prove $B_0(J)$.

For the analytic closure choose $M = \lfloor cT_J/a \rfloor + 1$. The inequality $\lfloor T_J \rfloor - \lfloor \eta T_J \rfloor \ge \lfloor cT_J/a \rfloor$ supplies every required ordinary input jet at every deleted node, for every output order under consideration. There are $2(q-1)(\lfloor S_J \rfloor + 1) > 2(q-1)S_J$ such nodes. Since $M > cT_J/a$ and $c_1 W_* S T \theta = qa$, their normalized mass is strictly greater than $2cq(q-1)$.

The exact cancellation (10.327) retains both the cardinal loss MB and the inverse/jet loss $(M-1)\max\{B, \beta - \vartheta, 0\}$. Lemma 10.123 applies to this rounded interval because $2q(\lfloor S_J \rfloor + 1) < 3q^r S_J$. Their sum is at most $[2cT_J/a + 1]H^{-J}H_1/\ln p$. Use $1/T_J \le c/a$, the identity $c_1 W_* T_J H^{-J} H_1/\ln p \le 1/(c_3 Y_-)$, $W_* U = q^{r+1}$, and the strict coefficient-minimum bound from Lemma 10.123. Discard only the nonnegative term $(D_{\mathrm{a}} - u)\theta$. The remaining entry is strictly greater than $J_{\mathrm{c},r}$. This proves (10.382) using $cT_J/a \ge 1$, without the stronger successor multiplicity restriction. Orders $u > D_{\mathrm{a}}$ vanish identically and require no comparison. □

A continuous interval and an infinite rank range

Lemma 10.138 (all odd-prime strict comparisons). At every original odd-prime rank $r \ge 2$, throughout $1 \le I \le L_D$,

$$\Gamma_r - A_{\mathrm{f}}(I) > 3/4, \qquad J_{\mathrm{f},r} - A_{\mathrm{f}}(I) > 1. \tag{10.383}$$

Throughout the larger interval $1 \le J \le L_D + 1$,

$$2cq(q-1) - B_0(J) > 1/4, \qquad J_{\mathrm{c},r} - B_0(J) > 1. \tag{10.384}$$

Proof. Fix the actual r and D. The functions A_{f}, B_0 are convex: they are sums of a constant, an affine function and positive multiples of H^I, Q^{-I}. For example $(H^I)'' = (\ln H)^2 H^I > 0$. All the analytic lower bounds in (10.381)–(10.382) are independent of depth. Their gaps are therefore concave. A concave function on a closed interval is bounded below by the lesser of its two endpoint values: its defining chord inequality gives this directly. We will bound precisely those endpoints.

At ranks 2–7 put $\ell_- = \bar{\lambda}(347/500)/(5c_4)$ and $d_+ = 3\bar{\lambda}(347/500)/(c_4\mathcal{L}_{20}(Q))$, where $\mathcal{L}_{20}$ is the positive rational logarithm bound (10.370). Lemmas 10.122 and 10.129 give $D > 5$, $Q > 1$, $L_D > 1$, and $Q^{-L_D} = \exp(-3D) < 1/1000$. Consequently $\ell \le \ell_-$, $\ell L_D < d_+$, and $H^{L_D+1} \le H^2$. The following are upper bounds at the early and late endpoints, respectively:

$$\begin{aligned}
F_{\mathrm{E}} &= b + e_0 H^2 + 3\ell_- + \frac{H^{-1} + S_-^{-1}}{c_2 Q} + n/q^r, \\
F_{\mathrm{L}} &= b + e_0 H^2 + d_+ + 2\ell_- + \frac{H^{-1} + S_-^{-1}}{1000 c_2} + n/q^r, \\
C_{\mathrm{E}} &= b + n + e_0 H + 2\ell_- + \frac{q(1 + S_-^{-1})}{c_2 Q}, \\
C_{\mathrm{L}} &= b + n + e_0 H + d_+ + 2\ell_- + \frac{q(1 + S_-^{-1})}{1000 c_2}.
\end{aligned} \tag{10.385}$$

The closure late endpoint is $J = L_D + 1$. Its clearing cost is $\ell(L_D + 2)$, which explains both copies of ℓ_- in C_{L}. We have bounded H^{L_D+1} by the weaker H there. These substitutions only enlarge the arithmetic costs.

Here are strict lower bounds, divided by 1000, for all four endpoint gaps. The first table takes the minimum over ranks 2–7 and both endpoints. The second gives uniform bounds for every rank at least eight, justified below.

Case, ranks 2–7	Mass gap	Input gap	Closure gap	Closure input gap
I.1	1266	1204	320	9517
I.2	1231	1163	288	9472
II	1278	1220	347	9473
III.1	1200	1155	313	9438
III.2	1164	1136	284	9418
IV, $P = 1$	1219	1151	312	9449
IV, $P \ge 7$	1530	1553	713	10088

Case, every rank at least 8	Mass gap	Input gap	Closure gap	Closure input gap
I.1	868	1366	392	739955
I.2	815	1314	332	740095
II	913	1406	420	733980
III.1	815	1303	383	732361
III.2	780	1266	347	731256
IV, $P = 1$	811	1300	391	737433
IV, $P \ge 7$	1185	1675	836	737961

For each finite row, substitute the original rational c band, coupled constants, S_-, b_r and $C_r^\sharp$ in (10.379), (10.381), (10.382) and (10.385). Multiply the positive denominators. The editable exact rational calculation supplies all 84 individual finite substitutions and their integer numerators and denominators. Thus the finite columns are finite rational inequalities; the continuous depths follow from concavity.

We now justify the uniform columns without sampling ranks. For each row use its fixed c at ranks at least eight, and put

$$\begin{aligned} H_- &= (1-c/9)^9, \qquad E_-(c) = \sum_{j=0}^{10} \frac{c^j}{j!}, \qquad H_+ = E_-(c)^{-1}, \\ E_+(c) &= E_-(c) + \frac{c^{11}}{11!(1-c/12)}, \qquad Q_- = qH_-. \end{aligned} \tag{10.386}$$

The positive exponential series gives $E_- < \exp(c) < E_+$. Indeed the omitted terms start at $c^{11}/11!$; the ratio of each subsequent term to its predecessor is at most $c/12 < 1$, with a strict inequality after the first ratio. Summing the geometric majorant proves the upper inequality. Lemma 10.122 proves $H \geq H_-$; integration of $-1/(1-z) < -1$ gives $H < \exp(-c) < H_+$. Direct rational substitution in each original c row gives $Q_- > 1$.

The same integral also gives, for $0 < z < 1$, $\ln(1-z) > -z/(1-z)$, since $1/(1-t) < 1/(1-z)$ for $0 \leq t < z$. Apply it with $z = c/a$. Because $a - c > r$,

$$\ln(\eta^r) > -\frac{rc}{a-c} > -c, \qquad \eta^r > \exp(-c) > E_+(c)^{-1}. \tag{10.387}$$

Reverse the principal geometric sum in (10.381): $q^{-r}\sum_{j=0}^{r}(q\eta)^j = \sum_{k=0}^{r} q^{-k}\eta^{r-k}$. At ranks at least eight it contains its first four positive terms. Each of them is at least $q^{-k}\eta^r$. The negative floor bracket in (10.381) is less than $2(q+r-1)$, because every difference $\eta^j - H$ is less than one. Hence

$$\begin{aligned} \Gamma_r &> \frac{2cq}{E_+(c)} \sum_{k=0}^{3} q^{-k} - \frac{2qa(q+r-1)}{q^r S_-} \\ &\geq \frac{2cq}{E_+(c)} \sum_{k=0}^{3} q^{-k} - \frac{2q \cdot 9(q+7)}{q^8 S_8} =: \Gamma_+, \qquad S_8 = S_-(8). \end{aligned} \tag{10.388}$$

For the second inequality q=2 makes the subtracted numerator $4(r+1)^2$. The sequence $(r+1)^2/2^r$ decreases for $r \geq 8$: its consecutive ratio is $((r+2)/(r+1))^2/2 < 1$. The original S_- increases in each row, as is explicit in (10.331). Their combination proves the rank-eight upper bound on the loss.

The parameters b_r, S_- increase, $\epsilon_r, \bar{g}_{12}$ decrease, and $C_r^\sharp < 1$. All coefficients in (10.379) are positive. Consequently b is bounded above by b_+, its rank-eight value with $C_r^\sharp$ replaced by one; $e_+ = \bar{g}_9/(c_3 Y_-)$ is constant in this rank range. By Lemma 10.129, D exceeds six. Define $\ell_+ = \bar{\lambda}(347/500)/(6c_4)$ and $d_+^+ = 3\bar{\lambda}(347/500)/(c_4 \mathcal{L}_{20}(Q_-))$.

In the four endpoints (10.385), replace b, $e_0H^2, e_0H, H^{-1}, S_-^{-1}, Q^{-1}, \ell_-, d_+, n/q^r$ by the respective upper bounds $b_+, e_+H_+^2, e_+H_+, H_-^{-1}, S_8^{-1}, Q_-^{-1}, \ell_+, d_+^+, n/q^8$. The late torus factor is still 1/1000. Call the resulting rational upper bounds $F_{\mathrm{E}}^+, F_{\mathrm{L}}^+, C_{\mathrm{E}}^+, C_{\mathrm{L}}^+$. These are valid at every rank at least eight, in each row. Corresponding uniform analytic lower bounds are

$$J_{\mathrm{f},r} \ge c_1 q - \frac{c_1}{7 \cdot 9 b_8 q^8} - \frac{1 - H_- + c(1 + 2H_+)/9}{q^8 c_3 Y_-} =: J_{\mathrm{f},+},$$
$$J_{\mathrm{c},r} \ge c_1 q^9 - \frac{c_1}{7 \cdot 9 b_8} - \frac{3c}{9 c_3 Y_-} =: J_{\mathrm{c},+}. \tag{10.389}$$

For the first use $1 - H \le 1 - H_-$, $2cH/a \le 2cH_+/9$, $a \ge 9, q^r \ge q^8, b_r \ge b_8$. For the second use those last three bounds and $q^{r+1} \ge q^9$.

There are now only fourteen rational substitutions: seven coupled rows and two endpoints. Subtract their upper F bounds from $\Gamma_+, J_{\mathrm{f},+}$, and their upper C bounds from $2cq(q-1), J_{\mathrm{c},+}$. Multiplying positive denominators gives the uniform table displayed above. The same exact calculation file records all fourteen values. Over both tables, the minimum mass gap exceeds $780/1000 > 3/4$, minimum fractional input gap exceeds one, minimum closure gap exceeds $284/1000 > 1/4$, and minimum closure input gap exceeds one. Concavity extends these endpoint bounds to both entire depth intervals. This proves every assertion. □

The actual phase induction

Theorem 10.139 (the complete odd-prime first induction). Under the original contradiction and the initial integral kernel of Section 40, every odd-prime case has a nonzero family Φ_I for each $0 \le I \le I_0$. Each retains the initial coefficient-height bound and minimum δ, selected derivations and phase conditions, widths at most $q^{-I} D_i$, and additive argument $q^{-I} x$. It has every original full integer zero

$$\Phi_I(s; \boldsymbol{t}) = 0, \qquad |s| \le \lfloor q S_I \rfloor, \quad |\boldsymbol{t}| \le \lfloor \eta T_I \rfloor. \tag{10.390}$$

For $1 \le I \le I_0$, the stronger rounded radius $q(\lfloor S_I \rfloor + 1)$ is available at this full cutoff, and its q-deleted nodes have every zero through $\lfloor T_I \rfloor$. For $1 \le I < I_0$ the same family has all full integer outer blocks at radii $\lfloor q^j S_I \rfloor$, cutoffs $\lfloor \eta^j T_I \rfloor$, $2 \le j \le r$. Its full fractional output is

$$\Phi_I(s/q; \boldsymbol{t}) = 0, \qquad |s| \le q(\lfloor S_{I+1} \rfloor + 1), \quad |\boldsymbol{t}| \le \lfloor T_{I+1} \rfloor. \tag{10.391}$$

Proof. The initial family and all its integer extensions are Corollary 10.127. The complete first fractional block is Theorem 10.130; Theorem 10.131 gives the stage-one family, both its rounded full block and q-deleted block. The original initial multiplicity bound gives $I_0 \ge 1$. These earlier proofs include every original coupled row and rank.

Assume the stage-I family has its rounded full inner and deleted blocks, where $1 \le I < I_0$. Theorem 10.126 extends its original full integer blocks to every radius $\lfloor q^j S_I \rfloor$ with cutoff $\lfloor \eta^j T_I \rfloor$, $2 \le j \le r$. These comparisons use the same coefficient family throughout. The mixed multiplicities in (10.356) therefore apply to every target in (10.391) and every specified derivative order.

If such a value were nonzero, Lemma 10.137 would bound its normalized valuation above by $A_{\mathrm{f}}(I)$, in its actual support field. Lemma 10.138 says that both analytic entries exceed this bound. The principal exponential laws in Lemma 10.92 identify the analytic value with the prepared algebraic value, and the projected and unprojected values have the identical proved precision. Corollary 10.120, with the full global factor q^r, the selected local degrees and exact clearing, yields a contradiction. This proves (10.391) at every fractional node and derivative order.

Apply Theorem 10.51 to the nodes prime to q in that fractional block. Choose a phase class containing a coefficient of valuation δ; if q divides G_0, apply its required second phase partition as well. Each chosen class vanishes at all the prescribed nodes and orders. The invertible affine binomial transfer preserves the total-order cutoff, since q is a unit at p. The selected coefficient

subvector is integral, nonzero, of no larger height, and still has minimum δ. Its torus widths shrink by q, its additive argument is $q^{-(I+1)}x$, and Theorem 10.55 puts its actual integer prepared values in K. Thus it is Φ_{I+1}, with every required q-deleted zero through $\lfloor T_{I+1} \rfloor$.

Close that new deleted block with J=I+1. Even when $J = I_0$, the original indices give $J \le L_D + 1$ and $cT_J/a \ge 1$. The last part of Lemma 10.137 supplies both analytic entries; (10.384) makes each strictly exceed the nonzero integer bound $B_0(J)$. Corollary 10.120 forces every remaining integer value to vanish. Hence the stronger rounded full inner block is available through $\lfloor \eta T_J \rfloor$, including the terminal stage. This step uses the larger closure interval, without extending the fractional comparison beyond L_D.

Induction proves all the asserted families through I_0. Their reference coefficients attain δ at every phase passage, so all are nonzero. Every original integer and fractional radius and floor cutoff is retained. □

Corollary 10.140 (the first induction in every original prime and field case). For each original case I–V and every rank $r \ge 2$, the original parameters (10.186), coefficient bounds and minimum, selected derivations and both stopping indices (10.360) admit nonzero stage families through I_0, satisfying

$$|s| \le \lfloor qS_I \rfloor, \quad |\boldsymbol{t}| \le \lfloor \eta T_I \rfloor \implies \Phi_I(s;\boldsymbol{t}) = 0. \tag{10.392}$$

For positive stages their full rounded and deleted blocks, and before the terminal stage their full original outer and fractional blocks, have exactly the stronger ranges specified in Theorems 10.136 and 10.139.

Proof. Theorem 10.139 covers all odd primes, including both degree-one rows and both IV branches with their simultaneous field bounds. Theorem 10.136 covers p=2 with q=3. These exhaust the original constant table. Both theorems preserve the identical initial kernel data and stopping indices, so their conclusions combine directly. □

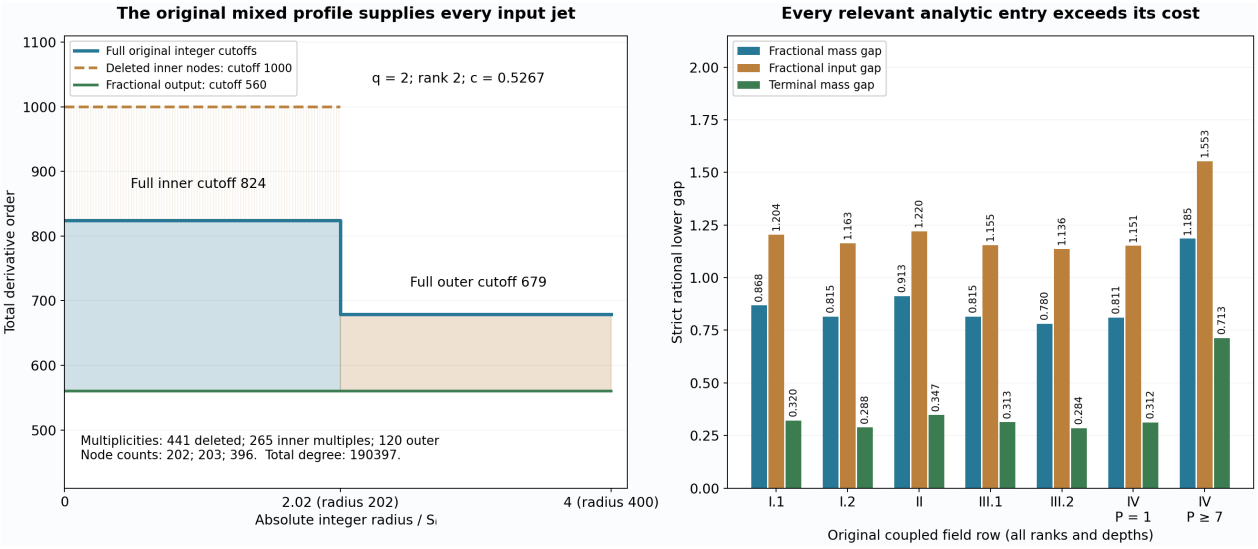


Figure 10.32. Left: the original rank-two III.2 profile, with $c = 0.5267, S_I = 100, T_I = 1000$. The full inner radius is 202, the outer radius 400, their derivative cutoffs 824 and 679, and the fractional output cutoff 560. Deleted inner nodes retain order 1000. The illustration records interpolation inputs and output orders; it does not plot auxiliary-function values. Solution 53 computes the exact multiplicities 441, 265, 120 and total interpolation degree 190397. Right: strict lower bounds for both fractional gaps and the terminal mass gap in Lemma 10.138, minimized over all finite-rank and analytically uniform endpoints for each original coupled row. The comparisons retain the global q^r field factor. Terminal closure is proved on its separate larger interval. Figure program, exact rational calculation. Human-source context: the original first induction and stopping ranges in Yu's freely accessible paper, Section 5, equations (5.13)–(5.20), (5.42)–(5.62); the full comparison and phase-induction proofs are given here.

45. Closing the geometric stopping branch

The first induction now supplies the actual zeros needed by the geometric reduction. Its last family retains the initial coefficient minimum, so its polynomial remains nonzero. The degree and rank-profile comparisons of Sections 34–35 therefore apply to that family with their original field and weights.

Theorem 10.141 (the original geometric stopping contradiction). Suppose the original contradiction parameters of Section 40 arise from the least admissible rank r in (10.277), with $r \geq 2$. Then the alternative $I^* \leq I_1$ is impossible.

Proof. If $I^* \leq I_1$, then $I_0 = I^*$. Set I equal to this integer. The exact stopping definition gives

$$Q^I > \exp(3D), \qquad \overline{S} = qS_I, \qquad \overline{T} = \eta T_I, \qquad \mathsf{D}_0 = kL, \quad \mathsf{D}_j = 2q^{\nu-I}D_j \ (1 \leq j \leq r). \quad (10.393)$$

Indeed $I = \lfloor 3D/\ln Q \rfloor + 1 > 3D/\ln Q$, including when the quotient is an integer. Corollary 10.140 gives the nonzero terminal family and all of its prepared zeros for integer $|s| \leq \lfloor \overline{S} \rfloor$, total order at most $\lfloor \overline{T} \rfloor$.

Its coefficients form a nonzero subvector of the original kernel, with one coefficient attaining δ. The additive basis still has k shifted polynomials and powers 1 through L. Its normalized torus exponent vectors are integral, distinct, and bounded by $q^{\nu-I}D_j$. Proposition 10.31 therefore constructs a nonzero ordinary polynomial with the coordinate bounds in (10.393). The triangular binomial change and Laurent monomial shift convert exactly these prepared zeros into ordinary jets through the same total order. The selected Euler directions still span the original hyperplane W: affine translations of their eigenvalues give invertible triangular changes, and their scalar rescalings are nonzero. Lemmas 10.23 and 10.30 prove these assertions.

All original field, height and parameter data remain unchanged. The independent torus bases and nonzero vector $\boldsymbol{B}$ are those of the minimum-rank representation. The original $\theta = \vartheta/(1+\varepsilon_n)$ satisfies the weaker restriction $0 \leq \tau \leq 1/100$ used in Theorem 10.100. Equation (10.393) is exactly its geometric depth hypothesis (10.264). Thus Theorem 10.100 verifies every degree inequality and every integer allocation required by Theorem 10.25, with the strict bounds $36/25, 8, 31$.

Corollary 10.101 yields independent integral vectors of some rank $1 \leq m < r$, representing the same nonzero form and producing local-unit bases in the original field. Theorem 10.102 preserves their exact residue-index-dependent rank profile, with the strict factor $1/700$. Corollary 10.103 then contradicts the defining least rank r. This rules out $I^* \leq I_1$, including equality. □

Figures 10.22–10.23 display the degree thresholds, allocation margins and preserved rank-profile factor used in this proof. Human-source context for the original branch is Yu's freely accessible paper, Section 6; the degree, multiplicity, jet and minimum-rank proofs used here are in this lesson.

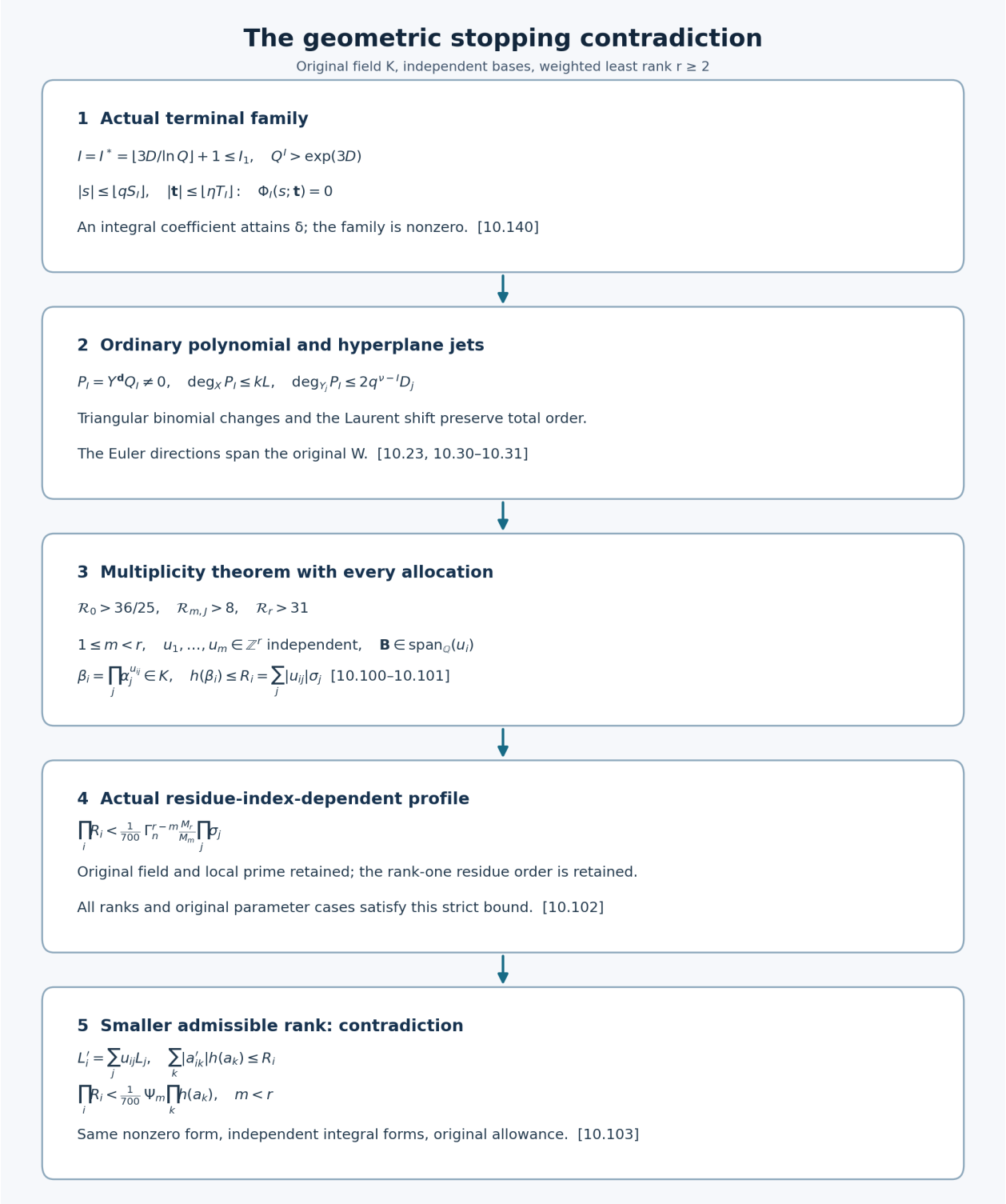


Figure 10.33. The branch $I^* \le I_1$ reaches its exact terminal integer, including $I^* = I_1$. Corollary 10.140 supplies the zeros and the preserved nonzero coefficient. Proposition 10.31 converts them to ordinary hyperplane jets without increasing their total order; the polynomial has additive degree at most kL and torus degrees at most $2q^{\nu-I}D_j$. Theorem 10.100 verifies all three allocation ratios. Corollary 10.101 produces independent integral vectors and new bases in the original field, and Theorem 10.102 retains the actual residue-index-dependent profile with factor $1/700$. Corollary 10.103 gives a smaller admissible rank, contradicting minimality. Arrows indicate proved implications, with their exact locators; no sampled parameter case is used. Figure program. Human-source context: Yu's free paper, Section 6. Solution 54 checks the integer endpoint and the weighted composition.

46. The multiplicity stop and the retained integer zeros

Retain the original contradiction and the actual stage families of Theorem 10.140. The remaining alternative to the geometric stop is $I_1 < I^*$. Write $a = r+1, c = c_5, H = \eta^a, Q = qH, D = A + \nu \ln q$. In this section $S_1 = S_{I_1}, T_1 = T_{I_1}$: the subscript denotes the stopping stage, which need not be the first stage. We first determine the exact last integer derivative level and the subsequent depth. We also prove what a family with the required final zeros would imply. The terminal integer zeros themselves are constructed in Theorem 10.144 below.

Lemma 10.142 (exact parameters of the fixed-order branch). There is a unique integer r_1 with $0 \le r_1 \le r$ and

$$1 \le \frac{c\eta^{r_1} T_1}{a} < \frac{1}{\eta}. \tag{10.394}$$

Set $T_\flat = \eta^{r_1+1} T_1$, $R = q\lfloor q^{r_1} S_1 \rfloor$, and

$$I_2 = \left\lfloor \frac{3D - I_1 \ln Q}{\ln q} \right\rfloor + 1, \qquad I_3 = I_1 + I_2. \tag{10.395}$$

Then $I_1 \ge 1, I_2 \ge 1$, $\eta \le cT_\flat/a < 1$, and $Q^{I_1} q^{I_2} > \exp(3D)$, including a possible integer endpoint in the quotient defining I_2.

Proof. Lemma10.70 gives $T \ge g_4$. In (10.284), $F_r \ge 9$ and $(7/2)^{r-2} \ge 1$, so Lemma10.104 gives $g_4 > (9000/29)a > 6a$. Thus $cHT/a > 1$, using $c > 1/2, H > 1/3$. Thus stage1 is allowed and $I_1 \ge 1$. The definition and maximality of I_1 give $cT_1/a \ge 1$ and $cHT_1/a < 1$. The positive sequence $c\eta^j T_1/a$ decreases strictly to zero because $0 < \eta < 1$. Its last term at least one has an integer index r_1. The next term is less than one, proving the stated two-sided inequality. Since the term at $j = a$ is less than one, $r_1 < a$, hence $r_1 \le r$. Strict decrease proves uniqueness. Multiplying by η gives the assertion for $T_\flat$.

From $I_1 < \lfloor 3D/\ln Q \rfloor + 1$ and integrality, $I_1 \le \lfloor 3D/\ln Q \rfloor \le 3D/\ln Q$. Thus the numerator defining I_2 is nonnegative and $I_2 \ge 1$. The floor plus one is strictly larger than its real argument, even when that argument is an integer. Multiplication by $\ln q > 0$ and exponentiation prove $Q^{I_1} q^{I_2} > \exp(3D)$. □

Proposition 10.143 (the final fixed-order zeros force an additive polynomial contradiction). Suppose an admissible stage family at I_3 retains a nonzero coefficient subvector of the original kernel, the selected derivations and integral normalized torus exponents of Proposition10.31, with widths bounded by $q^{-I_3} D_j$, and has its prepared zeros for every integer $|s| \le R$ through $\lfloor T_\flat \rfloor$. Then this family cannot exist. In fact, every torus coordinate degree of its ordinary polynomial is less than $145/14336 < 1$, and its number of distinct integer roots exceeds $(171/58)kL$.

Proof. The first inequality in Lemma10.142 gives

$$q^{I_3} H^{I_1} > \exp(3D). \tag{10.396}$$

For the following degree estimate use $T_1 < a/(cH)$, the ratio of the two original parameters in (10.186), and $c_2 qP/(e\theta) \ge 7/2$:

$$\begin{aligned} q^{\nu-I_3}D_j &< q^{\nu}H^{I_1}D_j\exp(-3D) \\ &= q^{\nu}T_1\frac{D_j}{T}\exp(-3D) \\ &< \frac{q^{\nu}e\theta t}{cc_2qrPHd\sigma_j}\exp(-3D) \\ &\le \frac{2}{7crH}\frac{t}{d\sigma_j}\exp(-3A). \end{aligned} \tag{10.397}$$

In the last line $q^{\nu}\exp(-3D)=\exp(-3A-2\nu\ln q)\le\exp(-3A)$, with $\nu\ge 0$. Each original base is non-torsion, so the one-base case of Theorem10.18 gives $d\sigma_j>1/(87d^2)$, as proved in Theorem10.100. Also $c>1/2, r\ge 2, H>1/q\ge 1/3, t\le A$. Therefore

$$q^{\nu-I_3}D_j<\frac{522}{7}d^2A\exp(-3A). \tag{10.398}$$

Now $A\ge g_1=4+\ln(ad)>5$. Hence $d^2\exp(-2A)\le\exp(-8)/a^2$. The function $y\exp(-y)$ decreases for $y\ge 5$, because its derivative is $(1-y)\exp(-y)<0$. Using $a\ge 3$ and $\exp(1)>2$, proved in Theorem10.100, gives

$$2q^{\nu-I_3}D_j<\frac{5220}{7\cdot 9\cdot 2^{13}}=\frac{145}{14336}<1. \tag{10.399}$$

The normalized torus exponent differences in Proposition10.31 are integers of absolute value at most $q^{\nu-I_3}D_j<145/28672<1$. Every such integer is zero. Thus the nonzero ordinary polynomial constructed there depends only on the additive coordinate X and has degree at most $D_{\mathrm{a}}=kL$. Equivalently, an ordinary polynomial whose every torus degree is less than one is independent of those variables. The remaining coefficient polynomials cannot cancel: the distinct shifted powers in Lemma10.29 are independent, as used in Proposition10.31.

It remains to count its roots. We have $S_1\ge S>58$, and $r_1\ge 0$, so

$$2R+1>2q(S_1-1)+1>\frac{57}{29}qS_1. \tag{10.400}$$

Put $\lambda=1+g_5^{-1}<5/4$. The parameter bounds proved in Lemma10.70 and Theorem10.100 give $D_{\mathrm{a}}\le\lambda S\mathcal{D}/(c_1c_4dD)$. Use $S_1/S=T/T_1$, the formula for T, and $T_1<a/(cH)$:

$$\begin{aligned} \frac{2R+1}{D_{\mathrm{a}}} &> \frac{57}{29}\frac{qc_1c_4dD}{\lambda\mathcal{D}}\frac{T}{T_1} \\ &= \frac{57}{29}\frac{q^2ac_4dD}{\lambda e\theta tT_1} \\ &> \frac{57}{29}\frac{q^2cc_4H}{\lambda\theta}\frac{d}{e}\frac{D}{t} \\ &> \frac{57}{29}\frac{2(1/2)(15/4)}{(5/4)\,2}=\frac{171}{58}>1. \end{aligned} \tag{10.401}$$

Here $qH>1, c_4>15/4, d\ge e, D\ge t$, and $\theta\le 2$ are the original proved bounds. Prepared vanishing includes total order zero, since $T_\flat>0$; Proposition10.31 therefore gives ordinary zeros at the $2R+1$ distinct integers. A nonzero polynomial of degree at most D_{a} has at most that many roots: divide successively by $X-s$ at distinct roots, or use the root-count proof already in Theorem10.98. The strict count above is impossible. □

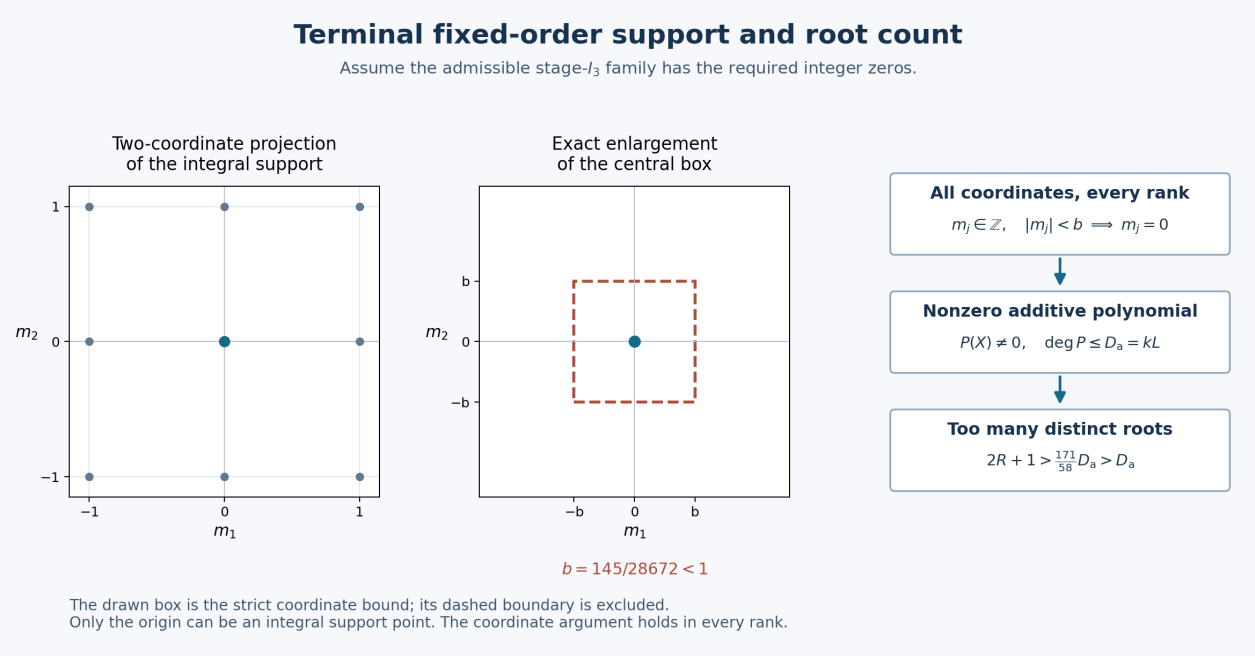


Figure 10.34. A two-coordinate projection of the strict integral-support bound in Proposition 10.143, with its exact enlargement. Every coordinate satisfies $|m_j| < 145/28672$; the dashed boundary is excluded. Only the origin can be integral. The right-hand arrows give the resulting nonzero additive polynomial of degree at most kL, and the excessive number of distinct roots. The hypothesis is the admissible stage-I_3 family with the stated integer zeros. Figure program. Human-source context: the final stopping alternative in Section 7 of Yu's freely accessible paper. The coordinate bound, additive nonzeroness and root count are proved above.

Completing the integer zeros at the multiplicity stop

The previous integer theorem assumed that the next full contraction still had enough derivatives. At the multiplicity stop that assumption can fail. We use the derivatives that remain at the last integer level, and reduce the number of Hermite conditions on a short outer interval. The output radius and derivative order remain the original ones.

Theorem 10.144 (the original terminal integer zero block). Assume the original contradiction and initial kernel of Theorem 10.140. Suppose $I_1 < I^*$, and use $S_1, T_1, r_1, T_\flat, R$ of Lemma 10.142. The actual stage-I_1 family produced by Theorem 10.140 has all its prepared zeros at

$$|s| \leq R = q\lfloor q^{r_1} S_1 \rfloor, \qquad |\boldsymbol{t}| \leq \lfloor T_\flat \rfloor. \tag{10.402}$$

This holds in every original prime, field and rank case. The coefficient minimum, coefficient subvector and original phase condition are retained.

Proof. Theorem 10.140 supplies the actual stage family and its zeros at $|s| \leq \lfloor qS_1 \rfloor$ through order $\lfloor \eta T_1 \rfloor$. When $r_1 = 0$, these include the stated zeros, since $q\lfloor S_1 \rfloor \leq \lfloor qS_1 \rfloor$. Assume henceforth $r_1 \geq 1$. Write $a = r + 1, c = c_5, H = \eta^a$, and retain $b_r, S_-, Y_-, \bar{\lambda}, \mathcal{R}_r(j)$ from Section 40. The stopping inequalities are $c\eta^{r_1} T_1/a \geq 1$ and $cHT_1/a < 1$. Also $I_1 \leq 3(A + \nu \ln q)/\ln(qH)$, by Lemma 10.142.

First consider an integer level $1 \leq j \leq \min\{r_1, r-1\}$. Suppose the zeros at radius $\lfloor q^j S_1 \rfloor$ through order $\lfloor \eta^j T_1 \rfloor$ have been obtained. For a target order at most $\lfloor \eta^{j+1} T_1 \rfloor$, choose $\mu = \lfloor c\eta^j T_1/a \rfloor + 1$ at every input node. The floor inequality used in Theorem 10.126 supplies all these ordinary jets by Theorem 10.118. Put $N_* = (2\lfloor q^j S_1 \rfloor + 1)\mu$. The weaker stopping condition suffices for the following upper bound:

$$\mu \leq \frac{c(\eta^j + \eta^{r_1})T_1}{a} \leq \frac{2cT_1}{a}. \tag{10.403}$$

Indeed the added one is at most $c\eta^{r_1}T_1/a$. The coefficient and separation estimates in the proof of Lemma 10.123 use only $S_1 = S\eta^{-aI_1}$, $T_1 = T\eta^{aI_1}$, and the original height bounds. They still give $\beta = v_p(b_n)$, $\max\{B, \beta - \vartheta, 0\} \le \eta^{-aI_1}H_1/\ln p$, and the same required projection depth $U - \beta > 1$. Using $S_1T_1 = ST$ and $c_1W_*ST\theta = qa$, the full input sum, multiplied by c_1W_*/q^a, is bounded above by

$$\frac{c_1W_*}{q^a}\left[N_*\theta + \beta + (\mu - 1)\max\{B, \beta - \vartheta, 0\}\right] \le \frac{4c}{q} + \frac{2c}{q^rS_-} + \frac{c_1}{7ab_rq^a} + \frac{c}{c_3Y_-aq^a}. \tag{10.404}$$

Here the first term uses $j \le r - 1$, and both powers of η are at most one. At odd primes insert $c \le 14/25, q = 2, S_- > 58, c_1 < 3, b_r \ge 7200/29, c_3Y_- > 19/10$. The resulting rational bound is less than $23/20 < c_1$. In V use $c \le 827/1000, q = 3, c_3Y_- > 19/20$; the bound is less than $9/8 < c_1$. These exact substitutions are included in the certificate below. Dropping the nonnegative term $(D_{\mathrm{a}} - u)\theta$ from the left side of the input criterion therefore proves its strict inequality for every additive order $u \le D_{\mathrm{a}}$. Larger additive orders vanish identically.

Theorem 10.119 and the scalar identity (10.327) give analytic precision at least $N_*\theta$ in the original quantity $v_p(\Phi) + G_u - \delta$. Its strict lower bound (10.330) follows from $\mu > c\eta^jT_1/a$; it does not use the stronger next-contraction condition. All integer target values lie in the original field K by the retained phase identity (10.151). Lemma 10.122 applies at the same depth I_1, with $j \ge 1$, and charges the full q-denominator. Thus their original arithmetic bound is $\mathcal{R}_r(j)$. Lemma 10.125 gives $c_1W_*N_*\theta > \mathcal{R}_r(j) + 1/200$. Corollary 10.120 forces each value to zero, with the actual degree and local degrees of K. Successive applications complete all required levels unless $r_1 = r$. In that last case they give the full radius $\lfloor q^rS_1 \rfloor$ through order $\lfloor \eta^rT_1 \rfloor$.

It remains to extend this last level when $r_1 = r$. Put $z = c\eta^rT_1/a$, so $1 \le z < 1/\eta < 2$. The last inequality follows from $\eta \ge 1 - 827/3000 > 1/2$. The same floor inequality gives $\lfloor \eta^rT_1 \rfloor - \lfloor HT_1 \rfloor \ge \lfloor z \rfloor = 1$. Consequently every input node provides a value and its first ordinary jet for every target selected order at most $\lfloor HT_1 \rfloor$. We retain the extra jet only in an inner interval. Set

$$R_b = \lfloor q^rS_1 \rfloor, \qquad R_0 = \lfloor 24R_b/25 \rfloor, \qquad \mu(s) = \begin{cases} 2, & |s| \le R_0, \\ 1, & R_0 < |s| \le R_b. \end{cases} \tag{10.405}$$

Since $R_b > q^rS_1 - 1 > 231$, these are two strictly nested nonempty intervals. Their Hermite mass is exactly

$$N_* = 2R_b + 2R_0 + 2, \qquad \frac{98}{25}R_b < N_* \le \frac{98}{25}R_b + 2. \tag{10.406}$$

The inequalities follow respectively from $R_0 > 24R_b/25 - 1$ and $R_0 \le 24R_b/25$. Lemma 10.94 and Theorem 10.119 apply with $M = 2, E = 0$. In particular the outer nodes lose one separation factor in their cardinal polynomial; that loss is already included in $(M - 1)B$. There is no deletion of nodes and no q-deleted cardinal loss.

Because $1 \le z$, the upper bound on N_* and the same coefficient and separation bounds give the following upper bound on the entire input sum, after multiplication by c_1W_*/q^a:

$$P_r = \frac{98}{25}c\eta^r + \frac{2c\eta^r}{q^rS_-} + \frac{c_1}{7ab_rq^a} + \frac{c\eta^r}{c_3Y_-aq^a}. \tag{10.407}$$

For the last term use $M - 1 = 1 \le z$; this pays the outer cardinal loss and all retained jet costs, rather than dropping them. On the other side $T_1 < a/(cH)$ and the strict lower mass inequality yield

$$\frac{c_1 W_* N_* \theta}{q^a} > G_r := \frac{98}{25} cH \left(1 - \frac{1}{q^r S_-}\right). \tag{10.408}$$

The upper arithmetic bound for a nonzero integer target at $|s| \le \lfloor q^{r+1} S_1 \rfloor$ and order $\lfloor HT_1 \rfloor$ is the corrected $\mathcal{R}_r(r)$ of (10.332). Its full normalized cost is

$$C_r = \frac{1}{q^a}\left[A_r + c_1\left(\frac{1}{300} + \frac{1}{273 \cdot 999 b_r}\right) + \frac{\bar{g}_9 H}{c_3 Y_-} + \frac{\bar{\lambda}}{c_4}\left(1 + \Omega_+ + \frac{\ell_q}{5}\left(r + \frac{1}{q-1}\right)\right)\right] + \frac{1}{c_2}. \tag{10.409}$$

Here $\ell_q = 347/500$ at $q = 2$ and $1099/1000$ at $q = 3$ are the proved logarithm upper bounds. In particular the final $1/c_2$, the full depth-denominator indicator, the Euler/lcm term, and the separate scalar remainder all survive.

We now prove the needed strict inequalities for every rank, using the coupled field rows of Lemma 10.121. For ranks 2–7 substitute the exact decimal rationals of that table and the rank band for c into these three formulas. For ranks at least eight, c is constant within each case. Put $\eta_8 = 1 - c/9, H_8 = \eta_8^9, S_8 = S_-(8), b_8 = b_r(8)$. The following bounds hold simultaneously for all such ranks:

$$\begin{aligned} P_r &\le \frac{98}{25} c\eta_8^8 + \frac{2c}{q^8 S_8} + \frac{c_1}{7 \cdot 9 b_8 q^9} + \frac{c}{9 c_3 Y_- q^9}, \\ G_r &\ge \frac{98}{25} c H_8 \left(1 - \frac{1}{q^8 S_8}\right), \\ C_r &\le \frac{O_8}{q^9} + \frac{\bar{\lambda}\ell_q(8 + 1/(q-1))}{5 c_4 q^9} + \frac{1}{c_2}. \end{aligned} \tag{10.410}$$

In the last line O_8 is the bracket before the linear-in-r term in the formula for C_r, evaluated at rank eight with $C_r^\sharp$ and H enlarged to one and $\bar{g}_9 = 107/103$. This is exactly the `high=True` output of the preceding integer certificate. To verify the enclosing bounds, b_r, S_- increase with rank, while $\epsilon_r, \bar{g}_{12}$ decrease; $C_r^\sharp < 1$ by Lemma 10.121 and $\bar{g}_9 = 107/103$ for every rank at least eight. The positive constant terms divided by q^{r+1} decrease. So does r/q^{r+1}, since its successive ratio is $(r+1)/(qr) < 1$.

For the remaining powers, $(1 - c/y)^y$ increases for $y \ge 3$, as proved in Lemma 10.122. The function $(1 - c/y)^{y-1}$ decreases: with $w = c/y$, its logarithmic derivative is

$$\ln(1 - w) + \frac{w}{1 - w} - \frac{w}{y(1 - w)} \le \frac{w^2}{2(1 - w)^2} - \frac{w}{y(1 - w)} < 0. \tag{10.411}$$

The non-strict bound integrates $t/(1-t)^2$ from zero to w. The strict sign uses $c < 2(1 - w)$, which follows from $c \le 827/1000 < 2(1 - c/3)$. Thus $H \ge H_8$ and $\eta^r \le \eta_8^8$; replacing the remaining numerator powers by one only enlarges the two input remainders. This proves all the uniform bounds, without inferring them from sampled ranks.

The exact rational certificate gives the following strict lower bounds, each divided by 1000. Each row includes its six finite ranks and its one uniform bound for every rank at least eight.

Original case	$c_1 - P_r$	$G_r - C_r$
I.1	21/1000	493/1000
I.2	21/1000	489/1000
II	20/1000	492/1000
III.1	20/1000	487/1000

Original case	$c_1 - P_r$	$G_r - C_r$
III.2	19/1000	481/1000
IV, first prime branch	19/1000	490/1000
IV, larger prime branch	26/1000	540/1000
V	872/1000	475/1000

All entries are exact rational substitutions with downward-rounded lower bounds; the editable certificate retains their exact numerators and denominators. In particular $P_r < c_1$ and $G_r > C_r$. The first inequality proves the complete input criterion with $W_*U = q^a$. The second gives strictly more analytic precision than the full arithmetic bound in the original field K, so Corollary 10.120 forces every target value to zero.

We have therefore obtained the radius $\lfloor q^{r_1+1}S_1 \rfloor$ and derivative order $\lfloor \eta^{r_1+1}T_1 \rfloor$ in every case, including $r_1 = r$. Finally $q\lfloor q^{r_1}S_1 \rfloor \le \lfloor q^{r_1+1}S_1 \rfloor$, so these include exactly the required original terminal integer zeros. □

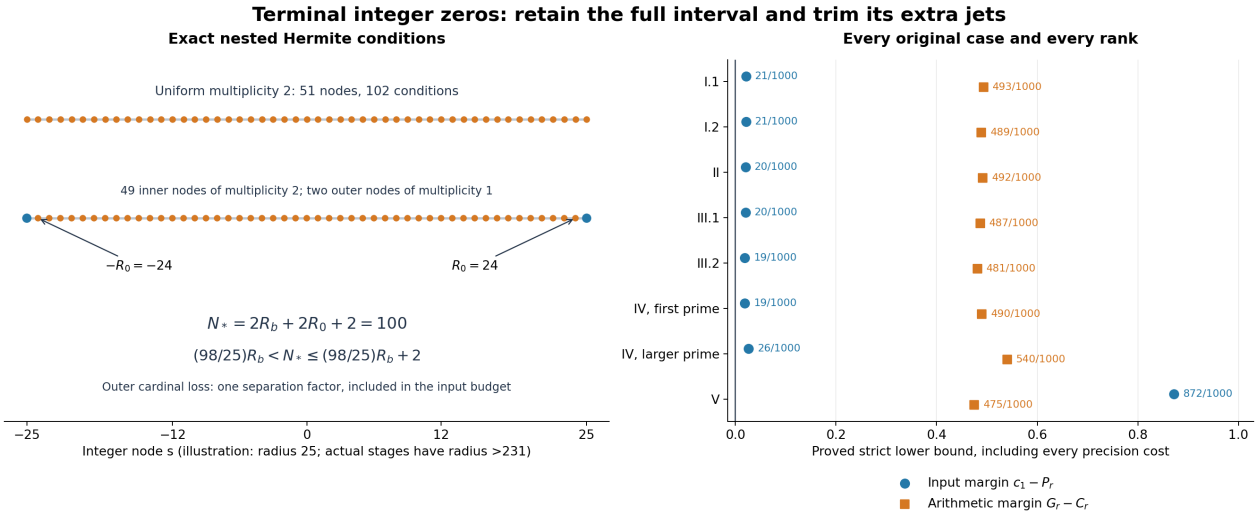


Figure 10.35. Left: an illustrative nested interval with $R_b = 25$, $R_0 = 24$. Its 49 inner nodes carry multiplicity two and its two outer nodes carry multiplicity one, so $N_* = 100$ instead of 102. This small example illustrates the exact cardinal construction; actual terminal stages satisfy $R_b > 231$. For every integer radius the retained mass lies strictly above $(98/25)R_b$ and at most two above that bound. Right: the proved lower margins in the table, for all original rank and field cases. The input margins include the coefficient minimum and the outer cardinal loss; the arithmetic margins include the full global degree, denominator, Euler/lcm and scalar costs. Figure program. Human-source context: the terminal integer range in Lemma 5.5 of Yu's freely accessible paper. The complete nested interpolation and both strict comparisons are proved above.

Full-node interpolation at the fixed derivative order

Proposition 10.145 (the full fixed-order analytic input). Use the original multiplicity-stop parameters of Lemma10.142, and the original explicit depth choices in Section26. Put $O = \lfloor T_\flat \rfloor$, $R = q\lfloor q^{r_1}S_1 \rfloor$, $N = 2R + 1$. Suppose an admissible stage family at any depth $I \ge I_1$ retains the initial coefficient bound, minimum coefficient and phase, and has all its prepared zeros at the integers $|s| \le R$ through selected order O. Under the original contradiction, for every selected order $\boldsymbol{t}$ with $|\boldsymbol{t}| \le O$ and additive order $u \le D_{\mathrm{a}}$,

$$v_p(\Phi^{(I)}(x;\boldsymbol{t})) + G_u - \delta \ge N\theta \qquad (x \in \mathbb{Z}_p \subset \mathbb{Q}_p). \tag{10.412}$$

In particular this holds at each fractional argument $x = s/q$, with $s \in \mathbb{Z}$, since $q \ne p$. No additional ordinary jet beyond the given selected order is required. Larger additive orders vanish by polynomial degree. This is a local analytic precision assertion.

Proof. Normality and projection depth are those of Lemma10.91 and Theorems10.117–10.119. Multiplication of the torus slopes and additive argument by q^{-I} does not change their p-adic depths, because $q \neq p$. The retained coefficient bound and minimum provide the same $\beta = v_p(b_n)$, U, and δ. Thus these results apply at the given depth without its being inside the earlier geometric range.

Apply the full-node case of Theorem10.119 with $M = 1, E = 0$. Every node has multiplicity one, so the available prepared zero of the target order itself is sufficient. The complete integer-interval cardinal polynomials are integral on $\mathbb{Z}_p$, by Lemma10.94. Their inverse Taylor polynomials here have degree zero. Consequently there is neither a q-deleted loss nor a higher-jet separation loss. By (10.319), $K_u(1, B) = M_u(0) = \gamma_u$, and (10.327) at $M = 1$ gives $G_u + \gamma_u \geq (D_{\mathrm{a}} - u)\theta$. The full precision therefore is

$$\mathcal{P}_u \geq \min\{N\theta,\ U - \beta + (D_{\mathrm{a}} - u)\theta\}. \tag{10.413}$$

It remains to prove $U - \beta > N\theta$ for every original case and rank. We retain the actual large initial multiplicity instead of using only the coarser lower bound on $\mathcal{D}$. From the formula for S, $\beta \ln p \leq h \leq H_1$, and $ef \leq d$,

$$c_1 W_* \beta \leq \frac{f}{c_3 d\theta T} \leq \frac{1}{c_3 YT} \leq \frac{1}{c_3 Y_- g_4}, \qquad Y = e\theta. \tag{10.414}$$

Here $T \geq g_4$ is Lemma10.70, and the simultaneous field bounds of Lemma10.121 give $Y \geq Y_-$. The endpoint condition $c\eta^{r_1} T_1/a \geq 1$, together with $c_1 W_* S_1 T_1 \theta = qa$, bounds the node input by

$$\frac{c_1 W_* N\theta}{q^a} \leq 2c\eta^{r_1} q^{r_1+1-r} + \frac{c\eta^{r_1}}{q^r S_1}. \tag{10.415}$$

Since $q\eta > 1$ and $r_1 \leq r$, the first term is at most $2cq\eta^r$. The other stopping inequality gives $T_1 < a/(cH)$. Using $S_1/S = T/T_1$, $T \geq g_4$ and $H > 1/3$ therefore gives $S_1 > Sg_4 cH/a > S_- g_4 c/(3a)$. The second term is less than $3a/(q^r S_- g_4)$. Thus a sufficient exact input bound is

$$B_r := 2cq\eta^r + \frac{3a}{q^r S_- g_4} + \frac{1}{c_3 Y_- g_4 q^a} < c_1. \tag{10.416}$$

This bound charges the coefficient minimum in full. It follows for all ranks from the following explicit lower bound on the actual g_4. Write $F_r = r^r a^r/(r!)^2$. The formulas (10.185) give

$$g_4 \geq L_r := \frac{2c_0 c_4 qa}{\rho\widehat{\vartheta}} a_*^r F_r v_0, \tag{10.417}$$

where the simultaneous original choices are

Case	ρ	$\widehat{\vartheta}$	a_*	v_0
I.1	58	3/2	7	1
I.2	17	3/2	7	5
II	58	5/4	7	1
III.1	58	1	7/2	5
III.2	17	1	7/2	5
both IV branches	58	7/6	7	1
V	58	2	26/3	1

For I.2 and III the definition of g_4 retains its g_1 factor, which exceeds five; using $v_0 = 5$ is a lower bound. In the other rows that factor cancels or can be enlarged, so $v_0 = 1$ suffices. The values of ρ follow from the same degree cases as Lemma10.121; the depth and a_* choices are exactly (10.184) and its ensuing explicit upper depths. Thus this uses the actual field row, not independently combined extrema.

For ranks 2–7, substitute L_r for g_4, the original rational constant row and its exact c band into B_r. For ranks at least eight, c is constant and η^r decreases by the derivative proved in Theorem10.144. The ratio $F_{r+1}/F_r > 4$ is proved in Lemma10.104; consequently $L_{r+1}/L_r > 14(r+2)/(r+1)$. Both a/L_r and $1/L_r$ decrease. Since S_- and q^r increase, each of the two positive remainders also decreases. Therefore the rank-eight substitution bounds every larger rank.

The exact rational certificate `fixed_order_input_endpoints.py` gives these strict lower bounds for $c_1 - B_r$, including each case's six finite rows and one uniform row:

Case	Strict input margin
I.1	77/1000000
I.2	39/1000000
II	80/1000000
III.1	89/1000000
III.2	23/1000000
IV, first prime branch	56/1000000
IV, larger prime branch	67/1000000
V	54/1000000

Thus $c_1 W_*[N\theta + \beta] < c_1 q^a = c_1 W_* U$, proving the required strict input. Since $(D_{\mathrm{a}} - u)\theta \geq 0$, the displayed minimum is at least $N\theta$. This proves the proposition. □

The argument includes the source's zero-additional-jet convention: one value condition at each node corresponds to $M = 1$ in our Hermite notation. This distinction prevents accidentally requesting an extra derivative or discarding the coefficient cost.

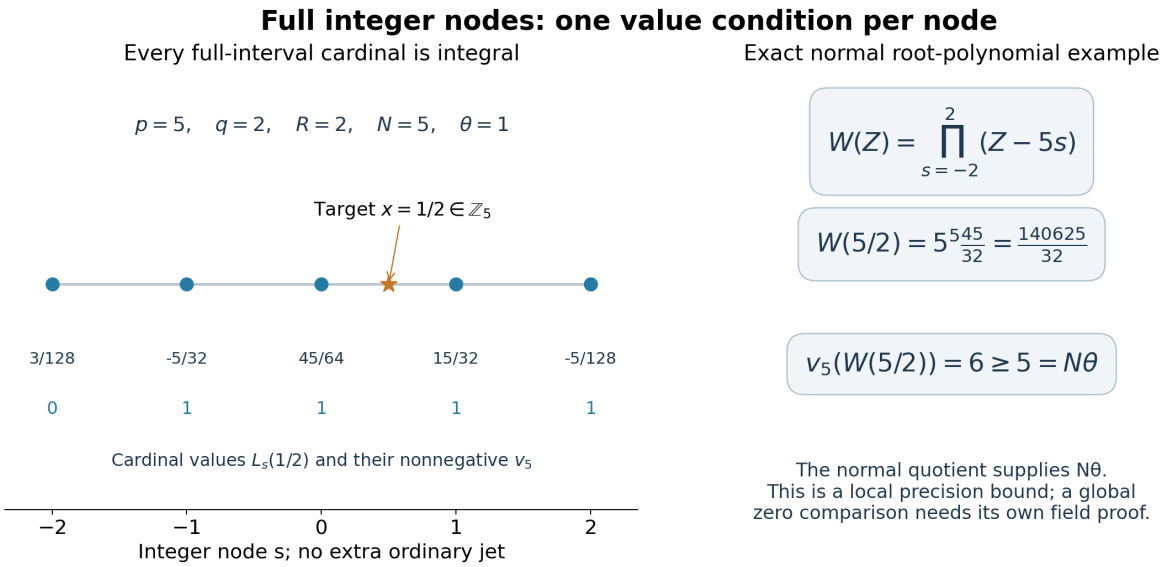


Figure 10.36. An exact illustration at $p = 5, q = 2, R = 2, \theta = 1$. The five integer nodes each carry one value condition, with no additional ordinary jet. At $x = 1/2 \in \mathbb{Z}_5$ their cardinal values are $3/128, -5/32, 45/64, 15/32, -5/128$, with nonnegative valuations $0, 1, 1, 1, 1$. Right: the monic normal root polynomial $W(Z) = \prod_{s=-2}^{2}(Z - 5s)$ has $W(5/2) = 140625/32$, of valuation six, exceeding its normal bound $N\theta = 5$. This example illustrates the full-node interpolation mechanism; it is not a complete choice of logarithmic contradiction parameters. Proposition 10.145 proves the full original input bound, and Solution 57 checks the example. Figure program and all-rank input certificate. Human-source context: the zero-additional-jet argument in Section 7 of Yu's freely accessible paper. The normality, cardinal construction and simultaneous input bounds are proved above.

Closing successor integers without lowering the derivative order

Theorem 10.146 (the full fixed-order successor integer block). Retain the original multiplicity-stop parameters of Lemma10.142, including $I_1 < I^*, R = q\lfloor q^{r_1} S_1 \rfloor$ and $O = \lfloor T_\flat \rfloor$. Suppose $I_1 + 1 \le J \le I_3$, and an admissible stage-J coefficient family retains the initial coefficient bound and minimum, the phase condition, and all prepared zeros through total order O at the integers $|s| \le R$ prime to q. Then that same family has every prepared zero through order O at all integers $|s| \le R$.

The proof uses one value condition at each q-deleted node, without any additional ordinary jet, and keeps the full q-denominator at depth J.

Proof. Put $a = r + 1, c = c_5, H = \eta^a, Q = qH, D = A + \nu \ln q$, and use $C, Y_-, \Omega_+, S_-, b_r, A_r, \bar{g}_9$ of Section40. Let L_r be the explicit lower bound on the initial g_4 in (10.417). Thus $T \ge g_4 \ge L_r$, while $cT_1/a \ge 1 > cHT_1/a$. Let m_r be the largest nonnegative integer satisfying

$$\frac{cL_r H^{m_r}}{a} \ge 1. \tag{10.418}$$

Then $I_1 \ge m_r$, since otherwise strict decrease would contradict $cTH^{I_1+1}/a < 1$. In particular $Q^{I_1} > q$. Here is a uniform proof of this last fact. Within each constant band H increases with rank, as proved in Lemma10.122. At the three band starts, exact substitution gives $H > 11/20$ for odd p and $H > 5/12$ in V. Lemma10.142 gives $T > (9000/29)a$. At odd p, $c > 1/2$, and $(11/20)^8(4500/29) > 1$; hence $I_1 \ge 8$, so $Q^{I_1} > (11/10)^8 > 2 = q$. In V, $c > 3/4$, and $(5/12)^6(6750/29) > 1$; hence $I_1 \ge 6$, so $Q^{I_1} > (5/4)^6 > 3 = q$. The sharper m_r below will retain the actual initial multiplicity rather than only these coarse depths.

There are exactly $N_* = 2(q-1)\lfloor q^{r_1} S_1 \rfloor$ q-deleted nodes. For each target prepared index of order at most O and additive order $u \le D_{\mathrm{a}}$, Theorem10.119 with $M = 1, E = 1$ gives

$$\mathcal{P}_u \ge \min\{N_*\theta,\ U-\beta+(D_{\mathrm{a}}-u)\theta-B\},\qquad B=\lfloor \log_p(2R+1)\rfloor. \tag{10.419}$$

Indeed $K_u(1,B)=\gamma_u$ and $G_u+\gamma_u\ge (D_{\mathrm{a}}-u)\theta$, by (10.327). The single deleted-cardinal loss is B; there is no higher-jet inverse loss. Normality and projection apply at J because q is a p-adic unit, just as in Proposition10.145. Beyond additive degree D_{a}, the value is identically zero.

We compare both entries with the arithmetic bound for that same normal quantity in the original field K. The phase identity places every integer prepared value in K. Set

$$\begin{aligned} E_r &= c_1\left(\frac{1}{300}+\frac{1}{273\cdot 999 b_r}\right),\\ K_r &= A_r+E_r+\frac{\bar g_9 a}{cc_3Y_-L_r}+\frac{\bar\lambda}{c_4}\left[1+\Omega_+ +\frac{3\ell_q}{\mathcal{L}(Q)}+\frac{(1+1/(q-1))\ell_q}{5}\right],\\ F_r(x) &= K_r+\frac{\bar\lambda x\ell_q}{5c_4}+\frac{q^x}{c_2Q^{m_r}}. \end{aligned} \tag{10.420}$$

Here $\bar\lambda=189/188$, $\ell_2=347/500, \ell_3=1099/1000$, and $\mathcal{L}(Q)>0$ is the five-term positive-series lower bound

$$\mathcal{L}(Q)=2\sum_{j=0}^{4}\frac{z^{2j+1}}{2j+1},\qquad z=\frac{Q-1}{Q+1};\qquad \ln Q>\mathcal{L}(Q). \tag{10.421}$$

The logarithm series and its positivity are proved in Solution34. For a nonzero target V, Corollary10.116 and its same-order normal bound therefore give

$$c_1W_*[v_p(V)+G_u-\delta]<F_r(r_1). \tag{10.422}$$

To verify every term, the coefficient bound is A_r, as in (10.332). The complete Euler remainder is at most E_r, by (10.310) and $(r-1)/Z<1/(273b_r)$. Since $O<T_\flat<a/c$ and $T\ge L_r$, the main Euler term is at most $\bar g_9 a/(cc_3Y_-L_r)$. The additive factorial estimate in Lemma10.122 applies directly: its physical radius satisfies

$$q^{-J}R<q^{r_1+1}SQ^{-I_1}q^{-(J-I_1)}\le q^{r_1}SQ^{-I_1}<q^{r_1-1}S. \tag{10.423}$$

Enlarging this radius to $q^{r_1+1}S$ in that lemma's factorial proof bounds the scalar by $\bar\lambda[1+r_1\ell_q/5]/c_4$; this step uses no geometric-depth absorption. The normal term G_u is at most $D_{\mathrm{a}}(\theta+1/(p-1))$, giving the separate $\bar\lambda\Omega_+/c_4$. The complete denominator is $\Xi_{J,0,u}\le D_{\mathrm{a}}[J+1/(q-1)]$. Lemma10.142 gives $I_1\le 3D/\ln Q$ and

$$I_3\ln q\le 3D+I_1(\ln q-\ln Q)+\ln q\le \frac{3D\ln q}{\ln Q}+\ln q. \tag{10.424}$$

Together with $D>5, J\le I_3$, this gives precisely the remaining depth and constant-denominator terms in K_r. Finally the full torus cost is at most $q^{r_1}/(c_2Q^{I_1})$, hence at most $q^{r_1}/(c_2Q^{m_r})$. Nothing has been dropped at primes above q, or from the normal scalar or coefficient minimum.

For the first analytic entry, $T_1<a/(c\eta^{r_1+1})$, $c_1W_*ST\theta=qa$, and $ST=S_1T_1$ give

$$c_1W_*N_*\theta>2c(q-1)(q\eta)^{r_1+1}-\frac{2q(q-1)a}{S_-L_r}. \tag{10.425}$$

The floor loss follows from $c_1 W_* \theta = qa/(ST)$, $S \geq S_-, T \geq L_r$. The difference of this lower bound and $F_r(r_1)$ increases for every real $0 \leq r_1 \leq r$. In fact $Q^{m_r} > q$ and $\eta^{x+1} \geq H$ give the derivative lower bound

$$q^{x-1}\left[2c(q-1)q^2 H \ln(q\eta) - \frac{\ell_q}{c_2}\right] - \frac{\bar{\lambda}\ell_q}{5c_4} > 0. \tag{10.426}$$

For odd p, use $c \geq 5267/10000, H > 11/20$, $\ln(q\eta) > 12/25, c_4 > 18, c_2 = 7/4, q = 2$. The bound exceeds 1/3, since $q^{x-1} \geq 1/q$. For V use $c > 3/4, H > 5/12, \ln(q\eta) > 3/4, c_4 > 15/4, c_2 = 13/9, q = 3$; the bound exceeds two. Indeed $\eta \geq 61/75$ at odd p and $\eta \geq 2173/3000$ in V; the first two terms of the positive logarithm series give $\ln(122/75) > 12/25$ and $\ln(2173/1000) > 3/4$. The exact endpoint substitutions below also check $Q^{m_r} > q$. Thus it suffices to compare at $r_1 = 0$.

For the second entry, $R \leq q^{r+1} S_1$. The separation argument of Lemma10.123 gives $\ln(2R+1) < H_1 H^{-I_1} + \ln q$. Since $\ln q / H_1 < 1/3$, and $T_1 \geq a/c$,

$$c_1 W_* B < \frac{4c}{3c_3 Y_- a}, \qquad c_1 W_* \beta \leq \frac{1}{c_3 Y_- L_r}. \tag{10.427}$$

Consequently its normal input is greater than or equal to the conservative bound $c_1 q^a - 1/(c_3 Y_- L_r) - 4c/(3c_3 Y_- a)$. For its opposite arithmetic envelope, $F_r(x)$ increases, and $Q^{m_r} > q$; hence

$$F_r(r_1) < K_r + \frac{q^{r-1}}{c_2} + \frac{\bar{\lambda} r \ell_q}{5c_4}. \tag{10.428}$$

For ranks2–7, choose the largest integer m_r for which $cL_r H^{m_r}/a \geq 1$, and substitute the coupled original row in these formulas. The exact rational certificate gives the following strict margins; every entry is divided by1000.

Case	First-entry margin at $r_1 = 0$	Second-entry margin
I.1	38	8808
I.2	84	8769
II	123	8776
III.1	87	8723
III.2	55	8650
IV, first prime branch	105	8801
IV, larger prime branch	232	9035
V	1405	60699

The same table also includes one analytic enclosure for every rank at least eight. Here c is constant, H and Q increase, while $L_{r+1}/L_r > 14(r+2)/(r+1)$, as proved in Proposition10.145. Therefore L_r/a and S_- increase. The same m_8 is available for every larger rank, and $Q^{I_1} \geq Q_8^{m_8} > q$. Bound A_r by its rank-eight envelope with $C_r^\sharp$ replaced by one. All its other positive components decrease; $\bar{g}_9 = 107/103$. Thus $K_r \leq K_8^+$, using the rank-eight lower Q in $\mathcal{L}$, and its lower L_r/a. The mass at $r_1 = 0$ increases, and its floor loss decreases. The rank-eight mass comparison therefore encloses every larger rank.

For the remaining input comparison, divide both bounds by q^{r+1}. The constant torus term becomes $1/(q^2 c_2)$; the remaining positive constant costs decrease after division, and r/q^{r+1} decreases. The losses involving $1/L_r$ and $1/a$ also decrease. Hence its rank-eight divided comparison proves every larger rank, with at least the unscaled positive margin displayed in the table. These monotonicity arguments, not a finite rank experiment, certify the infinite range.

Both analytic entries strictly exceed the original-K arithmetic bound for every target. Corollary10.120 forces each remaining integer value to zero. This closes the entire original R-block at exactly O, for the same coefficient family, including $J = I_3$. □

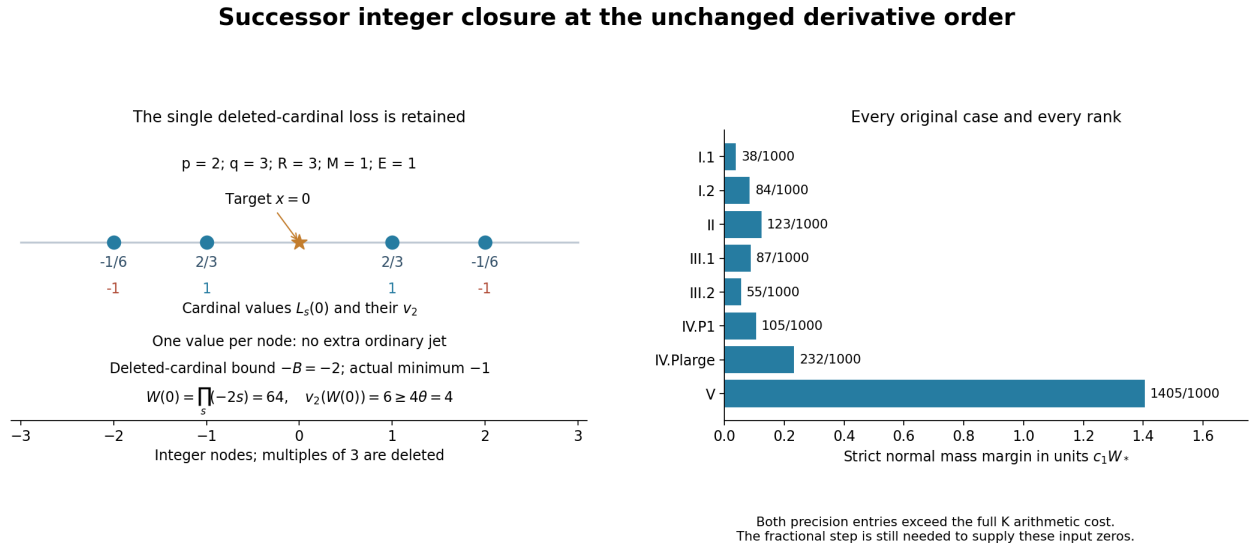


Figure 10.37. Left: at $p = 2, q = 3, R = 3$, the four q-deleted integer nodes carry one value condition each. Their cardinals at zero are $-1/6, 2/3, 2/3, -1/6$, of valuations $-1, 1, 1, -1$. The uniform cardinal loss is $B = 2$, while the actual loss here is one. The normal root polynomial $W(Z) = \prod_s (Z - 2s)$ has $W(0) = 64$, of valuation six, above $N_*\theta = 4$ at $\theta = 1$. These exact values illustrate the interpolation, rather than a full original logarithmic-contradiction parameter choice. Right: proved strict first-entry margins for every original case, minimized over all ranks; every second-entry margin also exceeds eight in the same normalization. Both entries retain the original field K, coefficient minimum, full denominator and deleted-cardinal loss. The actual q-deleted family is the hypothesis of Theorem10.146; Theorem10.149 and the phase construction supply it in Corollary10.150. Solution58 checks the example and endpoint. Figure program, exact all-rank certificate. Human-source context: Lemma7.2 of Yu's freely accessible paper. The full successor integer argument and both strict comparisons are proved above.

The exact field of a fractional value

The field generated by all fractional monomials can be larger than the field generated by a particular prepared value. We can determine the latter field from the components which survive in that value. This gives an exact arithmetic factor without identifying distinct local primes.

Theorem 10.147 (the surviving Kummer components). Retain Theorem10.56, and write $h = h_\Lambda$. Choose its independent principal monomials $U_1, \dots, U_h$, with $U_j^q \in K$. For $a \in \{0, \dots, q-1\}^h$, put $U^a = \prod_j U_j^{a_j}$. Every fractional prepared value has a unique expansion

$$V = \sum_a A_a U^a, \qquad A_a \in K, \qquad W_V = \langle a : A_a \neq 0 \rangle_{\mathbb{F}_q}, \qquad t_V = \dim W_V. \tag{10.429}$$

If $V \neq 0$, then

$$[K(V) : K] = q^{t_V}, \qquad K(V)_{\mathfrak{P}} = K_{\mathfrak{p}}. \tag{10.430}$$

In particular $t_V = 0$ precisely when $V \in K$. A nonzero constant component contributes the zero vector and does not increase t_V. The arithmetic upper bounds (10.156), (10.244) and (10.313) for this value may use the exact factor $q^{t_V} d/(ef \ln p)$, while retaining their other terms and the same coefficient normalization. All those terms remain present; the normal summand G_u in (10.313) is added after the arithmetic bound, as before.

Proof. Theorem10.56 gives the basis U^a of $E = K(U_1, \dots, U_h)$, of degree q^h. Its exact monomial identities collect every prepared term into the stated expansion; the rational scalars and coefficients lie in K. Fix a primitive q-th root $\omega \in K$. For $b \in \mathbb{F}_q^h$, define

$$\sigma_b\left(\sum_a A_a U^a\right) = \sum_a A_a \omega^{b\cdot a} U^a. \tag{10.431}$$

These are K-automorphisms. Indeed products of basis monomials are reduced by $U_j^q \in K$, and $\omega^q = 1$, so the formula respects multiplication and addition. Its inverse is σ_{-b}. Uniqueness of the expansion shows that $\sigma_b(V) = V$ if and only if $b \cdot a = 0$ for every surviving component. Row reduction over $\mathbb{F}_q$ gives exactly q^{h-t_V} such b. Thus the orbit of V has q^{t_V} distinct elements.

Every orbit element is a root of V's minimal polynomial over K, because each σ_b fixes K. Conversely, the product of $X - V'$ over the distinct orbit elements is fixed by every σ_b. Its coefficients belong to K: in the basis expansion of any fixed element, a nonzero exponent vector a is detected by some b with $b \cdot a \neq 0$, forcing its coefficient to vanish. The orbit polynomial therefore belongs to $K[X]$. These two degree inequalities prove the exact degree without an additional Galois-theory input. The principal roots lie in $K_{\mathfrak{p}}$, so $K \subset K(V) \subset K_{\mathfrak{p}}$. Its closure is exactly $K_{\mathfrak{p}}$, as in Theorem10.56; hence the selected local indices are the original e,f.

For the arithmetic assertion, let $F = K(V)$, and divide V by the same nonzero reference coefficient of minimum valuation δ. This quotient belongs to F. The individual torus monomials may still belong only to E, so we justify the height comparison over F explicitly. At each place of F extend its absolute value to E. The triangle or ultrametric bound for the displayed prepared sum holds in that extension. The absolute value of every fractional monomial is determined by its q-th power in K. Changing the extension therefore changes none of its absolute values; its root-of-unity factors also have absolute value one. Coefficient norms and rational scalar bounds likewise depend only on the restricted place of K. Consequently the same local bound used in Theorems10.41 and10.92 bounds $V/c_{\min}$ at the place of F.

When these bounds are summed, the local weights over a place of K sum to $[F : K]$ times its original weight, by the extension and norm formula proved in the prerequisite lesson on places. The coefficient height, scalar maximum, q-denominator term and signed torus height hence acquire the factor $[F : K] = q^{t_V}$. The selected local term has weight ef, as just proved. The complete product-formula proof therefore gives exactly the claimed arithmetic bound. Adding the same G_u proves its normal version. This argument does not require the other primes to have the selected valuation. □

For s prime to q, multiplication of every support residue vector by s is invertible over $\mathbb{F}_q$. This observation alone does not reduce t_V: reductions require exact vanishing of components A_a, rather than a change of representative or a choice of local embedding. At different prepared indices the surviving components may be different, so their fields must be checked for the value in question.

Here is an exact phase example. Set $K = \mathbb{Q}, p = 73, q = 2, \theta_1 = 74, \theta_2 = 147, P = 1$. Take the roots $U_1 = \sqrt{74}, U_2 = \sqrt{147}$ which are congruent to one at73. They are the principal roots of Lesson9. The phase data may be $G_0 = 36, d_1 = d_2 = 0, \alpha_0 = -1$, so the normalized residue relation is the genuine relation $w = 0$. The square classes of $-1, 74, 147$ are independent: prime valuations at37 and3 first force the exponents of74 and147 to be even, and then the sign forces the remaining exponent to be even. Thus the full root field has degree eight, and the two principal monomials generate a degree-four field. For $V = 2 - U_1 - U_2$, the surviving vectors are $(0,0), (1,0), (0,1)$, which span dimension two. Theorem10.147 gives degree four. Its minimal polynomial and norm are

$$P_V(X) = X^4 - 8X^3 - 418X^2 + 1736X + 3577, \qquad N_{K(V)/\mathbb{Q}}(V) = 3577 = 49 \cdot 73. \tag{10.432}$$

The four sign embeddings above73 have valuations $1, 0, 0, 0$, with the selected principal-root embedding first. Thus the weighted sum is one, whereas four times the selected valuation is four. These are explicit algebraic values with the phase relation and principal roots; they are an illustration of the field and valuation mechanisms, rather than an auxiliary family with all the integer zeros of the second induction. Solution59 proves these computations. Theorem10.149 verifies the strict arithmetic comparison for the actual fractional targets by strengthening the integer input.

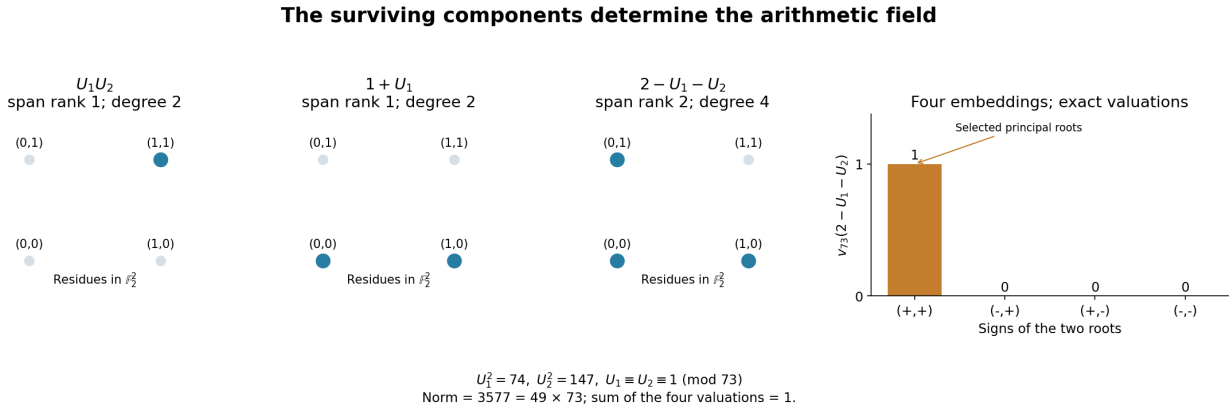


Figure10.38. Left: the nonzero residue components of U_1U_2, $1+U_1$ and $2-U_1-U_2$ have span dimensions one, one and two, giving exact degrees two, two and four over $\mathbb{Q}$. A constant component contributes the origin. Right: at73, the four sign embeddings of $2-\sqrt{74}-\sqrt{147}$ have valuations $1,0,0,0$. The norm has valuation one; the selected embedding is marked. The bars are exact valuations, rather than height upper bounds. Theorem10.147 proves the component and field assertions; Solution59 verifies the polynomial, principal roots and norm. Figure program, exact calculations. Human-source context: the fractional phase and product-formula passage in Yu's freely accessible paper, Section7. The exact field and local comparisons used here are proved above.

Increasing the integer input before the fractional step

The prescribed integer interval is retained throughout the second induction. Before taking a fractional value, we can temporarily prove more integer zeros at the same derivative order. Two such extensions give enough analytic precision to keep the full global field degree.

Lemma 10.148 (the two-radius arithmetic comparisons). Retain the original fixed-order parameters, constants and notation of Theorems10.145–10.146. Set $\xi = 24/25$, $a = r+1$, and

$$\begin{aligned} &\mu_r = q^r/Q^{m_r}, && f_r = \max\{1, \xi c_1/(2cqH)\}, \\ &j_r = \min\{j \in \mathbb{Z}_{\geq 0} : \mu_r f_r \leq q^j\}, && \Gamma = \bar{\lambda}\Omega_+/c_4, && \sigma_r = qa/(S_-L_r). \end{aligned} \tag{10.433}$$

Use the same full-depth K_r as in Theorem10.146; for ranks at least eight use its enclosing coefficient bound with $C_r^{\sharp} \leq 1$. The following quantities are strictly positive in every original case and at every rank $r \geq 2$:

$$\begin{aligned} P_1 &= 2cq^2\eta - 2q\sigma_r - K_r - \frac{\bar{\lambda} j_r \ell_q}{5c_4} - \frac{\mu_r f_r}{c_2}, \\ P_2 &= \xi c_1 q^r - 2\sigma_r - K_r - \frac{\bar{\lambda}(j_r+1)\ell_q}{5c_4} - \frac{q\mu_r f_r}{c_2}, \\ P_3 &= \xi c_1 q - K_r + \Gamma - \frac{\bar{\lambda} j_r \ell_q}{5c_4} - \frac{\mu_r}{c_2} - \frac{\Gamma + 2\sigma_r}{q^r}. \end{aligned} \tag{10.434}$$

Also $j_r \leq 2$, and $(1-\xi)c_1 q^a > 1/(c_3 Y_- L_r)$.

Proof. All terms are rational endpoints of the complete bounds in Theorem10.146. Its m_r is the last integer with $cL_rH^{m_r}/a \geq 1$. We give an enclosure for the unbounded rank range which also retains the factor q^r.

Put $E_x = (1 + 1/x)^x$, $x > 0$. Differentiating its logarithm gives $\ln(1 + 1/x) - 1/(x + 1) > 0$: the first term is the integral of $1/(1 + t)$ from zero to $1/x$, strictly greater than the interval length times its endpoint minimum. Thus E_x increases. The exact factorial ratio of $F_r = r^r(r + 1)^r/(r!)^2$ is $F_{r+1}/F_r = E_rE_{r+1}$. At rank eight the rational product $(9/8)^8(10/9)^9 > 13/2$. Therefore $F_{r+1}/F_r > 13/2$ for every $r \geq 8$.

For $r \geq 32$, the c band is constant, and both H and Q increase by Lemma10.122. The formula (10.417) gives

$$\frac{L_{r+2}/(r + 3)}{L_r/(r + 1)} > a_*^2(13/2)^2. \tag{10.435}$$

The following three rational substitutions at $H_{32} = (1-c/33)^{33}$, $Q_{32} = qH_{32}$, verify $a_*^2(13/2)^2H_{32}^{d_0} > 1$ and $Q_{32}^{d_0} > q^2$:

Rank band	c used in the enclosure	a_*	q	d_0
odd cases other than III	14/25	7	2	12
III	11/20	7/2	2	11
V	827/1000	26/3	3	9

In the first row the other actual c values are smaller, so their H and Q are larger. Multiplying the inequality defining m_r by the displayed two-rank ratio and by $H_{32}^{d_0}$ proves $m_{r+2} \geq m_r + d_0$. Consequently

$$\mu_{r+2} \leq \mu_r\, q^2/Q_r^{d_0} < \mu_r \qquad (r \geq 32). \tag{10.436}$$

This proves decrease along both parity sequences. It is not an enclosure obtained by holding m_r fixed while rank grows. Furthermore f_r decreases because H increases. Thus j_r is nonincreasing along each sequence. The enclosing K_r decreases: its coefficient and Euler remainders decrease, a/L_r decreases, and the increasing Q enlarges the positive logarithm lower bound in its denominator term. These are exactly the monotonicities used in Theorem10.146, now without a growing r_1 scalar cost. Also σ_r decreases, while η and q^r increase. Each displayed P_i therefore increases along each parity sequence from32 or33. The input comparison likewise improves.

Substitute ranks2–31 and parity endpoints32,33 in each original row, using exact integer comparisons for m_r, j_r and the margins. The following strict margins are divided by1000000.

Case	P_1	P_2	P_3
I.1	1028198	2077317	437081
I.2	1376333	2808629	735003
II	1237102	2462590	678380
III.1	1212788	2458197	615506
III.2	1309556	2722070	711872
IV, first prime branch	1190595	2368885	589777
IV, larger prime branch	1318170	2496460	748768
V	2912809	10621246	82270

All256 substitutions also give $j_r \le 2$ and the strict input comparison. Their rational formulas and integer calculations are provided with Figure10.39. Together with the parity argument, these checks include every rank. □

Theorem 10.149 (the actual fixed-order fractional zeros). Suppose $I_1 \le I < I_3$ and an admissible stage-I family has all its prepared zeros at integers $|s| \le R$ through the original order $O = \lfloor T_\flat \rfloor$. Then it has all its prepared zeros at s/q for integers $|s| \le R$, through that same order O. The original coefficient minimum, support widths and phase are retained.

Proof. Put

$$R_{\mathrm{s}} = \left\lfloor \frac{\xi U/(q\theta) - 1}{2} \right\rfloor, \qquad R_{\mathrm{l}} = \left\lfloor \frac{\xi U/\theta - 1}{2} \right\rfloor. \tag{10.437}$$

The original parameter lower bounds make both radii positive. Indeed $U/\theta = c_1 q^r ST/a$, by (10.186), while $S \ge S_- > 58, T \ge g_4 > (9000/29)a$, $c_1 > 7/5$, and $q^r \ge 4$. For $x \in \mathbb{R}$, $\lfloor x \rfloor > x - 1$. Therefore, writing $N_{\mathrm{s}} = 2R_{\mathrm{s}} + 1$, $N_{\mathrm{l}} = 2R_{\mathrm{l}} + 1$,

$$\begin{aligned} &N_{\mathrm{s}}\theta > \xi U/q - 2\theta, \quad N_{\mathrm{s}}\theta \le \xi U/q, \\ &N_{\mathrm{l}}\theta > \xi U - 2\theta, \qquad N_{\mathrm{l}}\theta \le \xi U, \qquad qR_{\mathrm{s}} \le R_{\mathrm{l}}. \end{aligned} \tag{10.438}$$

For the last inequality use $qR_{\mathrm{s}} \le (\xi U/\theta - q)/2 \le (\xi U/\theta - 1)/2$, and integrality. Lemma10.148 and (10.414) give $\beta < (1 - \xi)U$. Thus both new intervals satisfy the full-node input inequality $N\theta < U - \beta$. The original interval already satisfies it by Proposition10.145. Taking the larger of it and either new interval preserves that inequality.

We first show that every integer $|s| \le R_{\mathrm{s}}$ is a zero, without changing the family. The old interval supplies the local normal precision $(2R + 1)\theta$ at every $x \in \mathbb{Z}_p$, by Proposition10.145. The same calculation as in Theorem10.146, now using all nodes, gives

$$c_1 W_*(2R + 1)\theta > 2cq^2\eta - 2q\sigma_r. \tag{10.439}$$

Here $q\eta > 1$ bounds the original r_1 mass below by its value at zero; the floor loss uses $c_1 W_*\theta = qa/(ST)$. If $u > D_{\mathrm{a}}$ the target is identically zero. Otherwise its integer value belongs to K by the phase identity, so it has the original-K arithmetic bound.

For precision we bound the physical real radius rather than absorbing the denominator into a geometric stage restriction. From $ST = S_1 T_1$, $T_1 < a/(cH)$, and $I \ge I_1 \ge m_r$,

$$q^{-I} R_{\mathrm{s}} \le q^{-I} R_{\mathrm{l}}/q < \frac{\xi c_1 q^{r-1}}{2cHQ^{I_1}} S \le \mu_r f_r S \le q^{j_r} S. \tag{10.440}$$

The factorial proof of (10.325), applied directly with this physical radius enlarged to $q^{j_r+1}S$, bounds the scalar by $\bar{\lambda}[1 + j_r \ell_q/5]/c_4$ in the normalization $c_1 W_*$. This uses only that factorial proof, not the separate geometric absorption (10.326). The full coefficient, Euler and normal terms and the entire denominator through I_3 remain those in K_r. The torus contribution is at most $\mu_r f_r/c_2$. The same-order normal arithmetic cost is therefore less than $K_r + \bar{\lambda} j_r \ell_q/(5c_4) + \mu_r f_r/c_2$. The strict P_1 comparison forces every proposed nonzero integer value to zero, by Corollary10.120. Integers already in the old interval retain their zeros. We now have the full interval of radius $\max\{R, R_{\mathrm{s}}\}$ at exactly O.

Apply the full-node normal interpolation theorem10.119 to that larger interval. Its input inequality was just verified, so its precision is at least $N_{\mathrm{s}}\theta$. In normalization this is greater than $\xi c_1 q^r - 2\sigma_r$. At an integer target of radius R_{l}, the physical radius and torus height can be larger by at most q

than the previous bounds. The factorial scalar may conservatively use $j_r + 1$; all other costs still belong to the same K_r. The normal arithmetic cost is less than $K_r + \bar{\lambda}(j_r + 1)\ell_q/(5c_4) + q\mu_r f_r/c_2$. The strict P_2 comparison forces every such integer target to zero. Thus the original family has all its zeros through O on the full interval of radius $\max\{R, R_1\}$.

Its normal analytic precision, by the same full-node theorem and its verified input, is greater than $\xi U - 2\theta$ at every point of $\mathbb{Z}_p$. In normalization it is greater than $\xi c_1 q^{r+1} - 2\sigma_r$. Consider now only the prescribed fractional targets $|s| \le R$, $x = s/q$. Their physical radius satisfies $q^{-I} R/q \le q^r S/Q^{I_1} \le \mu_r S \le q^{j_r} S$. Their torus cost is consequently at most μ_r/c_2, and their factorial scalar uses the already justified j_r envelope.

For a nonzero fractional value, Theorems10.56 and10.147 permit a global degree factor at most q^r, with the selected e,f unchanged. Its complete arithmetic bracket therefore has normal upper bound, in the scale $c_1 W_*$,

$$q^r \left(K_r - \Gamma + \frac{\bar{\lambda} j_r \ell_q}{5c_4} + \frac{\mu_r}{c_2} \right) + \Gamma. \tag{10.441}$$

The Γ term bounds G_u, which is added once after the product-formula arithmetic bracket. All q-denominators are charged at depth $I + 1 \le I_3$; every coefficient, Euler and torus cost remains. The difference between the analytic lower bound and this upper bound, divided by q^r, is exactly $P_3 > 0$. Hence the value must be zero. The local exponential branch is the algebraic principal-root branch of Corollary10.52. Indices beyond additive degree vanish already. This proves every required fractional zero through O. Neither the two extra integer intervals nor either interpolation changes any coefficient, phase, minimum, support width or selected derivative order. □

Corollary 10.150 (the original second induction and its contradiction). Under the original contradiction parameters at least admissible rank $r \ge 2$, the alternative $I_1 < I^*$ is impossible.

Proof. The actual terminal family at I_1 and all of its original integer R-block zeros through O are supplied by Theorem10.144. Suppose a family with those properties has reached depth $I_1 \le I < I_3$. Theorem10.149 supplies every original fractional zero. The full saturated Kummer and phase construction of Theorem10.51 then extracts a nonzero successor family with its q-deleted integer zeros for $|s| \le R$ through the same order O. It retains a coefficient attaining the original minimum, never increases the coefficient height, preserves the exact phase and divides the support widths by q. Its rational scalar is the stage $I + 1$ scalar; the affine binomial changes in that theorem preserve the total-order cutoff. The selected derivations and integral normalized torus exponents therefore remain the ones required by Proposition10.31.

Theorem10.146 applies at $J = I + 1 \le I_3$ and closes all the missing integers on the original R-block at the unchanged O. This is the induction hypothesis for the next depth. Finite induction reaches the actual family at I_3, including its endpoint. That family has every hypothesis of Proposition10.143, which makes its integral torus support constant and yields more distinct integer roots than the degree of its nonzero additive polynomial. This is a contradiction. The alternative is therefore impossible. Together with Theorem10.141, both original stopping branches, including their integer endpoints, have now been ruled out. □

The full global factor is retained in Theorem10.149. Separate primes are never assigned the same valuation. The stronger intermediate integer zeros supply the additional local precision before phase extraction. The required original radius and order remain the induction invariant.

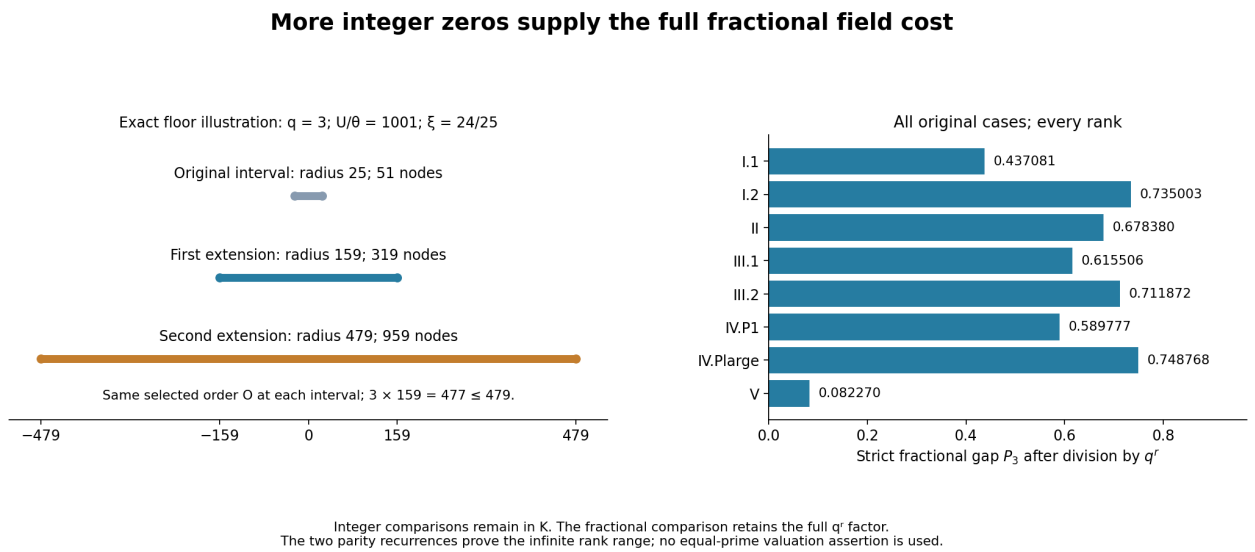


Figure10.39. Left: exact illustrative floor data q=3, theta=1, U=1001, xi=24/25 give radii159 and479, with319 and959 integer nodes. The old radius25 is shown only to illustrate the two extensions. Their node masses lie above xi U/q-2 and xi U-2, and3 times159 is at most479. These data illustrate the floor mechanism, rather than a full original contradiction parameter choice. The selected order O is unchanged. Right: proved strict lower fractional gaps P3 after division by the full q^r field factor, for every original case and every rank. Lemma10.148 proves the two parity enclosures, Theorem10.149 proves both integer extensions and all original fractional zeros, and Corollary10.150 performs the actual phase induction and stopping contradiction. Solution60 checks the floors and recurrence. Figure program, exact all-rank certificate. Human-source context: Section7 of Yu's freely accessible paper. The stronger intermediate integer intervals and the complete full-field comparison are proved above.

47. The first explicit Yu estimate

The two stopping contradictions now give an explicit estimate for the original multiplicative form. The bases used during the proof can have smaller rank than the original list. We keep their height allowance until it cancels against the auxiliary bound. This also requires an elementary argument when the least rank is one.

The constants are uniform in the rank

Use the seven cases and their constants from Section 36. Write $a = r+1$, $F_r = r^r(r+1)^r/(r!)^2$, and let a_*, ρ and $\widehat{\vartheta}$ have their meanings there. For the two IV branches use the same constants. Define

$$L_r = \frac{2c_0c_4q(r+1)a_*^rF_r}{\rho\widehat{\vartheta}}\,v, \qquad v = \begin{cases} 5, & \text{I.2, III.1,} \\ 509/100, & \text{III.2,} \\ 1, & \text{I.1, II, IV, V.} \end{cases} \tag{10.442}$$

These are lower bounds for the actual g_4. In the branches where g_1 remains in (10.185), use $g_1 > 5$. For III.2, $d = 1$ gives $g_1 \geq 4 + \ln 3 > 509/100$. Indeed the first three positive terms of the logarithm series at $(3-1)/(3+1) = 1/2$ sum to $1 + 1/12 + 1/80 > 109/100$. In the other branches the factor g_1 cancels, and $v = 1$ is valid.

Define the following lower bounds for the actual g_2:

$$G_r = c_3 q (r+1)^2 \begin{cases} (39r/10 + 36/5)/(11/10), & \text{I.1, I.2,} \\ (39r/10 + 36/5)\, 20/17, & \text{II,} \\ 2, & \text{III.1, IV,} \\ 1, & \text{III.2,} \\ (39r/10 + 36/5)/(347/500), & \text{V.} \end{cases} \tag{10.443}$$

For I and V, these follow from (10.185), (10.288), $e \geq 1$, $\ln 3 < 11/10$ and $\ln 2 < 347/500$. The logarithm bounds were proved in the exact series calculations of Section 40 and Solution 48. For II, $e \geq 2$ and $\ln 5 < 17/10$; the positive exponential series through degree four at $17/10$ already exceeds five. For III and IV use $d \geq 2$ except in III.2, where $d = 1$. Thus these inequalities apply to every field in the corresponding case.

Lemma 10.151 (closing every numerical constant). Put

$$F_0 = (1+\epsilon)(2 + g_2^{-1}) c_0 c_1 c_3 c_4 q^2. \tag{10.444}$$

For every rank $r \geq 2$, $(1 + 10^{-26})F_0 < c$, where c is the following exact case constant. The last column gives strict lower bounds for the gap.

Case	c	Gap greater than
I.1	939	$165185/10^6$
I.2	636	$262798/10^6$
II	505	$146558/10^6$
III.1	1794	$305548/10^6$
III.2	1790	$4059/10^6$
IV	2680	$656871/10^6$
V	206	$724409/10^6$

Proof. Set $\Delta_r = (r+1)/(2L_r)$. The exact formula (10.282) gives $F_{r+1}/F_r > 4$, and $a_* \geq 7/2$, so

$$0 < \Delta_{r+1} < \Delta_r/14, \qquad \Delta_2 < 1. \tag{10.445}$$

The second inequality follows by direct substitution in the seven rows. For $0 < z < 1$, $(1 + z/14)^2 < 1 + z$: after subtracting one and dividing by z, its left side is $1/7 + z/196 < 1$. Since $r + 1 \leq 2r$, it follows that $(1 + \Delta_{r+1})^{r+1} < (1 + \Delta_r)^r$. Each displayed G_r increases with r. By (10.185), therefore,

$$(1+10^{-26})F_0 \leq (1+10^{-26})(1+\Delta_r)^r (2+G_r^{-1}) c_0 c_1 c_3 c_4 q^2 \leq (1+10^{-26})(1+\Delta_2)^2 (2+G_2^{-1}) c_0 c_1 c_3 c_4 q^2. \tag{10.446}$$

All entries of the final expression are rational. Subtracting it from the seven integers c, and multiplying positive denominators, gives the table. The exact fractions and the two positive-series checks are recorded in the reproducible calculation. The decrease just proved supplies the quantifier over every rank; the table alone would only check rank two. □

The bound for the original independent bases

Let $a_1, \ldots, a_n$ be multiplicatively independent $\mathfrak{p}$-adic units in a number field K, with $n \geq 2$. Write $d = [K : \mathbb{Q}]$, $e = e_{\mathfrak{p}}$, $f = f_{\mathfrak{p}}$, $t = f \ln p$ and $\ell_d = \max\{1, \ln d\}$. At odd p put $q = 2$. At $p = 2$, assume

$\zeta_3 \in K$ and put $q = 3$. Let q^u be the order of the q-primary roots of unity in K. Use the saturated residue index $\delta(\boldsymbol{a})$ from (10.49), as in (10.272), for the original list of bases.

Let $b_1, \dots, b_n \in \mathbb{Z}$, not all zero. Renumber so that $b_n \neq 0$ and $v_p(b_n) \leq v_p(b_j)$ for every j, with $v_p(0) = +\infty$. Put

$$\begin{aligned} \Xi &= \prod_{j=1}^{n} a_j^{b_j}, & B_\dagger &= \min_{b_j \neq 0} |b_j|, \\ R_{\max} &= \max_{j<n} \left\{ \frac{|b_n|}{h(a_j)} + \frac{|b_j|}{h(a_n)} \right\}, & A_n &= \max\{4 + \ln((n+1)d), e, t\}. \end{aligned} \tag{10.447}$$

Independence implies $\Xi \neq 1$ and every $h(a_j) > 0$. Use $W(d)$ from (10.222). The constants for the statement are

$$\mathfrak{a} = \begin{cases} 14, & \text{I, II, IV}, \\ 7(p-1)/(p-2), & \text{III}, \\ 26, & \text{V}, \end{cases} \qquad a_0 = \begin{cases} 2 + \ln 14, & \text{I, II, IV}, \\ 2 + \ln 7, & \text{III}, \\ 2 + \ln 26, & \text{V}. \end{cases} \tag{10.448}$$

Use c from Lemma 10.151 and the exact a_1, a_2 from (10.287): $a_1 = 4.03, 4.79, 3.44, 4.71, 5.84, 5.12, 2.52$ in the seven cases, $a_2 = a_1$ in I.2 and III.2, and $a_2 = a_1 + \ln 2$ otherwise. Define

$$\begin{aligned} G_1(n,d) &= (n+1)(a_0 n + a_1 + \ln(a_0 n + a_2) + \ln d), \\ \mathsf{h} &= \max\left\{ \ln \frac{R_{\max}}{dW(d)},\ \ln B_\dagger,\ G_1(n,d),\ (n+1)t \right\}, \\ C_1 &= c\mathfrak{a}^n \frac{n^n (n+1)^{n+1} d^{n+2} \ell_d}{n! q^u t} \max\left\{ \frac{p^f}{\delta(\boldsymbol{a}) t^{n+1}}, \frac{\mathrm{e}^n}{n^n} \right\} A_n. \end{aligned} \tag{10.449}$$

Here $\mathrm{e} = \exp(1)$; e is the ramification index. The cases are I: $p = 3$, with I.1 for $d > 1$ and I.2 for $d = 1$; II: $p = 5, e \geq 2$; III: $p \geq 5, e = 1$, with the same degree subdivisions; IV: $p \geq 7, e \geq 2$; V: $p = 2, \zeta_3 \in K$.

Theorem 10.152 (Yu's first explicit estimate, with proof). Under these hypotheses,

$$\operatorname{ord}_{\mathfrak{p}}(\Xi - 1) < C_1 \mathsf{h} \prod_{j=1}^{n} h(a_j). \tag{10.450}$$

Proof. Choose $\tau = 10^{-26}/(2n)$ in (10.184). Keep this same $\theta = \vartheta/(1+\tau)$ and the same original n at every smaller rank. Direct substitution gives

$$qH_* = \mathfrak{a}(1+\tau), \qquad (1+\tau)^n < 1 + 10^{-26}. \tag{10.451}$$

For the second inequality, the binomial expansion and $\binom{n}{k} \leq n^k$ give $(1+\tau)^n \leq (1 - n\tau)^{-1}$. Here $n\tau = 10^{-26}/2$, and $(1-x)^{-1} < 1 + 2x$ for $0 < x < 1/2$.

Take the least admissible rank r of (10.277), with integral forms $L_0 = z_0, L_1, \dots, L_r$, rational coefficients B_i, weights σ_i and local unit bases α_i. This rank exists by the explicit rank-n representation in Section 35. The weighted height condition there and the original integers b_j supply precisely (10.218). The function $G_1(j,d)$ increases for integers $j \geq 1$, because all its positive linear and logarithmic factors increase. Consequently h meets the original height conditions (10.223), (10.287), and $\mathsf{h} \geq (r+1)t$.

Suppose first $r \ge 2$. Form the complete original parameters (10.183)–(10.186) with $h = \mathsf{h}$ and this least-rank representation. Write $U_r = q^{r+1}S\mathcal{D}/(et)$. If $v_p(\Xi - 1) \ge U_r$, the depth and phase construction of Sections 20 and 28, the minimum-valuation derivation choice of Lemma 10.23, and the integral kernel of Theorem 10.71 give the original contradiction setting of Section 40. The evaluated linear form is $P \log \Xi$, whose valuation is at least U_r. This evaluation is legitimate on the principal unit disc: (10.280) gives $U_r > q^{r+1}Sb_r > 1 \ge 1/(p-1)$, because $A_r + x \ge e$. Lemma 9.5 and Proposition 9.4 identify its valuation with that of $\Xi - 1$. The original minimum-valuation coefficient hypothesis is retained; no bound on newly chosen rational B_i replaces it.

Corollary 10.140 constructs the first induction. If $I^* \le I_1$, Theorem 10.141 gives the least-rank contradiction, including equality. If $I_1 < I^*$, Corollary 10.150 constructs the second induction and gives its contradiction. These exhaustive alternatives prove $v_p(\Xi - 1) < U_r$.

Now use $\gamma(\mathsf{h} + x)(A_r + x)/q^\nu = \mathsf{h}A_r$ in (10.186). Exact cancellation, with M_r and H_* as in (10.272), gives

$$eU_r = F_0(qH_*)^r \frac{r^r (r+1)^{r+1} d^{r+2} \ell_d}{r! q^u t^2} M_r(\boldsymbol{\alpha}) A_r \mathsf{h} \prod_i \sigma_i. \tag{10.452}$$

The profile (10.277) bounds its last weight product by $(\mathrm{e}H_*(n+1)d)^{n-r}(M_n(\boldsymbol{a})/M_r(\boldsymbol{\alpha}))\prod_j h(a_j)$. Thus the actual smaller-rank residue factor cancels exactly. The ratio between the remaining rank factor and the desired rank-n factor is at most

$$\left(\frac{\mathrm{e}}{q}\right)^{n-r} \frac{r^r/r!}{n^n/n!} \left(\frac{r+1}{n+1}\right)^{r+1} \le 1. \tag{10.453}$$

Indeed consecutive ratios of $k^k/k!$ are $(1 + 1/k)^k \ge 2$, by the binomial expansion or its increasing logarithm. Hence the middle quotient is at most 2^{r-n}. Also $\mathrm{e} < 3$, $q \ge 2$, and $r \le n$. Finally $A_r \le A_n$, and $M_n/t^2 = t^{-1}\max\{p^f/(\delta(\boldsymbol{a})t^{n+1}), \mathrm{e}^n/n^n\}$. Lemma 10.151 and the small-τ bound therefore give the claimed strict inequality.

It remains to prove the result for $r = 1$. Write $B_1 = p_1/q_1$ in lowest terms, $q_1 > 0$. Comparison of the integral coefficients in $q_1\mathcal{L} = q_1B_0z_0 + p_1L_1$ gives $q_1B_0 \in \mathbb{Z}$, $p_1 \mid b_j$ for every j, and $|p_1| \le B_\dagger$. Set $\beta = \alpha_1$ and let g be its actual residue order, as defined in Theorem 10.21. It is non-torsion by the independence in (10.277), and

$$\Xi^{q_1q^u} = \beta^{p_1q^u} \ne 1, \qquad \mathrm{ord}_{\mathfrak{p}}(\Xi - 1) \le \frac{d}{t}\left(2\mathsf{h} + 2eg\sigma_1\right). \tag{10.454}$$

The power identity kills the z_0 torsion factor. A positive integer power of a local unit never decreases the valuation of its difference from one, by the factorization of $X^m - 1$. Apply the proved bound (10.60) to that power. We have $q^u \le p^f - 1 < p^f$, $t > 1$, $\mathsf{h} \ge (n+1)t$, and $\mathsf{h} \ge \ln B_\dagger$; hence $\ln(2q^uB_\dagger) < 2\mathsf{h}$. Also $p/(p-1) \le 2$ and $h(\beta) \le \sigma_1$. These facts prove the displayed inequality without a logarithm estimate for a rank-one form.

Let $\mathcal{T} = C_1\mathsf{h}\prod_j h(a_j)$. The height-product lower bound (10.55), with $w_K \ge q^u$, and $M_n \ge \mathrm{e}^n t/n^n$, imply

$$\mathcal{T} \ge \frac{d\mathsf{h}}{t} \frac{c\mathfrak{a}^n n^n (n+1)^{n+1} A_n}{\rho (n!)^2} > 800\frac{d\mathsf{h}}{t}. \tag{10.455}$$

For the last inequality use $c \ge 200$, $\mathfrak{a} \ge 7$, $n \ge 2$, $A_n > 5$, $\rho \le 58$, and $n^n(n+1)^{n+1}/(n!)^2 \ge 1$. Thus the first summand in the elementary rank-one bound is less than $\mathcal{T}/400$.

At rank one, $\delta(\beta) = (p^f - 1)/g$, so $M_1(\beta) \geq p^f g/((p^f - 1)t) > g/t$. The profile gives $\sigma_1 \leq \Gamma_n^{n-1}(M_n/M_1)\prod_j h(a_j)$. Consequently its second summand is less than $2ed\Gamma_n^{n-1}M_n\prod_j h(a_j)$. By Lemma 10.73, $eq^u \leq 2d$. Dividing this summand by $\mathcal{T}$ and substituting $H_* = \mathfrak{a}(1+\tau)/q$ gives an upper bound

$$\frac{4t^2(\mathrm{e}/q)^{n-1}(n!/n^n)(1+\tau)^{n-1}}{c\mathfrak{a}(n+1)^2 d\ell_d A_n \mathsf{h}} < \frac{8}{c\mathfrak{a}(n+1)^3 d\ell_d} \leq \frac{8}{200\cdot 7\cdot 27} < \frac{1}{4000}. \tag{10.456}$$

Here $n!/n^n \leq 2^{1-n}$, by the same consecutive-ratio argument; $\mathrm{e}/q < 3/2$, $(1+\tau)^{n-1} < 2$, $A_n \geq t$, and $\mathsf{h} \geq (n+1)t$. The two summands together are strictly smaller than $(1/400 + 1/4000)\mathcal{T} < \mathcal{T}$. This completes the rank-one case and the proof. □

This is the first estimate in Theorem I of Yu's freely accessible 2013 paper, with its actual residue index, ramification, degree-one constants, coefficient choice and height parameter. The proof above derives it from the complete programme arguments, including both stopping branches.

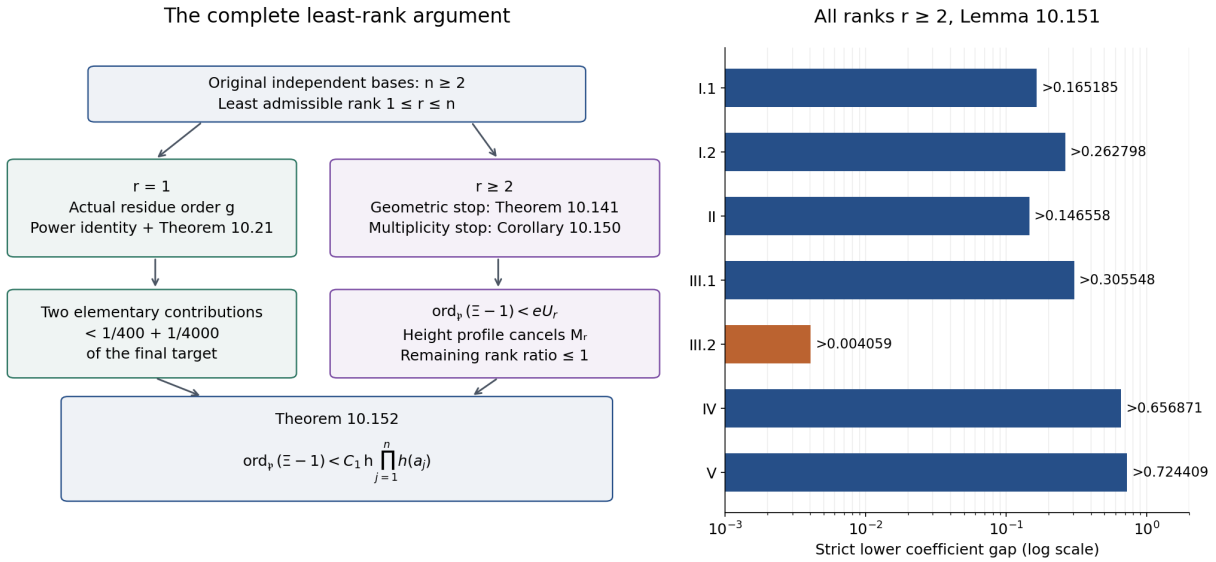


Figure 10.40. The least admissible rank divides the proof into the elementary residue-order bound at rank one and the two exhaustive stopping contradictions at larger rank. The actual smaller-rank residue factor cancels through (10.277); the remaining rank factor is at most one. The bars give strict lower bounds for the seven coefficient gaps in Lemma 10.151, on a logarithmic scale; every bar applies to all ranks, including the small III.2 gap. Theorem 10.152 proves the final original-base estimate and Solution 61 checks the cancellation and rank-one margins. Figure program, exact rational calculation. Human-source context: Sections 1.1, 1.3, 2 and 3.2 of Yu's free paper.

48. An application bound in terms of the largest coefficient

Keep the field cases and constants of Theorem 10.152. Let $a_1, \ldots, a_n \in K^*$ be independent local units, $n \geq 1$, and let $b_1, \ldots, b_n \in \mathbb{Z}$ be not all zero. Set $\Xi = \prod_j a_j^{b_j}$ and put $\Omega = \prod_j h(a_j)$, $B \geq \max\{3, |b_1|, \ldots, |b_n|\}$, $H = \max\{\ln B, t\}$, and $C_1^* = (n+1)C_1$. When $n = 1$, use the same displayed formula for C_1, with $\delta(a_1) = (p^f - 1)/g$, where g is the original residue order. The height parameter can be replaced by $(n+1)H$ after a simple local-height alternative. This replacement will then be useful for dependent lists of bases.

An explicit coefficient-height threshold

For $n \geq 2$, set

$$W_B = \frac{7899}{7900}\frac{t}{nd}\,C_1^*\frac{\Omega}{\max_j h(a_j)}. \tag{10.457}$$

Lemma 10.153 (the large-coefficient branch supplies every height condition). For every original case and every $n \ge 2$,

$$\ln W_B > a_0 n + a_1 + \ln d, \qquad \ln W_B > a_0 n + a_2. \tag{10.458}$$

Proof. Remove one base of maximal height from the independent list, and apply (10.55) to the remaining $n-1$ bases. Use $w_K \ge q^u$, $M_n \ge \mathrm{e}^n t/n^n$, and the exact formula for C_1^*. All powers of q^u, t and ℓ_d cancel, giving

$$W_B \ge \frac{7899}{7900}\frac{c\mathrm{e}}{\rho}\mathfrak{a}^n\frac{(n+1)^{n+2}(n-1)^{n-1}}{(n!)^2}\,dA_n. \tag{10.459}$$

In all cases $\mathfrak{a} \ge \exp(a_0 - 2)$. For $2 \le n \le 511$, taking logarithms in this lower bound and subtracting $a_0 n + a_1 + \ln d$ leaves at least

$$\begin{aligned}&\ln\frac{7899c}{7900\rho} + 1 + (n+2)\ln(n+1) + (n-1)\ln(n-1)\\ &\quad - 2\ln(n!) - 2n + \ln(4 + \ln((n+1)d_0)) - a_1,\end{aligned} \tag{10.460}$$

where $d_0 = 1$ in I.2 and III.2, and $d_0 = 2$ in the other five cases. All 3570 expressions have strictly positive rigorous lower endpoints. Their minimum is greater than $5/1000$, in case IV at $n = 14$. To reproduce this finite certificate, reduce each positive logarithm argument to $[1, 2]$ by powers of two, then use 32 terms of the positive series $2\sum_{j\ge0} z^{2j+1}/(2j+1)$, $z = (x-1)/(x+1)$. The omitted tail is at most $2z^{65}/(65(1-z^2))$. Round every lower endpoint down and every upper endpoint up to 10^{-24}. Form $\ln(n!)$ by adding the integer logarithm intervals; use its upper endpoint in the subtracted term and the lower endpoint of the positive $\ln A_n$. These directions preserve the lower bound at every operation. The reproducible exact calculation records the seven minima and their rational endpoints.

For the infinitely many $n \ge 512$, concavity of $\ln x$ puts its graph above the chord on each interval $[k, k+1]$. Summing their trapezoid integrals gives

$$\ln(n!) - \tfrac{1}{2}\ln n \le \int_1^n \ln x\,dx = n\ln n - n + 1. \tag{10.461}$$

Hence $n! \le \mathrm{e}\sqrt{n}(n/\mathrm{e})^n$. The integral identities $\ln(1+x) > x/(1+x)$ and $\ln(1-x) > -x/(1-x)$, for $x > 0$ and $0 < x < 1$ respectively, give $(1+1/n)^{n+2}(1-1/n)^{n-1} > 1$. Thus the preceding lower bound implies

$$\frac{W_B}{d\exp(a_0 n)} > \frac{7899c}{7900\rho\mathrm{e}}A_n > \frac{7899c}{7900\rho(11/4)}\,10 > \exp(a_1). \tag{10.462}$$

Here $\mathrm{e} < 11/4$, $\exp 6 < 513$, and therefore $A_n \ge 4 + \ln(n+1) > 10$. The final seven inequalities follow from exact positive-exponential series upper bounds: sum through degree 64 and bound the remaining tail by its first term divided by $1 - x/66$, at each $x = a_1 < 6$. Each difference is positive; the smallest exceeds $47/100$, in II. The same series checks $\mathrm{e} < 11/4$ and $\exp 6 < 513$. This proves the first assertion for all ranks. In the degree-one cases $a_2 = a_1$; in the others $\ln d \ge \ln 2$ and $a_2 = a_1 + \ln 2$. The second assertion follows. □

Conversion to the maximum coefficient

Theorem 10.154 (independent-base application bound). For the independent units just described, if $\Xi \neq 1$, then

$$\operatorname{ord}_{\mathfrak{p}}(\Xi - 1) < C_1^* \Omega H. \tag{10.463}$$

At $n = 1$, its right side may be multiplied by $1/2100$.

Proof. First let $n \geq 2$, and write $\mathcal{T} = C_1^* \Omega H$. The height-product lower bound (10.55), and the second term in M_n, give

$$\mathcal{T} \geq \frac{dH}{t} \frac{c\mathfrak{a}^n n^n (n+1)^{n+2} A_n}{\rho (n!)^2} > 7900 \frac{d}{t} \ln 2. \tag{10.464}$$

Indeed consecutive ratios of $k^k/k!$ are at least two, so $n^n/n! \geq 2^{n-1}$ and $(n+1)^n/n! \geq 2^n$. Thus its rank factor is at least 72. Use $c \geq 200$, $\mathfrak{a} \geq 7$, $A_n > 5$, $\rho \leq 58$, $H > 1$ and $\ln 2 < 1$.

The local height inequality in the proof of Theorem 10.21 gives

$$\operatorname{ord}_{\mathfrak{p}}(\Xi - 1) \leq \frac{d}{t} (nB \max_j h(a_j) + \ln 2). \tag{10.465}$$

If $B/\ln B \leq W_B$, its first term is at most $(7899/7900)\mathcal{T}$, and its second is strictly less than $\mathcal{T}/7900$. This proves the desired strict bound.

Otherwise $B/\ln B > W_B$. Renumber the bases so a nonzero coefficient of minimum p-adic valuation is last; C_1 and Ω are unchanged. Since $\ln B > 1$, we have $B > W_B$ and then $B > W_B \ln W_B$. Lemma 10.153 yields $(n+1)\ln B > G_1(n,d)$. The singleton case of (10.55), with $w_K \geq 2$, gives $\min_j h(a_j) \geq 2/(\rho d^2 \ell_d \mathrm{e})$. Also $W(d) > 1$ and $\ell_d \leq d$; thus

$$\frac{R_{\max}}{dW(d)} \leq \rho \mathrm{e} d^2 B. \tag{10.466}$$

We have $\ln \rho + 1 < 6$, since the positive exponential series at five already exceeds 58. Lemma 10.153 gives $n \ln B > 2 \ln d + 6$, using $a_0 > 39/10$, $a_1 > 0$ and $n \geq 2$. Therefore $\ln(R_{\max}/(dW(d))) < (n+1)\ln B$. The other two entries in h are bounded by $(n+1)H$ directly. Apply Theorem 10.152 to obtain $\operatorname{ord}_{\mathfrak{p}}(\Xi - 1) < C_1(n+1)H\Omega = \mathcal{T}$. This argument uses the proved programme singleton height bound in place of a separate unproved lower-height theorem.

Now take $n = 1$. Let g be the original residue order, $\Omega = h(a_1)$, and $\mathcal{T} = C_1^* \Omega H$, with the rank-one definition of M_1. Formula (10.60) bounds the valuation by $(d/t)\ln(2B) + 2edg\Omega/t$. At two, the roots -1 and the q-primary root together have order $2q^u$, so $w_K/q^u \geq 2$; at odd primes it is at least one. Directly in the seven rows, $c\mathfrak{a} w_K/(\rho q^u) > 120$ and $c\mathfrak{a} > 5000$. Use the singleton height bound, $M_1 \geq \mathrm{e}t$, and $A_1 \geq 4 + \ln 2 > 14/3$, to get

$$\mathcal{T} > 4480 \frac{dH}{t}. \tag{10.467}$$

Since $2^3 < 3^2$ and $B \geq 3$, $\ln(2B) < (5/3)H$. Its summand is therefore less than $\mathcal{T}/2688$. For the other summand, retain the actual residue term $M_1 > g/t$ and $eq^u \leq 2d$ from Lemma 10.73. Exact division by $\mathcal{T}$ gives

$$\frac{2edg\Omega/t}{\mathcal{T}} < \frac{t^2}{2c\mathfrak{a} d \ell_d A_1 H} \leq \frac{1}{2c\mathfrak{a} d \ell_d} < \frac{1}{10000}. \tag{10.468}$$

Both A_1 and H are at least t. Finally $1/2688 + 1/10000 < 1/2100$, since $12688 \cdot 2100 < 26880000$. This proves the stronger rank-one assertion. □

For example, let $K = \mathbb{Q}, p = 3, a_1 = 4, b_1 = 3^k$, $k \geq 1$, and choose $B = 3^k$. The original residue order is $g = 1$, so $\delta(a_1) = 2$; the roots in $\mathbb{Q}$ give $q^u = 2$. Here $t = \ln 3$, $H = k \ln 3$, $A_1 = 4 + \ln 2$, and $M_1 = \mathrm{e} \ln 3$: its other entry is $3/(2 \ln 3) < 3/2$, whereas $\mathrm{e} \ln 3 > 2$. Substituting the I.2 constants gives

$$v_3(4^{3^k} - 1) = k + 1 < \frac{4 \cdot 636 \cdot 14}{2100} \mathrm{e}(4 + \ln 2) k \ln 4. \tag{10.469}$$

The valuation equality follows from $v_3(4 - 1) = 1$ and the proved deep-unit power law, Lemma 9.5. The right side is greater than $128k$, using $4 \cdot 636 \cdot 14/2100 > 16$, $\mathrm{e} > 2$, $4 + \ln 2 > 4$, and $\ln 4 > 1$; hence it strictly exceeds $k + 1$. The application bound is deliberately uniform rather than sharp in this simple example. It uses the original rank-one residue order, which is different from the residue group of a saturated basis.

This is the independent-list part of Theorem 1 in Yu's free 2013 paper, Section 8. Independence supplies positive heights and the height-product bounds for every sublist. For a dependent list, a change of basis must also control the new coefficients, its weighted height product and torsion. Those additional conditions are distinct from the coefficient-height conversion proved here.

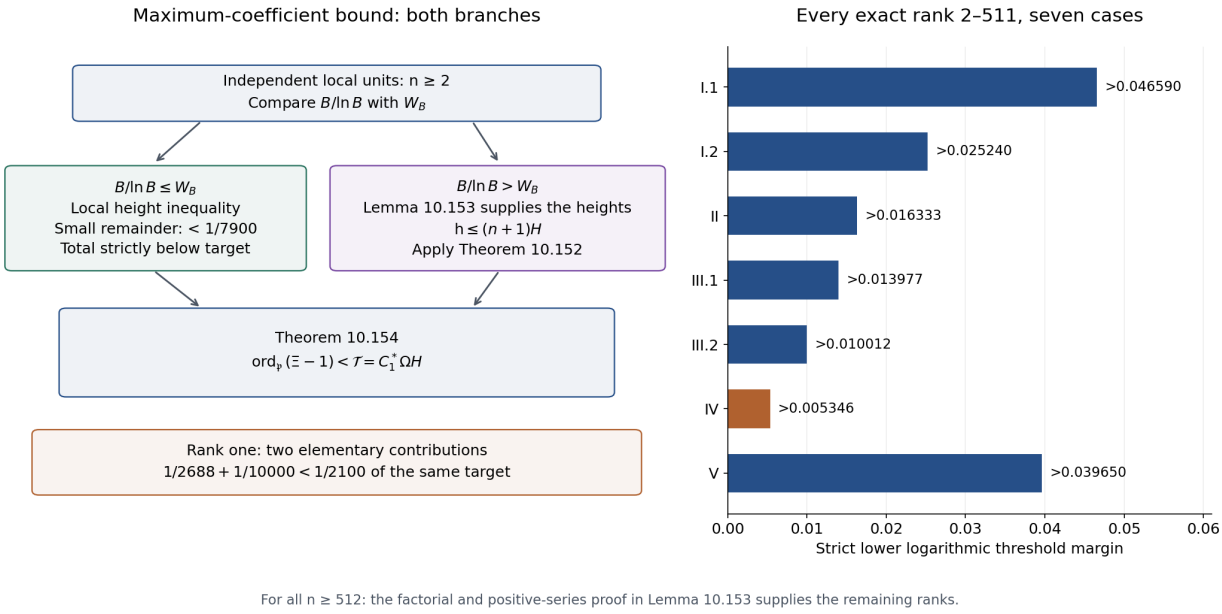


Figure 10.41. For $n \geq 2$, the small-coefficient branch proves the bound directly by the local height inequality; the large-coefficient branch supplies every entry of h *and applies Theorem 10.152. Both give* $\mathrm{ord}_{\mathfrak{p}}(\Xi - 1) < \mathcal{T} = C_1^* \Omega H$. *The bars are strict lower bounds for* $\ln W_B - (a_0 n + a_1 + \ln d)$, *minimized over all exact ranks 2–511 in each field case. The infinite range* $n \geq 512$ *is proved separately in Lemma 10.153, rather than inferred from the bars. At rank one the two elementary contributions sum to less than* $1/2100$ *of* $\mathcal{T}$, *as shown below the diagram. Solution 62 checks the factorial bound, branch conversion and rational example. Figure program, exact calculation. Human-source context: Section 8 of Yu's free 2013 paper; the height conversion here uses the proved programme singleton bound.*

49. A weighted application bound for dependent bases

Keep the field and prime hypotheses of Theorem 10.154. Let $a_1, \ldots, a_n \in K^*$ be local units of multiplicative rank $r \geq 1$ modulo roots of unity. They can now be dependent. Let $b_i \in \mathbb{Z}$, $\Xi = \prod_i a_i^{b_i} \neq 1$, $B \geq \max\{3, |b_1|, \ldots, |b_n|\}$, and $H = \max\{\ln B, t\}$. Sort the indexed list by nondecreasing height, keeping a fixed ordering at equal heights. Define

$$\varkappa=\begin{cases}18, & \mathrm{I},\mathrm{II},\mathrm{IV},\\ 9(p-1)/(p-2), & \mathrm{III},\\ 34, & \mathrm{V},\end{cases}\qquad (10.470)$$
$$X_n=\max\{n,t\},\qquad L_n=\frac{X_n}{\varkappa(n+5)d},\qquad H_n(a)=\max\{h(a),L_n\}.$$

Choose $\boldsymbol{\beta}$ by reading this sorted list and retaining a base exactly when it increases the multiplicative rank. This greedy basis has r elements. Its residue index δ_β is saturated when $r\ge 2$, and is the original rank-one residue index when $r=1$, as in (10.272). All products below are over the indicated indices, so repeated bases are counted correctly in the complementary product. Set

$$\begin{aligned}\Omega_n&=\prod_{a\in\boldsymbol{\beta}}h(a)\prod_{a\notin\boldsymbol{\beta}}H_n(a),\\ C_n(\boldsymbol{\beta})&=c\mathfrak{a}^n\frac{n^n(n+1)^{n+2}d^{n+2}\ell_d}{n!q^ut}\max\left\{\frac{p^f}{\delta_\beta t^{n+1}},\frac{\mathrm{e}^n}{n^n}\right\}A_n,\\ \mathcal{T}_n&=C_n(\boldsymbol{\beta})\Omega_nH,\qquad \lambda=2099/2100,\\ \mathcal{W}_n&=\lambda\frac{t}{nd}C_n(\boldsymbol{\beta})\frac{\Omega_n}{h(a_n)}.\end{aligned}\qquad (10.471)$$

The n in the constant is the length of the original list. Its residue index belongs to the specified r-element basis. We will preserve that basis and index during the proof.

Lemma 10.155 (weighted height and coefficient bounds). The greedy basis minimizes Ω_n over all independent r-element sublists. For $n\ge 2$,

$$\mathcal{T}_n>2100\frac{d}{t}\ln 2,\qquad \mathcal{W}_n>\mathrm{e}^nd,\qquad \mathcal{W}_n>w_K.\qquad (10.472)$$

Suppose additionally $r<n$, a_1 is non-torsion and $B/\ln B>\mathcal{W}_n$. Let x be the first index discarded by the greedy procedure. Its minimal circuit has an exact integer relation with positive coefficient at a_x, and every coefficient has absolute value at most K, where

$$K<\begin{cases}B/8, & x<n,\\ BdH_n(a_x)/8, & x=n.\end{cases}\qquad (10.473)$$

The rank factor needed after removal satisfies, for every $n\ge 2$,

$$J_n:=\frac{(n+1)^{n+2}}{(n-1)^{n-1}n^2}\left(\frac{n+4}{n+5}\right)^{n-2}>\mathrm{e}(n+5).\qquad (10.474)$$

Proof. Rank is ordinary linear rank in the rational space of formal exponent vectors modulo rational relations whose products are roots of unity. Clearing denominators proves that these relations form a subspace. Independence modulo torsion is the same as multiplicative independence, because a torsion product can be raised to its finite order. This supplies the elementary linear algebra used by the greedy procedure.

Write its selected indices as $i_1<\cdots<i_r$. Any independent sublist with indices $j_1<\cdots<j_r$ has $j_k\ge i_k$: otherwise it would give k independent vectors before the greedy list first attained rank k. The function $h/\max\{h,L_n\}$ increases with h. Factoring every candidate product as the common product of all $H_n(a_i)$, times these r ratios, proves the minimum assertion. Removing a discarded

dependent base or a torsion base leaves the ordered spans unchanged. Thus it also leaves the greedy basis unchanged, including when the floor becomes L_{n-1}.

For an independent sublist of length $j \ge 1$, (10.55) gives

$$\prod h(a) \ge \frac{w_K}{\rho d^{j+1}\ell_d}\frac{j^j}{j!\mathrm{e}^j}. \tag{10.475}$$

We will also use this at j=0, with the empty product and the last ratio both defined to be one. To justify it, apply the untwisted count of Theorem 10.17 at $H_0 > 1$ to the height-zero roots, then let H_0 decrease to one. It gives $w_K \le 30d\ln d$ for $d \ge 2$, and $w_K \le 17$ for $d = 1$. In both cases $w_K \le \rho d\ell_d$, precisely the required empty-list bound.

To bound $\mathcal{W}_n$, remove a_n. If it is selected, put k=r; otherwise put k=r+1 and use $H_n(a_n)/h(a_n) \ge 1$. In each case $1 \le k \le n$. Apply the height bound at length k-1, and the floors to the n-k remaining complementary elements. The second entry of the maximum in C_n yields

$$\begin{aligned}\mathcal{W}_n \ge \lambda\frac{c}{\rho}\frac{w_K}{q^u}\frac{dA_n}{n}\,\mathfrak{a}^n\mathrm{e}^{n-k+1}\frac{(n+1)^{n+2}}{n!}\\ \times\left(\frac{X_n}{\varkappa(n+5)}\right)^{n-k}\frac{(k-1)^{k-1}}{(k-1)!}.\end{aligned} \tag{10.476}$$

The case constants give $\mathfrak{a}/\varkappa \ge 13/17 > 3/4$, and $\mathrm{e} > 8/3$, so $\mathfrak{a}\mathrm{e}/\varkappa > 2$. Also $[n/(n+5)]^{n-k} > \mathrm{e}^{-5}$, since $\ln(1+5/n) < 5/n$. Use the factorial bound of Lemma 10.153 and $(n+1)^2/n^{3/2} \ge 3$ to obtain

$$\mathcal{W}_n > \frac{\lambda(200/58)\,105}{(11/4)^5}\frac{w_K}{q^u}d\mathrm{e}^n > 2\frac{w_K}{q^u}d\mathrm{e}^n. \tag{10.477}$$

For the rank factor just used, the derivative of $(x+1)^2/x^{3/2}$ is $(x+1)(x-3)/(2x^{5/2})$. Its minimum at three is $16/(3\sqrt{3}) > 3$, because $256 > 243$. The displayed rational prefactor is greater than two by exact multiplication. Since $w_K \ge q^u$ and Lemma 10.73 gives $q^u \le 2d$, both assertions about $\mathcal{W}_n$ follow.

For $\mathcal{T}_n$, use the height bound at length r and all n-r complementary floors. Put $z_n = 2n/(n+5)$. Since $7 > z_n$, $\mathfrak{a}\mathrm{e}/\varkappa > 2$ and $X_n \ge n$, the resulting bound is

$$\frac{\mathcal{T}_n}{dH/t} \ge \frac{200}{58}\,5\cdot 7\,\frac{(n+1)^{n+2}}{n!}\,z_n^{n-1}. \tag{10.478}$$

Consecutive ratios of $(n+1)^{n+2}/n!$ exceed two, so it is at least $(81/2)2^{n-2}$. At n=2 the remaining factor $2^{n-2}z_n^{n-1}$ is $4/7$; at three and four it is $9/8$ and $2048/729$; at all n at least five it is at least eight. Thus the preceding coefficient is at least $81000/29 > 2100$. Use $H > 1$ and $\ln 2 < 1$ to prove the assertion about $\mathcal{T}_n$.

For the circuit assertion, let $a_1, \dots, a_m$ be the longest initial independent block, and set x=m+1. Choose a minimal dependent sublist containing a_x and v earlier selected bases, with $1 \le v \le r$. Let S be these v bases. Theorem 10.20 gives an exact integer relation, and its sign can make the coefficient at x positive. Every earlier height is at most $h(a_x)$, so its coefficients satisfy

$$K \le \rho d^{v+1}\ell_d\frac{v!\mathrm{e}^v}{v^v}\frac{h(a_x)\prod_{a\in S}h(a)}{\min_{a\in S}h(a)}. \tag{10.479}$$

Put j=r-v+1 and s=n-r-1. After dividing by Ω_n, the height denominator is the smallest-height member of S and the basis elements outside S, an independent list of length j. Therefore

$$\frac{K}{\Omega_n} \le \frac{\rho^2 d^{r+3}\ell_d^2}{w_K}\,\mathrm{e}^{r+1}\frac{v!j!}{v^v j^j}\frac{h(a_x)}{H_n(a_x)}L_n^{-s}. \tag{10.480}$$

Normalize using $B > \mathcal{W}_n \ln B$. In the terminal case x=n, divide additionally by $dH_n(a_x)$. The normalized ratio of 8K to the claimed coefficient bound is at most

$$\frac{8n\rho^2}{\lambda c\mathfrak{a}^{r+1}(n+1)^2}\left[\frac{\varkappa}{\mathfrak{a}\mathrm{e}}\left(1+\frac{5}{n}\right)\right]^s. \tag{10.481}$$

This uses $q^u/w_K \le 1$, $\ell_d/A_n \le 1$, $\ln B > 1$, $n! \le n^n$, $v!j!/(v^v j^j) \le 1$ and $h(a_x)/H_n(a_x) \le 1$. Every degree power cancels, since r+3+s=n+2. If x<n and a_n is selected, it is outside S. Cancel its height against the $h(a_n)$ in $\mathcal{W}_n$, and use the height bound at length j-1. This yields the same envelope multiplied by 1/e; j=1 is covered by the empty-list bound. If a_n is not selected, cancel its height against its modified height. Use s-1 floors to obtain the envelope with exponent s-1 and the extra factor $1/(\mathfrak{a}\mathrm{e})$. These alternatives exhaust the internal case, since S lies before x<n.

We have $\varkappa/(\mathfrak{a}\mathrm{e}) < 1/2$. For $0 \le s \le n-2$, $[(1+5/n)/2]^s \le 4/3$: the exponent is zero at n=2; at n=3 it is at most one; at n=4 use $(9/8)^2 < 4/3$; at n at least five the base is at most one. Also $n/(n+1)^2 \le 2/9$. Every normalized ratio is therefore strictly less than

$$\frac{64\cdot 58^2}{27\lambda\cdot 200\cdot 7^2} = \frac{107648}{132237} < 1. \tag{10.482}$$

This proves both circuit bounds, including the field height and all relation coefficients. The exact calculation records these rational endpoints.

Finally, for $2 \le n \le 15$, direct rational evaluation gives $J_n/(n+5) > 11/4 > \mathrm{e}$. For n at least sixteen put y=1/n. Integrating the geometric-series remainders gives $\ln(1+x) \ge x - x^2/2$, $\ln(1+x) \le x - x^2/2 + x^3/3$ for x nonnegative, and $-\ln(1-y) \ge y + y^2/2$. Substitute them into

$$\begin{aligned}\ln\frac{J_n}{n+5} &= (n+2)\ln(1+y) - (n-1)\ln(1-y)\\ &\quad + (n-2)\ln(1+4y) - (n-1)\ln(1+5y).\end{aligned} \tag{10.483}$$

The result is at least $1 + (5/2)y - (119/3)y^2 + (125/3)y^3$. Since $y \le 1/16$, it is greater than $1 + y/48$. Exponentiation proves the rank-factor assertion for all remaining n. □

Theorem 10.156 (the dependent-list main application estimate). For the stated local units, coefficients and specified greedy basis,

$$\mathrm{ord}_{\mathfrak{p}}(\Xi - 1) < C_n(\boldsymbol{\beta})\Omega_n \max\{\ln B, t\}. \tag{10.484}$$

Proof. Induct on n. The full-rank case r=n, including n=1, is Theorem 10.154. Suppose r<n and n at least two. If $B/\ln B \le \mathcal{W}_n$, the local height inequality gives $\mathrm{ord}_{\mathfrak{p}}(\Xi-1) \le (d/t)(nBh(a_n)+\ln 2) < \mathcal{T}_n$: its two contributions are at most $\lambda\mathcal{T}_n$ and strictly less than $\mathcal{T}_n/2100$, by Lemma 10.155.

Otherwise $B > \mathcal{W}_n \ln B > \mathcal{W}_n$, so $\ln B > n + \ln d$, $B > w_K$ and $H \ge X_n$. First suppose a_1 is non-torsion. Use the circuit at the first discarded index x, with positive coefficient k_x at x and coefficients bounded by K. Raising Ξ to k_x eliminates a_x. Its new integer coefficients have absolute value at most 2BK. The greedy basis and its residue index are unchanged. If $\Xi^{k_x} = 1$, then Ξ is a root of unity; Theorem 10.21 gives $\mathrm{ord}_{\mathfrak{p}}(\Xi - 1) \le (d/t)\ln 2 < \mathcal{T}_n$.

Otherwise put $z = dH_n(a_x)$. The new coefficients are at most $B^2/4$ for x<n, and $B^2 z/4$ for x=n. For every positive u, $\ln u \le u/\mathrm{e}$: differentiating $\ln u/u$ locates its maximum at $u = \mathrm{e}$. A valid new bound is therefore $B' = B^2 \exp(z/(4\mathrm{e}))$, and its corresponding parameter satisfies

$$H' \le H\left(2 + \frac{z}{4\mathrm{e}X_n}\right). \tag{10.485}$$

The two entries in the maximum defining C_n have ratios $1/t$ and $\mathrm{e}(n-1)^{n-1}/n^n > 1/n$ compared with C_{n-1}. The latter inequality follows from $(n-1)\ln(1-1/n) > -1$. Their maximum thus has ratio at least $1/X_n$, and $A_n \ge A_{n-1}$. Exact cancellation gives

$$\frac{C_n(\boldsymbol{\beta})}{C_{n-1}(\boldsymbol{\beta})} \ge \mathfrak{a}\frac{d}{X_n}\frac{(n+1)^{n+2}}{(n-1)^{n-1}n^2}. \tag{10.486}$$

Let Ω' be the modified product of the remaining list with its new floor and the same basis. Each surviving modified height has ratio at least $(n+4)/(n+5)$, because $X_n \ge X_{n-1}$. Consequently

$$\frac{\Omega_n}{\Omega'} \ge H_n(a_x)\left(\frac{n+4}{n+5}\right)^{n-r-1} \ge H_n(a_x)\left(\frac{n+4}{n+5}\right)^{n-2}. \tag{10.487}$$

The product of these two ratios exceeds $\mathfrak{a}\mathrm{e}(n+5)z/X_n$, by Lemma 10.155. Since $z/X_n \ge 1/(\varkappa(n+5))$,

$$\begin{aligned}\left[\mathfrak{a}\mathrm{e}(n+5) - \frac{1}{4\mathrm{e}}\right]\frac{z}{X_n} &\ge \frac{\mathfrak{a}\mathrm{e}}{\varkappa} - \frac{1}{4\mathrm{e}\varkappa(n+5)} \\ &> \frac{104}{51} - \frac{1}{672} > 2.\end{aligned} \tag{10.488}$$

This uses $\mathfrak{a}/\varkappa \ge 13/17$, $\mathrm{e} > 8/3$, $\varkappa \ge 9$ and n+5 at least seven. Thus the original $C_n\Omega_n$ absorbs the complete increase of the coefficient-height parameter. Apply induction to Ξ^{k_x}, and the factored-power inequality $\mathrm{ord}_{\mathfrak{p}}(\Xi - 1) \le \mathrm{ord}_{\mathfrak{p}}(\Xi^{k_x} - 1)$, to prove the claim.

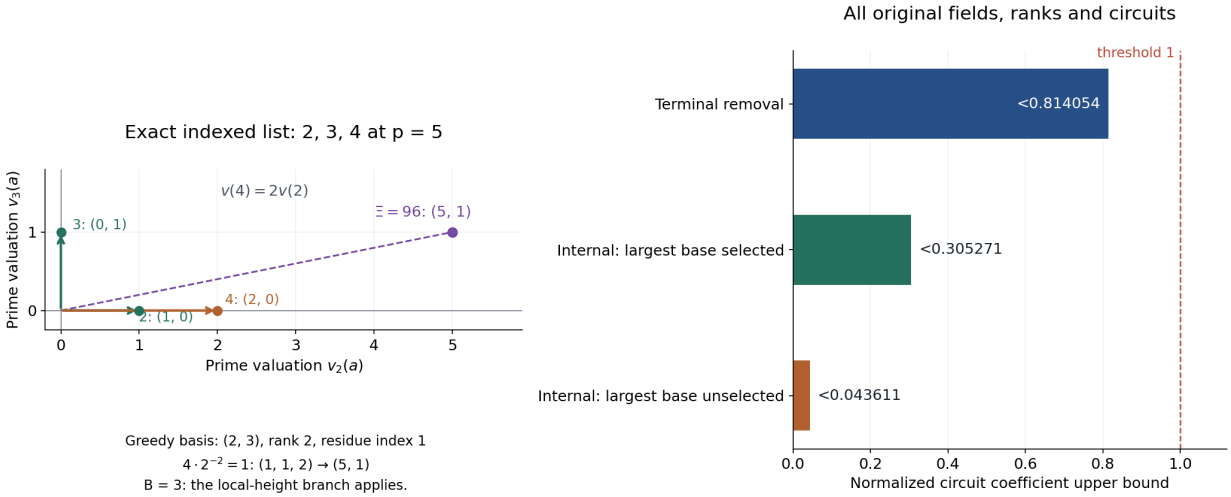
If a_1 is torsion, raise Ξ to its order $w \le w_K$. This removes a_1 and preserves the basis. If the resulting power is one, use the torsion bound above. Otherwise its coefficients are bounded by $wB < B^2$, so $H' \le 2H$. The product-ratio argument, now with deleted height $H_n(a_1) = L_n$, gives a ratio greater than $\mathfrak{a}\mathrm{e}/\varkappa > 104/51 > 2$. Induction and the same factored-power inequality conclude the proof. □

For an indexed example, take $K = \mathbb{Q}, p = 5$ and the sorted list $(2, 3, 4)$. Prime valuations at two and three give vectors $(1, 0), (0, 1), (2, 0)$; thus the greedy basis is $(2, 3)$ and r=2. Its q-primary saturation has no extra rational exponent, again by prime valuations. The residue of two generates all four nonzero classes at five, so $\delta_\beta = 1$. Here $\varkappa = 12$, $L_3 = 1/32$, and every actual height exceeds this floor. Hence

$$\Omega_3 = (\ln 2)(\ln 3)(\ln 4), \qquad 4 \cdot 2^{-2} = 1, \qquad 2^1 3^1 4^2 = 2^5 3^1 = 96. \tag{10.489}$$

Both sides have valuation $v_5(96 - 1) = 1$. With B=3, $B/\ln B < 3 < 8 < \mathcal{W}_3$, by Lemma 10.155, so this example is in the local-height branch. The exact circuit identity also illustrates how the coefficient transfer keeps the selected basis and its residue index. The full list still has length three in C_3, while its basis has rank two. For a list of rank zero, $\Xi \ne 1$ is torsion and Theorem 10.21 directly gives $\mathrm{ord}_{\mathfrak{p}}(\Xi - 1) \le (d/t)\ln 2$.

Theorem 10.156 proves the main dependent-list inequality corresponding to Theorem 1 of Yu's free 2013 paper, Section 8. Its statement uses the specified greedy basis, which attains the minimum weighted product. The factor 1/2100 established in Theorem 10.154 applies to an independent list of length one.



Lemma 10.155 proves the three envelopes; Theorem 10.156 performs the complete main-bound induction.

Figure 10.42. Left: exact prime-valuation vectors of the rational list $(2,3,4)$ *give the greedy basis* $(2,3)$*; the circuit* $4\cdot 2^{-2}=1$ *transfers exponents* $(1,1,2)$ *to* $(5,1)$*. Its actual value is 96, of valuation one at five. This B=3 example uses the local-height branch. Right: proved upper bounds for the normalized circuit coefficient cost in the large-coefficient branch, for all original fields, ranks and circuit sizes. Lemma 10.155 treats terminal removal, internal removal with the largest-height base selected, and internal removal with that base unselected. Each lies below one. Theorem 10.156 then performs the rank-preserving induction, including torsion. Solution 63 checks the coefficient, height-floor and scalar-rank cancellations. Figure program, exact universal envelopes. Human-source context: Section 8 of Yu's freely accessible paper.*

50. The sharper bound for a dependent list of rank one

Retain the units, sorted indexed list, greedy basis, heights and constants of Section 49. Suppose its multiplicative rank is one. Write i_0 for the selected index, $\beta=a_{i_0}$ for its first non-torsion base, and $g=|\langle\overline{\beta}\rangle|$. Its residue index is the original one, $\delta_\beta=(p^f-1)/g$. Put

$$R=\prod_{i\neq i_0}\frac{H_n(a_i)}{L_n}\geq 1,\qquad U=\frac{t}{d}\mathcal{T}_n. \tag{10.490}$$

The product is over indices other than the selected index of β, including roots of unity and repetitions. A direct reduction to one base will also handle coefficients that are too small for the large-coefficient branch of Theorem 10.156.

Two uniform budgets

For $n\geq 2$ define

$$\begin{aligned}
F_n&=\frac{c}{\rho}\frac{w_K}{q^u}\mathfrak{a}^n\mathrm{e}^{n-1}\frac{(n+1)^{n+2}}{n!}\left(\frac{n}{\varkappa(n+5)}\right)^{n-1},\\
E_n&=\frac{4n!\varkappa^{n-1}(n+5)^{n-1}}{c\mathfrak{a}^n n^n(n+1)^{n+2}},\\
\alpha_n&=\left(\frac{1}{2100}-\frac{1}{40000}\right)F_n.
\end{aligned} \tag{10.491}$$

Lemma 10.157 (rank-one scalar budgets). These quantities satisfy

$$U \geq F_n A_n H R, \qquad \frac{2egh(\beta)}{U} < \frac{1}{40000}, \qquad F_{n+1} > 3F_n. \tag{10.492}$$

At $n = 2$, the following bounds hold. In the last column the field degree and ℓ_d are already included.

Field case	Lower bound for α_2	Upper bound for $E_2/(d\ell_d)$
I.1	9/4	1/118314
I.2	9/4	1/40068
II	4/3	1/63630
III.1	9/4	1/113022
III.2	9/4	1/56385
IV	9/4	1/337680
V	19/10	119/5639868

Every lower bound is strict, and every last entry is less than 1/40000. For $n \geq 3$, $\alpha_n > 4n/3$.

Proof. The singleton case of (10.55) gives $h(\beta) \geq w_K/(\rho e d^2 \ell_d)$. Insert this and $\Omega_n = h(\beta) L_n^{n-1} R$ into U, using the second entry of the maximum in C_n and $X_n \geq n$. All degree and height-floor powers cancel to give the first inequality.

For the second inequality use the first entry instead. The original residue order gives $p^f/\delta_\beta > g$, while $L_n \geq t/[\varkappa(n+5)d]$. Exact division yields

$$\frac{2egh(\beta)}{U} < \frac{2eq^u n! \varkappa^{n-1}(n+5)^{n-1} t^2}{c\mathfrak{a}^n n^n (n+1)^{n+2} d^2 \ell_d R A_n H} \leq \frac{E_n}{d\ell_d R}. \tag{10.493}$$

The last step uses $eq^u \leq 2d$ from Lemma 10.73, and $t^2 \leq A_n H$. At two, $E_2 = 14\varkappa/(81c\mathfrak{a}^2)$. In I.1, II, III.1, IV and V use $d \geq 2$; in the other two cases use $d = 1$. Also $\ell_d \geq 1$. Direct substitution of the seven case constants gives the last column. In III, $\mathfrak{a}^2/\varkappa \geq 49/9$, including every prime in that case.

For the middle column, use $\mathrm{e} > 27/10$, obtained by summing its positive exponential series through degree five. At odd primes $w_K/q^u \geq 1$. At two, -1 together with the q-primary roots has order $2q^u$, so $w_K/q^u \geq 2$. Thus

$$F_2 > \frac{c}{\rho} \frac{w_K}{q^u} \frac{\mathfrak{a}^2}{\varkappa} \frac{2187}{70}. \tag{10.494}$$

Use $\rho = 17$ in degree one and $\rho = 58$ otherwise. Substitution gives the stated strict rational bounds on α_2. The exact scalar calculations include each difference as a positive fraction.

The ratio of consecutive F_n is

$$\frac{F_{n+1}}{F_n} = \frac{\mathfrak{a}\mathrm{e}}{\varkappa} \left(1 + \frac{1}{n+1}\right)^{n+3} \frac{n+1}{n+6} \left(\frac{(n+1)(n+5)}{n(n+6)}\right)^{n-1}. \tag{10.495}$$

We have $\mathfrak{a}\mathrm{e}/\varkappa > 104/51$. For $n = 2, 3, 4, 5$, the rational product after this factor is, respectively,

$$\frac{56}{27}, \quad \frac{15625}{6561}, \quad \frac{1594323}{625000}, \quad \frac{11529602}{4348377}. \tag{10.496}$$

Multiplying each by 104/51 gives a number greater than three. For all $n \geq 6$, the first power in the ratio is greater than e, by $\ln(1+x) > x/(1+x)$. The last power is greater than one and

$(n+1)/(n+6) \geq 7/12$. The ratio is therefore greater than $(104/51)(8/3)(7/12) = 1456/459 > 3$. This proves the claim at every rank.

Direct cancellation also gives $E_{n+1}/E_n = enF_n/[(n+1)F_{n+1}] < 11/12$. Consequently the last column bounds every $E_n/(d\ell_d R)$. Finally $\alpha_2 > 4/3$ in every row, so $\alpha_n > 4 \cdot 3^{n-3} \geq 4n/3$ for $n \geq 3$. The latter inequality follows by induction, starting at three. □

Simultaneous reduction to the selected base

Theorem 10.158 (the rank-one application refinement). Under the hypotheses of Theorem 10.156, if the list has rank one, then

$$\operatorname{ord}_{\mathfrak{p}}(\Xi - 1) < \frac{1}{2100} C_n(\beta)\Omega_n \max\{\ln B, t\}. \tag{10.497}$$

Proof. At $n = 1$ this is Theorem 10.154. Let $n \geq 2$. Put

$$D_n = \rho e d^2 \ell_d L_n, \qquad D_0 = \max\{1, D_n\}. \tag{10.498}$$

For each non-torsion base at an index $i \neq i_0$, Theorem 10.20 applied to the dependent pair (β, a_i) supplies an exact relation $\beta^{u_i} a_i^{k_i} = 1$, with k_i a positive integer and $k_i \leq \rho e d^2 \ell_d h(a_i)$. Here $h(\beta) \leq h(a_i)$, so the smaller height in that theorem is $h(\beta)$. Taking heights in the relation gives $|u_i|/k_i = h(a_i)/h(\beta)$.

Let m be the number of these non-torsion complementary indices. If the original list has no torsion bases, put $P = \prod k_i$. Otherwise put $P = w_K \prod k_i$. Every torsion base then disappears, and the relations give

$$\Xi^P = \beta^M, \qquad M \in \mathbb{Z}, \qquad |M| \leq nBP \frac{h(a_n)}{h(\beta)}. \tag{10.499}$$

This is an identity in K, with no change in the residue index. If the power is one, Ξ is torsion. Theorem 10.21 and $\mathcal{T}_n > 2100(d/t)\ln 2$ from Lemma 10.155 give the required bound. Hence assume M is nonzero.

Each $k_i \leq D_0 H_n(a_i)/L_n$, so the product of these coefficients is at most $D_0^m R$. If there is a non-torsion complementary base, the largest height occurs among the complementary indices, or equals the selected height. The singleton height bound then gives

$$\frac{h(a_n)}{h(\beta)} \leq \max\{1, D_n/w_K\} R. \tag{10.500}$$

When there is no such base, this ratio is one. Since $\ln B \leq H$, the resulting bounds for $\ln(2|M|)$ are

$$\begin{cases} Z_0 + 2\ln R, & \text{no torsion bases,} \\ Z_1 + 2\ln R, & \text{torsion and a non-torsion complement,} \\ H + \ln(2w_K), & \text{only torsion in the complement,} \end{cases} \tag{10.501}$$

where

$$\begin{aligned} Z_0 &= H + \ln(2n) + (n-1)\ln D_0 + \ln\max\{1, D_n/w_K\}, \\ Z_1 &= H + \ln(2n) + (n-2)\ln D_0 + \ln\max\{w_K, D_n\}. \end{aligned} \tag{10.502}$$

We show that all three are less than $\alpha_n A_n H R$. The empty-list height bound in Lemma 10.155 gives $w_K \le \rho d \ell_d$. Since $D_0 \ge D_n$,

$$\frac{w_K}{D_0} \le \frac{\varkappa(n+5)}{\mathrm{e} X_n} < 45. \tag{10.503}$$

Indeed $\varkappa \le 34$, $X_n \ge n$, $(n+5)/n \le 7/2$ and $\mathrm{e} > 8/3$. Every displayed alternative is thus at most $Z_n + 2\ln R$, where

$$Z_n = H + \ln(2n) + n \ln D_0 + \ln 45. \tag{10.504}$$

For $n \ge 3$, use $X_n \le nH$, $H > 1$, and $\rho \mathrm{e}/\varkappa < 18$. Concavity of the logarithm at eight gives $\ln \ell_d \le (\ln d)/8 + 11/10$: if $d \ge 3$ use $\ln x \le x/8 + \ln 8 - 1$, and if $d = 1, 2$ the left side is zero. Also $\ln H \le H - 1$, $\ln 18 < 3$ and $\ln 45 < 4$. Consequently

$$Z_n < (n+1)H + \frac{31n}{10} + \ln(2n) + 4 + \frac{9n}{8} \ln d. \tag{10.505}$$

By Lemma 10.157, $\alpha_n > 4n/3$; since $H > 1$, its coefficient of $\ln d$ is greater than $9n/8$. For the remaining terms use $A_n \ge 4 + \ln(n+1) + \ln d$, $\ln(n+1) \ge \ln 4 > 4/3$, and $\ln(2n) < n$. At H=1 their comparison reduces to

$$\frac{64n}{9} > \frac{51n}{10} + 5 \qquad (n \ge 3). \tag{10.506}$$

The difference is $181n/90 - 5 > 0$. Increasing H only increases the difference, because $\alpha_n[4 + \ln(n+1)] > n + 1$. Thus $Z_n < \alpha_n A_n H$ throughout this infinite range.

At $n = 2$ there are just two possibilities. If the complementary base is non-torsion, the first alternative is at most $H + \ln 4 + 2\ln D_0 + 2\ln R$. Here $D_0 \le s d \ell_d H$, where $s = \max\{1, 2\rho\mathrm{e}/(7\varkappa)\}$. In II, $s < 319/126 < \mathrm{e}$; in every case $s < 319/63 < \exp(5/3)$. The logarithmic tangent at sixteen gives $\ln \ell_d \le (\ln d)/16 + 9/5$. Using $\ln 4 < 7/5$ and $\ln H \le H - 1$ bounds its part independent of R by

$$\begin{cases} 3H + 5 + (17/8)\ln d, & \text{II}, \\ 3H + 19/3 + (17/8)\ln d, & \text{all other cases}. \end{cases} \tag{10.507}$$

In II, $H \ge \ln 5 > 8/5$ and $\alpha_2 > 4/3$. In V, $f \ge 2$, hence $H \ge 2\ln 2 > 4/3$, and $\alpha_2 > 19/10$. In all remaining cases $H > 1$ and $\alpha_2 > 9/4$. These give $\alpha_2 H > 17/8$. Since $4 + \ln 3 > 5$, the constant comparisons in II and the other cases follow from $(20/3 - 3)(8/5) > 5$ and $19/2 - 3 > 19/3$, respectively. Thus this alternative is less than $\alpha_2 A_2 H + 2\ln R$.

If the complement is torsion, its bound is $H + \ln(2w_K)$. The same tangent, with $\ln \rho < 5$, gives at most $H + 15/2 + (17/16)\ln d$. The coefficient comparison already established is stronger than needed. For the constant part, in II use $(20/3 - 1)(8/5) > 15/2$, and otherwise use $19/2 - 1 > 15/2$. This proves the comparison in the second possibility too. All numerical logarithm comparisons used here follow from the positive series in Solution 64; they require no approximation to be accepted as an equality.

In every case the bound before the $2\ln R$ term is less than $\alpha_n A_n H$, which is greater than two. The inequality $\ln R \le R - 1$, for $R \ge 1$, therefore proves

$$\ln(2|M|) < \alpha_n A_n H R \le \left(\frac{1}{2100} - \frac{1}{40000}\right) U. \tag{10.508}$$

Apply Theorem 10.21 to β^M, using its actual residue order g. Factoring the P-th power difference gives

$$\begin{aligned} \operatorname{ord}_{\mathfrak{p}}(\Xi - 1) &\le \operatorname{ord}_{\mathfrak{p}}(\beta^M - 1) \\ &\le \frac{d}{t}\left(\ln(2|M|) + 2egh(\beta)\right) < \frac{\mathcal{T}_n}{2100}, \end{aligned} \tag{10.509}$$

by the two budgets. This includes every size of B and every torsion pattern. □

For example, at five in $K = \mathbb{Q}$, the indexed list $(4, 16)$ has rank one and selected base four, with original residue order two and $\delta_\beta = 2$. Its floor is $L_2 = 1/42$, and $\Omega_2 = (\ln 4)(\ln 16)$. For coefficients $(0, 5^k)$, $k \ge 0$, choose $B = \max\{3, 5^k\}$. The exact relation $16 \cdot 4^{-2} = 1$ permits P=1 and gives $\Xi = 4^{2\cdot 5^k}$. Lemma 9.5 proves $v_5(\Xi - 1) = 1 + k$. The bound of Theorem 10.158 uses the length-two constant and this weighted product, while keeping the raw residue order of four. It does not replace it by the order of a saturated base.

The rank-one factor is also asserted in Theorem 1 of Yu's freely accessible paper. The proof above uses the exact integer relations and the proved one-base power bound for the whole list; it therefore also covers the small-coefficient case directly.

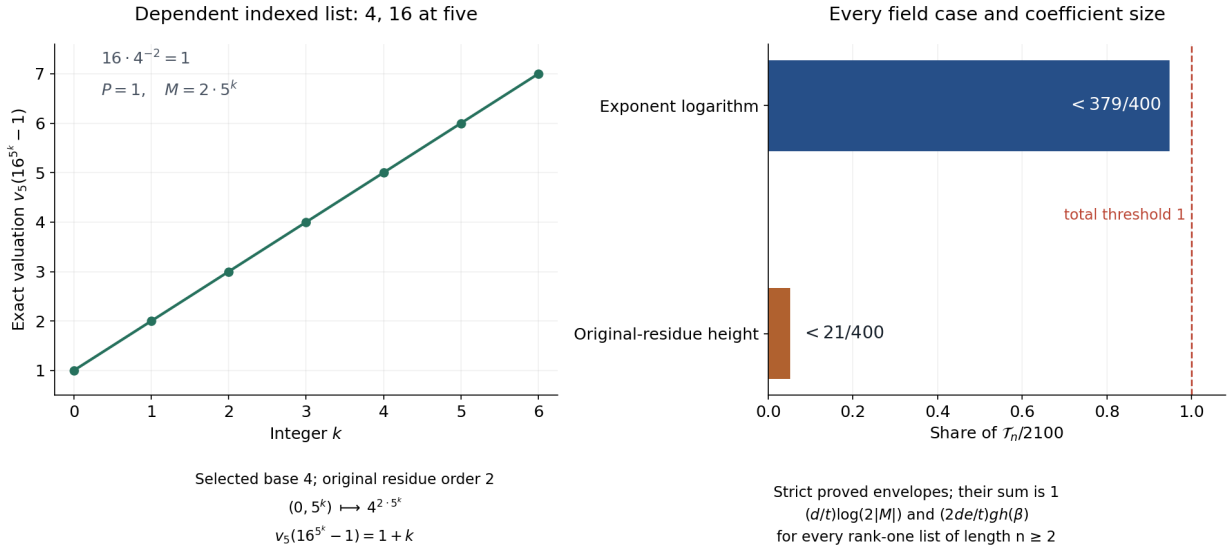


Figure 10.43. Left: the dependent indexed list $(4, 16)$ *at five, with coefficients* $(0, 5^k)$*, reduces exactly to* $4^{2\cdot 5^k}$*; the plotted valuations* $1 + k$ *are exact values, proved by the deep-unit power law. Right: the normalized logarithmic exponent cost is strictly below* $1/2100 - 1/40000$*, and the original-residue height cost is strictly below* $1/40000$*. These are universal proved envelopes for every rank-one list of length at least two, every field case and every coefficient size, whose sum is* $1/2100$*. Lemma 10.157 proves the scalar budgets and Theorem 10.158 proves the full reduction, including torsion. Figure program, exact scalar calculations. Human-source context: Theorem 1 and Section 8 of Yu's freely accessible paper.*

51. An effective p-adic distance between integral S-units

Let S be a finite set of rational primes. An integral S-unit here is a nonzero integer whose prime factors all belong to S; either sign is allowed. Let x and y be distinct integral S-units, set $M = \max\{|x|, |y|\}$, and let p be any rational prime. Put

$$g_0 = \gcd(|x|, |y|), \qquad x_0 = x/g_0, \qquad y_0 = y/g_0, \qquad S_p = S \setminus \{p\}. \tag{10.510}$$

The common factor must be retained. For example, $x = 3^N 25, y = 3^N 7$ have difference $3^N 18$, of valuation $N + 2$ at three. Their coprime parts have fixed distance of valuation two, however large N is.

Write $S_p = \{q_1 < \cdots < q_n\}$. For $n \geq 1$ the following explicit constants will bound the distance of the coprime parts. Use the field

$$K_p = \begin{cases} \mathbb{Q}, & p > 2, \\ \mathbb{Q}(\zeta_3), & p = 2, \end{cases} \qquad (d, e, f, q^u) = \begin{cases} (1, 1, 1, 2), & p > 2, \\ (2, 1, 2, 3), & p = 2. \end{cases} \tag{10.511}$$

Here ζ_3 is a primitive cube root of unity, and q is the auxiliary prime of the earlier logarithm estimates, distinct from the listed primes q_i. Put $t = f \ln p$. The rational prime bases are independent, so set $\boldsymbol{\beta} = (q_1, \ldots, q_n)$. Use their saturated residue index when $n \geq 2$, and the raw index at $n = 1$.

These indices are entirely explicit. At odd p and $n \geq 2$, let G be the subgroup of $\mathbb{F}_p^*$ generated by $-1, q_1, \ldots, q_n$; then $\delta_\beta = (p-1)/|G|$. At odd p and $n = 1$, use $\delta_\beta = (p-1)/\operatorname{ord}_p(q_1)$. At two, the indices are one for $n \geq 2$ and three for $n = 1$. All these finite orders can be found by enumerating a finite multiplicative group.

Let $\epsilon = \operatorname{sgn}(x_0/y_0)$, and define

$$\begin{aligned} m &= \begin{cases} n, & \epsilon = 1, \\ n+1, & \epsilon = -1, \end{cases} \\ \Omega_\epsilon &= \begin{cases} \displaystyle\prod_{i=1}^{n} \ln q_i, & \epsilon = 1, \\ \displaystyle L_m \prod_{i=1}^{n} \ln q_i, & \epsilon = -1, \end{cases} \\ \eta_n &= \begin{cases} 1/2100, & n = 1, \\ 1, & n \geq 2, \end{cases} \\ K_{S,p}^{\epsilon} &= \eta_n C_m(\boldsymbol{\beta}) \Omega_\epsilon, \\ B_M &= \max\{3, \ln M / \ln 2\}, \qquad H_{S,p}(M) = \max\{\ln B_M, t\}. \end{aligned} \tag{10.512}$$

The L_m, C_m, $\varkappa$ and case constants are those of Section 49, computed in the specified field K_p. Thus C_m uses the original list length m and the index of its n-element basis. The constant depends on S, p and the sign, but not on the exponents occurring in x and y.

Theorem 10.159 (integral S-unit distance). If $n \geq 1$, then

$$v_p(x - y) < v_p(g_0) + K_{S,p}^{\epsilon} H_{S,p}(M). \tag{10.513}$$

If $n = 0$, the exact excess $v_p(x - y) - v_p(g_0)$ is zero unless $x_0 = -y_0 = \pm 1$, in which case it is $v_p(2)$.

Proof. Coprimality of $|x_0|$ and $|y_0|$ implies that at most one is divisible by p. If one is divisible by p, their difference is a p-adic unit, so $v_p(x - y) = v_p(g_0)$. For $n \geq 1$ the constant is positive, proving the strict inequality in this case. At $n = 0$, all prime factors of the coprime parts must be p. If neither part is divisible by p, their absolute values are both one; distinctness gives the stated opposite sign case. This proves the entire $n = 0$ assertion.

Now assume $n \geq 1$ and both coprime parts are p-adic units. Their ratio is

$$\Xi = \frac{x_0}{y_0} = \epsilon \prod_{i=1}^{n} q_i^{b_i} \neq 1, \qquad b_i \in \mathbb{Z}, \qquad |b_i| \leq \frac{\ln M}{\ln q_i} \leq \frac{\ln M}{\ln 2}. \tag{10.514}$$

Indeed the nonnegative exponents in each of $|x_0|, |y_0|$ are at most $\ln M / \ln q_i$, and their difference has absolute value at most their maximum. Rational prime valuations show that the listed bases are independent, even modulo roots of unity: a positive rational number that is a root of unity is one, and each prime valuation then makes its exponent zero. Their heights are $h(q_i) = \ln q_i$.

We verify the field and index data. At odd p these are the rational field data. Its q-primary saturation of prime bases has no additional exponents: if a rational q-power root of their product exists, prime valuations make all the indicated exponents integral. Root choices only introduce the already available sign. Consequently the saturated residue group is precisely G. The raw singleton index follows from its actual residue order, as in (10.272).

At two the polynomial $X^2 + X + 1$ has no rational root, so $[K_p : \mathbb{Q}] = 2$. Its reduction is irreducible over $\mathbb{F}_2$, since it vanishes at neither zero nor one. Thus the residue degree is at least two. The degree inequality $ef \le d = 2$ gives $e = 1, f = 2$. Its root reduces to an element of order three in $\mathbb{F}_4^*$, so including the q-primary torsion generator makes every saturated residue group at rank at least two equal to that whole group, irrespective of a choice of saturated basis. Every odd rational prime reduces to one, so a singleton's raw order is one and its index is three. The field contains ζ_3, and cannot contain ζ_9, whose degree six is proved by the cyclotomic argument of Lesson 9. Hence $q^u = 3$. These also verify the two-adic field hypothesis required by Theorems 10.154 and 10.156.

If $\epsilon = 1$, apply Theorem 10.154 to the n independent prime bases and bound $B = B_M$. Its weighted product is $\prod_i \ln q_i$. The factor $\eta_1 = 1/2100$ is included by that theorem's singleton conclusion.

If $\epsilon = -1$, use the indexed list $(-1, q_1, \dots, q_n)$, whose length is m=n+1 and rank is n. Its first base has height zero, so the greedy basis is exactly $\boldsymbol{\beta}$, and its weighted product is $L_m \prod_i \ln q_i$. The coefficient of the sign base is one, which is also bounded by B_M. Apply Theorem 10.156, and apply its sharpened rank-one form, Theorem 10.158, when n=1. Neither argument changes the selected basis's residue index.

The fields used here both have ramification index e=1. Restricting their normalized prime-ideal valuation to the rational numbers is therefore exactly v_p. Since y_0 is a p-adic unit,

$$v_p(x - y) - v_p(g_0) = v_p(x_0 - y_0) = v_p(\Xi - 1) < K_{S,p}^{\epsilon} H_{S,p}(M). \tag{10.515}$$

This proves the claimed effective bound in every case. □

Corollary 10.160 (distance after removing the common factor). With $|z|_p = p^{-v_p(z)}$ and $n \ge 1$,

$$\frac{|x - y|_p}{|g_0|_p} > \min\left\{ p^{-K_{S,p}^{\epsilon} t}, B_M^{-K_{S,p}^{\epsilon} \ln p} \right\}. \tag{10.516}$$

Proof. Exponentiate the strict valuation inequality of Theorem 10.159. The negative exponential of the maximum of two real numbers is the minimum of their negative exponentials. This gives exactly the two terms displayed. □

For fixed S and p, this lower bound is a fixed negative power of $\ln M$ once $\ln B_M \ge t$. The normalization by the common factor is necessary; the family at the start of the section proves that directly.

For the positive example $S = \{3, 5, 7\}, p = 3$, the prime bases are five and seven. Together with minus one their residues generate $\mathbb{F}_3^*$, so $\delta_\beta = 1$. The data are $d = e = f = 1, q^u = 2, c = 636, \mathfrak{a} = 14$, $t = \ln 3$, $\Omega = (\ln 5)(\ln 7)$, and $A_2 = 4 + \ln 3$. Thus the completely specified constant is

$$K_{S,3}^{+} = \frac{636 \cdot 14^2 \cdot 81}{\ln 3} \max\left\{ \frac{3}{(\ln 3)^3}, \frac{\mathrm{e}^2}{4} \right\} (4 + \ln 3)(\ln 5)(\ln 7). \tag{10.517}$$

For $x = 3^N 25, y = 3^N 7$, the coprime ratio is $25/7$, with prime exponents $(2, -1)$. Its difference from one is $18/7$, of exact valuation two. The original difference has valuation N+2. The theorem bounds the excess two by a constant times $H_{S,3}(3^N 25)$, while retaining the exact contribution N of the common factor.

For a signed example take $x = 25, y = -7, p = 2$ and $S = \{5, 7\}$. The ratio is $-25/7$, represented by the list $(-1, 5, 7)$ with coefficients $(1, 2, -1)$. Here $d = 2, e = 1, f = 2, q^u = 3, \delta_\beta = 1, \varkappa = 34, m = 3$, and $L_3 = 3/544$. The weighted product is $(3/544)(\ln 5)(\ln 7)$, and the exact difference $25 + 7 = 32$ has valuation five at two. The sign requires the dependent-list theorem rather than being silently discarded from the exponent vector.

The logarithmic-form input is the complete programme proof in Theorems 10.154, 10.156 and 10.158. Its scholarly context is Theorem 1 of Yu's freely accessible paper.

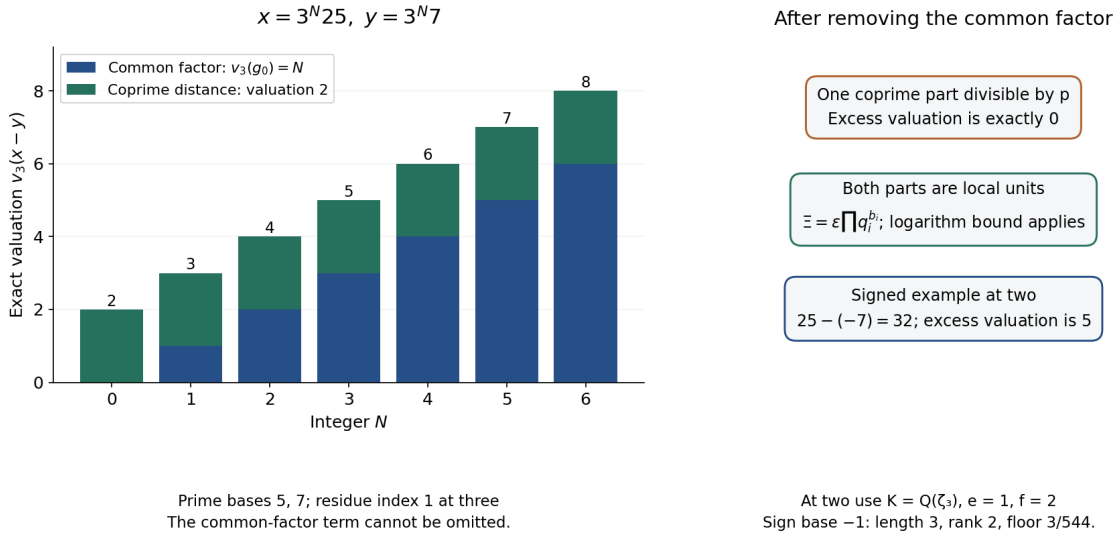


Figure 10.44. Left: for $x = 3^N 25, y = 3^N 7$, the exact valuation is N+2. Its two components are the common-factor valuation N and the fixed coprime distance of valuation two. The bars show exact integer values, while Theorem 10.159 bounds the second component in general. Right: a coprime nonunit case has zero excess valuation; the unit case is reduced to $\epsilon \prod q_i^{b_i}$. The example $25 - (-7) = 32$ at two requires the sign base minus one, the field $\mathbb{Q}(\zeta_3)$, and the length-three weighted height floor $3/544$. Theorem 10.159 proves the full sign and field conditions, Corollary 10.160 gives the normalized distance, and Solution 65 checks both examples. Reproducible exact calculation and figure. Human-source context: Theorem 1 of Yu's freely accessible paper.

52. The one-base case of Yu's second constant

The second constant in Yu's free 2013 paper has a different rank dependence from the first. Its one-base estimate follows directly from the power bound already proved here. We give its full numerical constant and keep the original residue order throughout.

Let K be a number field, $\mathfrak{p}$ a prime above p, and $\beta \in K^*$ a non-torsion unit at $\mathfrak{p}$. Write $d = [K : \mathbb{Q}]$, $e = e_\mathfrak{p}$, $f = f_\mathfrak{p}$, $t = f \ln p$, and $\ell_d = \max\{1, \ln d\}$. At odd p put $q = 2$. At $p = 2$, assume $\zeta_3 \in K$ and put $q = 3$. Let q^u be the order of the q-primary roots of unity in K, and let w_K be the order of all its roots of unity. The order g of the reduction of β is its original residue order, without replacing β by a saturated base. Put

$$\delta_\beta = \frac{p^f - 1}{g}, \qquad A_1 = \max\{4 + \ln(2d), e, t\}, \qquad a^{(2)} = \begin{cases} 7, & p > 2, \\ 13, & p = 2. \end{cases} \tag{10.518}$$

Use the following exhaustive field cases. Their constants are the freely accessible parameters (1.29)–(1.30) in Yu's paper.

Case	Field and prime hypotheses	$c^{(2)}$
I.1	$p = 3, d > 1$	1438
I.2	$p = 3, d = 1$	648
II	$p = 5, e \geq 2$	690
III.1	$p \geq 5, e = 1, d > 1$	$495(p-1)/(p-2)$
III.2	$p \geq 5, e = 1, d = 1$	$557(p-1)/(p-2)$
IV	$p \geq 7, e \geq 2$	2418
V	$p = 2, \zeta_3 \in K$	406

Take a nonzero integer b and $B \geq \max\{3, |b|\}$. Set $H = \max\{\ln B, t\}$ and

$$\mathcal{T}_2 = \frac{8c^{(2)}a^{(2)}\mathrm{e}d^3\ell_d}{q^u t^3} \max\left\{\frac{p^f}{\delta_\beta}, \mathrm{e}t^2\right\} A_1 h(\beta) H. \tag{10.519}$$

Here $\mathrm{e} = \exp(1)$, distinct from the ramification index e. The notation $\mathcal{T}_2$ refers to the second Yu constant at list length one. Substituting $n = 1$ into his formula (1.10) and multiplying by $n+1$ gives this expression: the factor $P = p^\kappa$ cancels and $0! = 1$.

Theorem 10.161 (the second one-base application estimate). Under these hypotheses,

$$\mathrm{ord}_{\mathfrak{p}}(\beta^b - 1) < \frac{\mathcal{T}_2}{4000}. \tag{10.520}$$

Proof. Put $U_2 = (t/d)\mathcal{T}_2$. The singleton height bound (10.55) is $h(\beta) \geq w_K/(\rho \mathrm{e} d^2 \ell_d)$, with $\rho = 17$ for $d = 1$ and $\rho = 58$ for $d \geq 2$. Using the second entry of the maximum gives

$$U_2 \geq J A_1 H, \qquad J = \frac{8c^{(2)}a^{(2)}\mathrm{e}}{\rho}\frac{w_K}{q^u}. \tag{10.521}$$

For the other contribution, the first entry gives $p^f/\delta_\beta = p^f g/(p^f - 1) > g$. Exact division yields

$$\begin{aligned} \frac{2egh(\beta)}{U_2} &< \frac{eq^u t^2}{4c^{(2)}a^{(2)}\mathrm{e}d^2\ell_d A_1 H} \\ &\leq \frac{1}{2c^{(2)}a^{(2)}\mathrm{e}d\ell_d}. \end{aligned} \tag{10.522}$$

The last step uses $t^2 \leq A_1 H$ and $eq^u \leq 2d$. The latter follows from Lemma 10.73: $d/e \geq q^{u-1}(q-1)$, while $q/(q-1) \leq 2$.

At odd primes $w_K/q^u \geq 1$. At two, the roots -1 and the q-primary roots generate a group of order $2q^u$, so $w_K/q^u \geq 2$, as in Lemma 10.157. The positive exponential series gives $\mathrm{e} > 27/10$. Also $\ln 2 > 2/3$, so $A_1 > 14/3$ at degree one and $A_1 > 16/3$ at degree at least two.

Since $B \geq 3$ and $2^3 < 3^2$, $\ln(2B) < (5/3)H$. In III, $H \geq \ln 5 > 8/5$ and $\ln 2 < 7/10$, so the sharper inequality is $\ln(2B) < (23/16)H$. These logarithm comparisons follow from the positive series proved in Section 40 and Solution 48; equivalently, the first four terms of the series for $\ln 5$, with argument $2/3$, already exceed $8/5$.

The following table supplies all numerical endpoints. The column L is a strict lower bound for U_2/H. The last column is an upper bound for the sum of the two contributions after division by U_2.

Case	L	Logarithm multiplier μ	Total upper bound
I.1	2899008/145	5/3	115/1242432
I.2	2286144/85	5/3	235/2286144
II	278208/29	5/3	1/5184
III.1	199584/29	23/16	2257/9580032
III.2	1965096/85	23/16	10345/94324608
IV	4874688/145	5/3	115/2089152
V	104832/5	5/3	295/3040128

To obtain each entry, replace e by 27/10, ℓ_d by one, and d by its lower bound: one in I.2 and III.2, two in every other case. Cases II and IV have $d \geq e \geq 2$; V has $d \geq 2$ because it contains a primitive cube root. Use the corresponding value of ρ, the stated torsion ratio, and the lower bound for A_1. In III, $(p-1)/(p-2) > 1$, so replacing this factor by one covers every prime in that case. The entry in the last column is exactly

$$\frac{\mu}{L} + \frac{1}{2c_{\min}^{(2)} a^{(2)}(27/10) d_{\min}}. \tag{10.523}$$

All seven entries are less than 1/4000. Their smallest positive difference from 1/4000 occurs in III.1 and is 17251/1197504000. Multiplying positive denominators proves each comparison; the exact calculation retains every fraction and every positive difference.

The complete one-base power estimate (10.60) now gives

$$\begin{aligned} \operatorname{ord}_{\mathfrak{p}}(\beta^b - 1) &\leq \frac{d}{t}(\ln(2B) + 2egh(\beta)) \\ &< \frac{d}{t}\frac{U_2}{4000} = \frac{\mathcal{T}_2}{4000}. \end{aligned} \tag{10.524}$$

Non-torsion ensures $\beta^b \neq 1$, so every quantity in this argument is defined. □

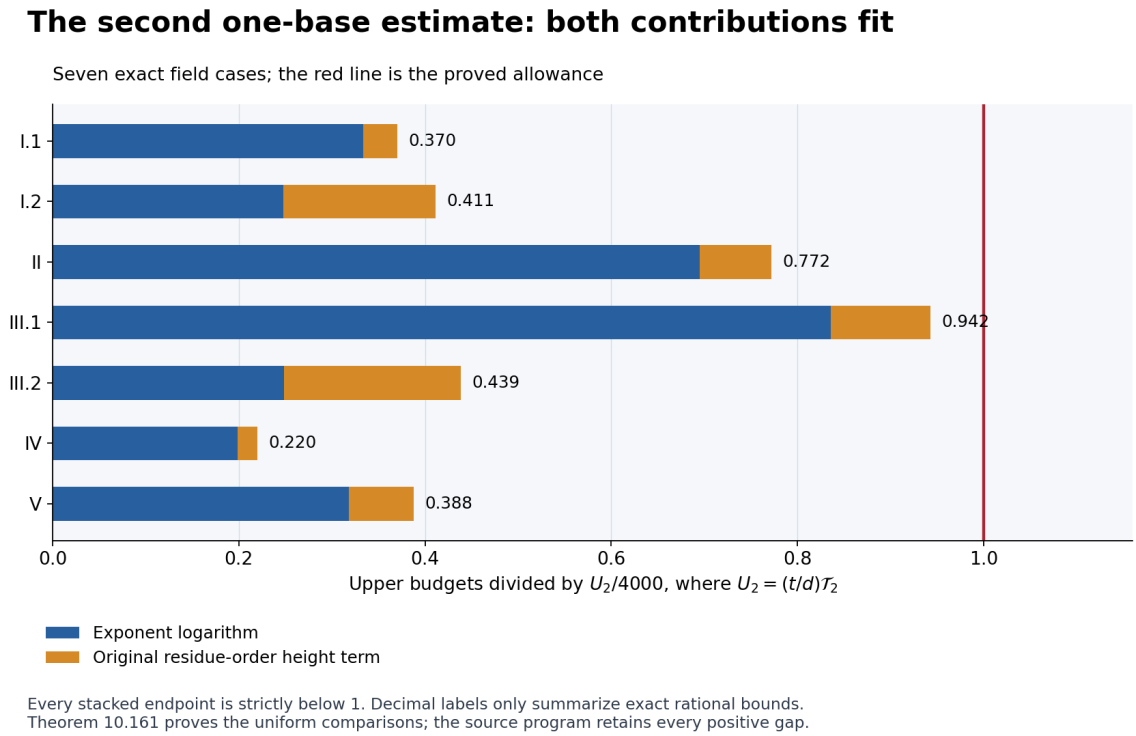


Figure 10.45. The logarithm and original residue-order height contributions are stacked after normalization by $U_2/4000$. The bars are uniform rational upper bounds for the seven exhaustive field cases, and the red line is the allowance. The displayed decimals summarize the exact fractions in the table. Proof: Theorem 10.161. Human source for the target constant: equations (1.10), (1.29)–(1.30) of Yu's free 2013 paper, cited below. Reproducible figure and exact certificate.

For $K = \mathbb{Q}, p = 3, \beta = 4, b = 3^k$, $k \geq 1$, take $B = 3^k$. The raw residue order is one, $\delta_\beta = 2$, $q^u = 2$, $A_1 = 4 + \ln 2$, and $H = k \ln 3$. The second entry of the maximum dominates: $\mathrm{e}(\ln 3)^2 > 8/3 > 3/2 = p^f/\delta_\beta$. Theorem 10.161 therefore reads

$$v_3(4^{3^k} - 1) = k + 1 < \frac{4 \cdot 648 \cdot 7}{4000} \mathrm{e}^2 (4 + \ln 2) k \ln 4. \tag{10.525}$$

The exact valuation is the proved power identity in Lemma 9.5. The right side exceeds $126k$: its successive factors exceed $9/2$, seven, four, and one. This theorem proves the singleton case. The second constant at greater list lengths, with its different modified-height floors, requires its own proof.

53. The second constant for every rank-one list

The singleton estimate extends to a dependent list of any length whose multiplicative rank modulo roots of unity is one. The height floor of the second constant differs from that in Section 49, so we retain it explicitly. The proof again reduces the whole list to one exact power in K.

Use the field and prime conventions of Section 52. Take an indexed list $a_1, \ldots, a_n$ of local units, of multiplicative rank one modulo roots of unity. Order its indices by nondecreasing height, with a fixed order at equal heights, and let $\beta = a_{i_0}$ be the first non-torsion member. Let $b_i \in \mathbb{Z}$, $\Xi = \prod_i a_i^{b_i} \neq 1$, $B \geq \max\{3, |b_1|, \ldots, |b_n|\}$, and $H = \max\{\ln B, t\}$. The residue index δ_β belongs to this original base, as defined in Section 52.

Let $\kappa \geq 0$ be the integer for which $p^{\kappa-1}(p-1) \leq 2e < p^\kappa(p-1)$, and put $P = p^\kappa$. Then $1 \leq P \leq 2pe/(p-1) \leq 4d$. Set

$$\begin{aligned}
\varkappa_2 &= \begin{cases} 25, & p>2, \\ 48, & p=2, \end{cases} & X_n &= \max\{n,t\}, \\
L_n^{(2)} &= \frac{X_n}{\varkappa_2 P dt}, & H_n^{(2)}(a) &= \max\{h(a), L_n^{(2)}\}, \\
\Omega_n^{(2)} &= h(\beta)\prod_{i\neq i_0} H_n^{(2)}(a_i), & A_n &= \max\{4+\ln((n+1)d), e, t\}.
\end{aligned} \tag{10.526}$$

This is precisely the floor in (1.25) and (1.29) of Yu's free paper. Among the non-torsion choices of β, the specified choice minimizes $\Omega_n^{(2)}$: factor out the common product of all modified heights and use that $h/\max\{h, L_n^{(2)}\}$ increases with h. Repeated bases are separate indices in the product. Define the second application constant and its complete budget by

$$\begin{aligned}
C_n^{(2)}(\beta) &= \frac{c^{(2)}(a^{(2)}\mathrm{e}P)^n}{P}\frac{(n+1)^{n+2}d^{n+2}\ell_d}{(n-1)!q^u t^3} \\
&\quad \times \max\left\{\frac{p^f}{\delta_\beta}, \frac{\mathrm{e}^n t^{n+1}}{n^n}\right\} A_n, \\
\mathcal{T}_n^{(2)} &= C_n^{(2)}(\beta)\Omega_n^{(2)} H.
\end{aligned} \tag{10.527}$$

The constant is $(n+1)$ times formula (1.10) of that paper. Its residue index is that of the one-element selected basis, even though n is the length of the entire indexed list. At $n=1$, $\mathcal{T}_1^{(2)} = \mathcal{T}_2$ of Section 52.

Two scalar budgets

Suppose $n \geq 2$, and abbreviate $c = c^{(2)}$, $a = a^{(2)}$, and $\varkappa = \varkappa_2$. Put

$$\begin{aligned}
R &= \prod_{i\neq i_0} \frac{H_n^{(2)}(a_i)}{L_n^{(2)}} \geq 1, & U &= \frac{t}{d}\mathcal{T}_n^{(2)}, \\
F_n &= \frac{c}{\rho}\frac{w_K}{q^u}\frac{a^n \mathrm{e}^{2n-1}(n+1)^{n+2}}{n!\varkappa^{n-1}}, & & \\
E_n &= \frac{4(n-1)!\varkappa^{n-1}}{c a^n \mathrm{e}^n (n+1)^{n+2}}, & \alpha_n &= \left(\frac{1}{4000} - \frac{1}{100000}\right) F_n.
\end{aligned} \tag{10.528}$$

Lemma 10.162 (second-constant rank-one scalar bounds). For every $n \geq 2$,

$$U \geq F_n A_n H R, \qquad \frac{2egh(\beta)}{U} < \frac{1}{100000}, \qquad F_{n+1} > 3F_n, \qquad \alpha_n > \frac{3n}{2}. \tag{10.529}$$

In particular $\alpha_2 > 3$.

Proof. Substitute $\Omega_n^{(2)} = h(\beta)(L_n^{(2)})^{n-1}R$ into U. Using the singleton height bound (10.55) and the second entry of its maximum gives

$$U \geq \frac{c}{\rho}\frac{w_K}{q^u}\frac{a^n \mathrm{e}^{2n-1}(n+1)^{n+2}}{(n-1)!n^n}\left(\frac{X_n}{\varkappa}\right)^{n-1} A_n H R \geq F_n A_n H R. \tag{10.530}$$

For the other term use $p^f/\delta_\beta > g$, $L_n^{(2)} \geq 1/(\varkappa P d)$, $eq^u \leq 2d$, and $t^2 \leq A_n H$. Exact cancellation gives

$$\frac{2egh(\beta)}{U} < \frac{2eq^{u}(n-1)!\varkappa^{n-1}t^{2}}{ca^{n}\mathrm{e}^{n}(n+1)^{n+2}d^{2}\ell_{d}RA_{n}H} \leq \frac{E_{n}}{d\ell_{d}R}. \tag{10.531}$$

The next table gives strict lower bounds on α_2 and upper bounds on $E_2/(d\ell_d)$. Use $\mathrm{e} > 27/10$, the degree and torsion bounds from Section 52, and $\ell_d \geq 1$. In III replace $(p-1)/(p-2)$ by one.

Case	α_2 greater than	$E_2/(d\ell_d)$ less than
I.1	9	2500/2080355319
I.2	14	1250/234365481
II	4	500/199644669
III.1	3	1000/286446699
III.2	12	10000/1611624357
IV	15	2500/3498121809
V	9	1600/675264681

Every last entry is less than 1/100000, by multiplication of positive denominators. The exact calculation retains each positive gap. For growth, direct division gives

$$\frac{F_{n+1}}{F_{n}} = \frac{a\mathrm{e}^{2}}{\varkappa}\left(1+\frac{1}{n+1}\right)^{n+3} > 3. \tag{10.532}$$

Indeed $a\mathrm{e}^2/\varkappa > 9477/4800 > 3/2$, and the positive binomial terms give the last power greater than two. Thus $\alpha_n > 3\cdot 3^{n-2} \geq 3n/2$; the elementary inequality $2\cdot 3^{n-2} \geq n$ follows by induction starting at two. Finally,

$$\frac{E_{n+1}}{E_{n}} = \frac{\varkappa}{a\mathrm{e}}\frac{n}{n+2}\left(\frac{n+1}{n+2}\right)^{n+2} < \frac{\varkappa}{a\mathrm{e}^{2}} < \frac{2}{3}. \tag{10.533}$$

Here $\ln(1-y) < -y$ for $0 < y < 1$, by integrating $1/(1-y) > 1$. The last numerical comparison uses $4800/9477 < 2/3$. Hence the height-budget bound at two holds at every larger n. □

Exact reduction and the exponent budget

Theorem 10.163 (the full second-constant rank-one refinement). For every rank-one list just specified, including all coefficient sizes and torsion patterns,

$$\mathrm{ord}_{\mathfrak{p}}(\Xi-1) < \frac{\mathcal{T}_{n}^{(2)}}{4000}. \tag{10.534}$$

Proof. For $n = 1$, use Theorem 10.161. Suppose $n \geq 2$, and put $D = \rho\mathrm{e}d^{2}\ell_{d}L_{n}^{(2)}$, $D_0 = \max\{1, D\}$. As in the exact pair reduction in Theorem 10.158, Theorem 10.20 gives, for every non-torsion complementary index i, a relation $\beta^{u_i}a_i^{k_i} = 1$, with $k_i \in \mathbb{Z}_{>0}$, $k_i \leq \rho\mathrm{e}d^{2}\ell_{d}h(a_i)$, and $|u_i|/k_i = h(a_i)/h(\beta)$. Let m count these indices. Set $Q = \prod k_i$ if no torsion base occurs, and $Q = w_K\prod k_i$ otherwise. Then

$$\Xi^{Q} = \beta^{M}, \qquad M \in \mathbb{Z}, \qquad |M| \leq nBQ\frac{h(a_n)}{h(\beta)}. \tag{10.535}$$

Each $k_i \leq D_0H_n^{(2)}(a_i)/L_n^{(2)}$. If a non-torsion complementary base occurs, its modified height and the singleton height bound also give $h(a_n)/h(\beta) \leq \max\{1, D/w_K\}R$. Otherwise that height ratio is

one. Consequently, for $M \neq 0$, the three bounds (10.501)–(10.502) apply with this D, D_0, and R. This follows by multiplying the exact relation coefficients; it uses no first-constant estimate.

The empty-list height bound gives $w_K \leq \rho d \ell_d$. Thus

$$\frac{w_K}{D_0} \leq \frac{\varkappa_2 P t}{\mathrm{e} X_n} \leq 192 d. \tag{10.536}$$

We used $\varkappa_2 \leq 48$, $P \leq 4d$, $t \leq X_n$, and $\mathrm{e} > 1$. The factor d is retained in the exponent budget. Each of the three alternatives is at most $Z_n + 2 \ln R$, where

$$Z_n = H + \ln(2n) + n \ln D_0 + \ln 192 + \ln d. \tag{10.537}$$

For odd primes $t \geq \ln 3 > 1$. At two, the primitive cube root reduces injectively, so $3 \mid 2^f - 1$, $f \geq 2$, and $t \geq \ln 4 > 1$. Therefore $X_n / t \leq n$ and $D_0 < 7 n d \ell_d$, since $\rho \mathrm{e} / \varkappa_2 \leq 58(11/4)/25 < 7$. For $n \geq 3$, the logarithmic tangent from Section 50, $\ln \ell_d \leq (\ln d)/8 + 11/10$, together with $\ln 7 < 2$, $\ln 192 < 6$, and $\ln(2n) < n$, gives

$$Z_n < H + n \ln n + \frac{41n}{10} + 6 + \left(\frac{9n}{8} + 1 \right) \ln d. \tag{10.538}$$

Lemma 10.162 gives $\alpha_n > 3n/2$. The coefficient of $\ln d$ in $\alpha_n A_n H$ is larger than $9n/8+1$, since $H > 1$ and $n \geq 3$. At H=1 the constant difference is larger than $77n/30 - 7 > 0$: use $A_n \geq 4 + \ln(n+1) + \ln d$, $\ln(n+1) \geq \ln 4 > 4/3$, and $\ln n < \ln(n+1)$. Increasing H increases the difference, because its slope is greater than $6n - 1 > 0$. Thus $Z_n < \alpha_n A_n H$ for every $n \geq 3$.

For $n = 2$, a non-torsion complement gives the sharper bound $H + \ln 4 + 2 \ln D_0 + 2 \ln R$. Here $D_0 < 13 d \ell_d$, since $2\rho \mathrm{e} / \varkappa_2 < 13$. Using $\ln 13 < 8/3$, $\ln 4 < 7/5$, and the tangent $\ln \ell_d \leq (\ln d)/16 + 9/5$, its part before $2 \ln R$ is less than $H + 31/3 + (17/8) \ln d$. This is less than $\alpha_2 A_2 H$, because $\alpha_2 > 3$, $A_2 > 5$, $H > 1$, $14 > 31/3$, and $3 > 17/8$. If the complement is torsion, its bound is $H + \ln(2 w_K) < H + 15/2 + (17/16) \ln d$, which is absorbed by the same lower bound. All numerical logarithm comparisons follow from positive exponential series, or the logarithm series already proved in Solution 64.

In every case the bound before $2 \ln R$ is less than $\alpha_n A_n H > 2$. Since $\ln R \leq R - 1$,

$$\ln(2|M|) < \alpha_n A_n H R \leq \left(\frac{1}{4000} - \frac{1}{100000} \right) U. \tag{10.539}$$

Apply the complete one-base estimate (10.60) to β^M, retaining its actual residue order g. Since $\Xi^Q - 1 = (\Xi - 1)(1 + \Xi + \cdots + \Xi^{Q-1})$ and Ξ is a local unit,

$$\begin{aligned} \operatorname{ord}_{\mathfrak{p}}(\Xi - 1) &\leq \operatorname{ord}_{\mathfrak{p}}(\beta^M - 1) \\ &\leq \frac{d}{t} (\ln(2|M|) + 2egh(\beta)) < \frac{\mathcal{T}_n^{(2)}}{4000}, \end{aligned} \tag{10.540}$$

by Lemma 10.162. If M=0, then $\Xi \neq 1$ is torsion. Theorem 10.21 bounds its valuation by $(d/t) \ln 2$. Meanwhile $U/4000 > \alpha_n A_n H R > 2 > \ln 2$, proving the same strict inequality. This exhausts every case. $\square$

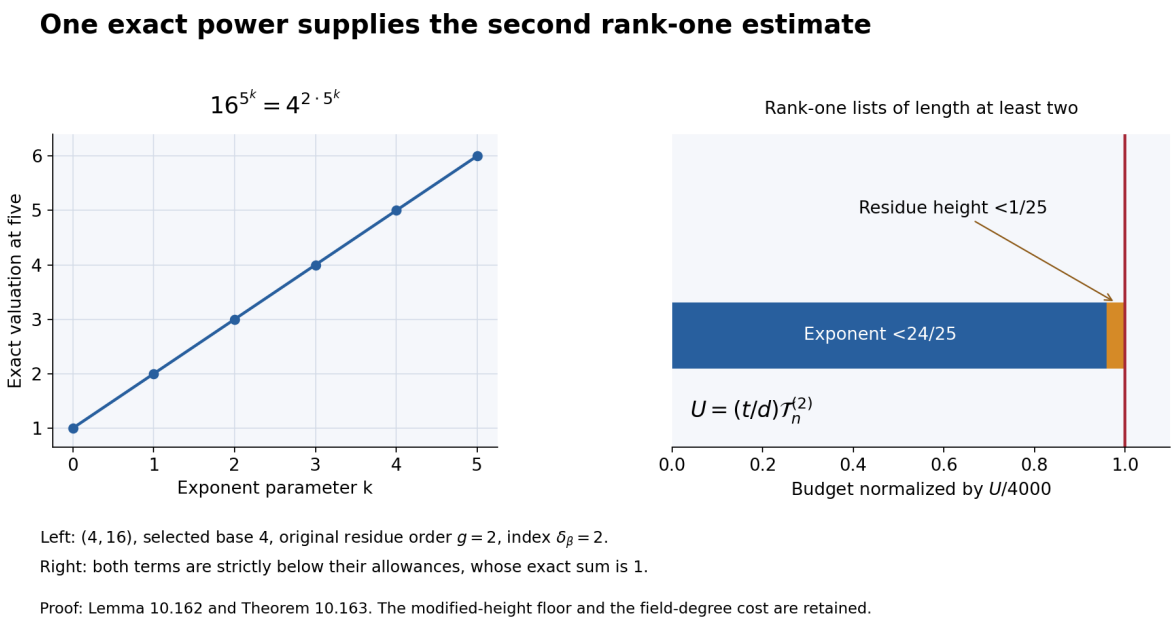


Figure 10.46. Left: the exact rank-one identity for the indexed list $4, 16$ *at five gives valuation* $1 + k$*, proved by the power law. Right: for every list of length at least two, the exponent cost and original residue-order height cost are strictly below* $24/25$ *and* $1/25$*, respectively, after normalization by* $U/4000$*. These exact allowances sum to one. The second modified-height floor and the degree term in the exponent bound are retained. Proof: Lemma 10.162 and Theorem 10.163. Human source for the target: Theorem 2 and equations (1.10), (1.25), (1.29)–(1.30) of Yu's free 2013 paper, cited below. Reproducible figure and exact calculation.*

For $K = \mathbb{Q}, p = 5, (a_1, a_2) = (4, 16)$, the selected base is four, $g = 2$, and $\delta_\beta = 2$. Here $P = 1$, $\varkappa_2 = 25$, and $L_2^{(2)} = 2/(25 \ln 5)$. The two actual heights exceed this floor. With coefficients $(0, 5^k)$, $k \geq 0$, the exact reduction is $\Xi = 4^{2 \cdot 5^k}$, and the proved valuation is $1 + k$. Theorem 10.163 uses the length-two second constant and $(\ln 4)(\ln 16)$, while preserving the original residue order two.

The numerical target is the rank-one refinement in Theorem 2 of Yu's free paper. The proof here supplies this full rank-one application independently of the omitted general second-constant proof. Rank at least two and the general coefficient profile remain separate requirements.

54. Sharp two-logarithm constants with the E-parameter

The determinant comparison (10.40) contains considerably more information than the parameter choice used in Theorem 10.11. We now use smaller rectangles to obtain the constants 36.1 and 53.8, while keeping the coefficient ordering and the exponential domain explicit. The two constants accompany different cutoffs; a smaller numerical constant need not give the better bound for a particular pair.

Lemma 10.164 (factorials and the common p-part). For an integer $K \geq 2$, put

$$Q_K = \left(\prod_{j=1}^{K-1} j! \right)^{-2/(K(K-1))}. \tag{10.541}$$

Then

$$Q_K \leq \frac{\mathrm{e}^{3/2}}{K-1}. \tag{10.542}$$

If $a \geq 1$ and $b \geq 1$, then

$$\ln b \leq a \max\{\ln(b/a), 1\}. \tag{10.543}$$

Consequently, for $a_i \geq 1$, $g \geq 1$, $q = E \ln p > 0$, $z = g a_1 a_2$, positive integer coefficients b_i, and

$$\mathcal{D} = \frac{b_1}{a_2} + \frac{b_2}{a_1}, \qquad qB \geq \max\{\ln \mathcal{D} + 2/5, 5\}, \tag{10.544}$$

the integer $u = v_p(\gcd(b_1, b_2))$ satisfies

$$u \le EzB. \tag{10.545}$$

Proof. Write $n = K - 1$. Monotonicity of $\ln x$ gives $\ln(j!) \ge \int_1^j \ln x\, dx = j \ln j - j + 1$. The function $x \ln x$ is convex on $[1, \infty)$, so its trapezoid sums dominate its integral:

$$\sum_{j=1}^{n} j \ln j \ge \int_1^n x \ln x\, dx + \frac{n \ln n}{2} = \frac{n(n+1)}{2} \ln n - \frac{n^2}{4} + \frac{1}{4}. \tag{10.546}$$

Summing the factorial bounds therefore gives

$$\sum_{j=1}^{n} \ln(j!) \ge \frac{n(n+1)}{2} \ln n - \frac{3n^2}{4} + \frac{n}{2} + \frac{1}{4} \ge \frac{n(n+1)}{2}(\ln n - 3/2). \tag{10.547}$$

Exponentiation proves the first assertion, including $n = 1$.

If $b \le \mathrm{e}a$, then $\ln b \le 1 + \ln a \le a$. If $b > \mathrm{e}a$, the inequality

$$(a-1)\ln b - a \ln a > (a-1)(1 + \ln a) - a \ln a = a - 1 - \ln a \ge 0 \tag{10.548}$$

proves the second assertion for $a > 1$; at $a = 1$ it is immediate. Finally $p^u \mid b_1$, and $\ln(b_1/a_2) \le \ln \mathcal{D} < qB$, with $qB \ge 5$. Thus

$$u \le \frac{\ln b_1}{\ln p} \le \frac{a_2 qB}{\ln p} = Ea_2 B \le EzB. \tag{10.549}$$

□

Theorem 10.165 (sharp rational E-parameter estimates). Let α_1, α_2 be multiplicatively independent nonzero rational p-adic units. Let g be the least positive integer such that

$$v_p(\alpha_1^g - 1) > 0, \qquad v_p(\alpha_2^g - 1) > 0. \tag{10.550}$$

For odd p, this integer divides $p - 1$; for $p = 2$ it is one. Suppose

$$v_p(\alpha_1^g - 1) \ge E > \frac{1}{p-1}. \tag{10.551}$$

Let b_1, b_2 be nonzero integers, put $u = v_p(\gcd(b_1, b_2))$, and suppose

$$p \nmid b_2/p^u. \tag{10.552}$$

When $p = 2$, also suppose $v_2(\alpha_2 - 1) \ge 2$. Choose real numbers $A_i > 1$ with

$$\ln A_i \ge \max\{h(\alpha_i), E \ln p\}, \qquad B' = \frac{|b_1|}{\ln A_2} + \frac{|b_2|}{\ln A_1}. \tag{10.553}$$

For $\nu = 6, 4$, define

$$H_\nu = \max\{\ln B' + \ln(E \ln p) + 2/5,\ \nu E \ln p,\ 5\}. \tag{10.554}$$

Then both estimates hold:

$$\begin{aligned} v_p(\alpha_1^{b_1} - \alpha_2^{b_2}) &\le 36.1\, \frac{g}{E^3 (\ln p)^4}\, \ln A_1 \ln A_2\, H_6^2, \\ v_p(\alpha_1^{b_1} - \alpha_2^{b_2}) &\le 53.8\, \frac{g}{E^3 (\ln p)^4}\, \ln A_1 \ln A_2\, H_4^2. \end{aligned} \tag{10.555}$$

The coefficient-ordering requirement may be omitted if **both** $v_p(\alpha_i^g - 1) \ge E > 1/(p-1)$. In that case the pair can be reordered to make b_2/p^u a unit; for $p = 2$ both bases are already in the required exponential domain.

Proof. Negative coefficients are made positive by inverting the corresponding bases. This preserves heights, residue orders, independence, and all stated precision conditions. Use the notation

$$q = E \ln p, \qquad a_i = \frac{\ln A_i}{q} \ge 1, \qquad z = g a_1 a_2 \ge 1, \qquad B = \frac{H_\nu}{q}. \tag{10.556}$$

Then $B \ge \nu$, $qB \ge 5$, and $qB \ge \ln \mathcal{D} + 2/5$, where $\mathcal{D} = B'q$. Lemma 10.164 gives $u \le EzB$.

The principal-unit and error-term argument leading to (10.40) uses only $p \nmid g$, the stated first-base precision, the integral ratio b_1'/b_2', and the exponential domain of the second base. It therefore works with this exact common residue order, rather than the possibly larger $p-1$. Its sufficient inequality is still (10.40), and its conclusion under the two cardinalities is

$$v_p(\alpha_1^{b_1} - \alpha_2^{b_2}) < E(N - 1/2) + u. \tag{10.557}$$

In particular no assumption of precision E for the second logarithm is silently inserted.

For the two choices use the following parameters:

ν	k	ℓ
6	47/10	12/5
4	11/2	5/2

$$L = \lfloor \ell B \rfloor + 2, \quad K = \lfloor kzL \rfloor + 1. \tag{10.558}$$

Set

$$\begin{aligned} R_1 &= \lfloor \sqrt{gLa_2/a_1} \rfloor + 1, & S_1 &= \lfloor \sqrt{gLa_1/a_2} \rfloor + 1, \\ R_2 &= \lfloor \sqrt{g(K-1)La_2/a_1} \rfloor + 1, & S_2 &= \lfloor \sqrt{g(K-1)La_1/a_2} \rfloor + 1. \end{aligned} \tag{10.559}$$

Put $R = R_1 + R_2 - 1$, $S = S_1 + S_2 - 1$, $N = KL$. Since $\lfloor x \rfloor + 1 > x$, the first rectangle has more than gL points, and the second has more than $g(K-1)L$. Independence makes its multiplicative labels injective. Pigeonholing the first rectangle by its g residue classes provides the first cardinality in (10.18). If the second rectangle's linear labels are injective in every residue class, pigeonholing also provides the second cardinality.

We first treat this injective case. The rounded sides satisfy

$$(R-1)a_1,\ (S-1)a_2 \le \sqrt{zL}(1 + \sqrt{K-1}). \tag{10.560}$$

Thus Lemma 10.164 and the definition (10.15) imply

$$\beta \le \frac{\mathrm{e}^{3/2}}{2\sqrt{k}} \, \frac{(1+\sqrt{K-1})\sqrt{K}}{K-1} \, \mathcal{D}. \tag{10.561}$$

Here $L \ge 16, K \ge 76$ when $\nu = 6$, and $L \ge 12, K \ge 67$ when $\nu = 4$. For $K \ge 67$,

$$\frac{(1+\sqrt{K-1})\sqrt{K}}{K-1} = \left(1 + \frac{1}{\sqrt{K-1}}\right)\sqrt{1 + \frac{1}{K-1}} < \frac{9}{8}\,\frac{81}{80} < \frac{6}{5}. \tag{10.562}$$

The elementary bounds $\mathrm{e}^{11/10} < 10/3$ and $k > 4$ consequently give

$$\ln \beta < \ln \mathcal{D} + 2/5 \le qB. \tag{10.563}$$

These exponential bounds can also be verified directly from their series: after term M, the tail is at most $(x^{M+1}/(M+1)!)/(1 - x/(M+2))$, when $0 < x < M+2$.

The height contribution in (10.40), divided by q, is at most

$$\frac{L}{2}((R-1)a_1 + (S-1)a_2) \le \sqrt{k}\, zL^2 + L\sqrt{zL}. \tag{10.564}$$

After removing $(K-1)B$ from the comparison's two sides and dividing by zL^2, it therefore suffices to prove

$$k\left(1-\frac{B+1}{L}\right)>\sqrt{k}+\frac{1}{\sqrt{L}}+\frac{3B\ln(kL^2+L)}{5L^2}. \tag{10.565}$$

Indeed $K>kzL$, $L-1-B>0$, and $qB\ge 5$. Also $\ln(kzL^2+L)/z$ decreases for $z\ge 1$: its derivative has the sign of $kzL^2/(kzL^2+L)-\ln(kzL^2+L)<0$. This justifies replacing z by one only in this upper envelope.

From $L>\ell B+1$ and $B\ge\nu$ we obtain

$$\frac{B+1}{L}<\frac{\nu+1}{\ell\nu+1}. \tag{10.566}$$

Furthermore $B<L/\ell$, and $\ln(kL^2+L)/L$ decreases throughout either range of L. Its derivative has the sign of $(2kL+1)/(kL+1)-\ln(kL^2+L)<0$, since the first term is less than two and the second exceeds two. The following exact bounds finish the sufficient comparison:

ν	left lower bound	$\sqrt{k}$ upper	$L^{-1/2}$ upper	logarithmic upper
6	141/55	217/100	1/4	9/80
4	3	47/20	3/10	67/500

(10.567)

For the logarithmic entries use $\ln((47/10)16^2+16)<36/5$ and $\ln((11/2)12^2+12)<67/10$, respectively. The square comparisons and the exponential-series bound just given establish every entry. The sum of the three upper bounds is $1013/400<141/55$ in the first row, and $348/125<3$ in the second. Thus (10.40) holds.

The resulting valuation bound satisfies

$$\frac{EN+u}{EzB^2}\le k\left(\ell+\frac{2}{\nu}\right)^2+\frac{\ell+2/\nu}{\nu}+\frac{1}{\nu}=\begin{cases}80407/2250<36.1, & \nu=6,\\ 101/2<53.8, & \nu=4.\end{cases} \tag{10.568}$$

This covers the injective alternative for every real height scale, not only a finite parameter sample.

It remains to treat a collision of the linear labels within a residue class. Put $n=\gcd(b_1,b_2)$, $r_0=b_1/n$, $s_0=b_2/n$, and $\sigma=\alpha_1^{r_0}\alpha_2^{-s_0}$. If q_0 is the order of the residue of σ, the two colliding points give

$$q_0\mid g,\qquad q_0r_0\le R_2-1,\qquad q_0s_0\le S_2-1. \tag{10.569}$$

To see the latter inequalities, their differences are $t(r_0,-s_0)$ for a nonzero integer t. Equality of their residue classes forces $q_0\mid t$, while both coordinate differences lie within the rectangle.

If $q_0\nmid n$, the difference has valuation zero. Otherwise σ^{q_0} is a deep principal unit, and the power law of Lesson 9, with $p\nmid q_0$, gives

$$v_p(\alpha_1^{b_1}-\alpha_2^{b_2})=u+v_p(\sigma^{q_0}-1). \tag{10.570}$$

Independence ensures $\sigma^{q_0}\neq 1$. The one-place height bound applied to $X-1$ yields

$$v_p(\alpha_1^{b_1}-\alpha_2^{b_2})\le u+E((R_2-1)a_1+(S_2-1)a_2)+\frac{\ln 2}{\ln p}\le u+2E\sqrt{k}\,zL+\frac{\ln 2}{\ln p}. \tag{10.571}$$

Consequently its quotient by EzB^2 is at most

$$\frac{1}{\nu}+\frac{2\sqrt{k}}{\nu}\left(\ell+\frac{2}{\nu}\right)+\frac{\ln 2}{5\nu}<\begin{cases}3, & \nu=6,\\ 4, & \nu=4.\end{cases} \tag{10.572}$$

Here we used $z\ge 1$, $qB\ge 5$, and $\ln 2<7/10$. Both desired constants exceed these bounds.

Finally

$$EzB^2 = \frac{g}{E^3(\ln p)^4} \ln A_1 \ln A_2 \, H_\nu^2. \tag{10.573}$$

This proves the two stated estimates. If both bases have precision E, one of the coefficients after removal of the common p-part is a unit; reordering makes it the second coefficient without losing any precision or convergence hypothesis. □

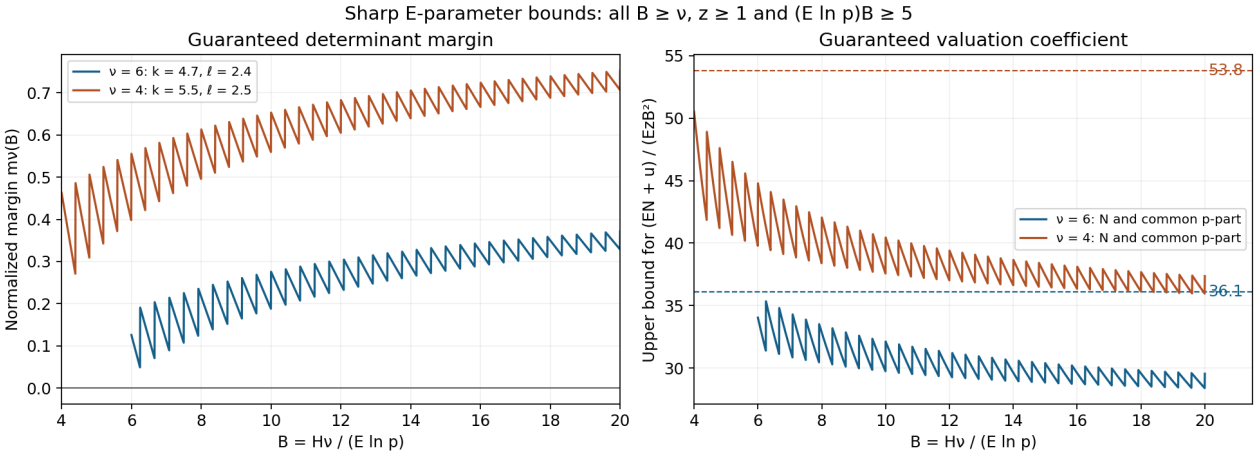


Figure 10.47. Theorem 10.165 uses $k = 47/10, \ell = 12/5$ for $\nu = 6$, and $k = 11/2, \ell = 5/2$ for $\nu = 4$. Left: the proved lower envelope $m_\nu(B) = k(1 - (B+1)/L) - \sqrt{k} - L^{-1/2} - 3B\ln(kL^2 + L)/(5L^2)$, with $L = \lfloor \ell B \rfloor + 2$, remains positive for all $B \geq \nu$. Right: the proved upper envelope $k(L/B)^2 + L/B^2 + 1/B$ for $(EN + u)/(EzB^2)$ stays below the displayed constants. The teeth come from the integer rectangle size. The plots display $4 \leq B \leq 20$; the monotonicity and rational inequalities in the proof establish the whole infinite ranges, including every $z \geq 1$ and $qB \geq 5$. The collision alternative has the separate bounds three and four. Human-source context: Bugeaud–Laurent's freely accessible 1996 paper and Bugeaud's freely accessible 1999 paper; the complete parameter proof and coefficient-ordering correction are given here. Figure program.

For a concrete family, take $p = 5$, $\alpha_1 = 1 + 5^m$, $\alpha_2 = 1 + 2 \cdot 5^m$, $b_1 = b_2 = 1$, $m \geq 1$. The two positive integer bases are coprime, since their second minus twice their first is -1; primes dividing either base then prove independence. Their common residue order is $g = 1$, both have precision $E = m$, and the difference has valuation exactly m. With $\ln A_i = h(\alpha_i)$, both height cutoffs are attained by $\nu E \ln 5$. Thus the two bounds give

$$m \leq 1299.6 \, \frac{\ln(1 + 5^m)\ln(1 + 2 \cdot 5^m)}{m(\ln 5)^2}, \qquad m \leq 860.8 \, \frac{\ln(1 + 5^m)\ln(1 + 2 \cdot 5^m)}{m(\ln 5)^2}. \tag{10.574}$$

The bound involving 53.8 is stronger in this family because its height cutoff is four rather than six.

55. The rational 2-adic bound across both unit classes

The 2-adic boundary does not require pretending that every principal unit belongs to the exponential domain. There are two rational unit classes modulo four. A sign change moves either class into that domain, and the sign remaining in the linear form determines whether a logarithmic estimate is needed.

Corollary 10.166 (the rational 2-adic estimate). Let α_1, α_2 be multiplicatively independent nonzero rationals with $v_2(\alpha_i) = 0$. Let b_1, b_2 be nonzero integers, choose $A_i > 1$ with

$$\ln A_i \geq \max\{h(\alpha_i), \ln 2\}, \qquad B' = \frac{|b_1|}{\ln A_2} + \frac{|b_2|}{\ln A_1}, \qquad T = \max\{\ln B' + 1/25, 10\}. \tag{10.575}$$

Then

$$v_2(\alpha_1^{b_1} - \alpha_2^{b_2}) \leq 208 \, \ln A_1 \ln A_2 \, T^2. \tag{10.576}$$

There is no coefficient-ordering requirement and no additional restriction on either residue class modulo four.

Proof. A rational 2-adic unit has an odd numerator and odd denominator, so its reduction modulo four is either one or three. Choose $\eta_i \in \{1, -1\}$ such that

$$\beta_i = \eta_i \alpha_i \in 1 + 4\mathbb{Z}_2, \qquad \eta = \eta_1^{b_1} \eta_2^{-b_2}, \qquad \frac{\alpha_1^{b_1}}{\alpha_2^{b_2}} = \eta \, \frac{\beta_1^{b_1}}{\beta_2^{b_2}}. \tag{10.577}$$

The new bases have the same heights and are independent: squaring a relation among the β_i would give a relation among the original bases, with both exponents doubled. Inverses and integer powers preserve $1 + 4\mathbb{Z}_2$.

If $\eta = -1$, the ratio on the right is three modulo four. Subtracting one gives valuation exactly one. This already satisfies the asserted bound, since $\ln A_i \geq \ln 2 > 69/100$ and $T \geq 10$.

Suppose $\eta = 1$. First enlarge each prescribed logarithmic height by the same $\varepsilon > 0$, writing $a_{i,\varepsilon} = \ln A_i + \varepsilon$, and define

$$E_\varepsilon = \min\{2, a_{1,\varepsilon}/\ln 2, a_{2,\varepsilon}/\ln 2\} > 1, \qquad q_\varepsilon = E_\varepsilon \ln 2. \tag{10.578}$$

Both β_i have precision at least two, hence at least E_ε. The common residue order is one. The both-deep alternative of Theorem 10.165 applies to these bases and all coefficient sizes, after the necessary reordering. In particular its bound already includes the common p-part of the coefficients.

Let $B'_\varepsilon = |b_1|/a_{2,\varepsilon} + |b_2|/a_{1,\varepsilon}$, and $T_\varepsilon = \max\{\ln B'_\varepsilon + 1/25, 10\}$. Since $q_\varepsilon \leq 2 \ln 2 < 7/5 < \mathrm{e}^{2/5}$, the cutoff for $\nu = 6$ satisfies

$$\begin{aligned} H_{6,\varepsilon} &= \max\{\ln B'_\varepsilon + \ln q_\varepsilon + 2/5, \ 6q_\varepsilon, \ 5\} \\ &\leq T_\varepsilon + 19/25 < \frac{11}{10} T_\varepsilon. \end{aligned} \tag{10.579}$$

Indeed the logarithmic term is less than $T_\varepsilon + 19/25$, whereas $6q_\varepsilon < 42/5 < 10 \leq T_\varepsilon$ and $5 \leq T_\varepsilon$.

The first estimate in Theorem 10.165 consequently gives

$$\begin{aligned} v_2(\alpha_1^{b_1} - \alpha_2^{b_2}) &\leq \frac{36.1(11/10)^2}{E_\varepsilon^3 (\ln 2)^4} a_{1,\varepsilon} a_{2,\varepsilon} T_\varepsilon^2 \\ &< \frac{4368100000}{22667121} a_{1,\varepsilon} a_{2,\varepsilon} T_\varepsilon^2 < 208 \, a_{1,\varepsilon} a_{2,\varepsilon} T_\varepsilon^2. \end{aligned} \tag{10.580}$$

The rational coefficient uses $E_\varepsilon > 1$ and $\ln 2 > 69/100$; its gap below 208 is $346661168/22667121 > 0$. The bound on $\ln 2$ follows by comparing the exponential series at $69/100$ and $7/10$ with two, using the tail estimate in Theorem 10.165. Now let ε decrease to zero. The reciprocal heights, logarithm and maximum defining T_ε are continuous at the positive prescribed heights. This yields the desired inequality even when a prescribed height is exactly $\ln 2$. □

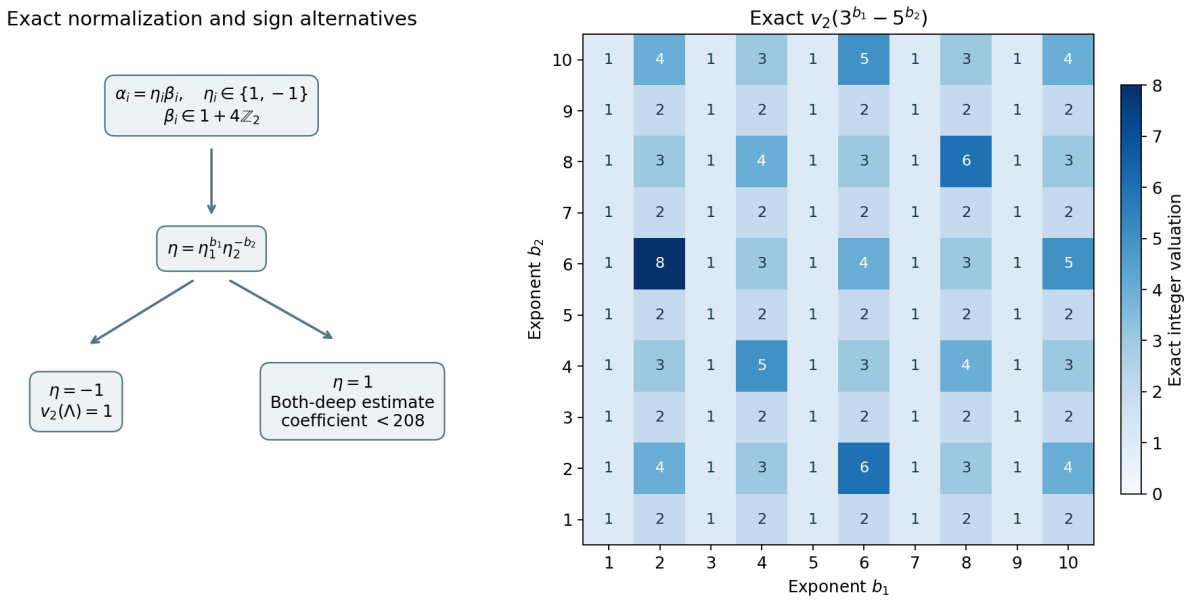


Figure 10.48. Left: the exact sign decomposition in Corollary 10.166 sends both bases into $1 + 4\mathbb{Z}_2$. An odd remaining sign gives valuation one; an even sign uses the fully proved both-deep estimate with coefficient below 208. Right: exact integer valuations $v_2(3^{b_1} - 5^{b_2})$, $1 \le b_i \le 10$. Here $\eta_1 = -1, \eta_2 = 1$, so every odd b_1 gives valuation one, while the even columns require the deep-unit argument. Every cell is calculated by dividing the nonzero integer difference by two until it becomes odd. This finite example illustrates the two cases; the proof covers all independent rational units, signed nonzero exponents, and prescribed heights at their lower boundary. Human-source context: the rational 2-adic constant in Bugeaud's freely accessible 1999 Theorem 2; the complete sign-normalization proof is given here. Figure program.

56. The constant 283 for Fermat differences and arbitrary powers

For the special coefficients $p-1, p-1$, their common p-part is zero and their logarithmic size is small enough to use a fixed rectangle height. This supplies the constant 283 directly from the proved determinant machinery, including the one-base case.

Theorem 10.167 (Fermat differences and arbitrary powers). Let $a > b \ge 1$ be coprime integers, and let p be an odd prime not dividing ab. Then

$$v_p(a^{p-1} - b^{p-1}) \le \frac{1631}{6} \frac{p-1}{(\ln p)^2} \ln(5a) \ln(5b) < 283 \frac{p}{(\ln p)^2} \ln(5a) \ln(5b). \tag{10.581}$$

For every integer $n \ge 1$,

$$v_p(a^n - b^n) \le 283 \frac{p}{(\ln p)^2} \ln(5a) \ln(5b) + v_p(n). \tag{10.582}$$

Proof. We first prove a determinant bound for any independent rational p-adic units α_1, α_2 whose chosen logarithmic height bounds H_i satisfy $H_1 > 2$, $H_2 > 3/2$. Set

$$g = p - 1, \quad q = \ln p > 1, \quad a_i = H_i/q > 0, \quad z = g a_1 a_2, \quad b_1 = b_2 = g, \quad E = 1, \quad u = 0. \tag{10.583}$$

Here $z > 3$. Indeed $\ln^2 p < p - 1$: for $x > 1$, the function $\sqrt{x} - 1/\sqrt{x} - \ln x$ vanishes at one and has derivative $(\sqrt{x} - 1)^2/(2x^{3/2}) > 0$. Hence $\ln^2 x < (x-1)^2/x < x - 1$. Now $g/(\ln p)^2 > 1$ and $H_1 H_2 > 3$.

Fermat's theorem and the principal-unit decomposition of Lesson 9 put both principal factors in $1 + p\mathbb{Z}_p$, with precision at least $E = 1 > 1/(p-1)$. The coefficient $b_2 = g$ is a p-adic unit. Thus the analytic and arithmetic determinant comparison (10.40) applies. Its derivation requires positive chosen height bounds above the actual heights; the separate restriction $a_i \ge 1$ in Theorem 10.165 was used to simplify that theorem's parameter envelopes, and is not imposed here.

Take

$$L = 7, \qquad k = 11/2, \qquad K = \lfloor kzL \rfloor + 1, \tag{10.584}$$

and use the four rounded square-root sides from Theorem 10.165 with these positive a_i. Its cardinality argument depends only on the products of these sides, so it still supplies the first residue-class cardinality and, in the injective alternative, the second one. Its two side-height bounds and factorial calculation also apply unchanged:

$$(R-1)a_1,\ (S-1)a_2 \le \sqrt{zL}(1+\sqrt{K-1}), \qquad \ln\beta < \ln\mathcal{D} + 2/5, \tag{10.585}$$

where $\mathcal{D} = gq(1/H_1 + 1/H_2)$. For the latter inequality, $K \ge 116 > 67$ and $k > 4$, so the exact factorial and square-root comparisons in Lemma 10.164 and Theorem 10.165 apply.

Since $1/H_1 + 1/H_2 < 7/6 < 6/5$ and $\ln(6/5) < 1/5$,

$$\ln\beta < \ln g + \ln q + 3/5 < q + \ln q + 3/5 < 2q. \tag{10.586}$$

The last inequality uses $q - \ln q \ge 1$, proved by differentiating from its minimum at $q = 1$. Dividing (10.40) by q, we may therefore remove $(K-1)B$ with $B = 2$. After division by zL^2, a sufficient comparison is

$$\frac{22}{7} > \sqrt{11/2} + \frac{1}{\sqrt{zL}} + \frac{3\ln(kzL^2 + L)}{qzL^2}. \tag{10.587}$$

The logarithmic quotient decreases with $z \ge 3$, by the derivative calculation in Theorem 10.165. Using $q > 1$, $z > 3$, $L = 7$, $\sqrt{11/2} < 47/20$, $1/\sqrt{21} < 1/4$, and $\ln((11/2)3 \cdot 7^2 + 7) < 7$, its right side is less than

$$\frac{47}{20} + \frac{1}{4} + \frac{1}{7} = \frac{96}{35} < \frac{22}{7}. \tag{10.588}$$

The gap is exactly $2/5$. The logarithmic comparison follows from the exponential series at seven, whose finite sum already exceeds $(11/2)3 \cdot 7^2 + 7 = 1631/2$. Thus the sufficient inequality holds and

$$v_p(\alpha_1^g - \alpha_2^g) < N \le kzL^2 + L < \left(\frac{539}{2} + \frac{7}{3}\right) z = \frac{1631}{6} z. \tag{10.589}$$

In the collision alternative, the residue-order argument in Theorem 10.165 gives instead

$$v_p(\alpha_1^g - \alpha_2^g) \le 2\sqrt{k}\, zL + \frac{\ln 2}{q} < \left(\frac{329}{10} + \frac{7}{30}\right) z < 34z < \frac{1631}{6} z. \tag{10.590}$$

It uses $u = 0$, $z > 3$, $q > 1$, and $\ln 2 < 7/10$, and includes the possibility of valuation zero. This proves the determinant bound in both alternatives.

If $b > 1$, the coprime positive integer bases a, b are independent, as proved in Corollary 10.9. Choose $H_1 = \ln(5a)$, $H_2 = \ln(5b)$; these dominate their heights and both exceed two. The bound just proved gives the first asserted inequality.

If $b = 1$, use the auxiliary prime from Corollary 10.10: some $c \in \{2, 3, 5\}$ is different from p and independent of a, and the rational bases $a/c, 1/c$ are independent p-adic units. Choose $H_1 = \ln(5a) > 2$ and $H_2 = \ln 5 > 3/2$. They dominate the two rational heights. The exact identity

$$(a/c)^{p-1} - (1/c)^{p-1} = c^{-(p-1)}(a^{p-1} - 1) \tag{10.591}$$

preserves valuation because $p \nmid c$. This proves the first inequality with $b = 1$ as well. Its replacement by the constant 283 follows from $1631/6 < 283$ and $p - 1 < p$.

Finally let d_0 be the order of a/b in the multiplicative group of the residue field. If $d_0 \nmid n$, the difference has valuation zero. Otherwise $d_0 \mid p - 1$, and the odd-prime deep power law from Lesson 9 gives

$$\begin{aligned} v_p(a^n - b^n) &= v_p((a/b)^{d_0} - 1) + v_p(n), \\ v_p(a^{p-1} - b^{p-1}) &= v_p((a/b)^{d_0} - 1). \end{aligned} \tag{10.592}$$

The powers of b are units, and $(p-1)/d_0$ is prime to p, which justifies the second identity. Apply the first part to conclude. □

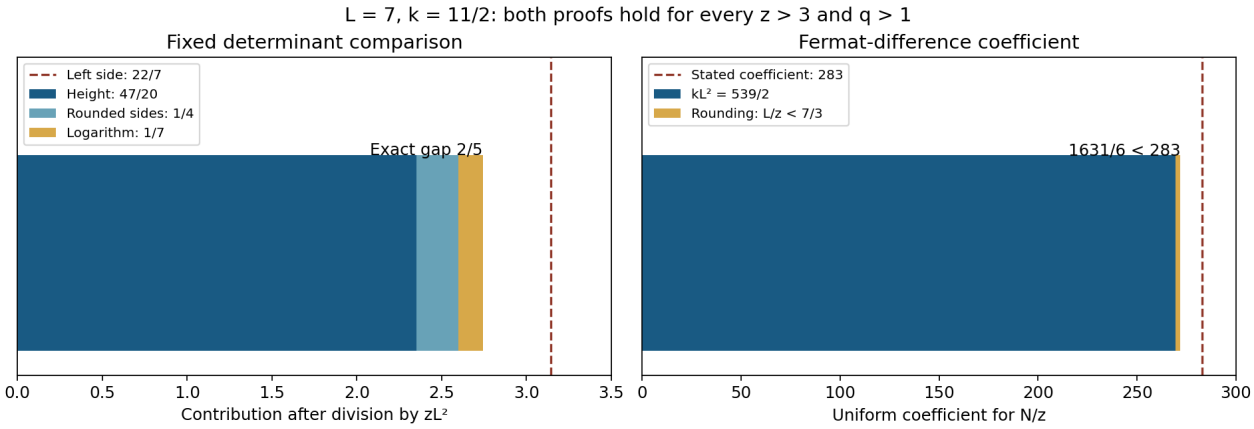


Figure 10.49. Theorem 10.167 uses $E = 1$, $g = p - 1$, $L = 7$, $k = 11/2$, $z = gH_1H_2/(\ln p)^2 > 3$, and $q = \ln p > 1$. Left: the sufficient determinant comparison has left lower bound $22/7$ and right upper bound $47/20 + 1/4 + 1/7 = 96/35$, with exact gap $2/5$. The three colored pieces bound the height contribution, rounded-side error and logarithmic term. Right: N/z is at most $539/2 + 7/3 = 1631/6 < 283$; the collision case has the smaller bound 34. These are proved uniform budgets, not measured valuations. The coprime two-base case and the $b = 1$ auxiliary-prime construction use the same parameters and both satisfy $H_1 > 2, H_2 > 3/2$. Human-source context: the free Bugeaud–Laurent and Bugeaud determinant papers and Yamada's freely accessible note; the complete proof, including the one-base case, is given here. Figure program.

57. Residue occupancy and the number-field estimate

The residue order of a pair of units is often smaller than the order of the entire residue-field multiplicative group. Using that smaller order does more than shrink the rectangles: a single residue congruence also limits how often a coordinate can occur. The centered occupancy theorem from Lesson 6 then improves the arithmetic height of the determinant.

Lemma 10.168 (occupancy in one residue class). Let K, L, R, S, g be positive integers, $N = KL$, and suppose $\gcd(m_1, m_2, g) = 1$. Let (r_j, s_j), $1 \le j \le N$, be distinct integer pairs satisfying

$$0 \le r_j < R, \quad 0 \le s_j < S, \quad m_1 r_j + m_2 s_j \equiv c \pmod{g}. \tag{10.593}$$

Let $l_1, \dots, l_N$ contain each integer $0, \dots, L-1$ exactly K times. Put

$$\begin{aligned} M_1 &= \frac{L-1}{2} \sum_j r_j, \quad G_1 = \frac{NL(R+g-1)}{8} - \frac{gN^2L}{24(S+g-1)}, \\ M_2 &= \frac{L-1}{2} \sum_j s_j, \quad G_2 = \frac{NL(S+g-1)}{8} - \frac{gN^2L}{24(R+g-1)}. \end{aligned} \tag{10.594}$$

Then $G_1, G_2 \ge 0$, and for every permutation π,

$$\left| \sum_i l_i r_{\pi(i)} - M_1 \right| \le G_1, \qquad \left| \sum_i l_i s_{\pi(i)} - M_2 \right| \le G_2. \tag{10.595}$$

Proof. We use the completely proved centered occupancy lemma, Lemma 6.4. Put $g_0 = \gcd(m_2, g)$, $g_1 = g/g_0$. Since $\gcd(m_1, g_0) = 1$, every r_j is in the same class $c_0 \pmod{g_0}$, with $0 \le c_0 < g_0$. Write $r_j = c_0 + g_0 r'_j$. These nonnegative integers lie in a range of length at most

$$R' = \left\lfloor \frac{R-1}{g_0} \right\rfloor + 1. \tag{10.596}$$

For a fixed r_j, division of the original congruence by g_0 determines s_j uniquely modulo g_1, because m_2/g_0 is invertible modulo g_1. Therefore each r'_j is repeated at most

$$S' = \left\lfloor \frac{S-1}{g_1} \right\rfloor + 1 \tag{10.597}$$

times. There are N actual selected pairs, so $N \le R'S'$. Lemma 6.4 applies to this compressed coordinate and capacity:

$$\left| \sum_i \left(l_i - \frac{L-1}{2} \right) r_{\pi(i)} \right| \le g_0 \left(\frac{NLR'}{8} - \frac{N^2 L}{24S'} \right). \tag{10.598}$$

The constant c_0 disappears since the centered row weights sum to zero. Now

$$g_0 R' \le R + g_0 - 1, \qquad \frac{g_0}{S'} \ge \frac{g}{S + g_1 - 1}. \tag{10.599}$$

Inserting these two inequalities, and then $g_0, g_1 \le g$, gives the asserted G_1. The compressed occupancy bound is nonnegative: $N \le R'S'$ implies its value is at least $NLR'/12 > 0$. Its successive upper bounds are nonnegative as well. Interchanging the two coordinates proves the assertion for G_2. $\square$

For the rest of this section, let g be the order of the subgroup generated by the residues of α_1, α_2, rather than the full $p^f - 1$ of (10.1). This subgroup is cyclic by the proof in Section 1. Lift its generator to a root of unity η of order g, and write $\alpha_i = \eta^{m_i} \epsilon_i$. The minimality of this subgroup gives

$$g \mid p^f - 1, \qquad \gcd(m_1, m_2, g) = 1. \tag{10.600}$$

The principal factors and the integer t from (10.2) are unchanged. All analytic estimates and residue-class zero arguments proved earlier apply to this subgroup.

Theorem 10.169 (the refined determinant parameter theorem). Retain the local and global data d, e, f, t of Section 1, now with the subgroup order g just defined. Set

$$D = d/f, \qquad \gamma_1 = \frac{R+g-1}{2R} - \frac{gN}{6R(S+g-1)}, \qquad \gamma_2 = \frac{S+g-1}{2S} - \frac{gN}{6S(R+g-1)}. \tag{10.601}$$

Let $K \ge 3, L \ge 2$, $R = R_1 + R_2 - 1$, $S = S_1 + S_2 - 1$, $N = KL$, and let β be (10.15). Suppose both residue-class cardinalities (10.18) hold, and

$$\begin{aligned} p^t K(L-1) \ln p > (D+2e) \ln N + D(K-1) \ln \beta \\ + Dp^t L(\gamma_1 Rh(\alpha_1) + \gamma_2 Sh(\alpha_2)). \end{aligned} \tag{10.602}$$

Then

$$v_p(\alpha_1^{b_1} - \alpha_2^{b_2}) < \frac{p^t}{e}(N - 1/2) + u, \qquad u = v_p(\gcd(b_1, b_2)). \tag{10.603}$$

Proof. As in Theorem 10.7, the zero lemma supplies a nonzero determinant Δ with all selected columns in one class. In the polynomial $P(X, Y)$ used in Lemma 10.6, every first exponent lies between $M_1 - G_1$ and $M_1 + G_1$, and every second exponent lies between $M_2 - G_2$ and $M_2 + G_2$, by Lemma 10.168. Removing the smallest actual powers gives an integer polynomial of partial degrees at most $2G_1, 2G_2$. Its torus norm is still bounded by (10.17). The complete one-place height proof in Lemma 10.6 therefore gives

$$v_p(\Delta) \le \frac{DN}{2e \ln p} \left(\ln N + (K-1) \ln \beta + p^t L(\gamma_1 Rh(\alpha_1) + \gamma_2 Sh(\alpha_2)) \right), \tag{10.604}$$

because $4G_1/N = \gamma_1 LR$ and $4G_2/N = \gamma_2 LS$. The constants γ_i are nonnegative by Lemma 10.168. The analytic estimate (10.10) is unchanged. If the asserted valuation inequality failed, comparing

these two bounds and multiplying by $2e\ln p/N$ would give the opposite weak inequality to the stated strict parameter condition. □

Theorem 10.170 (the explicit number-field bound). Let α_1, α_2 be multiplicatively independent nonzero algebraic numbers, with a fixed p-adic embedding and $v_p(\alpha_i) = 0$. Put

$$d = [\mathbb{Q}(\alpha_1, \alpha_2) : \mathbb{Q}], \qquad F = \mathbb{Q}_p(\alpha_1, \alpha_2), \qquad [F : \mathbb{Q}_p] = ef, \qquad D = d/f. \tag{10.605}$$

Let g be the order of the subgroup generated by their nonzero residues. For nonzero integer coefficients b_1, b_2, choose $A_i > 1$ with

$$\ln A_i \geq \max\{h(\alpha_i), \ln p/D\}, \qquad B' = \frac{|b_1|}{D \ln A_2} + \frac{|b_2|}{D \ln A_1}. \tag{10.606}$$

Define

$$H = \max\{\ln B' + \ln\ln p + 2/5,\ 10\ln p/D,\ 10\}. \tag{10.607}$$

Then

$$v_p(\alpha_1^{b_1} - \alpha_2^{b_2}) \leq 24\,\frac{pg}{(p-1)(\ln p)^4}\,D^4 \ln A_1 \ln A_2\, H^2. \tag{10.608}$$

In particular $g \leq p^f - 1$ gives the same bound with $p(p^f - 1)$ in place of pg. Neither unramifiedness nor an ordering of the coefficient valuations is assumed.

Proof. Inverting bases handles negative coefficients, preserving all relevant data; multiplying the resulting difference by a local unit preserves its valuation. We may assume positive coefficients. Write

$$D_0 = D/\ln p, \qquad a_i = D\ln A_i/\ln p \geq 1, \qquad z = ga_1a_2 \geq 1, \qquad B = D_0 H. \tag{10.609}$$

Then $D \geq e \geq 1$, $B \geq 10$, and $B \geq 10D_0$. Use

$$k = 3, \qquad \ell = 13/5, \qquad L = \lfloor \ell B \rfloor + 2, \qquad K = \lfloor kzL \rfloor + 1 \tag{10.610}$$

and the four rounded square-root sides from Theorem 10.165, with this g and these a_i. Their products exceed gL and $g(K-1)L$. Independence gives injective multiplicative labels in the first rectangle, so pigeonholing its residue classes supplies the first cardinality.

First assume the linear labels in the second rectangle are injective within each residue class. Pigeonholing supplies the second cardinality. Theorem 10.169 applies once its parameter inequality is checked.

Let $\mathcal{D} = b_1/a_2 + b_2/a_1 = B' \ln p$. The factorial proof in Lemma 10.164 and the side bounds in Theorem 10.165 give

$$\beta \leq \frac{\mathrm{e}^{3/2}}{2\sqrt{3}}\,\frac{(1+\sqrt{K-1})\sqrt{K}}{K-1}\,\mathcal{D}. \tag{10.611}$$

Here $L \geq 28$ and $K \geq 85$. Since $K - 1 \geq 84 > 81$,

$$\frac{(1+\sqrt{K-1})\sqrt{K}}{K-1} < \frac{10}{9}\frac{81}{80} = \frac{9}{8}. \tag{10.612}$$

The exponential-series bound $\mathrm{e}^{11/10} < 61/20$, together with $(9 \cdot 61)^2 < 3(16 \cdot 20)^2$, implies

$$\ln\beta < \ln\mathcal{D} + 2/5 \leq H. \tag{10.613}$$

The square comparison is exact: $301401 < 307200$.

We next bound the refined height terms. Put

$$X = \sqrt{g(K-1)La_1a_2}, \qquad Y = \sqrt{gLa_1a_2}. \tag{10.614}$$

The rounded sides give $(R+g-1)a_1 \le X+Y+ga_1$. Also

$$\frac{gKLa_1}{S+g-1} \ge \frac{X^2}{X+(S_1+g-1)a_2} \ge X-(S_1+g-1)a_2 \ge X-Y-ga_2. \tag{10.615}$$

The middle inequality is $x^2/(x+y) \ge x-y$ for $x,y \ge 0$. Inserting these estimates in $\gamma_1 Ra_1$, and then interchanging the two coordinates, yields

$$\begin{aligned} \gamma_1 Ra_1 &\le X/3+2Y/3+ga_1/2+ga_2/6, \\ \gamma_2 Sa_2 &\le X/3+2Y/3+ga_2/2+ga_1/6. \end{aligned} \tag{10.616}$$

Since $a_i \ge 1$, their sum is at most

$$\frac{2}{3}\sqrt{k}\,zL+\frac{4}{3}\sqrt{zL}+\frac{4}{3}z. \tag{10.617}$$

The nonnegativity of γ_i justifies replacing the actual heights by their prescribed bounds.

Divide the parameter condition by $p^t \ln p$. Because $(D+2e)/p^t \le 3D$, $\ln\beta \le H$, and $B = D_0 H$, it suffices to prove

$$K(L-1) > 3D_0 \ln N + (K-1)B + L(\gamma_1 Ra_1 + \gamma_2 Sa_2). \tag{10.618}$$

This replacement remains valid if $\ln\beta < 0$, since $H > 0$. Set $\lambda = \ell + 2/10 = 14/5$; then $\ell B < L-1 < L \le \lambda B$. After dividing by zB^2, the left side minus the coefficient term is larger than

$$k\ell^2 - k\ell - \frac{2k}{B} \tag{10.619}$$

and the height terms are bounded by

$$\frac{2}{3}\sqrt{k}\lambda^2 + \frac{4}{3}\frac{\lambda^{3/2}}{\sqrt{B}} + \frac{4}{3}\frac{\lambda}{B}. \tag{10.620}$$

Moreover $N \le kzL^2+L$, $D_0 \le B/10$. The function $\ln(kz\lambda^2B^2+\lambda B)/z$ decreases for $z \ge 1$, and $\ln(k\lambda^2B^2+\lambda B)/B$ decreases for $B \ge 10$. For the first assertion differentiate and use $N > \mathrm{e}$; for the second, the logarithmic derivative times B is less than two, while the logarithm exceeds two. Thus

$$\frac{3D_0 \ln N}{zB^2} \le \frac{3\ln(k\lambda^2B^2+\lambda B)}{10B} \le \frac{3\ln 2380}{100} < \frac{117}{500}. \tag{10.621}$$

Using $\sqrt{3} < 26/15$, $\lambda^{3/2}/\sqrt{10} < 149/100$, and $\ln 2380 < 39/5$, the sufficient comparison follows from the exact rational inequality

$$3(13/5)^2 - 3(13/5) - \frac{2}{3}\frac{26}{15}(14/5)^2 > \frac{3}{5}+\frac{149}{75}+\frac{28}{75}+\frac{117}{500}. \tag{10.622}$$

Its two sides are $3848/1125$ and $4791/1500$, with positive difference $1019/4500$. The square and logarithmic bounds follow by squaring and the exponential-series certificate already used above. This establishes the strict parameter condition.

To include the common coefficient p-part, use Lemma 10.164's logarithmic inequality with $a = a_2$, $b = b_1$. Since $\ln(b_1/a_2) \le \ln \mathcal{D} < H$ and $H \ge 10$,

$$u \le \frac{\ln b_1}{\ln p} \le \frac{a_2 H}{\ln p} = \frac{a_2 B}{D} \le zB. \tag{10.623}$$

Theorem 10.169, $p^t/e \le C = p/(p-1)$, and $L \le (14/5)B$ now give

$$\begin{aligned} v_p(\alpha_1^{b_1} - \alpha_2^{b_2}) &< CN + u \\ &\le C\left(3(14/5)^2 + \frac{14/5}{10} + \frac{1}{10}\right) zB^2 \\ &= \frac{239}{10} CzB^2 < 24CzB^2. \end{aligned} \tag{10.624}$$

It remains to handle a collision of linear labels inside a residue class. Write $n = \gcd(b_1, b_2)$, $r_0 = b_1/n$, $s_0 = b_2/n$, and $\sigma = \alpha_1^{r_0}\alpha_2^{-s_0}$. The residue of σ has order $q_0 \mid g$. The colliding differences give

$$q_0 r_0 \le R_2 - 1, \qquad q_0 s_0 \le S_2 - 1. \tag{10.625}$$

If $q_0 \nmid n$, the valuation is zero. Otherwise the prime-to-p and p-power laws of Lesson 9 give

$$v_p(\alpha_1^{b_1} - \alpha_2^{b_2}) \le u + v_p(\sigma^{q_0 p^t} - 1). \tag{10.626}$$

For $u \ge t$ the deep-unit law gives the sharper exact subtraction $u - t$; for $u < t$, further p-th powers do not decrease positive valuation. Thus this inequality covers both sides of the exponential boundary. Independence ensures $\sigma^{q_0 p^t} \ne 1$.

The one-place height bound from Lemma 10.6, applied to $X - 1$, gives

$$\begin{aligned} v_p(\alpha_1^{b_1} - \alpha_2^{b_2}) &\le u + \frac{p^t}{e}\bigl((R_2 - 1)a_1 + (S_2 - 1)a_2\bigr) + \frac{D_0}{e}\ln 2 \\ &\le zB + 2C\sqrt{3}\,zL + \frac{B}{10}\ln 2 < \frac{28}{25}CzB^2 < 24CzB^2. \end{aligned} \tag{10.627}$$

For the displayed coarse bound, divide by CzB^2, use $B \ge 10, z \ge 1, C \ge 1$, $\sqrt{3} < 7/4$, $\ln 2 < 7/10$, and $L \le (14/5)B$. The sum is at most $1/10 + 49/50 + 7/1000 = 1087/1000 < 28/25$. No root of unity outside the given local field is required.

Finally

$$zB^2 = \frac{gD^4}{(\ln p)^4}\ln A_1 \ln A_2\, H^2. \tag{10.628}$$

This proves the claimed constant 24 in both alternatives, for all ramification indices and coefficient sizes. □

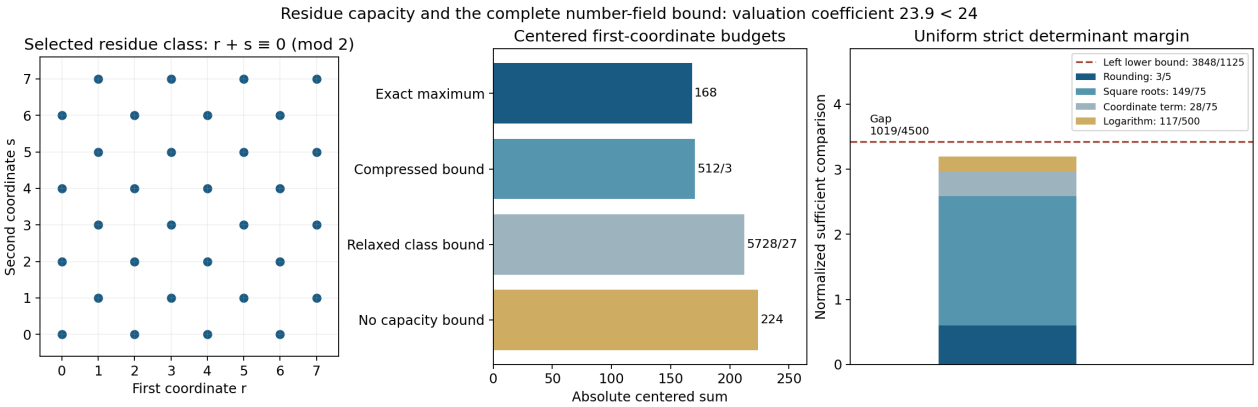


Figure 10.50. Left: all 32 selected pairs in the 8-by-8 rectangle satisfying $r + s \equiv 0 \pmod 2$. Each first coordinate occurs four times; the congruence reduces the capacity from eight to four. With $K = 4, L = 8$, arrange first coordinates increasingly against the row labels to attain the exact centered sum 168. Middle: the compressed Lemma 6.4 bound is 512/3; its relaxed residue-class bound in Lemma 10.168 is 5728/27; ignoring coordinate capacity gives 224. These are proved bounds for every permutation, while 168 is the exact maximum for this example. Right: Theorem 10.170's uniform sufficient comparison has lower side 3848/1125, upper side 4791/1500, and exact positive margin 1019/4500. The infinite ranges $B \ge 10$, $z \ge 1$, $D_0 \le B/10$ are covered by the proof's monotonicity arguments. The resulting valuation coefficient is $23.9 < 24$, including the common coefficient p-part; the collision bound is separately smaller. Human-source context: Bugeaud–Laurent's freely accessible 1996 paper, Lemmas 10–13 and Theorem 3; all used occupancy, height, analytic and numerical arguments are proved in this programme. Figure program.

Corollary 10.171 (the complete cutoff table). Under the algebraic, local-unit, independence and height hypotheses of Theorem 10.170, put

$$H_{\nu,\eta} = \max\{\ln B' + \ln\ln p + 2/5,\ \nu \ln p/D,\ \eta\}. \tag{10.629}$$

For each entry in the table,

$$v_p(\alpha_1^{b_1} - \alpha_2^{b_2}) \le c(\nu,\eta)\,\frac{pgD^4}{(p-1)(\ln p)^4}\ln A_1 \ln A_2\, H_{\nu,\eta}^2, \tag{10.630}$$

where the constants are exactly

	$\nu=4$	$\nu=6$	$\nu=8$	$\nu=10$	$\nu=15$
$\eta=5$	53.8	36.1	28.1	24	18.6
$\eta=10$	51.7	34.8	27.3	23.2	18

. (10.631)

Proof. Use Theorem 10.170's complete proof with $B = D_0 H_{\nu,\eta}$. Now $B \ge \nu$ and $D_0 \le B/\eta$. Use the following parameters and literal minimum integer $K_0 = \lfloor k(\lfloor \ell\nu \rfloor + 2)\rfloor + 1$:

η	ν	k	ℓ	K_0	δ
5	4	4.3	3.0	61	0.170
5	6	3.6	2.8	65	0.093
5	8	3.4	2.6	75	0.004
5	10	3.0	2.6	85	0.011
5	15	3.1	2.3	112	0.049
10	4	3.9	3.1	55	0.119
10	6	3.7	2.7	67	0.044
10	8	3.3	2.6	73	0.043
10	10	2.9	2.6	82	0.002
10	15	3.0	2.3	109	0.043

. (10.632)

Here δ is a strict lower bound for the sufficient-comparison margin, verified below, rather than a measured valuation. Put $L = \lfloor \ell B \rfloor + 2$, $K = \lfloor kzL \rfloor + 1$, and $\lambda = \ell + 2/\nu$. The cardinalities, refined side-height inequalities and common p-part bound remain exactly those proved above.

The factorial logarithmic cutoff is valid in all ten cases. The factor

$$f(K) = \left(1 + \frac{1}{\sqrt{K-1}}\right)\sqrt{\frac{K}{K-1}} \tag{10.633}$$

decreases with K. For $K \ge 82$, it is less than $9/8$, by the same comparison as above with $K - 1 \ge 81$. All such rows have $k \ge 2.9$, and

$$\left(\frac{601}{200}\frac{9}{8}\right)^2 < 4(2.9). \tag{10.634}$$

The remaining rows have $K \ge 55$, $k \ge 3.3$, and

$$f(K) < \left(1 + \frac{7}{50}\right)\frac{101}{100} < \frac{29}{25}, \qquad \left(\frac{601}{200}\frac{29}{25}\right)^2 < 4(3.3). \tag{10.635}$$

The square comparisons are exact; $1/\sqrt{54} < 7/50$ and $\sqrt{55/54} < 101/100$. Since $\mathrm{e}^{11/10} < 601/200$, every row therefore has $\ln\beta < \ln\mathcal{D} + 2/5 \le H_{\nu,\eta}$.

The uniform determinant comparison reduces, exactly as in Theorem 10.170, to

$$k\ell^2 - k\ell - \frac{2}{3}\sqrt{k}\lambda^2 > \frac{2k}{\nu} + \frac{4}{3}\sqrt{\lambda^3/\nu} + \frac{4\lambda}{3\nu} + \frac{3}{\eta\nu}\ln(k\lambda^2\nu^2 + \lambda\nu). \tag{10.636}$$

The same two monotonicity arguments cover the entire ranges $z \ge 1$, $B \ge \nu$; their logarithmic arguments exceed e^2 already at the minimum parameters. To verify this comparison explicitly, the

following rational upper endpoints suffice:

η	ν	$\sqrt{k}$	$\sqrt{\lambda^3/\nu}$	$\ln(k\lambda^2\nu^2+\lambda\nu)$
5	4	2.07365	3.27396	6.75321
5	6	1.89737	2.26431	7.16332
5	8	1.84391	1.70108	7.49012
5	10	1.73206	1.48163	7.77486
5	15	1.76069	0.98007	8.33483
10	4	1.97485	3.41526	6.71309
10	6	1.92354	2.15678	7.12592
10	8	1.81660	1.70108	7.46065
10	10	1.70294	1.48163	7.74136
10	15	1.73206	0.98007	8.30233

. (10.637)

Each finite decimal means the exact rational with denominator 100000, and is a strict upper bound. The first two columns are checked by squaring. For a logarithmic endpoint x, use the positive partial sum $\sum_{j=0}^{64} x^j/j!$, which exceeds the displayed argument $k\lambda^2\nu^2+\lambda\nu$ in every row. Substituting these endpoints makes the left side minus the right side strictly larger than the corresponding positive δ in the first table. These are ten exact rational calculations. Together with the monotonicity proof they establish the comparison for every allowed real parameter, not just the tabulated minima.

In the injective alternative, the total valuation coefficient, including u, is at most

$$k(\ell+2/\nu)^2+\frac{\ell+2/\nu}{\nu}+\frac{1}{\nu}. \tag{10.638}$$

For the ten rows in order, its exact values are

η	ν	coefficient upper bound
5	4	269/5
5	6	40537/1125
5	8	112391/4000
5	10	239/10
5	15	55753/3000
10	4	25847/500
10	6	104149/3000
10	8	54571/2000
10	10	5779/250
10	15	16193/900

. (10.639)

Each is at most the stated $c(\nu,\eta)$; equality in the first row is harmless because the determinant valuation comparison is strict. In the collision alternative, the proof above gives a coefficient at most

$$\frac{1}{\nu}+\frac{2\sqrt{k}\lambda}{\nu}+\frac{\ln 2}{\eta\nu}<\frac{1}{4}+\frac{189}{50}+\frac{7}{200}=\frac{813}{200}<5. \tag{10.640}$$

Here $\nu\ge 4,\eta\ge 5,\lambda\le 18/5$, $\sqrt{k}<21/10$ and $\ln 2<7/10$. Every stated constant is at least 18, so this alternative satisfies each estimate as well. The final identity for zB^2 proves the asserted table. □

58. The second constant and coefficient imbalance

Two features of an explicit logarithmic-form estimate can be separated: the constant multiplying the height product, and the parameter measuring the coefficients. We first compare Yu's two constants, with their own coefficient parameters. We then give a two-branch estimate in which the last coefficient has a separate bound.

Comparing the two constants

Retain every hypothesis and definition of Theorem 10.152. In particular, the bases are independent local units, the last nonzero coefficient has minimum p-adic valuation, and the original saturated residue index is used. Write C_1 for (10.449), $P = p^{\kappa}$, where

$$p^{\kappa-1}(p-1) \le 2e < P(p-1), \qquad t = f \ln p. \tag{10.641}$$

Here κ can be zero. Put $a^{(2)} = 7$ at odd primes and $a^{(2)} = 13$ at two. Define

$$C_2 = \frac{c^{(2)}}{P}(a^{(2)}\mathrm{e}P)^n \frac{(n+1)^{n+1}d^{n+2}\ell_d}{(n-1)!q^u t^3} \max\left\{\frac{p^f}{\delta(\boldsymbol{a})}, \frac{\mathrm{e}^n t^{n+1}}{n^n}\right\} A_n. \tag{10.642}$$

The constants that specify its coefficient parameter are as follows; $x = (p-1)/(p-2)$ in the two III rows.

Case	$c^{(2)}$	$a_0^{(2)}$	$a_1^{(2)}$
I.1	1438	$2 + \ln 21$	1.94
I.2	648	$2 + \ln 21$	2.76
II	690	$2 + \ln 35$	0.71
III.1	$495x$	$2 + \ln 7$	1.99
III.2	$557x$	$2 + \ln 7$	3.32
IV	2418	$2 + \ln 7$	3.58
V	406	$2 + \ln 52$	1.48

Set

$$\begin{aligned} G_2(n,d) &= (n+1)(a_0^{(2)}n + a_1^{(2)} + \ln(n+1) + \ln d), \\ \mathsf{h}_2 &= \max\left\{\ln\frac{R_{\max}}{dW(d)}, \ln B_{\dagger}, G_2(n,d), (n+1)t\right\}. \end{aligned} \tag{10.643}$$

These are equations (1.10), (1.12), (1.13), (1.29) and (1.30) of Yu's free paper. The following comparison uses the programme proof of Theorem 10.152.

Lemma 10.172 (uniform comparison of coefficient parameters). For every $n \ge 2, d \ge 1$ in each of the seven cases,

$$G_1(n,d) < \frac{3}{2}G_2(n,d), \qquad \mathsf{h} \le \frac{3}{2}\mathsf{h}_2. \tag{10.644}$$

Proof. In this proof write a_0, a_1, a_2 for the first constant's parameters in (10.448)–(10.449). Their displayed values give $a_0 < 7$ and $a_2 < 7$, so $\ln(a_0 n + a_2) < \ln(n+1) + \ln 7$. For example, $\mathrm{e} > 27/10$ gives $\ln 26 < 4$, and $\ln 2 < 7/10$ checks every entry involving $\ln 2$. Consequently $(3G_2/2 - G_1)/(n+1)$ is greater than

$$\left(\frac{3}{2}a_0^{(2)} - a_0\right)n + \frac{3}{2}a_1^{(2)} - a_1 + \frac{1}{2}\ln(n+1) + \frac{1}{2}\ln d - \ln 7. \tag{10.645}$$

The coefficient of n is positive: in every row $a_0 \le a_0^{(2)} + \ln 2$ and $a_0^{(2)} > 3$. The displayed expression therefore has its least value at n=2,d=1. Substitution and cancellation of logarithms gives, respectively,

$$\begin{array}{c|c} \text{I.1} & 22/25+\ln(27/4)+\frac{1}{2}\ln 3 \\ \text{I.2} & 27/20+\ln(27/4)+\frac{1}{2}\ln 3 \\ \text{II} & -3/8+\ln(125/4)+\frac{1}{2}\ln 3 \\ \text{III.1} & 11/40+\frac{1}{2}\ln 3 \\ \text{III.2} & 57/50+\frac{1}{2}\ln 3 \\ \text{IV} & 9/4-2\ln 2+\frac{1}{2}\ln 3 \\ \text{V} & 17/10+\ln(208/7)+\frac{1}{2}\ln 3. \end{array} \tag{10.646}$$

Every entry is positive. For II use $125/4 > \mathrm{e}$, which follows from $\mathrm{e} < 3$; for IV use $\ln 2 < 7/10$. The elementary bounds on e follow from its positive power series: the partial sum through degree four exceeds $27/10$, while $k! \geq 2^{k-1}$, with strict inequality for k at least four, bounds the full sum above by three. This proves the first assertion. The other entries of the two maxima agree. Since $G_2 > 0$, each of them is bounded by $3\mathsf{h}_2/2$, including entries that are negative. This proves the second assertion. □

Theorem 10.173 (the second estimate when $n \leq Pt$). Under the hypotheses of Theorem 10.152, if $n \leq p^{\kappa} f \ln p$, then

$$\mathrm{ord}_{\mathfrak{p}}(\Xi - 1) < C_2 \mathsf{h}_2 \prod_{j=1}^{n} h(a_j). \tag{10.647}$$

Proof. Multiplying the maximum in C_1 by t^{n+1} gives exactly the maximum in C_2. Cancelling all common factors, including $n!/(n-1)! = n$, yields

$$\frac{C_2}{C_1} = \frac{c^{(2)}}{c} \left(\frac{a^{(2)}\mathrm{e}}{\mathfrak{a}} \right)^{n} \left(\frac{Pt}{n} \right)^{n-1}. \tag{10.648}$$

The last factor is at least one. Outside III, the middle base is $\mathrm{e}/2 > 27/20 > 1$, so its least power for n at least two is its square. In III the first two factors are $(c_*/c)\mathrm{e}^n x^{1-n}$, with $1 < x \leq 4/3$. For fixed x their least value is at n=2, and at n=2 their least value over x is bounded below by $(c_*/c)(27/10)^2(3/4)$. The resulting strict rational lower bounds for C_2/C_1 are

Case	Strict lower bound for C_2/C_1
I.1	174717/62600
I.2	19683/10600
II	50301/20200
III.1	72171/47840
III.2	1218159/716000
IV	881361/536000
V	147987/41200

All exceed 3/2. The smallest gap is the exact positive fraction 411/47840 in III.1. Lemma 10.172 and Theorem 10.152 now give $C_1\mathsf{h} < C_2\mathsf{h}_2$, and the asserted valuation bound follows. No height floor, residue index, field extension or coefficient ordering has been changed. □

For example, over $\mathbb{Q}$ at three, $P = 3$ and $Pt = 3\ln 3 > 3$. Thus the theorem applies to two or three independent rational local units. In contrast, for an unramified prime $p \geq 5$, $P = 1$: the condition is $n \leq f \ln p$. The stated range is part of the theorem's hypothesis.

A separate bound for the last coefficient

We use ε for the coefficient-imbalance parameter; it is unrelated to the saturated residue index $\delta(\boldsymbol{a})$. The two alternatives below allow a small last coefficient B_n to be retained explicitly.

Lemma 10.174 (a logarithm versus a linear term). Let $T, H_0 > 0$, $X, B_n, r \geq 1$, and $0 < \varepsilon \leq 1$. Put

$$\eta = \frac{23}{500}, \qquad \Phi = (1+\eta)T, \qquad Z = \max\{8\Phi X, \exp(H_0/(1+\eta))\}. \tag{10.649}$$

Then

$$T \max\{\ln(B_n X r), H_0\} \leq \max\left\{\Phi \ln \frac{ZB_n}{\varepsilon}, \varepsilon r\right\}. \tag{10.650}$$

Proof. The positive exponential series gives

$$\mathrm{e} > \sum_{j=0}^{6} \frac{1}{j!} = \frac{1957}{720} > \frac{125}{46} = \frac{1}{8\eta}; \qquad \frac{1957}{720} - \frac{125}{46} = \frac{11}{16560} > 0. \tag{10.651}$$

For any $s > 0$, differentiation of $u\exp(-su)$ on $u \geq 0$ proves $\ln y \leq y^s/(\mathrm{e}s)$ for $y \geq 1$. Let $Y = B_n X r$ and $A = ZB_n/\varepsilon$. Then $A \geq \exp(H_0/(1+\eta)) > 1$. If $Y < A^{1+\eta}$, both entries on the left are at most $\Phi \ln A$, proving the inequality in this alternative. For $Y \geq A^{1+\eta}$, use $s = \eta/(1+\eta)$. Since $Z/X \geq 8\Phi$,

$$\varepsilon r = \frac{Z}{X}\frac{Y}{A} \geq 8\Phi Y^{\eta/(1+\eta)} > T \ln Y. \tag{10.652}$$

The last strict comparison follows from $8\eta\mathrm{e} > 1$. Also $T \ln Y \geq \Phi \ln A \geq T H_0$. This proves the weak scalar inequality, and proves it strictly in the second alternative. □

Theorem 10.175 (a two-branch coefficient estimate). Keep the hypotheses of Theorem 10.152. Let $B \geq \max_j |b_j|$, $1 \leq B_n \leq B$, and $|b_n| \leq B_n$. Define

$$\begin{aligned} h_{\min} &= \min_j h(a_j), & T &= C_1 \prod_{j=1}^{n} h(a_j), \\ H_0 &= \max\{G_1(n,d), (n+1)t\}, & X &= \max\left\{1, \frac{1}{dW(d)}\left(\frac{1}{h_{\min}} + \frac{1}{h(a_n)}\right)\right\}. \end{aligned} \tag{10.653}$$

With Φ, Z as in Lemma 10.174, for every $0 < \varepsilon \leq 1$,

$$\mathrm{ord}_{\mathfrak{p}}(\Xi - 1) < \max\left\{\Phi \ln \frac{ZB_n}{\varepsilon}, \varepsilon \frac{B}{B_n}\right\}. \tag{10.654}$$

Proof. Put $r = B/B_n$. The nonzero integer b_n implies $B_\dagger \leq |b_n| \leq B_n$. For every $j < n$,

$$\frac{1}{dW(d)}\left(\frac{|b_n|}{h(a_j)} + \frac{|b_j|}{h(a_n)}\right) \leq B_n X r. \tag{10.655}$$

It follows that $\mathsf{h} \leq \max\{\ln(B_n X r), H_0\}$. Theorem 10.152 is strict. Combining it with the weak scalar inequality of Lemma 10.174 therefore proves the strict valuation estimate, including its scalar equality case. □

Every constant in this result is explicit and independent of B and B_n. The factor 1.046 measures the logarithmic slack; the price for that small slack is the fixed factor Z. The proof applies to arbitrary signed coefficients and allows zero coefficients other than the designated last one. Its transcendence input is exactly the proved first estimate, not a citation standing in for a proof.

For $(a_1, a_2) = (2, 5)$ over $\mathbb{Q}$ at three, take $(b_1, b_2) = (3^k, 1)$, with $k \geq 1$. The bases are independent by their prime valuations. Here $B_n = 1, B = 3^k$, and X=1: substituting W(1) reduces its other entry to $(\ln 3)(\ln 10)/((\ln 5)(\ln 6)) < 1$, since $(\ln 2)(\ln 3) < (\ln 2)(\ln 5)$. Theorem 10.175 reads

$$v_3(5 \cdot 2^{3^k} - 1) < \max\{\Phi \ln(Z/\varepsilon), \varepsilon 3^k\}. \tag{10.656}$$

The exact valuation is one: modulo nine, $3^k \equiv 3 \pmod 6$, $2^{3^k} \equiv 8$, and $5 \cdot 8 - 1 \equiv 3$. Thus B_n has stayed fixed while the other coefficient grows.

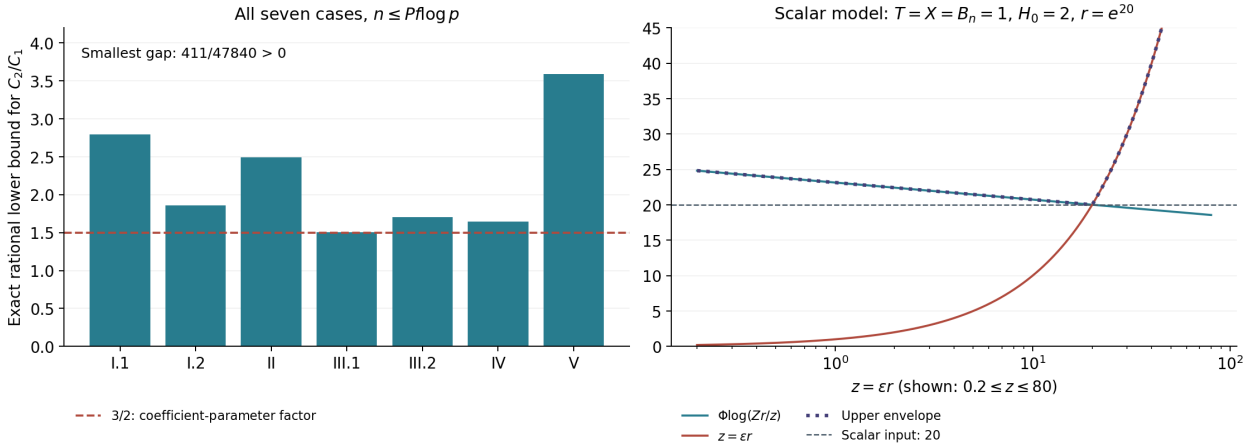


Figure 10.51. Left: exact rational lower bounds for C_2/C_1 under $n \leq Pt$; the dashed line is 3/2, the factor in Lemma 10.172. Theorem 10.173 proves the resulting second estimate in that range. Right: the scalar model $T = X = B_n = 1, H_0 = 2, r = \exp 20$ of Lemma 10.174. With $z = \varepsilon r$, the two curves are $\Phi \ln(Zr/z)$ and z. Their upper envelope bounds 20, the original scalar estimate. The curves illustrate the proved inequality; they are not sampled prime-ideal valuations. Theorem 10.175 transfers this mechanism to the logarithmic form. Reproducible calculation and figure. Human source for the two constants: equations (1.9)–(1.13), (1.28)–(1.30) of Yu's freely accessible paper.

59. Congruence selection and a larger analytic disc

A congruence between the unit parts of two monomials gives extra precision for their quotient. Selecting many such monomials costs a finite index. The useful balance is between that index and the enlarged disc on which the auxiliary function is normal. We keep the polynomial indices separate from the monomial exponents throughout the selection.

Selecting the monomials

Let L be a finite extension of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}_L$, uniformizer π, ramification index e and residue field of cardinality p^f. Use $v_p(p) = 1$, so that $v_p(\pi) = 1/e$. For an integer $j \geq 1$, write $U_j = 1 + \pi^j \mathcal{O}_L$.

Theorem 10.176 (congruence selection with all polynomial indices). Let $u_1, \dots, u_r \in U_j$, let $S \subset \mathbb{Z}^r$ be finite and nonempty, and let E be a finite nonempty set of auxiliary indices. For an integer $m \geq 0$, put

$$H_m = \operatorname{im}\left(\mathbb{Z}^r \longrightarrow U_j/U_{j+m}, \quad \lambda \longmapsto \prod_{i=1}^{r} u_i^{\lambda_i}\right), \qquad h_m = |H_m|. \tag{10.659}$$

There is a nonempty $S_0 \subset S$ such that

$$h_m \le p^{fm}, \qquad |S_0| \ge \left\lceil \frac{|S|}{h_m} \right\rceil, \qquad |E \times S_0| \ge \frac{|E \times S|}{p^{fm}}, \tag{10.660}$$

and, for every $\lambda, \mu \in S_0$,

$$v_p\left(\prod_i u_i^{\lambda_i - \mu_i} - 1 \right) \ge \frac{j+m}{e}. \tag{10.661}$$

Every $\lambda^0 \in S_0$ supplies the full reference fibre $E \times \{\lambda^0\}$. Moreover, let q be an integer prime to p, with $q \ge 2$. Restrict S_0 to any nonempty class $a + q\mathbb{Z}^r$, and replace $\lambda = a + q\mu$ by μ. The displayed congruence remains true for the differences of the new exponents, with the same units, j and m. This remains true after every finite sequence of such restrictions and divisions.

Proof. For $s \ge 1$, the map

$$U_s/U_{s+1} \longrightarrow \mathcal{O}_L/\pi\mathcal{O}_L, \qquad 1 + \pi^s a \longmapsto a \pmod{\pi} \tag{10.662}$$

is an isomorphism of groups, where the target is additive. It is induced by the surjective map on U_s with kernel U_{s+1}, and the product of two representatives introduces only the extra term $\pi^{2s}ab$, which lies in $\pi^{s+1}\mathcal{O}_L$. Multiplying the successive indices proves $|U_j/U_{j+m}| = p^{fm}$, including m=0. Negative exponents are allowed because every U_s is a group: the inverse identity is $(1+x)^{-1} - 1 = -x/(1+x)$.

The monomial map is therefore a homomorphism with at most p^{fm} values. One of its fibres on S has at least $\lceil |S|/h_m \rceil$ elements. Choose that fibre as S_0. Equal images mean exactly that the quotient belongs to U_{j+m}. The selection is independent of E, so every auxiliary index remains available and the reference fibre has the asserted form.

For the final assertion, let two retained exponents be $a + q\mu, a + q\nu$. Put $w = \prod_i u_i^{\mu_i - \nu_i} \in U_j$. The original congruence says $w^q \in U_{j+m}$. But

$$w^q - 1 = (w-1)(1 + w + \cdots + w^{q-1}), \qquad 1 + w + \cdots + w^{q-1} \equiv q \pmod{\pi}. \tag{10.663}$$

The second factor is a unit, giving $v_p(w^q - 1) = v_p(w - 1)$. Thus $w \in U_{j+m}$. Induction proves persistence through every subsequent restriction and division. □

In particular, E can be $\{0, \dots, D_{-1}\} \times \{0, \dots, D_0\}$, while S already satisfies a prescribed residue-phase condition. The phase subclasses of S_0 are intersections with the original subclasses; their number cannot increase. The selected coefficient set retains both complete polynomial-index intervals.

For a concrete example, take $L = \mathbb{Q}_3$, $u_1 = 10, u_2 = 19, j = 2, m = 1$, and $S = \{0, \dots, 5\}^2$. Since

$$10^{\lambda_1} 19^{\lambda_2} \equiv 1 + 9(\lambda_1 + 2\lambda_2) \pmod{27}, \tag{10.664}$$

there are three classes of twelve exponents each. The zero class contains the three even-coordinate exponents $(0,0), (2,2), (4,4)$. Division of those coordinates by two gives $(0,0), (1,1), (2,2)$, whose monomials are still one modulo 27.

Choosing the congruence level

Let $t = f \ln p$, and let A be a positive real number. For a positive integer r put $\rho = t/r$ and

$$\chi(x) = (A + x) \exp(-\rho x), \qquad x_0 = 1/\rho - A. \tag{10.665}$$

The quantity $p^{fm}(A + m)^{-r} = \chi(m)^{-r}$ combines the worst-case index loss with r powers of the analytic gain.

Lemma 10.177 (integer optimization with explicit rounding losses). Suppose $A \geq A_0$ and $x_0 > 3/5$. Choose m by

$$\begin{cases} m = 1, & 3/5 < x_0 \leq 1, \\ m = k, & x_0 > 1, \quad k = \lfloor x_0 \rfloor, \quad x_0 - k \leq 1/\rho - 1/(\exp \rho - 1), \\ m = k + 1, & \text{otherwise.} \end{cases} \tag{10.666}$$

This m maximizes χ among the positive integers. For each row in the following table it also satisfies

$$A + m < \varsigma_1 \frac{r}{t}, \qquad \chi(m) > \frac{\chi(x_0)}{\varsigma_2}. \tag{10.667}$$

A_0	ς_1	ς_2
3/2	1.2133	1.0202
3/4	1.3128	1.0415
1/2	1.3701	1.0568
2	1.176	1.014

Consequently, throughout the same ranges,

$$\frac{p^{fm}}{(A + m)^r} < \varsigma_2^r \left(\frac{\mathrm{e}t}{r}\right)^r p^{-fA}. \tag{10.668}$$

Proof. Differentiation gives $\chi'(x) = \exp(-\rho x)(1 - \rho(A + x))$. If $x_0 \leq 1$, the greatest value at a positive integer is at one. Otherwise it suffices to compare k and k+1. Their quotient is $\exp(-\rho)(A + k + 1)/(A + k)$, and comparison with one gives exactly the threshold in the statement. The threshold is between zero and one. Positivity follows from $\exp \rho > 1 + \rho$. For $\rho \geq 1$ the upper bound is immediate; for $0 < \rho < 1$, the power series gives $\exp \rho < 1/(1 - \rho)$, which gives the upper bound as well.

For $x_0 > 1$, write $x_0 = k + v$, and define

$$z(y) = \frac{y}{\exp y - 1}, \qquad H(y) = y + z(y) = \frac{y}{1 - \exp(-y)}, \qquad F(y) = z(y) \exp(1 - z(y)). \tag{10.669}$$

The relative values at k and k+1 are $\exp(\rho v)(1 - \rho v)$ and $\exp(-\rho(1 - v))(1 + \rho(1 - v))$, respectively. The first decreases in v and the second increases. Their maximum therefore has its least value at their intersection, where it is $F(\rho)$. If m=k, then $\rho(A + m) \leq 1$. If m=k+1, the threshold gives $\rho(A + m) \leq H(\rho)$. Since $\rho < 1/(A + 1) \leq 1/(A_0 + 1)$, it follows that

$$\rho(A+m)\le H(1/(A_0+1)),\qquad \frac{\chi(m)}{\chi(x_0)}\ge F(1/(A_0+1)). \tag{10.670}$$

Here H increases, z decreases from one to zero, and F decreases. For H the derivative has the sign of $\exp y-1-y>0$. For z it has the sign of $\exp y(1-y)-1<0$. Finally $z\exp(1-z)$ increases for $0<z<1$. These derivative tests prove all the required monotonicity statements.

When $3/5<x_0\le 1$, m=1 and $\rho(A+1)<(A_0+1)/(A_0+3/5)$. Also

$$\frac{\chi(1)}{\chi(x_0)}=\exp(-c)(1+c),\qquad c=\rho(1-x_0)<\frac{2/5}{A_0+3/5}. \tag{10.671}$$

The function $(1+c)\exp(-c)$ decreases for c nonnegative. Thus both alternatives are reduced to four fixed endpoints. The following decimals are rational upper or lower endpoints with denominator 10^{12}:

A_0	Upper bound for H	Lower bound for F	Lower bound for the forced ratio
3/2	1.213297912688	0.980242084353	0.984006473362
3/4	1.312778225605	0.960181882841	0.963883250822
1/2	1.370098893207	0.946280756540	0.947923538725
2	1.175908824386	0.986228202757	0.989312214415

To verify these without a numerical exponential, for $0\le y\le 1$ use

$$E_{64}(y)=\sum_{i=0}^{64}\frac{y^i}{i!}\le\exp y\le E_{64}(y)+\frac{y^{65}}{65!}\frac{1}{1-y/66}. \tag{10.672}$$

The upper tail follows by bounding every subsequent term ratio by y/66. These rational bounds enclose z. Evaluate $z\exp(1-z)$ at its lower z endpoint, using its proved monotonicity, and round down; evaluate H with the upper z endpoint and round up. For the forced ratio divide $1+c$ by an upper bound for $\exp c$. This gives exactly the displayed endpoints. Each H endpoint is strictly below ς_1; each lower ratio is strictly above $1/\varsigma_2$, as multiplication of positive rational denominators verifies. The other height bounds are $25/21, 35/27, 15/11, 15/13$, respectively, and are also strictly below ς_1. The exact calculation records all rational gaps. The monotonicity argument supplies every allowed A and ρ, rather than a finite sampling of them.

Finally, $\chi(x_0)=\rho^{-1}\exp(-1+\rho A)$. Raising its strict reciprocal comparison to the rth power, and using $\rho r=f\ln p$, proves the last assertion. □

The four rows apply to the local radii used earlier. For $A=e\vartheta$, cases I and II of Theorem 10.152 have $A\ge 3/2$; case III has $A\ge 3/4$; case IV has $A\ge 1/2$; and case V has $A\ge 2$. Indeed $e\vartheta=P/2$ outside the unramified odd-prime case, while in that case $\vartheta=(p-2)/(p-1)$. The inequalities defining P give $P\ge 3$ at three, $P\ge 5$ at a ramified five, $P\ge 1$ at an odd prime at least seven, and $P\ge 4$ at two.

Normality and divided derivatives

The next statement includes a perturbation of the logarithmic frequencies. This permits the coefficient relation to be imposed without losing the enlarged disc.

Theorem 10.178 (normality from a congruence class). Use a class S_0 from Theorem 10.176 and a reference $\lambda^0\in S_0$. Suppose $j/e>1/(p-1)$, and put

$$\gamma_\lambda = \ln\left(\prod_i u_i^{\lambda_i-\lambda_i^0}\right). \tag{10.673}$$

Let $N \ge 0$ be rational with $N < (j+m)/e - 1/(p-1)$, and let $\epsilon_\lambda \in \mathbb{C}_p$ satisfy $v_p(\epsilon_\lambda) > N + 1/(p-1)$. Let $P_\lambda \in \mathbb{C}_p[z]$ have degree at most D and all coefficients of valuation at least $-C$, where $D \in \mathbb{Z}_{\ge 0}$, $C \in \mathbb{Q}_{\ge 0}$. Take coefficients $c_\lambda \in \mathbb{C}_p$, not all zero, and set $\beta = \min_\lambda v_p(c_\lambda)$. Then

$$f(z) = \sum_{\lambda \in S_0} c_\lambda P_\lambda(z) \exp\bigl(z(\gamma_\lambda + \epsilon_\lambda)\bigr) \tag{10.674}$$

converges on $|z|_p \le p^N$. Choose $\omega, \xi \in \mathbb{C}_p$ with $v_p(\omega) = -N$ and $v_p(\xi) = C + DN - \beta$. The function $\xi f(\omega z)$ is normal on the unit disc: its power-series coefficients are integral and tend to zero. Moreover, for every $a \in \mathbb{C}_p$ with $v_p(a) \ge 0$ and every integer $k \ge 0$,

$$v_p\left(\frac{f^{(k)}(a)}{k!}\right) \ge \beta - C - DN + kN. \tag{10.675}$$

Proof. The required scalar valuations are rational. Indeed, a nonzero element of $\mathbb{C}_p$ has an algebraic approximation close enough to have the same valuation; that approximation lies in a finite extension of $\mathbb{Q}_p$. Roots of integral powers of p therefore supply ω and ξ. The deep-unit logarithm law of Proposition 9.3 gives $v_p(\gamma_\lambda) \ge (j+m)/e$. Therefore $v_p(\omega(\gamma_\lambda + \epsilon_\lambda)) > 1/(p-1)$. For any w of valuation greater than $1/(p-1)$, the coefficients of $\exp(wz)$ satisfy

$$v_p(w^s/s!) \ge s\bigl(v_p(w) - 1/(p-1)\bigr), \tag{10.676}$$

and tend to infinity. Here $v_p(s!) = \sum_{i \ge 1} \lfloor s/p^i \rfloor \le s/(p-1)$, as proved in Section 3 of Lesson 9. Thus each rescaled exponential is normal. Every coefficient of $P_\lambda(\omega z)$ has valuation at least $-C - DN$. Multiplication by ξc_λ makes all coefficients integral. Finite sums and multiplication by polynomials preserve convergence and this coefficient bound, proving normality and the asserted convergence of f.

For a normal series $F(z) = \sum_s A_s z^s$, the kth divided derivative at an integral point b is $\sum_{s \ge k} A_s \binom{s}{k} b^{s-k}$. This series converges and every summand is integral. Apply this to $F(z) = \xi f(\omega z)$ at $b = a/\omega$, which is integral because N is nonnegative. The chain rule gives $F^{(k)}(a/\omega) = \xi \omega^k f^{(k)}(a)$. Taking valuations gives precisely the displayed divided-derivative bound. □

Corollary 10.179 (prepared polynomial factors). For each coefficient index, choose integers $k \ge 1, \ell \ge 1, t_0 \ge 0, I \ge 0, a \in \mathbb{Z}$ with $k\ell \le D$, and any finite list of integers γ_i, t_i, with $t_i \ge 0$. With $V(k) = \operatorname{lcm}(1, \dots, k)$, let

$$P(z) = \frac{V(k)^{t_0}}{t_0!}\left(\frac{d}{dz}\right)^{t_0}\bigl(\Delta(q^{-I}z + a; k)^\ell\bigr)\prod_i \Delta(\gamma_i; t_i). \tag{10.677}$$

These factors satisfy Theorem 10.178 with $C = D/(p-1)$. Thus its normalizing scalar can be chosen with valuation $D(N + 1/(p-1)) - \beta$. The same coefficient bound holds when the derivative is taken before the affine substitution.

Proof. The numerator of $\Delta(q^{-I}z + a; k)^\ell$ has integral p-adic coefficients, since q is prime to p. Its denominator is $(k!)^\ell$, of valuation at most $k\ell/(p-1) \le D/(p-1)$. Divided differentiation sends z^s to $\binom{s}{t_0} z^{s-t_0}$, so it preserves this coefficient bound. The factor $V(k)^{t_0}$ is an integer, as is every $\Delta(\gamma_i; t_i)$. For a negative integer γ_i, the latter assertion follows either from a zero factor or from $\Delta(-b; t) = (-1)^t \binom{b-1}{t}$ when $b > t$. Finally, the chain-rule difference between the two derivative

conventions is a power of q, which is a p-adic unit. The degree remains at most D, so Theorem 10.178 applies. □

In the parameters of Lemma 10.177, take $j = \lceil e(\vartheta + 1/(p-1)) \rceil$ and $N = (m + e\vartheta)/(e(1+\tau))$, with any positive rational τ for which N is rational. Then $N < (j+m)/e - 1/(p-1)$. Theorems 10.176 and 10.178 therefore give the enlarged disc and preserve it at every prime-to-p descent stage. Corollary 10.179 gives the full normalization of the prepared polynomial factors. Global height and product-formula estimates continue to use their original coefficient field and degree.

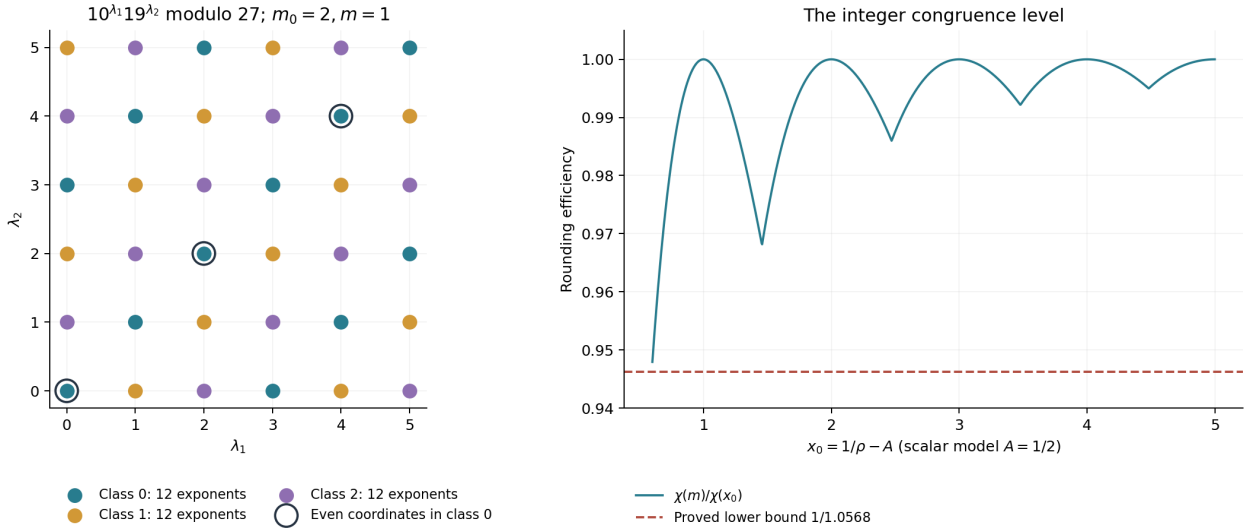


Figure 10.52. Left: the 36 exponents in $\{0,\dots,5\}^2$, colored by $\lambda_1 + 2\lambda_2 \pmod 3$. Each class has twelve points. The circled exponents $(0,0),(2,2),(4,4)$ contract to $(0,0),(1,1),(2,2)$, with the same quotient precision modulo 27. Theorem 10.176 proves the selection and its persistence, including every auxiliary polynomial index. Right: the scalar rounding efficiency for $A = 1/2$. The dashed line is the uniform bound 1/1.0568; the curve illustrates Lemma 10.177 and does not supply its infinite-parameter proof. Theorems 10.178–10.179 turn the retained congruence into a normal-function and divided-derivative estimate. Reproducible exact calculation and figure. Human-source context: Sections 2, 10 and 11 of Yu's free 1999 paper.

60. Projecting the selected auxiliary function

The congruence selection concerns a monomial, rather than each of its individual factors. We now impose the original coefficient relation on that monomial and carry the resulting precision through interpolation. The cost of selecting the monomials must also remain in the global coefficient count.

The original coefficient relation

Let K be a number field, let its chosen completion at p have ramification index e, and write $v_p(p) = 1$. Take local units $a_1,\dots,a_n \in K^*$, integers $b_1,\dots,b_n$ with $b_n \neq 0$, and put $\Xi = \prod_j a_j^{b_j}$. Let $P = p^\kappa$, $\kappa \in \mathbb{Z}_{\geq 0}$, and let ζ be a root of unity of order G prime to p in the completion. Suppose integers c_j make $u_j = a_j^P \zeta^{c_j}$ deep units: $v_p(u_j - 1) > 1/(p-1)$. Set $z_j = \log u_j$. For integers A_{ij}, let $L_i(Z) = \sum_j A_{ij} Z_j$, $1 \leq i \leq r$. Choose a prime $q \neq p$ and $\nu \in \mathbb{Z}_{\geq 0}$, and set $w_i = \exp(q^{-\nu} L_i(z))$. These exponentials lie in the chosen completion, by its completeness and the logarithmic convergence estimates of Lesson 9. Suppose $w_i \in U_j$, with $j/e > 1/(p-1)$, and select S_0 as in Theorem 10.176.

Theorem 10.180 (the coefficient relation on a selected class). Suppose $V = v_p(\Xi - 1) > 1/(p-1)$ is finite. Then

$$\sum_j c_j b_j \equiv 0 \pmod G, \qquad \mathcal{L} := \sum_j b_j z_j = P \log \Xi, \qquad v_p(\mathcal{L}) = \kappa + V. \tag{10.681}$$

Fix $\lambda^0 \in S_0$, write $d_i = \lambda_i - \lambda_i^0$, and let

$$M_i(Z) = L_i(Z) - \frac{A_{in}}{b_n}\sum_j b_j Z_j,$$
$$\gamma_\lambda = \sum_i d_i q^{-\nu} L_i(z) = \log\left(\prod_i w_i^{d_i}\right), \tag{10.682}$$
$$\epsilon_\lambda = -\frac{\mathcal{L}}{q^\nu b_n}\sum_i d_i A_{in}, \qquad H = \kappa + V - v_p(b_n).$$

The coefficient of Z_n in every M_i is zero, and

$$q^{-\nu}\sum_i d_i M_i(z) = \gamma_\lambda + \epsilon_\lambda, \qquad v_p(\gamma_\lambda) \geq \frac{j+m}{e}, \qquad v_p(\epsilon_\lambda) \geq H. \tag{10.683}$$

Consequently, whenever $N \geq 0$ is rational and

$$N < \frac{j+m}{e} - \frac{1}{p-1}, \qquad H > N + \frac{1}{p-1}, \tag{10.684}$$

the projected frequencies satisfy Theorem 10.178 on the whole closed disc $|y|_p \leq p^N$. This conclusion holds for every retained auxiliary polynomial index. It persists through every prime-to-p restriction and exponent division of Theorem 10.176.

Proof. The identity $\prod_j u_j^{b_j} = \Xi^P \zeta^{\sum_j c_j b_j}$ shows that the displayed root of unity reduces to one. Reduction is injective on roots of unity of order prime to p: if $t^G = 1$ and $v_p(t-1) > 0$, the factorization of $t^G - 1$ has second factor congruent to the p-adic unit G, forcing $t = 1$. This proves the phase congruence. All remaining factors are on the deep-unit disc. The logarithm product and power laws in Propositions 9.3–9.4 therefore give $\mathcal{L} = P \log \Xi$, with the stated exact valuation.

The definition of M_i cancels its last coefficient exactly. Summing its identity with coefficients $q^{-\nu} d_i$ gives the displayed perturbation. The exponential and logarithm are inverse on this disc, so their product law identifies γ_λ with the logarithm of the quotient monomial. Theorem 10.176 gives its valuation at least $(j+m)/e$. The sum $\sum_i d_i A_{in}$ is an integer and q is a p-adic unit. Thus the perturbation has valuation at least H, including a zero perturbation. The two strict disc conditions are precisely those of Theorem 10.178. Its proof applies to the finite sum over all auxiliary indices as well as over the exponent indices.

For a restriction $\lambda = a + q\mu$ and corresponding reference $\lambda^0 = a + q\mu^0$, both differences divide by q. Thus the new γ and ϵ are the old ones divided by q; their valuations are unchanged. Iteration proves the last assertion. No new global field or replacement coefficient relation has been used. □

In a contradiction argument with $V \geq U$, a bound $\log|b_n| \leq h$ gives $H \geq \kappa + U - h/\log p$. Indeed $p^{v_p(b_n)} \leq |b_n|$. This makes the strict perturbation condition explicit. In applications to Theorem 10.152 the original minimum-valuation choice of b_n is retained.

The error and its ordinary derivatives

Lemma 10.181 (Taylor precision of the projection error). Use the frequencies of Theorem 10.180 and a positive rational N satisfying its two disc conditions. Let P_λ have degree at most D and coefficients of valuation at least $-C$, with $D \geq 0$ integral and $C \geq 0$ rational. For a finite nonzero coefficient vector $c = (c_\lambda)$, put $\beta = \min_\lambda v_p(c_\lambda)$, and define

$$\Phi(y) = \sum_\lambda c_\lambda P_\lambda(y)\exp(y\gamma_\lambda), \qquad f(y) = \sum_\lambda c_\lambda P_\lambda(y)\exp(y(\gamma_\lambda + \epsilon_\lambda)). \tag{10.685}$$

Both functions have the normality conclusion of Theorem 10.178. For every integral $a \in \mathbb{C}_p$ and every $t \geq 0$,

$$v_p\left(\frac{(f-\Phi)^{(t)}(a)}{t!}\right) \geq \beta - C + H + \max\{0, (t-D-1)N\}. \tag{10.686}$$

If the P_λ are the prepared factors in (10.92), they are p-integral at every $a \in \mathbb{Z}_p$. For these factors the sharper order-zero error is

$$v_p(f(a) - \Phi(a)) \geq \beta + H \qquad (a \in \mathbb{Z}_p). \tag{10.687}$$

Proof. Normality follows from Theorem 10.178, first with zero perturbations and then with the actual perturbations. Write $\sigma = (j+m)/e$ and $L = \min\{\sigma, H\} > N + 1/(p-1)$. For one summand, consider $E(y) = \exp((\gamma+\epsilon)y) - \exp(\gamma y)$. At an integral a, $v_p(E(a)) \geq H$. For $l \geq 1$,

$$\frac{E^{(l)}(a)}{l!} = \frac{\exp(\gamma a)}{l!}\left[\gamma^l(\exp(\epsilon a) - 1) + \exp(\epsilon a)\sum_{h=1}^{l}\binom{l}{h}\gamma^{l-h}\epsilon^h\right]. \tag{10.688}$$

The bracket has valuation at least $H + (l-1)L$. The factorial identity of Lesson 9 gives $v_p(l!) = (l - s_p(l))/(p-1) \leq (l-1)/(p-1)$, where $s_p(l) \geq 1$ is the base-p digit sum. Therefore the displayed divided derivative has valuation at least $H + (l-1)N$. All divided derivatives of P_λ at an integral point have valuation at least $-C$, since their Taylor coefficients are sums of integral binomial multiples of its coefficients. They vanish above D. In the divided product rule the remaining exponential order is at least $t - D$. The bounds for orders zero and positive orders consequently give the asserted maximum. Multiplication by the coefficients and summation give the result.

For a prepared factor, the integer-valued Taylor proof in Lemma 10.28, extended to $\mathbb{Z}_p$ in Lemma 10.36, gives $v_p(P_\lambda(a)) \geq 0$. Its exponential error has valuation at least H, by Proposition 9.3. This proves the last bound. □

Interpolation on the enlarged disc

Theorem 10.182 (selected-support extrapolation with every jet cost). Use the full prepared family (10.92) on the selected support, with $k, L \geq 1$, $D = kL$, coefficient minimum β, and the actual selected derivations of Lemma 10.23. Let the unprojected and projected functions be $\Phi(y; \boldsymbol{t})$ and $f(y; \boldsymbol{t})$ from Theorem 10.180. Suppose N is positive and rational and satisfies that theorem's two strict disc conditions. Put

$$\begin{aligned} &C_0 = \max\{v_p(V(k)), v_p(b_n)\}, \qquad \Gamma = D\left(N + \frac{1}{p-1}\right) - \beta, \\ &M_0 = \max\{B, C_0\}, \qquad V(k) = \operatorname{lcm}(1, \ldots, k). \end{aligned} \tag{10.689}$$

Take one of the full or q-deleted integer node sets E of Theorem 10.34. Let $n_E = |E|$, let B be its separation bound, and let $\eta_E = 0$ for the full interval and $\eta_E = 1$ for the q-deleted interval. Suppose $\mu \geq 1$, $|\boldsymbol{t}| + \mu - 1 \leq T$, and $\Phi(s; \boldsymbol{t}') = 0$ at every node for every $|\boldsymbol{t}'| \leq T$. Then, for every $x \in \mathbb{Z}_p \subset \mathbb{Q}_p$, both f and Φ have valuation at least

$$\min\left\{n_E\mu N - \Gamma, \quad \beta + H - \eta_E\mu B - (\mu-1)M_0\right\}. \tag{10.690}$$

In particular, under the counterhypothesis $V \geq U$, the budget

$$\kappa + U + D\left(N + \frac{1}{p-1}\right) \geq n_E \mu N + (\eta_E + 1)\mu M_0 \tag{10.691}$$

gives

$$v_p(f(x;\boldsymbol{t})),\ v_p(\Phi(x;\boldsymbol{t})) \geq n_E \mu N - D\left(N + \frac{1}{p-1}\right) + \beta. \tag{10.692}$$

The strict perturbation condition must still hold. A sufficient version under this counterhypothesis is $\kappa + U - v_p(b_n) > N + 1/(p-1)$.

Proof. At a node, Lemma 10.181's prepared-factor estimate and the assumed zeros give $v_p(f(s;\boldsymbol{t}')) \geq \beta + H$. Each individual projected slope has valuation greater than $1/(p-1)$: the unprojected slope has that property by its definition from the deep z_j, and its correction has valuation at least H. Thus Lemma 10.38 applies with exactly the selected directions, yielding $v_p(f_l(s;\boldsymbol{t})) \geq \beta + H - lC_0$ for $0 \leq l < \mu$.

Theorem 10.178 and Corollary 10.179 supply a scalar ξ of valuation Γ for which $F(Z) = \xi f(\omega Z;\boldsymbol{t})$ is normal, where $v_p(\omega) = -N$. At the scaled node s/ω, its weighted jet precision is $\Gamma + \beta + H - lC_0$, by the chain rule. Apply Theorem 10.39 with $\rho = 1/\omega$, $\theta = N$, and its node indicator equal to η_E; subtracting Γ gives the displayed minimum for f. The prepared projection error at x is at least $\beta + H$, which is no smaller than the second entry of that minimum. The same minimum therefore applies to Φ.

Finally $H \geq \kappa + U - v_p(b_n)$, while

$$v_p(b_n) + \eta_E \mu B + (\mu - 1)M_0 \leq (\eta_E + 1)\mu M_0. \tag{10.693}$$

The budget makes the second entry of the minimum at least the first. This proves the last conclusion. In this proof only the combined monomial frequencies need the enlarged precision; the ordinary jet identity uses the original individual slopes on their original disc. □

Columns and heights after the selection

Retain the global monomial and height setting of Theorems 10.63–10.65. Let the exponent lattice have covolume Δ, let the original coordinate box have full widths $D_1, \ldots, D_r > 0$, and put $\mathcal{V} = \prod_i D_i$. Let the phase map have H_{ph} distinct possible values, as in Lemma 10.67. Let h_m be the actual finite unit-quotient image order in Theorem 10.176.

Proposition 10.183 (the selected coefficient surplus and height). The successive closed-box, phase and unit-congruence selections give a support S_0 and $N_c = kL|S_0|$ columns satisfying

$$N_c \geq kL\left\lceil \frac{\lceil (\lfloor \mathcal{V}/\Delta \rfloor + 1)/H_{\mathrm{ph}} \rceil}{h_m} \right\rceil > \frac{kL\mathcal{V}}{\Delta H_{\mathrm{ph}} h_m}. \tag{10.694}$$

Let $S \geq 1, T \geq 0$, $s_0 = \lfloor S \rfloor$, $t_0 = \lfloor T \rfloor$, and count the whole prepared system by

$$M = (2s_0 + 1)\binom{t_0 + r}{r}. \tag{10.695}$$

If $kL\mathcal{V}/(\Delta H_{\mathrm{ph}} h_m) \geq c_* M$, with $c_* > 1$, there is a nonzero integral coefficient vector giving all these zeros. For its actual row field E_0, of degree d_0, put $L_\alpha = \sum_i D_i h(\alpha_i)$. With the complete scalar bound $\mathcal{B}_0^*(s_0, t_0)$ of Corollary 10.64 it can be chosen with

$$h_2(c) \le \frac{1}{2}\log(N_c d_0) + (d_0-1)PL_\alpha + \frac{1}{c_*-1}\left[\frac{1}{2}\log N_c + \log \mathcal{B}_0^*(s_0,t_0) + \frac{Ps_0(s_0+1)}{2s_0+1}L_\alpha\right]. \tag{10.696}$$

An actual generator tower or discriminant may replace the corresponding upper bound as in Theorem 10.63. Every auxiliary pair $0 \le a < k, 1 \le \ell \le L$ remains available at every selected exponent; the resulting Laurent polynomial is nonzero.

Proof. Theorem 10.66 supplies $\lfloor \mathcal{V}/\Delta \rfloor + 1$ lattice points. Lemma 10.67 retains at least the inner ceiling, and Theorem 10.176 retains at least its quotient ceiling. Since $\lfloor \mathcal{V}/\Delta \rfloor + 1 > \mathcal{V}/\Delta$, the displayed strict comparison follows. The auxiliary pairs are independent of both selections, so their full factor is kL.

There are $2s_0+1$ nodes and $\binom{t_0+r}{r}$ indices of total degree at most t_0, by the divider count already proved in Theorem 10.32. Keep this entire list of M rows, including redundant or zero rows, rather than changing its average by a nonuniform deletion. The assumed inequality gives $N_c > c_* M > M$. The integral Siegel proof in Theorem 10.63 therefore applies.

The new support is a subset of the old box. After centering at one of its own points it has the same full-width bounds, so (10.173) still bounds the generator and both torus heights. The actual row field is generated by the new normalized monomials: the zero-derivative row at $s = 1, a = 0, \ell = 1$ contains $(k+1)W_\lambda$, and the reference monomial is one. This is the exact field-recovery argument of Theorem 10.63 and requires the retained full auxiliary product. It uses no ambient-degree replacement. Applying (10.174) to this full row list, and using $M/(N_c - M) < 1/(c_* - 1)$, proves the height bound. The mean torus term is unchanged because each integer node has the same full index list; $\sum_{s=-s_0}^{s_0} |s| = s_0(s_0+1)$. Finally Lemma 10.29 gives an independent binomial polynomial family for each of the distinct Laurent monomials. The nonzero coefficient vector therefore gives a nonzero Laurent polynomial. □

The factor $h_m \le p^{fm}$ in the column condition is essential. Lemma 10.177 controls how that finite-index cost trades against the larger analytic radius. A parameter choice for a final logarithmic estimate must satisfy both this coefficient inequality and the strict analytic and arithmetic comparisons in the zero induction.

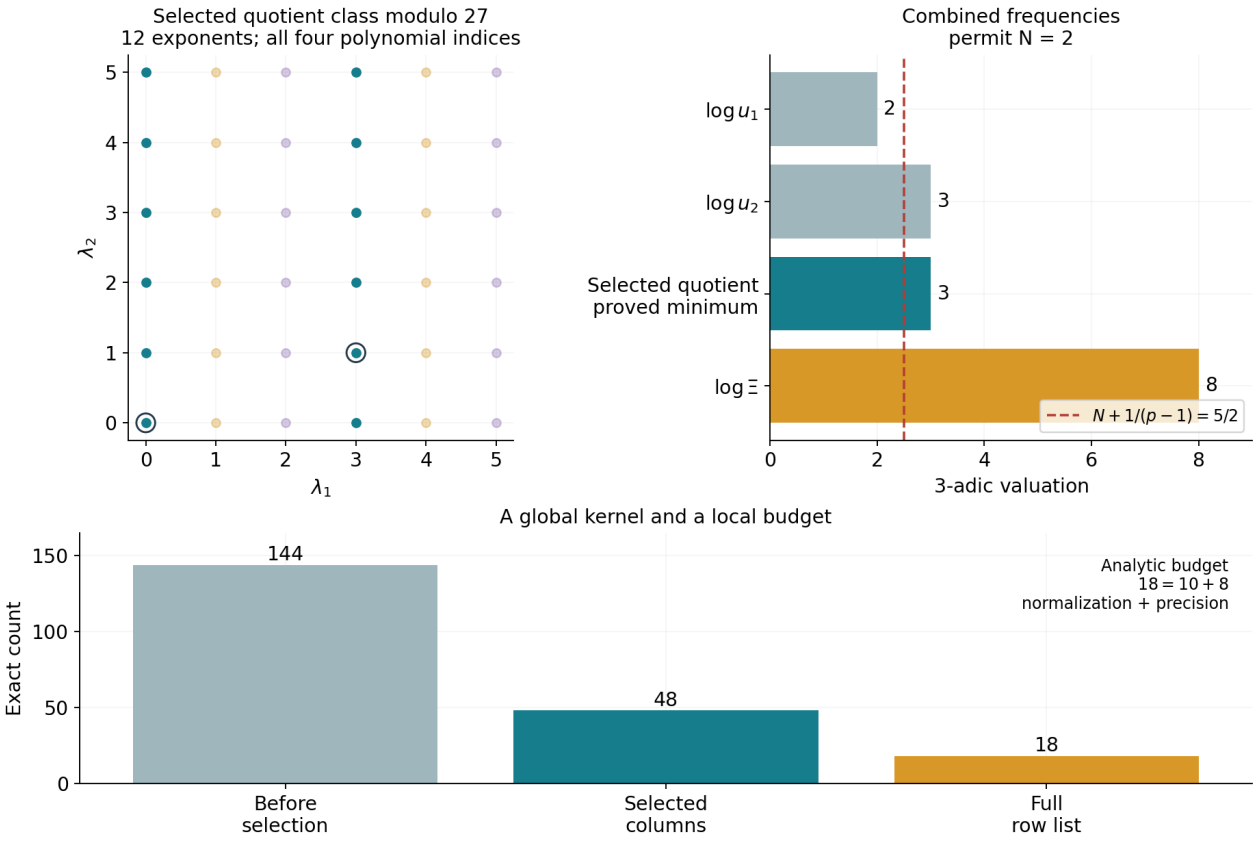


Figure 10.53. Take $p = 3$, $u_1 = 10$, $u_2 = 3281/500$, and $\Xi = u_1^3 u_2 = 1 + 3^8$. Upper left: the selected quotient class modulo 27 consists of the twelve points in $\{0, 3\} \times \{0, \dots, 5\}$. Each retains all four polynomial indices. The circled points have quotient Ξ and equal projected frequency. Upper right: the individual logarithm valuations are exactly two and three; the selected quotients have proved minimum three, attained at $(3, 0)$. The relation has exact valuation eight. The dashed threshold $N + 1/(p-1) = 5/2$ is strictly below both the quotient minimum and the perturbation precision, giving normality at $N = 2$. Below: selection leaves 48 of the original 144 columns, against eighteen full prepared rows. The uniform normalization cost is ten and the output precision is eight, with interpolation budget $18 = 10 + 8$. Theorems 10.180–10.182 and Proposition 10.183 prove the mechanism; Solution 74 verifies the exact rational example and its integral kernel. Reproducible figure and exact calculation. Human-source context: Sections 10–11 of Yu's freely accessible 1999 paper.

61. Constructing the auxiliary function at the second scale

We now choose the size of the selected support explicitly. Its defining inequality contains the actual unit-congruence index. Optimizing that index against the analytic radius removes the factor involving a large power of the rank. We keep the global residue index, the saturation index, the polynomial indices and the integer floors throughout.

Parameters and the selected support

Use a field case of Theorem 10.152. Write $t = f \log p$, $P = p^{\kappa}$, $\ell_d = \max\{1, \log d\}$, and let ϑ have its earlier value. Thus $t > 1$, $A_0 = e\vartheta \geq 1/2$, and $A_0 \leq P$. Let $3 \leq r \leq n$, and suppose

$$\frac{r}{t} > A_0 + \frac{3}{5}. \tag{10.700}$$

For independent local units $\alpha_1, \dots, \alpha_r$, take weights $\sigma_i \geq h(\alpha_i) > 0$. Put $\Omega_r = \prod_i \sigma_i$, let J be their q-primary saturation index, and let δ_r be their saturated residue index. Thus $J = q^{\nu}$ for an integer $\nu \geq 0$, the exponent lattice has covolume $1/J$, and its phase map has $(p^f - 1)/(q^u \delta_r)$ possible values. The saturation and phase constructions are those of Sections 8, 18 and 25.

Choose m, ζ_1, ζ_2 by Lemma 10.177, using its row for the current field case. Set

$$\begin{aligned}
\tau &= \frac{10^{-26}}{2n}, \qquad N = \frac{m + A_0}{e(1+\tau)}, \\
\mathcal{R}_r &= \max\left\{ \frac{p^f}{\delta_r}, \left(\frac{\mathrm{e}}{r}\right)^r t^{r+1} \right\}, \\
A_r &= \max\{4 + \log((r+1)d), e, t\}.
\end{aligned} \tag{10.701}$$

Here $\mathrm{e} = \exp(1)$. It is distinct from the ramification index e. Take a real $h \geq (r+1)t$, and put $H = h + \log J$. The following constants will be used:

Field cases	c_0	c_1	c_3	c_4	a
I.1, I.2, II	27/10	7/10	6/5	14	7
III.1, III.2	27/10	7/10	8/5	14	7
IV	27/10	7/10	3	14	7
V	27/10	41/50	3/5	6	13

Define the positive real parameters

$$\begin{aligned}
c_2 &= \frac{a}{q^2\zeta_2}, \quad g_S = c_3 q (r+1)^2 d, \quad K_S = \frac{101}{100}\left(2 + \frac{1}{g_S}\right), \\
S &= \frac{c_3 q (r+1) d H}{t}, \\
\mathcal{D} &= K_S c_0 c_1 c_4 \frac{N}{\vartheta} (\zeta_2 c_2 q \mathrm{e} P)^r \frac{(r+1)^r}{r!} \\
&\quad \times \frac{d^{r+1} \ell_d \Omega_r}{q^u J} \mathcal{R}_r A_r (1+\tau)^r, \\
T &= \frac{q(r+1)\mathcal{D}}{c_1 e N t}, \qquad D_i = \frac{\mathcal{D}}{c_1 c_2 r P d \sigma_i} \quad (1 \leq i \leq r).
\end{aligned} \tag{10.702}$$

The polynomial parameters are integers. More precisely, put

$$k = \lfloor H \rfloor, \qquad X = \frac{S\mathcal{D}}{c_1 c_4 (N/\vartheta) d A_r k}, \qquad L = \lfloor X \rfloor + 1, \qquad D_{\mathrm{a}} = kL. \tag{10.703}$$

The degree D_{a} of a prepared polynomial is not the real scale $\mathcal{D}$.

Theorem 10.184 (a selected kernel with explicit second-scale parameters). For these parameters,

$$T > 101 r^2, \qquad X > 58, \qquad S \geq g_S. \tag{10.704}$$

The closed-box, phase and unit-congruence selections of Proposition 10.183 give $N_c = kL|S_0|$ coefficients, with every polynomial pair retained, such that

$$N_c > c_0 (2\lfloor S \rfloor + 1) \binom{\lfloor T \rfloor + r}{r}. \tag{10.705}$$

In particular there is a nonzero global-integral coefficient vector whose unprojected prepared values vanish at every integer $|s| \leq S$ and every prepared index $|\boldsymbol{t}| \leq T$. The Laurent polynomial is nonzero, and its coefficients satisfy the actual row-field and height conclusions of Proposition 10.183.

Proof. First we establish the quantitative bounds that justify the floors. The saturation estimate (10.55), applied to the q-primary index J, gives

$$\Omega_r \geq \frac{w_K J r^r}{\rho\, d^{r+1}\ell_d\, r!\mathrm{e}^r}, \qquad \rho = \begin{cases} 17, & d = 1, \\ 58, & d \geq 2. \end{cases} \tag{10.706}$$

The weights dominate the individual heights, and the full saturation index dominates its q-primary part, so every factor has the indicated direction. Also $w_K \geq q^u$, $\mathcal{R}_r > 1$, and $K_S > 2$. Substitution in $\mathcal{D}$ yields

$$\mathcal{D} > \frac{2c_0c_1c_4}{\rho}\frac{N}{\vartheta}(c_2qP)^r\frac{r^r(r+1)^r}{(r!)^2}A_r. \tag{10.707}$$

The elementary inequalities $r!/r^r \leq 2^{1-r}$ and $(1+1/r)^r \geq 2$ follow from the consecutive-ratio and binomial proofs used in Theorem 10.152. Together they imply

$$\frac{r^r(r+1)^{r+1}}{(r!)^2} \geq (r+1)2^{2r-1}. \tag{10.708}$$

Each row of the parameter table has $c_0 \geq 2, c_3 \geq 1/3, c_4 \geq 6, c_2q > 3$. For the last assertion, at odd p use $2\zeta_2 \leq 2(1.0568) < 7/3$; at two use $3\zeta_2 = 3.042 < 13/3$. Using $A_r \geq t$, $A_0 = e\vartheta \leq P$, and $P \geq 1$, the preceding bound on $\mathcal{D}$ gives

$$T > \frac{12}{29}(r+1)12^r > 101r^2. \tag{10.709}$$

Indeed the first expression divided by r^2 increases for $r \geq 3$: its consecutive ratio is $12(r+2)r^2/(r+1)^3 > 1$. At three the required comparison is $82944/29 > 909$, an exact integer comparison after multiplication by 29.

Since $k \leq H$, the definitions of S and X give $X \geq c_3q(r+1)\mathcal{D}/(c_1c_4(N/\vartheta)A_rt)$. Use the lower bound for $\mathcal{D}$, the factorial comparison, and $r/t > A_0 + 3/5$. It follows that

$$X > \frac{c_0c_3q}{\rho}(A_0+3/5)\frac{r+1}{r}(4c_2qP)^r \geq \frac{8448}{145} > 58. \tag{10.710}$$

For the numerical lower bound use $A_0 \geq 1/2$ and the table's lower bounds just proved. The function $(r+1)12^r/r$ increases for $r \geq 3$, so its least value is at three. Finally $h \geq (r+1)t$ gives $S \geq g_S$, and k is a positive integer.

For the row count, $T > 101r^2$ implies

$$\binom{\lfloor T \rfloor + r}{r} \leq \frac{(T+r)^r}{r!} < \frac{101}{100}\frac{T^r}{r!}. \tag{10.711}$$

To see the last inequality without approximating a logarithm, bound $\binom{r}{j}(r/T)^j$ by $(r^2/T)^j$ and sum the geometric series. Its sum is less than $101/100$.

Let $V = \prod_i D_i$, and let h_m denote the actual unit-quotient image order. Proposition 10.183 gives $N_c > kLJV/(H_{\mathrm{ph}}h_m)$, where $H_{\mathrm{ph}} = (p^f-1)/(q^u\delta_r)$. Since $L > X$,

$$kL > \frac{S\mathcal{D}}{c_1c_4(N/\vartheta)dA_r}. \tag{10.712}$$

Divide this column lower bound by $(2S+1)(101/100)T^r/r!$, and substitute T and the widths. The powers of $\mathcal{D}$ cancel except for one. To obtain a ratio at least c_0, it is sufficient that

$$\mathcal{D} \geq \frac{101}{100}(2+1/S)c_0c_1c_4\frac{N}{\vartheta}\frac{(c_2qr(r+1)P)^r}{r!\,t^r}d^{r+1}\Omega_r A_r\frac{H_{\mathrm{ph}}}{J}\frac{h_m}{(eN)^r}. \tag{10.713}$$

Lemma 10.177 and $eN=(m+A_0)/(1+\tau)$ give

$$\frac{h_m}{(eN)^r} \leq (1+\tau)^r\frac{p^{fm}}{(m+A_0)^r} < (1+\tau)^r\zeta_2^r\left(\frac{\mathrm{e}t}{r}\right)^r p^{-fA_0}. \tag{10.714}$$

Now $S\geq g_S$, $\ell_d\geq 1$, and $\mathcal{R}_r\geq p^f/\delta_r > ((p^f-1)/\delta_r)p^{-fA_0}$. The asserted sufficient bound therefore follows from the definition of $\mathcal{D}$. This proves the strict coefficient surplus. Proposition 10.183 provides the integral vector, its height bound and the nonzero polynomial. Every field and residue index used in that proposition has been retained. □

Corollary 10.185 (the exact cost of the polynomial rounding). The integer degree of this kernel satisfies

$$\frac{S\mathcal{D}}{c_1c_4(N/\vartheta)dA_r} < D_{\mathrm{a}} < \frac{59}{58}\frac{S\mathcal{D}}{c_1c_4(N/\vartheta)dA_r}. \tag{10.715}$$

The complete normalizing valuation in Theorem 10.182 may consequently be bounded above by

$$\frac{59}{58}\frac{S\mathcal{D}}{c_1c_4dA_r}\left(\vartheta+\frac{\vartheta}{N(p-1)}\right)-\beta. \tag{10.716}$$

Proof. The lower bound is $kL>kX$. The upper bound follows from $L\leq X+1$ and $X>58$. Multiply by $N+1/(p-1)$ and subtract the coefficient minimum β to obtain the second assertion. This is the normalization of the combined monomials, justified by Theorem 10.180; it makes no enlarged disc assumption on an individual basis logarithm. □

The value of the counterhypothesis scale

Proposition 10.186 (comparison with the stated second constant). For the above kernel define the positive real number

$$U_r=\frac{q^{r+1}S\mathcal{D}}{et}. \tag{10.717}$$

Use the exact C_2 formula of Section 58 at rank r, with the same field and the actual residue index δ_r. Then

$$eU_r < C_2(r,d,p,\boldsymbol{\alpha})\,h\Omega_r. \tag{10.718}$$

This statement compares the parameter sizes. It does not assume, or assert by itself, the valuation inequality $v_p(\Xi-1)<U_r$.

Proof. Substitution of S and $\mathcal{D}$, followed by $\zeta_2c_2q^2=a$, gives

$$eU_r = K_Sc_0c_1c_3c_4q^2\frac{N}{\vartheta}(1+\tau)^r(aeP)^r\frac{(r+1)^{r+1}}{r!}\frac{d^{r+2}\ell_d\Omega_r}{q^uJt^2}\mathcal{R}_rA_rH. \tag{10.719}$$

The other rounding bound of Lemma 10.177 is $m + A_0 < \zeta_1 r/t$. Hence $N/\vartheta < \zeta_1 r/(A_0 t(1+\tau))$. Also $H/J \le h$: $\log J \le J - 1$ and $h > 1$ give $h + \log J \le hJ$. Thus

$$eU_r < \frac{F_r}{P}(aeP)^r \frac{(r+1)^{r+1}}{(r-1)!} \frac{d^{r+2}\ell_d}{q^u t^3} \mathcal{R}_r A_r h \Omega_r, \qquad F_r = K_S \zeta_1 c_0 c_1 c_3 c_4 q^2 \frac{P}{A_0}(1+\tau)^{r-1}. \tag{10.720}$$

For $r \le n$, the binomial argument in (10.451) gives $(1+\tau)^{r-1} < 1 + 10^{-26}$. The lower bounds for g_S in the seven field cases are respectively

$$\frac{384}{5}, \quad \frac{192}{5}, \quad \frac{384}{5}, \quad \frac{512}{5}, \quad \frac{256}{5}, \quad 192, \quad 90. \tag{10.721}$$

At two the positive-radius regime has $r \ge 4$, since $f \ge 2, A_0 \ge 2, t \ge \log 4 > 4/3$. For the other cases $r \ge 3$ is already an assumption. Outside case III, $P/A_0 = 2$. In case III it equals $1/\vartheta = (p-1)/(p-2)$. Substitution of these rational bounds and the parameter table shows

$$F_r < c^{(2)} \tag{10.722}$$

in each case, with the factor $1/\vartheta$ retained on both sides in case III. Here are convenient integer upper bounds, obtained by substituting the displayed rational constants in the formula for F_r. In the two case-III rows the common factor $1/\vartheta$ is divided out in both columns.

Field case	Strict upper bound for F_r	Stated coefficient
I.1	627	1438
I.2	631	648
II	627	690
III.1	452	495
III.2	454	557
IV	1763	2418
V	343	406

For example the I.2 entry is the exact rational comparison

$$\frac{101}{100}\left(2 + \frac{5}{192}\right)\frac{12133}{10000}\frac{27}{10}\frac{7}{10}\frac{6}{5}\,14 \cdot 4 \cdot 2\,(1 + 10^{-26}) < 631 < 648. \tag{10.723}$$

The other rows use the same multiplication with their displayed constants. The editable calculation records the exact rational differences as well as these upper bounds. This is a finite comparison of rational constants; the preceding monotonicity and small-τ arguments cover all ranks, degrees, primes and ramification indices. The remaining expression is exactly the displayed C_2 formula. □

Comparing two ranks

Lemma 10.187 (the rank factor of the second estimate). Put

$$F(j) = \frac{(j+1)^{j+1}}{(j-1)!} \quad (j \ge 1). \tag{10.724}$$

Then $F(j+1)/F(j) > \mathrm{e}$. In particular, define $\Gamma_2 = ae^2Pd/q$. Suppose the weights of a rank-r representation of an original n-base form satisfy

$$\Omega_r \le \Gamma_2^{n-r} \frac{\mathcal{R}_n(\boldsymbol{a})}{\mathcal{R}_r(\boldsymbol{\alpha})} \prod_{j=1}^{n} h(a_j). \tag{10.725}$$

For every $3 \le r \le n$ in the positive-radius regime, the scale from Proposition 10.186 then satisfies

$$eU_r < C_2(n, d, p, \boldsymbol{a})\, h \prod_{j=1}^{n} h(a_j). \tag{10.726}$$

All residue indices in these formulas are those of their indicated families. They need not be equal.

Proof. Direct division gives

$$\frac{F(j+1)}{F(j)} = \frac{j+2}{j} \left(1 + \frac{1}{j+1}\right)^{j+1}. \tag{10.727}$$

For $x > 0$, integrating $1/(1+t) > 1/(1+x)$ from zero to x gives $\log(1+x) > x/(1+x)$. The logarithm of the displayed ratio is consequently greater than $2/(j+2) + (j+1)/(j+2) > 1$. This proves the first assertion without a limiting factorial formula.

Insert the assumed weight product in Proposition 10.186. Its $\mathcal{R}_r$ cancels exactly, and $A_r \le A_n$. After division by the desired rank-n expression the remaining rank factor is bounded by

$$\left(\frac{\mathrm{e}}{q}\right)^{n-r} \frac{F(r)}{F(n)} \le 1. \tag{10.728}$$

For $r < n$ it is strictly smaller than q^{r-n}, by the first assertion. At $r = n$ it equals one, and Proposition 10.186 already supplies the strict comparison. This proves the claim. The assertion is conditional on the displayed weight allowance; preserving that allowance in a multiplicity argument is a separate mathematical requirement. □

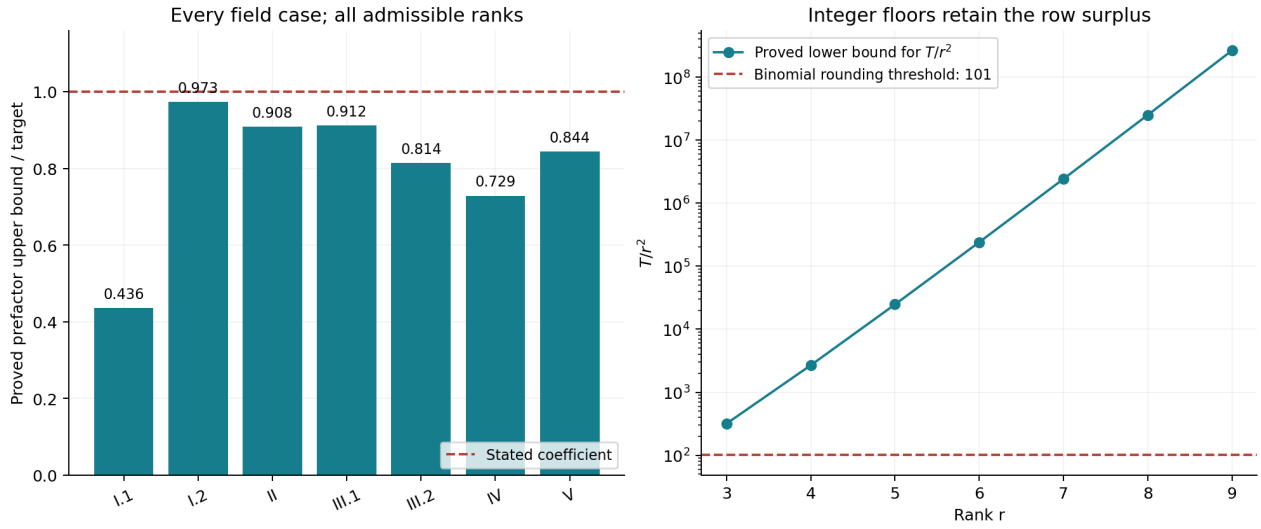


Figure 10.54.

On the left, Proposition 10.186 bounds the auxiliary scale below the stated coefficient in every field case. The common case-III factor cancels before forming either ratio. On the right, Theorem 10.184 bounds T/r^2 above the threshold needed to count all prepared rows; the consecutive-ratio proof covers every $r \ge 3$, while the plotted ranks are samples. These comparisons concern the constructed support and its parameters. They do not assert the global valuation estimate. The constant being compared is the one in Yu's freely accessible 2013 paper, p. 320.

62. Small optimizer cases and the first analytic step

The second-scale construction requires a positive optimizer. We first prove that the first estimate already handles the complementary regime for every list of at least three bases. We also settle almost all two-base cases by exact comparisons. We then verify the first analytic input for the new construction, retaining the original distinguished coefficient and all ordinary derivative costs.

When the first constant suffices

Keep the original independent bases, integer coefficients and field cases of Theorem 10.152, and the exact constants and coefficient parameters of Section 58. Write $A_0 = e\vartheta$, with e the ramification index.

Theorem 10.188 (the small-optimizer regime). Suppose $n \geq 3$. The second estimate

$$\operatorname{ord}_{\mathfrak{p}}(\Xi - 1) < C_2 \mathsf{h}_2 \prod_{j=1}^{n} h(a_j) \tag{10.736}$$

holds in each of the following ranges:

- outside case III, $n \leq (6/5)Pt$;
- in case III, $n \leq (\vartheta + 3/5)t$.

In particular it holds whenever $n/t - A_0 \leq 3/5$.

Proof. The cancellation (10.648) is valid without the restriction $n \leq Pt$. Outside III, put $y = n/(Pt)$. It becomes

$$\frac{C_2}{C_1} = \frac{c^{(2)}}{c} \frac{\mathrm{e}}{2} \left(\frac{\mathrm{e}}{2y} \right)^{n-1}. \tag{10.737}$$

All five relevant rows have $c^{(2)}/c \geq 2418/2680 = 1209/1340$. The fractions in Section 58 prove this by cross-multiplication; the other rows are $1438/939, 648/636, 690/505, 406/206$. If $y \leq 6/5$, then $\mathrm{e} > 27/10$ gives $\mathrm{e}/(2y) > 9/8 > 1$. Since $n - 1 \geq 2$,

$$\frac{C_2}{C_1} > \frac{1209}{1340} \frac{27}{20} \left(\frac{9}{8} \right)^2 > \frac{3}{2}. \tag{10.738}$$

In III write $c^{(2)} = c_*/\vartheta$, where $c_* = 495$ or 557, and put $z = n/t$. Here $\mathfrak{a} = 7/\vartheta$, $P = 1$, and $3/4 \leq \vartheta < 1$. The same exact cancellation gives

$$\frac{C_2}{C_1} = \frac{c_*}{c} \mathrm{e} \left(\frac{\mathrm{e}\vartheta}{z} \right)^{n-1}. \tag{10.739}$$

The function $v/(v + 3/5)$ increases for positive v: its derivative is $(3/5)/(v + 3/5)^2 > 0$. Thus the assumed $z \leq \vartheta + 3/5$ gives $\mathrm{e}\vartheta/z > (27/10)(3/4)/(27/20) = 3/2$. Also $c_*/c \geq 495/1794$, including the III.2 row $557/1790$. Consequently

$$\frac{C_2}{C_1} > \frac{495}{1794} \frac{27}{10} \left(\frac{3}{2} \right)^2 = \frac{8019}{4784} > \frac{3}{2}. \tag{10.740}$$

The two final rational comparisons are reproduced in the editable calculation for Figure 10.55. Lemma 10.172 now gives $C_1 \mathsf{h} < C_2 \mathsf{h}_2$, and Theorem 10.152 proves the claim.

Finally $A_0 = P/2$ outside III, and $P \geq 1$. Therefore $A_0 + 3/5 \leq (6/5)P$, since $3/5 \leq (7/10)P$. In III, $e = P = 1$, so $A_0 = \vartheta$. This proves the last assertion, including equality at the optimizer threshold. □

Theorem 10.189 (the remaining two-base comparisons). For $n = 2$ the second estimate also follows from Theorem 10.152 in every field case except possibly

$$p = 5, \qquad e = f = 1, \qquad d > 1. \tag{10.741}$$

Theorem 10.220 proves the second estimate in this remaining family. The comparison here already gives the conclusion whenever $\mathsf{h}_2 \geq G_1(2, d)$.

Proof. Theorem 10.173 applies if $2 \leq Pt$. Otherwise $P = 1$: if $P > 1$, it is an integer prime power at least two and $t > 1$. At two this also contradicts $P \geq 4$. Thus p is odd, $f = 1$, and $p < \exp 2 < 9$. The possible primes are three, five and seven. At three, $P \geq 3$, so only five and seven remain. The defining inequality $2e < P(p-1)$ gives $e = 1$ at five, and $e = 1$ or two at seven.

At seven, $\log 7 > 19/10$. In case III, $\vartheta = 5/6$, so the rank-two form of the preceding identity gives

$$\frac{C_2}{C_1} > \frac{495}{1794}\left(\frac{27}{10}\right)^2 \frac{5}{6}\frac{19}{20} > \frac{3}{2}. \tag{10.742}$$

In IV, necessarily $e = 2$ here, it instead gives

$$\frac{C_2}{C_1} > \frac{1209}{1340}\left(\frac{27}{20}\right)^2 \frac{19}{20} > \frac{3}{2}. \tag{10.743}$$

These strict rational comparisons and Lemma 10.172 settle seven.

At five, consider first $d = 1$, the III.2 row. A slightly sharper comparison of the coefficient parameters suffices:

$$\mathsf{h} < \frac{11}{8}\mathsf{h}_2, \qquad \frac{C_2}{C_1} > \frac{603231}{438550} > \frac{11}{8}. \tag{10.744}$$

To prove the first inequality, set $a_0 = 2 + \log 7$ and $x = 2a_0 + 146/25$. The parameters in Sections 47 and 58 give

$$\frac{G_1(2,1)}{3} = x + \log x, \qquad \frac{G_2(2,1)}{3} = 2a_0 + \frac{83}{25} + \log 3. \tag{10.745}$$

Using $19/10 < \log 7 < 2$, we have $x < 346/25$. Also $\exp(27/10) > (27/10)^2(1 + 7/10 + 49/200) > 346/25$, so $\log x < 27/10$. Hence

$$\frac{G_1(2,1)}{3} < \frac{827}{50}, \qquad \frac{G_2(2,1)}{3} > \frac{303}{25}, \qquad \frac{11}{8}\frac{303}{25} - \frac{827}{50} = \frac{1}{8} > 0. \tag{10.746}$$

All other entries of the coefficient maxima agree and $\mathsf{h}_2 > 0$, proving the coefficient inequality. For the second inequality use $\mathrm{e} > 19/7$, $\log 5 > 8/5$, and $\vartheta = 3/4$ in the exact rank-two quotient. This gives

$$\frac{C_2}{C_1} > \frac{557}{1790}\left(\frac{19}{7}\right)^2 \frac{3}{4}\frac{4}{5} = \frac{603231}{438550}. \tag{10.747}$$

Here its difference from 11/8 is $899/1754200 > 0$. Combining the two comparisons with Theorem 10.152 settles the rational field at five.

We justify all elementary logarithm bounds just used. The exponential series through degree six gives $\mathrm{e} > 1957/720 > 19/7 > 27/10$. Its sum through degree five is 163/60; the remaining terms have first term 1/720 and subsequent ratios at most 1/7, giving

$$\mathrm{e} < \frac{163}{60} + \frac{7}{4320} = \frac{11743}{4320} < \frac{87}{32} < \frac{11}{4}. \tag{10.748}$$

Now $\exp(19/10) = \mathrm{e}^2/\exp(1/10) < (11/4)^2/(11/10) = 55/8 < 7$, proving $\log 7 > 19/10$; $\mathrm{e}^2 > 7$ proves its upper bound. Similarly $\exp(2/5) > 1 + 2/5 + 2/25 + 4/375 = 559/375$. Therefore

$$\exp(8/5) < \frac{(87/32)^2}{559/375} < 5, \tag{10.749}$$

where the last rational comparison is $7569 \cdot 375 < 5 \cdot 1024 \cdot 559$. Thus $\log 5 > 8/5$. Finally $\log 3 > 1$ follows from $\mathrm{e} < 3$. These estimates prove every numerical inequality used above. The case $p = 5, e = f = 1, d > 1$ has not been inferred from the rational-field calculation. For its stated height subcase, all the entries in h are at most h_2, because only the G entry differs. Moreover

$$\frac{C_2}{C_1} > \frac{495}{1794}\left(\frac{19}{7}\right)^2 \frac{3}{4}\frac{4}{5} > 1. \tag{10.750}$$

Theorem 10.152 therefore proves the estimate in that subcase as well. □

The complete first local input budget

Return to the parameters and the actual kernel of Theorem 10.184. Suppose its original integer coefficient vector is nonzero and the distinguished $b_n \neq 0$ has minimum p-adic valuation. Take the height parameter to satisfy

$$h \geq \max\{G_2(r,d), (r+1)t, \log B_\dagger\}, \qquad B_\dagger = \min_{b_j \neq 0} |b_j|. \tag{10.751}$$

These conditions hold with the original h_2 when $r \leq n$, since $G_2(j,d)$ increases with j. Put $s_0 = \lfloor S \rfloor$, $\mu = \lfloor T/(q(r+1)) \rfloor + 1$, and

$$B = \lfloor \log_p(2s_0) \rfloor, \qquad C_0 = \max\{v_p(V(k)), v_p(b_n)\}, \qquad M_0 = \max\{B, C_0\}. \tag{10.752}$$

Theorem 10.190 (uniform first-step analytic budget). With these parameters,

$$\begin{aligned} \mu((2s_0+1)N + M_0) &< \frac{437}{2016} U_r < \frac{1}{4} U_r, \\ \kappa + U_r - v_p(b_n) &> N + \frac{1}{p-1}. \end{aligned} \tag{10.753}$$

Consequently the actual initial zeros of Theorem 10.184 meet the full input budget (10.691). Under the counterhypothesis $v_p(\Xi - 1) \geq U_r$, every prepared index $|\boldsymbol{t}| \leq T - \mu + 1$ satisfies, for every $x \in \mathbb{Z}_p$,

$$v_p(\Phi(x;\boldsymbol{t})) \geq (2s_0+1)\mu N - D_{\mathrm{a}}\left(N + \frac{1}{p-1}\right) + \beta. \tag{10.754}$$

This is a local analytic precision bound. Concluding that an algebraic value is zero still requires the strict global arithmetic comparison.

Proof. First we control all jet costs by the original height parameter. The constants give $c_3 \le 3, q \le 3, t > 1$, so $2S < 18(r+1)dH$. Also $a_0^{(2)} > 3$, $a_1^{(2)} + \log(r+1) > 1$, and hence

$$H \ge h \ge G_2(r,d) > (r+1)(3r + \log d + 1) \ge 40. \tag{10.755}$$

For $H \ge 2$, $\log H < H/2$: its difference has positive derivative after two and is positive at two. Furthermore $\log 18 < 3$ and $\log(r+1) \le r$. Thus

$$\log(2S) < \frac{H}{2} + 3 + r + \log d < H. \tag{10.756}$$

Indeed $(r+1)(3r + \log d + 1)/2 > 3 + r + \log d$: its left minus right is at least $5r - 1 > 0$ for $r \ge 3$, after keeping the nonnegative coefficient of $\log d$. The logarithm bounds follow respectively from $(27/10)^3 > 18$ and $\log(1+r) \le r$, proved by integrating $(1+x)^{-1} \le 1$.

The minimum-valuation coefficient hypothesis gives $v_p(b_n) \le \log B_\dagger / \log p \le h / \log p$. This uses the original integers, not newly chosen rational coefficients. Also $v_p(V(k)) = \lfloor \log_p k \rfloor$, by the proved least-common-multiple formula. Since $k \le H$ and $\log H < H$, all costs obey

$$M_0 \le \frac{H}{\log p}. \tag{10.757}$$

The local degree inequality $d \ge ef$, $c_3 \ge 3/5$, $q \ge 2$, $r + 1 \ge 4$, and $A_0 \ge 1/2$ give

$$\frac{SN}{H/\log p} = c_3 q (r+1) \frac{d}{f} N \ge c_3 q (r+1) \frac{m + A_0}{1 + \tau} > \frac{18}{5} > 3. \tag{10.758}$$

Here $m \ge 1$ and $0 < \tau < 1$. It follows that $M_0 < SN/3$ and $SN > 12$, the latter because $H/\log p \ge (r+1)f \ge 4$. Also $S \ge g_S \ge (3/5) \cdot 2 \cdot 16 > 16$.

By Theorem 10.184, $T > 101 r^2$. For $r \ge 3$, $q \le 3$ gives $q(r+1)/T < 3(r+1)/(101 r^2) \le 4/303 < 1/75$. The monotonic decrease of $(r+1)/r^2$ follows by differentiation, or multiplication of consecutive positive integers. Consequently

$$\mu < \frac{76}{75} \frac{T}{q(r+1)}. \tag{10.759}$$

Put $W = STN/(q(r+1))$. From $S > 16$ and the jet bound,

$$\mu\big((2s_0 + 1)N + M_0\big) < \frac{76}{75} \left(2 + \frac{1}{16} + \frac{1}{3}\right) W = \frac{437}{180} W. \tag{10.760}$$

On the other hand, substitution of T and U_r gives the exact identity $U_r = c_1 q^{r+1} W$. Every row has $c_1 \ge 7/10$; hence $U_r \ge (56/5)W$. Division proves the first bound, since $437/2016 < 1/4$.

For strict normality, the identity $U_r/(SN) = c_1 q^r T/(r+1) > 1$ follows from $T > 101 r^2$, $c_1 \ge 7/10$, and $r \ge 3$. Meanwhile $N < SN/16$, $1/(p-1) \le 1 < SN/12$, and $v_p(b_n) < SN/3$. Their sum is less than $(23/48)SN < SN < U_r$. Since $\kappa \ge 0$, the asserted strict inequality follows.

For the other strict disc inequality, use the phase and depth construction of Theorem 10.180. Its deep units lie in U_j with integer $j \ge A_0 + e/(p-1)$. Therefore

$$N = \frac{m + A_0}{e(1+\tau)} < \frac{j+m}{e} - \frac{1}{p-1}. \tag{10.761}$$

All hypotheses of Theorem 10.182 now hold for the full integer node set, with $\eta_E = 0$. The budget just proved is stronger than (10.691), whose left side also has the nonnegative terms $\kappa + D_{\mathrm{a}}(N+1/(p-1))$. Theorem 10.184 supplies every required initial prepared zero, and $|\boldsymbol{t}| + \mu - 1 \le T$ by assumption. Equation (10.692) proves the stated precision. □

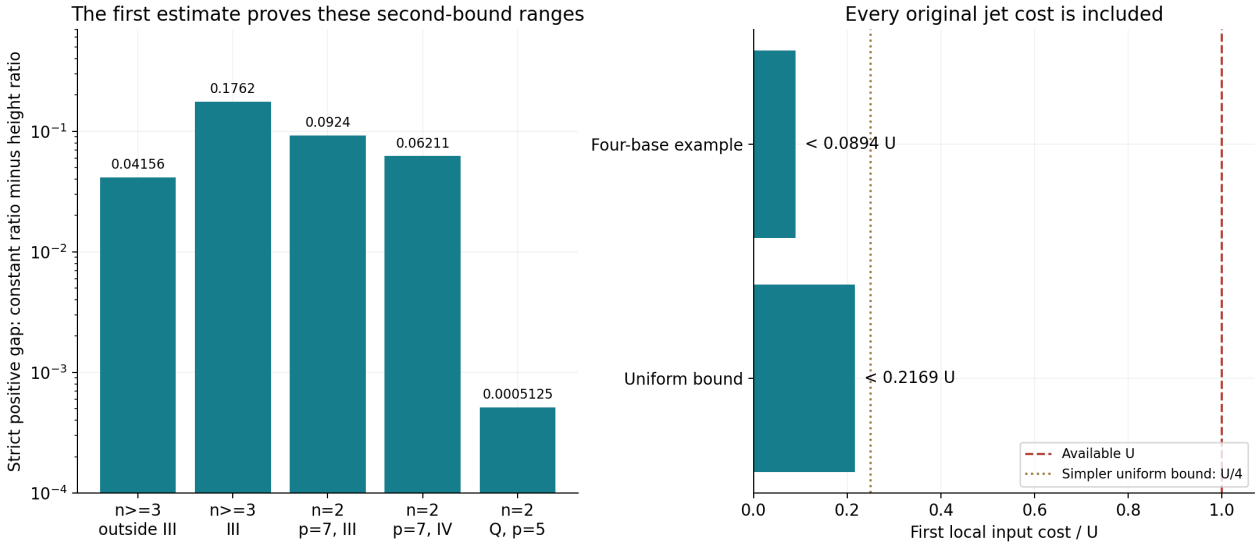


Figure 10.55.

Left: the rational differences in Theorems 10.188 and 10.189 are strictly positive. The comparison uses height ratio 3/2 in the first four bars and 11/8 in the rational five-adic case; the logarithmic vertical scale makes its small positive gap visible. Right: Theorem 10.190 bounds the full first local input by $(437/2016)U_r < U_r/4$, uniformly in all its parameters. Solution 76 computes the smaller cost for the actual four-base support of Solution 75 with a unit distinguished coefficient. Neither bar is a sampled prime-ideal valuation. Editable exact calculation and figure. Human-source context: the two estimates in Section 1.1 and the coefficient parameters in Section 1.3 of Yu's freely accessible paper.

63. The second-scale multiplicity stop and its height profile

The second-scale construction has its own polynomial degree and analytic radius. We verify the degree comparisons with these parameters before using the multiplicity theorem. The resulting height-product estimate has enough room to preserve the second constant through a reduction in rank. Its one-base endpoint follows from the elementary valuation estimate.

Degree and allocation inequalities

Retain the parameters of Theorem 10.184, including its independent $r \ge 3$ bases and $h \ge (r+1)t$. Write $A = A_r$, $x = \log J$, and $\lambda = 59/58$. Set

$$c_5 = \begin{cases} 27/50, & q = 2, \\ 4/5, & q = 3, \end{cases} \qquad \eta = 1 - \frac{c_5}{r+1}, \qquad Q = q\eta^{r+1}. \tag{10.766}$$

Choose any nonnegative integer I such that

$$E = Q^I > \exp(5(A+x)), \qquad \bar{S} = q\eta^{-(r+1)I}S, \qquad \bar{T} = \eta^{(r+1)I+1}T. \tag{10.767}$$

Use the coordinate-degree bounds

$$\mathsf{D}_0 = D_{\mathrm{a}} = kL, \qquad \mathsf{D}_j = 2Jq^{-I}D_j \quad (1 \le j \le r), \tag{10.768}$$

and the integer allocations

$$S_0 = \lfloor \bar{S}/3 \rfloor, \quad S_j = \lfloor 2\bar{S}/(3r) \rfloor \ (1 \le j \le r), \qquad T_j = \lfloor \bar{T}/(r+1) \rfloor \ (0 \le j \le r). \tag{10.769}$$

Theorem 10.191 (degree comparisons at the second-scale stop). These allocations satisfy (10.67), with total budgets $\lfloor \bar{S} \rfloor, \lfloor \bar{T} \rfloor$. Moreover,

$$\mathsf{D}_0 > 20\mathsf{D}_j \quad (1 \le j \le r), \qquad \mathsf{D}_0 > 4\bar{T}, \qquad \mathsf{D}_0 > 4r. \tag{10.770}$$

For the allocation ratios $\mathcal{R}_0, \mathcal{R}_{m,J}, \mathcal{R}_r$ defined in (10.267), with the present parameters,

$$\mathcal{R}_0 > \frac{32}{11}, \qquad \mathcal{R}_{m,J} > \frac{2^{20}}{29} \ (1 \le m < r), \qquad \mathcal{R}_r > 31. \tag{10.771}$$

In particular every allocation and coordinate-degree hypothesis of Theorem 10.25 holds.

Proof. We use $\mathrm{e} = \exp(1)$, reserving e for ramification. The established bounds give

$$\begin{gathered} A \ge 4 + \log((r+1)d) > 5, \quad t > 1, \quad H = h + x \ge (r+1)t > r+1, \\ c_3 \ge 3/5, \quad c_2 q > 3, \quad 6 \le c_4 \le 14, \quad A_0 = e\vartheta \ge 1/2, \quad \vartheta \le 2, \quad d \ge e, \\ eNt < \frac{\zeta_1 r}{1+\tau}, \quad \zeta_1 < 7/5, \quad \zeta_2 < 11/10, \quad K_S < 21/10, \quad \lambda < 11/10, \quad c_0 < 3. \end{gathered} \tag{10.772}$$

The bounds for ζ_1, ζ_2 are those of Lemma 10.177, and $K_S < 21/10$ follows from $g_S > 16$. The field-case definition of ϑ gives its displayed upper bound, as in Theorem 10.100. Also $2 < \mathrm{e} < 3$ and $\ell_d \le d$.

We have $\eta \ge 4/5$. The derivative argument in Theorem 10.100 shows that $(1 - c/y)^y$ increases for $y > c$. Consequently

$$Q \ge \begin{cases} 2(173/200)^4 > 1, & q = 2, \\ 3(4/5)^4 > 1, & q = 3. \end{cases} \tag{10.773}$$

On the other hand, $q\eta^{2(r+1)} < q\exp(-2c_5) < 1$. For the last comparison, the first three exponential terms give $\exp(27/25) > 2$ and $\exp(8/5) > 3$. It follows that

$$\eta^{-(r+1)I} > E, \qquad q^I/J > E/J > \exp(5A), \qquad E/J > \exp(5A). \tag{10.774}$$

In particular a finite I with the required property exists. These facts concern the size of a stopping depth; they do not assert that the auxiliary function has acquired its required zeros.

For an individual non-torsion base, Theorem 10.18 gives $\sigma_j \ge 2/(58\mathrm{e}d^2\ell_d)$, hence $d\sigma_j > 1/(87d^2)$. The lower bound in Corollary 10.185 and the definitions of S and D_j give

$$\begin{aligned} \frac{\mathsf{D}_0}{\mathsf{D}_j} &> \frac{c_3 c_2 q r (r+1) P H \vartheta}{2c_4 N t A} d\sigma_j \frac{q^I}{J} \\ &> \frac{9}{392}(r+1)^2 d\sigma_j \frac{\exp(5A)}{A}. \end{aligned} \tag{10.775}$$

In the second line use $eNt < \zeta_1 r$, $e\vartheta \ge 1/2$, $P \ge 1$, and $H > r+1$. Since $\exp(3A)/A$ increases for $A \ge 5$,

$$\frac{\exp(5A)}{A} > 256(r+1)^2 d^2 \frac{2^{15}}{5}. \tag{10.776}$$

Thus the degree ratio is greater than $9 \cdot 256^2 \cdot 2^{15}/(392 \cdot 87 \cdot 5) > 20$, where $(r+1)^4 \ge 4^4 = 256$ was used. Similarly,

$$\frac{\mathsf{D}_0}{\bar{T}} > \frac{c_3 A_0 H}{c_4 A} \eta^{-(r+1)I-1} > \frac{3}{35}\frac{2^{25}}{5} > 4. \tag{10.777}$$

Here $H > 4$, and $\exp(5A)/A$ increases for $A \ge 5$. Finally $k = \lfloor H \rfloor \ge r+1$ and $L > X > 58$, so $\mathsf{D}_0 > 58(r+1) > 4r$. These inequalities imply $\mathsf{D}_0 > \max_j \mathsf{D}_j$ and $\mathsf{D}_0 > \bar{T} + r \ge T_r + r$.

The integer allocations are nonincreasing and have sums at most their specified integer budgets. Their floors satisfy

$$S_0 + 1 > \bar{S}/3, \quad S_j + 1 > 2\bar{S}/(3r), \quad \binom{T_j + b}{b} > \frac{(\bar{T}/(r+1))^b}{b!} \quad (b \ge 1). \tag{10.778}$$

For the last inequality, each factor $T_j + l$, $l \ge 1$, is greater than $\bar{T}/(r+1)$. Put

$$B_0 = \frac{q^2 \eta c_4 d A}{3\lambda A_0 t}. \tag{10.779}$$

Direct cancellation using the upper bound for D_0 gives $\mathcal{R}_0 > B_0$. Since $d/A_0 \ge 1/2$, $A \ge t$, $q \ge 2$, $\eta \ge 4/5$, $c_4 \ge 6$, and $\lambda < 11/10$, we have $B_0 > 32/11$.

For every $1 \le m < r$ and m-element coordinate subset J', the same cancellation yields

$$\mathcal{R}_{m,J'} > \frac{2B_0}{r} \frac{\prod_{j \in J'} \sigma_j}{((m+1)!)^2} \left(\frac{\eta c_2 q r P d}{2eNt} \frac{E}{J} \right)^m. \tag{10.780}$$

The bracket is greater than $(6/7) d \exp(5A)$. Each selected family is independent, so Theorem 10.18 applies to it:

$$\prod_{j \in J'} \sigma_j \ge \frac{m^m}{29 m! \mathrm{e}^m d^{m+1} \ell_d}. \tag{10.781}$$

Using $B_0 > 2$, $m^m \ge m!$, $\mathrm{e} < 3$ and $(2/7)\exp(5A) > 2^{21}(r+1)^2 d^2$, we obtain

$$\mathcal{R}_{m,J'} > \frac{4}{29}(2^{21})^m \frac{(r+1)^{2m}}{r((m+1)!)^2} \frac{d^{2m-1}}{\ell_d} \ge \frac{4}{29} \frac{(2^{21})^m}{(m+1)^3} \ge \frac{2^{20}}{29}. \tag{10.782}$$

Indeed the exponential estimate follows by splitting its exponent into $2A + 3A$, using $A \ge 4 + \log((r+1)d)$ and $A > 5$. For the middle inequality, $d^{2m-1} \ge \ell_d$, $(m+1)! \le (m+1)^{m+1}$, and $(r+1)^{2m}/r$ increases for $r \ge m+1$. The final expression increases with m: its consecutive ratio is at least $2^{21}(2/3)^3 > 1$.

It remains to treat the full-rank allocation. Keep the entire formula for $\mathcal{D}$ in Theorem 10.184. Cancellation now gives

$$\mathcal{R}_r > \frac{2qq^u J^{1-r}\eta^r}{3r\lambda(r+1)!K_S c_0 \ell_d \mathcal{R}_r(\boldsymbol{\alpha})} q^{rI}\eta^{(r-1)(r+1)I} \times \left(\frac{r}{2eNt\zeta_2 \mathrm{e}(r+1)(1+\tau)}\right)^r. \tag{10.783}$$

The $\mathcal{R}_r$ on the left is the allocation ratio; $\mathcal{R}_r(\boldsymbol{\alpha})$ on the right is the residue maximum of Theorem 10.184. For that maximum,

$$\mathcal{R}_r(\boldsymbol{\alpha}) \le \exp(A)A. \tag{10.784}$$

Its first entry is at most $p^f = \exp t$; for the second, $(\mathrm{e}t/r)^r \le \exp t$ follows from $\log(t/r) \le t/r - 1$, and $t \le A$. Also $q^{rI}\eta^{(r-1)(r+1)I} = q^I E^{r-1} > E^r$, so $J^{1-r}E^r > \exp(5rA)$. Using the strict bounds already proved,

$$\frac{2qq^u}{3\lambda K_S c_0} > \frac{1}{6}, \qquad \frac{\eta}{2\zeta_1\zeta_2 \mathrm{e}} > \frac{1}{12}, \tag{10.785}$$

we conclude that

$$\mathcal{R}_r > \frac{\exp((5r-1)A)}{6r(r+1)![12(r+1)]^r \ell_d A} > \frac{16^{4r}}{6r12^r} > 31. \tag{10.786}$$

For the second inequality use $\exp((r-1)A) > A$, $\exp(4rA) > [16(r+1)d]^{4r}$, $(r+1)! \le (r+1)^{r+1}$, and $\ell_d \le d$. The last expression increases for $r \ge 3$, since its consecutive ratio is $(16^4/12)r/(r+1) > 1$; its value at three exceeds 31 by an integer comparison. This proves all assertions. □

The height product after a reduction in rank

Corollary 10.192 (the second height profile at a multiplicity stop). Suppose the contracted prepared family with the parameters of Theorem 10.191 is nonzero and vanishes at every integer $|s| \le \bar{S}$ through total prepared order $\bar{T}$. Suppose its independent bases and selected Euler directions give the hyperplane W of Theorem 10.25, with a nonzero vector $\boldsymbol{B}$. Then there are $1 \le m < r$ and independent integral vectors $u_1, \dots, u_m$, whose rational span contains $\boldsymbol{B}$. Put $\beta_i = \prod_j \alpha_j^{u_{ij}}$ and $R_i = \sum_j \sigma_j |u_{ij}|$. These are independent local units in the original field, with $h(\beta_i) \le R_i$, and

$$\prod_{i=1}^m R_i < 10^{-6}\Gamma_2^{r-m} \frac{\mathcal{R}_r(\boldsymbol{\alpha})}{\mathcal{R}_m(\boldsymbol{\beta})}\Omega_r, \qquad \Gamma_2 = a\mathrm{e}^2 Pd/q. \tag{10.787}$$

Here a is the case constant in Theorem 10.184, and each residue maximum uses its indicated family and its own saturated residue index. At rank one use the actual residue order of the base.

Proof. Proposition 10.31 converts the stipulated prepared zeros to ordinary W-jets, and clears the Laurent monomial with the coordinate degree bounds of Theorem 10.191. That theorem verifies every allocation and degree hypothesis of Theorem 10.25. As in the proof of Corollary 10.101, both multiplicity alternatives imply

$$(S_m+1)\binom{T_m+m}{m}\prod_{i=1}^m R_i \le (m+1)!r^m D_{\mathrm{a}} \left(\frac{2Jq^{-I}\mathcal{D}}{c_1c_2rPd}\right)^m. \tag{10.788}$$

Corollary 10.26 proves the independence, field and height assertions. If an initial multiplicative relation is expressed in the rational span of $\boldsymbol{B}$, clearing denominators and killing its root-of-unity factor gives the same identity in the new bases, as in (10.271).

Let $\mathcal{Q}$ be the product on the left of the claimed bound divided by $\Gamma_2^{r-m}(\mathcal{R}_r/\mathcal{R}_m)\Omega_r$.

Bound the denominator in the multiplicity output strictly below by $(2\bar{S}/(3r))(\bar{T}/(r+1))^m/m!$, since each floor plus one strictly exceeds the unrounded quotient. Insert the upper bound for D_{a} from Corollary 10.185, and divide by the desired profile. Exact substitution of $S, T, \mathcal{D}, \Gamma_2$ gives the following upper bound for the normalized quotient:

$$\begin{aligned}\mathcal{Q} < \frac{3r}{2q}\frac{(m+1)!m!}{r!}\lambda K_S c_0\,(r+1)^r \mathrm{e}^{-r}\frac{\ell_d \mathcal{R}_m}{q^u}(1+\tau)^{r-m} \\ \times\left(\frac{2\zeta_1\zeta_2 r\mathrm{e}^2}{\eta}\right)^m J^{m-1}\frac{\eta^{(r+1)I}}{E^m}.\end{aligned} \tag{10.789}$$

For clarity, the final inequality used the bound from Lemma 10.177 $eNt < \zeta_1 r/(1+\tau)$. Before that bound its bracket is $2\zeta_2 eNt\mathrm{e}^2/\eta$, and the power of $1+\tau$ is r. This retains the distinction between ramification e and Euler's number e. Both factorials from the multiplicity output and binomial count remain visible.

For every integer $m \geq 1$, $\mathcal{R}_m \leq p^f t \leq \exp(A)A$. Indeed its first entry is at most p^f, while $(\mathrm{e}t/m)^m \leq \exp t$ follows from $\log(t/m) \leq t/m - 1$. Multiplication by t treats its second entry. Also the integral bound for $\log(r!)$ gives

$$\frac{(r+1)^r}{r!\mathrm{e}^r} \leq \exp\left(r\log(1+1/r) - 1\right) < 1. \tag{10.790}$$

Equation (10.451) bounds $(1+\tau)^{r-m}$ by $101/100$. Also $g_S > 16$, hence $K_S < 21/10$. The other bounds are $\lambda < 11/10$, $c_0 < 3$, and $q \geq 2$. Thus the remaining constant factor is less than $(3/4)(11/10)(21/10)3(101/100) < 6$. Furthermore $\zeta_1 < 7/5, \zeta_2 < 11/10, \mathrm{e} < 3$ and $\eta \geq 4/5$ give $2\zeta_1\zeta_2\mathrm{e}^2/\eta < 693/20 < 35$.

From $q\eta^{2(r+1)} < 1$ we have $\eta^{(r+1)I} < E^{-1}$. The stopping inequality therefore implies

$$J^{m-1}\frac{\eta^{(r+1)I}}{E^m} < \exp(-5(m+1)A)J^{-4m-6} \leq \exp(-5(m+1)A). \tag{10.791}$$

Consequently

$$\mathcal{Q} < 6r(m+1)!m!\,\ell_d A\,(35r)^m \exp(-(5m+4)A). \tag{10.792}$$

Set $V = (r+1)d$. Since $A \geq 4 + \log V > 5$, $y\exp(-(5m+4)y)$ decreases for $y \geq 5$. The elementary inequalities $4 + \log V \leq 5V$ and $\ell_d \leq d$ bound the last display above by

$$30 \cdot 35^m \exp(-20m - 16)\frac{r^{m+1}(m+1)!m!}{(r+1)^{5m+3}d^{5m+2}}. \tag{10.793}$$

Here $m < r$, so $(m+1)!m! \leq (m+1)^{2m+1} \leq r^{2m+1}$. The last fraction is strictly smaller than one. Finally $\mathrm{e} > 2$, $35/2^{20} < 1$, and $m \geq 1$ give

$$\mathcal{Q} < \frac{30}{2^{16}}\left(\frac{35}{2^{20}}\right)^m \leq \frac{1050}{2^{36}} < 10^{-6}. \tag{10.794}$$

The last comparison is the integer inequality $1050 \cdot 10^6 < 2^{36}$. This proves the asserted product bound. In particular the stronger allowance of Theorem 10.187 survives this multiplicity output, including its actual change of residue index. □

The conclusion is conditional on the stated prepared zeros. The degree and height-product arguments establish what those zeros imply; their arithmetic production remains a separate step.

The one-base endpoint of the second height profile

Proposition 10.193 (one base with the stronger rank allowance). Use the original independent n-base setting, $n \geq 2$, and the exact C_2 constant in (10.642). Write $a = a^{(2)}$, $c_* = c^{(2)}$, $\Gamma_2 = a\mathrm{e}^2 Pd/q$, $F(n) = (n+1)^{n+1}/(n-1)!$, and $\Omega_n = \prod_j h(a_j)$. Suppose a rank-one integral-form representation satisfies the weighted conditions in (10.277), with its product allowance replaced by

$$\sigma \leq \Gamma_2^{n-1} \frac{\mathcal{R}_n(\boldsymbol{a})}{\mathcal{R}_1(\beta)} \Omega_n. \tag{10.795}$$

Here β is its non-torsion unit base, g is the actual residue order of β, and $\mathcal{R}_1(\beta) = \max\{p^f/\delta(\beta), \mathrm{e}t^2\}$, with $\delta(\beta) = (p^f - 1)/g$. In particular $\mathcal{R}_1 > g$. Let the original height parameter h obey $h \geq \max\{\log B_\dagger, (n+1)t\}$. Then

$$\mathrm{ord}_{\mathfrak{p}}(\Xi - 1) < \frac{1}{3000} C_2 h \Omega_n. \tag{10.796}$$

Proof. Clear the rational coefficient of the one independent integral form exactly as in (10.454). Its numerator divides every original nonzero integer coefficient, so its absolute value is at most $B_\dagger$. Kill the q-primary torsion by a positive power; the difference valuation cannot decrease. The elementary residue-order bound of Theorem 10.21, including signed powers, therefore yields

$$\mathrm{ord}_{\mathfrak{p}}(\Xi - 1) \leq \frac{d}{t}(2h + 2eg\sigma). \tag{10.797}$$

This is the elementary estimate proved in (10.454); no newly chosen rational coefficient is substituted for the original height parameter.

Put $\mathcal{T} = C_2 h \Omega_n$. The height-product estimate (10.55), with saturation index at least one and $w_K \geq q^u$, gives

$$\Omega_n \geq \frac{q^u n^n}{\rho d^{n+1} \ell_d n! \mathrm{e}^n}, \qquad \rho \leq 58. \tag{10.798}$$

Also $\mathcal{R}_n A_n / t^2 \geq 1/n$. If $t \leq n$, use $\mathcal{R}_n \geq 1, A_n \geq t$. If $t > n$, the second entry of the residue maximum gives $\mathcal{R}_n A_n / t^2 \geq (\mathrm{e}t/n)^n > 1/n$. Substituting the full C2 formula therefore proves

$$\mathcal{T} \geq \frac{c_* a^n P^{n-1} F(n) n^{n-1}}{\rho n!} \frac{dh}{t} \geq 9261 \frac{dh}{t}. \tag{10.799}$$

Indeed $c_* \geq 406, a \geq 7, P \geq 1, F(n) \geq F(2) = 27$, and $n! \leq n^{n-1}$. The exact last coefficient is $406 \cdot 49 \cdot 27/58 = 9261$. Thus the first summand in the elementary bound is less than $\mathcal{T}/4000$, since $8000 < 9261$.

For the second summand use the stipulated profile and $g/\mathcal{R}_1 < 1$. Exact cancellation against the full target gives the strict normalized upper bound

$$\frac{2eq^u \mathrm{e}^{n-2} t^2}{c_* a q^{n-1} d^2 F(n) \ell_d A_n h}. \tag{10.800}$$

The cyclotomic local-degree argument in Theorem 10.73 gives $eq^u \le 2d$. Also $A_n \ge t, h \ge (n+1)t$. The expression is therefore at most

$$\frac{4\mathrm{e}^{n-2}}{c_* a q^{n-1} d F(n) \ell_d(n+1)} \le \frac{4}{406 \cdot 7 \cdot 2 \cdot 27 \cdot 3} < \frac{1}{100000}. \tag{10.801}$$

For this last comparison, Theorem 10.187 proves $F(n) \ge 27\mathrm{e}^{n-2}$; all other factors have their displayed elementary lower bounds. The exact final integer comparison is $400000 < 460404$. Adding the two contributions gives $1/4000 + 1/100000 = 13/50000 < 1/3000$, proving the assertion. □

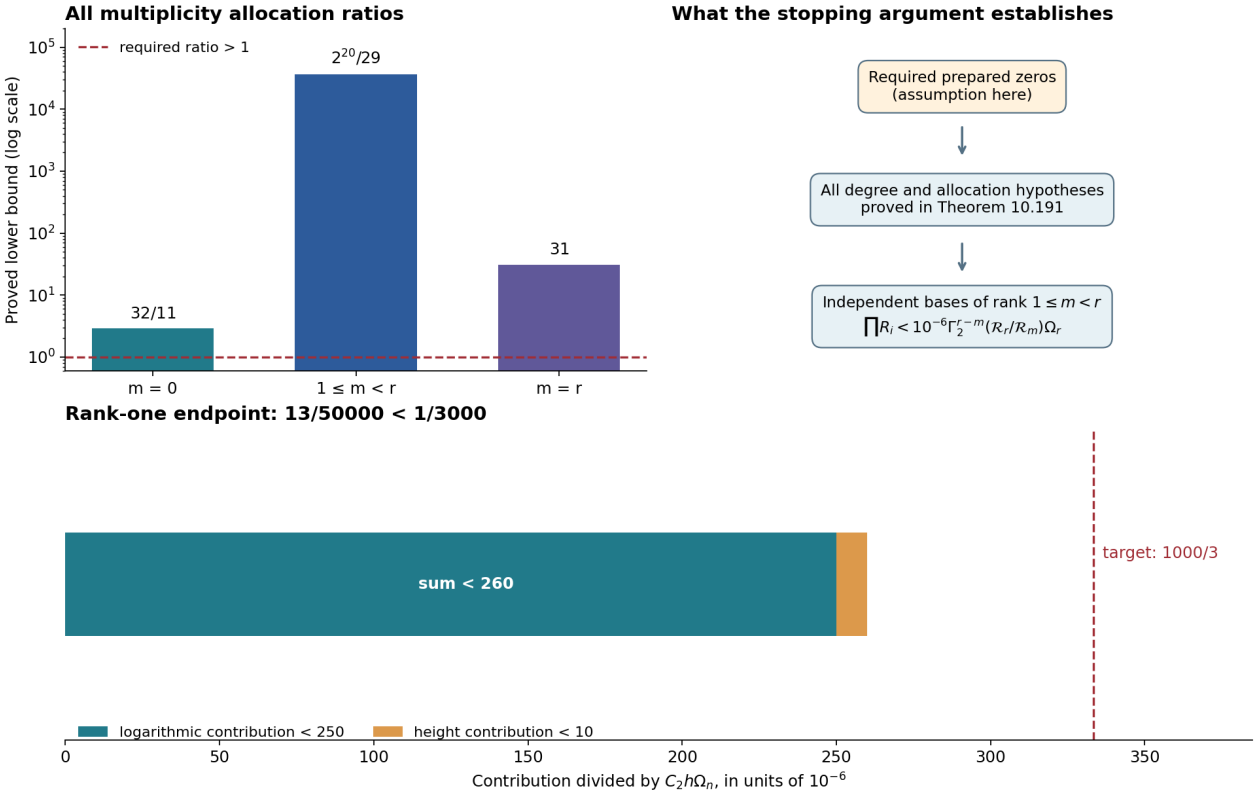


Figure 10.56. Left: proved uniform lower bounds for the three allocation ratios in Theorem 10.191, on a logarithmic scale. All exceed the required threshold one. Right: Corollary 10.192 uses prepared zeros as an explicit hypothesis and produces a smaller independent family with its actual residue index and the stronger height allowance. Below: the two rank-one contributions, each divided by $C_2 h\Omega_n$, have sum less than 260 in units of 10^{-6}; the target is $1000/3$ in the same units. Proposition 10.193 proves the inequality, and Solution 77 checks the algebra and exact endpoints. These are proved bounds, not sampled field parameters. Reproducible figure and exact endpoint calculation. Human-source context: the multiplicity method in Section 6 of Yu's freely accessible 2013 paper.

64. Coefficient heights and the first second-scale zero extension

The selected auxiliary function already vanishes on its initial integer interval. We estimate the height of its coefficients by averaging over the entire set of prepared equations. A positive generating series then bounds each individual value. The resulting strict comparison extends the zeros to an interval q times as large.

Retain all parameters of Theorem 10.184 and the original integer representation (10.218). The distinguished nonzero coefficient has minimum p-adic valuation. Throughout this section assume

$$h \ge \max\left\{G_2(r,d), (r+1)t, \log B_\dagger, \log \frac{R_{\max}}{dW(d)}\right\}. \tag{10.806}$$

Here $W(d)$ is (10.222), and the original coefficients define $R_{\max}$ and $B_{\dagger}$. In particular these hypotheses hold for the original second height parameter when $r \le n$. Write $Z = S\mathcal{D}/d$, $\lambda = 59/58$, $H = h + \log J$, and $y = 1/(r+1)$.

An individual scalar and its complete-simplex average

Lemma 10.194 (the second-scale generating remainder). At any contraction depth $I \ge 0$, let Ω_j be the actual nonnegative integer maxima of the Euler arguments, and set

$$\begin{aligned} m_I &= \sum_{j=1}^{r-1} \Omega_j + r - 2, \\ a_E &= H + 1 + \log W(d), \qquad a = \max\{107H/103, a_E\}, \\ E_I &= (m_I + 1)[-\log(1 - \exp(-a_E))]. \end{aligned} \tag{10.807}$$

For every nonnegative integer additive order u and Euler order v,

$$u \log V(k) + \log \binom{m_I + v}{v} \le a(u+v) + E_I, \qquad \frac{E_I}{Z} < \frac{q^{-I}}{79} + \frac{r-1}{999Z}. \tag{10.808}$$

Consequently, at real radius R and total order at most O, the logarithm of each prepared scalar is at most

$$D_{\mathrm{a}} \log(q^{-I}R + 2k) - L \log(k!) + aO + E_I. \tag{10.809}$$

At the initial full composition rows, the mean scalar logarithm is at most

$$D_{\mathrm{a}} \log(S + 2k) - L \log(k!) + C_r T H + E_0, \qquad C_r = \frac{r + 4/103}{r+1} + \frac{3(r-1)}{8(r+1)^2} \le 1 - \frac{483}{824(r+1)} < 1. \tag{10.810}$$

Proof. Theorem 10.112 gives $\log V(k) < 107k/103 \le 107H/103$. The positive series proved in Theorem 10.115 gives, for $0 < z < 1$,

$$\binom{m_I + v}{v} z^v \le (1-z)^{-m_I - 1}. \tag{10.811}$$

Use $z = \exp(-a_E)$. The Euler contribution is at most $a_E v + E_I$, while the lcm contribution is at most $(107H/103)u$. Both coefficients are at most a, proving the first inequality. For the remainder, Lemma 10.84, including its full-width contraction, gives

$$\frac{m_I + 1}{Z} \le \frac{q^{-I}(r-1)tW(d)\exp H}{c_1 c_2 c_3 q P (r+1) H} + \frac{r-1}{Z}. \tag{10.812}$$

There is no extra factor $1/r$ after the r widths are summed. The definition of a_E gives $W(d)\exp(H - a_E) = 1/\mathrm{e}$. Also $H > 40$ by (10.755), hence $z < 1/1000$. Integration gives $-\log(1-z) \le z/(1-z)$, and $t/H \le 1/(r+1)$. Therefore

$$\frac{E_I}{Z} < \frac{1000}{999} \frac{q^{-I}(r-1)}{\mathrm{e} c_1 c_2 c_3 q P (r+1)^2} + \frac{r-1}{999Z}. \tag{10.813}$$

Substitution in the second-scale table gives the following lower bounds for $c_1c_2c_3qP$, in cases I/II, III, IV and V respectively:

$$\frac{44100}{5101}, \qquad \frac{7840}{2083}, \qquad \frac{18375}{2642}, \qquad \frac{328}{39}. \tag{10.814}$$

They all exceed 15/4. Here use $P \geq 3$ in I/II, $P \geq 1$ in III/IV, and $P \geq 4$ in V; these follow from (10.641). Since $(r-1)/(r+1)^2 \leq 1/8$ and $1/\mathrm{e} < 3/8$, the first remainder coefficient is less than $25/1998 < 1/79$. This proves the individual bound at every depth and order.

In Theorem 10.92 the additive radius is $q^{-I}R + 2k - 1$. Its binomial coefficient satisfies $\binom{D_{\mathrm{a}}}{u}(q^{-I}R + 2k-1)^{D_{\mathrm{a}}-u} \leq (q^{-I}R + 2k)^{D_{\mathrm{a}}}$. This proves the asserted scalar bound and retains $-L\log(k!)$.

For the mean, retain the separate lcm and Euler coefficients just proved. We next show $a_E/H < 1 + 3/[8(r+1)]$. Every second-table $a_0^{(2)} > 39/10$ and $a_1^{(2)} \geq 71/100$. Since $r \geq 3$ and $\log 4 > 4/3$,

$$G_2(r,d) > (r+1)(40/3 + \log d). \tag{10.815}$$

For $d \geq 2$, the tangent bound in Lemma 10.106 proves $1 + \log W(d) < 5 + (\log d)/3$. This is at most $(3/8)(40/3 + \log d)$. For $d = 1$, that lemma proves $1 + \log W(1) < 2$, which gives the stronger conclusion. All logarithm estimates used here were proved by positive series in Sections 36 and 40.

At each node keep all $\binom{\lfloor T \rfloor + r}{r}$ indices, including zero rows. Permuting the r derivative coordinates and their slack shows that their exact coordinate means are $\lfloor T \rfloor/(r+1)$. The mean additive order is thus $\lfloor T \rfloor/(r+1)$, and the mean Euler order is $(r-1)\lfloor T \rfloor/(r+1)$. Apply the separate lcm bound and the Euler series with a_E. The resulting mean derivative contribution is at most $C_rTH + E_0$. Zero rows can be given the same positive majorant. Finally direct subtraction gives

$$1 - C_r = \frac{483r + 1101}{824(r+1)^2} \geq \frac{483}{824(r+1)}. \tag{10.816}$$

This proves the mean estimate. □

The number of columns and the coefficient vector

Lemma 10.195 (second-scale logarithmic column bounds). Set

$$b_r = 1500\,12^{r-3}, \qquad \varepsilon_r = \frac{8r+15}{9(r+1)b_r}, \qquad e_0 = \frac{1}{79} + \frac{1}{71928 \cdot 1500}. \tag{10.817}$$

For the actual selected number of columns N_c and any $1 \leq d_0 \leq d$,

$$\mathcal{D} > b_r A_r, \quad \frac{\log(N_c d_0)}{Z} < \varepsilon_r \leq \frac{13}{18000}, \quad \frac{E_0}{Z} < e_0. \tag{10.818}$$

The remainder is the one defined in Lemma 10.194.

Proof. The height-product substitution in Theorem 10.184 gives

$$\frac{\mathcal{D}}{A_r} > \frac{2c_0c_1c_4}{58}\frac{N}{\vartheta}(c_2qP)^r\frac{r^r(r+1)^r}{(r!)^2}. \tag{10.819}$$

Here $N/\vartheta > 99/100$, since $N/\vartheta = (m + A_0)/(A_0(1+\tau))$, $m \geq 1$ and $\tau < 1/100$. At $r = 3$ the factorial factor is 48. In I/II, III, IV and V, the resulting rational lower bounds are respectively

$$\frac{181946699550000000}{3849142313729},\quad \frac{431281065600000}{262099267823},\quad \frac{105293228906250}{66850775669},\quad \frac{6927360000}{63713}. \tag{10.820}$$

Each exceeds 1500. The consecutive factorial-factor ratio exceeds four by (10.282), and $c_2qP > 3$. Induction proves the first bound for all $r \geq 3$.

The singleton height estimate used in Lemma 10.105 gives $d\sigma_i > 1/(87d^2)$. Every lattice coordinate is in $J^{-1}\mathbb{Z}$. Counting coordinates in the original box, before either selection, and using $c_1c_2rP > 1$, gives $|S_0| < (88Jd^2\mathcal{D})^r$. Corollary 10.185 implies $D_{\mathrm{a}} < S\mathcal{D}$: use $\vartheta/N < 101/100$, $c_1c_4 > 4$ and $A_r > 5$. Consequently, writing $x = \log J$,

$$\log(N_c d_0) < \log S + (r+1)\log\mathcal{D} + (2r+1)\log d + rx + r\log 88. \tag{10.821}$$

Let $S_* = (9/5)(r+1)^2 d$. The table gives $S \geq S_*$. Both S_* and $5b_r$ exceed e, on whose right $\log z/z$ decreases. Also $\log S_* < 2A_r$, $\log 88 < 5 < A_r$, $\log d < A_r$, and $\log(5b_r) < 3r$. For the last comparison, $(27/10)^9 > 7500$ proves $\log 7500 < 9$, and $(27/10)^3 > 12$ proves $\log 12 < 3$. The bound $\mathrm{e} > 27/10$ was proved earlier. Dividing the four corresponding groups of terms by Z gives

$$\begin{aligned} \frac{\log S}{Z} &< \frac{10}{9(r+1)^2 b_r}, \\ \frac{(r+1)\log\mathcal{D}}{Z} &< \frac{r}{3(r+1)b_r}, \\ \frac{(2r+1)\log d + r\log 88}{Z} &< \frac{5(3r+1)}{9(r+1)^2 b_r}, \\ \frac{rx}{Z} &< \frac{5r}{9(r+1)b_r}. \end{aligned} \tag{10.822}$$

For the last line use the exact formula for S, $x/H \leq 1$, $c_3q \geq 9/5$, $\mathcal{D} > b_rA_r$, and $A_r \geq t$. Their sum is ε_r. Its prefactor $8 + 7/(r+1)$ decreases, while b_r increases, so its maximum is $13/18000$ at three. Finally $Z > 9(r+1)^2 b_r$. Thus $(r-1)/(999Z) < 1/(71928b_r)$, and Lemma 10.194 gives the stated remainder. □

Proposition 10.196 (the initial coefficient height at the second scale). The integral kernel in Theorem 10.184 can be chosen so that

$$\begin{aligned} \frac{h_2(c)}{Z} < \frac{\varepsilon_r}{2} &+ \frac{1}{c_1c_2c_3q(r+1)^2} \\ &+ \frac{10}{17}\left[\frac{\varepsilon_r}{2} + \frac{\lambda\vartheta}{c_1c_4N} + \frac{C_r}{c_1c_3eN} + e_0 + \frac{1}{2c_1c_2}\left(1 + \frac{1}{2g_S+1}\right)\right]. \end{aligned} \tag{10.823}$$

Here e in eN is the ramification index.

Proof. Apply the weighted integral Siegel argument of Proposition 10.183 to its full row list, using the mean scalar bound of Lemma 10.194 in the row sum. Its proof is a sum of individual row heights, so replacing their common maximum by their proved mean is valid. The coefficient surplus is $c_0 = 27/10$, whence the row ratio is smaller than $1/(c_0 - 1) = 10/17$.

For the additive part, integration gives $\log(k!) \geq k\log k - k + 1$. Also $k > 39H/40$, $c_3q \leq 6$ and $t > 1$, so

$$\frac{S+2k}{k} < 7(r+1)d, \qquad \log\frac{\mathrm{e}(S+2k)}{k} < A_r. \tag{10.824}$$

Indeed $2 \le (r+1)d/2$ and $(40/39)6 + 1/2 < 7$; then $1 + \log 7 < 4$. Corollary 10.185 bounds the normalized additive term by $\lambda\vartheta/(c_1c_4N)$. Direct substitution of S and T gives $TH/Z = 1/(c_1c_3eN)$. The coefficient-field generator term from Corollary 10.65 is at most $(d_0-1)/(c_1c_2S)$, which is smaller than $1/[c_1c_2c_3q(r+1)^2]$. The full-width mean torus term is precisely the one displayed; its factor is $1/(2c_1c_2)$. Use Lemma 10.195 for both logarithmic column terms and the remainder. This proves the bound while retaining every initial zero and the nonzero polynomial. □

The first extension gives exact algebraic zeros

Theorem 10.197 (the complete first integer extension). Choose the kernel as in Proposition 10.196. Set $c_5 = 27/50$ at $q = 2$ and $c_5 = 4/5$ at $q = 3$, and put $\eta = 1 - c_5/(r+1)$. Under the counterhypothesis $v_p(\Xi - 1) \ge U_r$,

$$\Phi(s;\boldsymbol{t}) = 0 \quad \text{for every integer } |s| \le qS \quad \text{and every } |\boldsymbol{t}| \le \eta T. \tag{10.825}$$

Proof. Put $\mu = \lfloor c_5T/(r+1)\rfloor + 1$ and $s_0 = \lfloor S\rfloor$. Since $T > 101r^2$,

$$\mu < \frac{101}{100}\frac{c_5T}{r+1}. \tag{10.826}$$

For example, the relative rounding error is smaller than $4/[(27/50)\,101\cdot 9] < 1/100$; the ratio $(r+1)/r^2$ decreases. The proof of Theorem 10.190 gives $M_0 < SN/3$, $S > 16$, and strict normality on the enlarged disc. Consequently

$$\frac{\mu[(2s_0+1)N + M_0]}{U_r} < \frac{101}{100}\frac{115}{48}\frac{c_5}{c_1q^r} < \frac{1}{4}. \tag{10.827}$$

For $q = 2$ and $q = 3$ the respective upper bounds, already using only $r \ge 3, c_1 \ge 7/10$, are $20907/89600$ and $2323/22680$. The exact floor inequality $\lfloor \eta T\rfloor + \mu - 1 \le \lfloor T\rfloor$ supplies every required input jet. Theorem 10.182 therefore yields

$$v_p(\Phi(s;\boldsymbol{t})) - \beta \ge (2s_0+1)\mu N - D_{\mathrm{a}}(N + 1/(p-1)). \tag{10.828}$$

Write $\mathcal{Z} = Zd/(et)$. We compare both terms with this same scale. Let d_* be the least field degree in the case, put $\gamma = c_3qd_*$, and retain $y = 1/(r+1)$. Since $S \ge \gamma/y^2$, the positive analytic term is strictly greater than

$$\mathcal{Z}\,\mathcal{L}(y), \qquad \mathcal{L}(y) = \frac{2c_5q}{c_1}\left(1 - \frac{y^2}{2\gamma}\right). \tag{10.829}$$

Suppose a target value is nonzero. It belongs to the original field K: these are integer nodes at depth zero, so Theorem 10.55 applies. Theorem 10.92 uses its actual degree d and local degrees e,f, and its q-denominator exponent is zero. The torus term divided by Z is at most $q/(c_1c_2)$. For the scalar at radius qS, Lemma 10.194 and the factorial estimate above give an additive contribution at most $\lambda\vartheta B_q/(c_1c_4N)$, where $B_2 = 57/50$, $B_3 = 61/50$. Indeed $qS + 2k \le q(S+2k)$, $A_r > 5$, $\log 2 < 7/10$, and $\log 3 < 11/10$.

Moreover

$$\frac{a\eta}{H} \le \frac{107}{103} - (c_5 - 3/8)y, \qquad C_r \le 1 - \frac{483}{824}y. \tag{10.830}$$

The first follows from $a/H \le 107/103 + 3y/8$ and multiplication by $1 - c_5 y$; discarding its negative quadratic term only increases the bound. All second-scale rows satisfy $c_1 c_2 c_3 q > 2$, as follows by direct substitution: the four products are $14700/5101, 7840/2083, 18375/2642, 82/39$. Combining Proposition 10.196 and the additional $(\log N_c)/2$ in Theorem 10.92 now bounds the arithmetic bracket, with its normalizing loss added, by $\mathcal{Z}\mathcal{A}(y, A_0)$, where

$$\begin{aligned} K &= \frac{107}{103} + \frac{10}{17}, \quad \Delta = c_5 - \frac{3}{8} + \frac{10}{17}\frac{483}{824}, \quad B = \lambda(B_q + 10/17), \\ \mathcal{A}(y, A_0) &= \frac{q}{c_1 c_2} + \frac{10}{17}\frac{1}{2c_1 c_2}\left(1 + \frac{y^2}{2\gamma}\right) + \frac{y^2}{2} + \frac{22}{17}\frac{13}{18000} + \frac{27}{17}e_0 + \mathcal{N} \\ &\quad + \frac{101}{100c_1}\frac{BA_0/c_4 + (K - \Delta y)/c_3}{1 + A_0}. \end{aligned} \tag{10.831}$$

The displayed rational factor follows from $eN = (m + A_0)/(1 + \tau) > (1 + A_0)/(101/100)$. It is not a substitution of ϑ for N.

Here are the remaining case data, including an upper bound $\mathcal{N}$ for the normalizing loss divided by $\mathcal{Z}$. Set $\xi = 101/100$.

Case	d_*	Lower bound a_* for A_0	$\mathcal{N}$
I.1	2	$3/2$	$\lambda(3/2 + \xi/2)/(c_1 c_4)$
I.2	1	$3/2$	$\lambda(3/2 + \xi/2)/(c_1 c_4)$
II	2	$5/2$	$\lambda(5/4 + \xi/4)/(c_1 c_4)$
III.1	2	$3/4$	$\lambda\xi/(c_1 c_4)$
III.2	1	$3/4$	$\lambda\xi/(c_1 c_4)$
IV	2	$1/2$	$\lambda(7/6 + \xi/6)/(c_1 c_4)$
V	2	2	$\lambda(1 + \xi/2)/(c_1 c_4)$

To prove the normalizing bounds, Corollary 10.185, $A_r \ge t$, $d \ge ef$, and $eN > A_0/\xi$ give

$$\frac{D_{\mathrm{a}}(N + 1/(p-1))}{\mathcal{Z}} < \frac{\lambda}{c_1 c_4}\left(\frac{\vartheta}{f} + \frac{\xi}{f(p-1)}\right). \tag{10.832}$$

Use the bounds for ϑ in Section 36. In III, $\vartheta + 1/(p-1) = 1$; in V, $f \ge 2$. The lower bounds for A_0 follow from (10.641) and its unramified case. This proves every entry in the table.

It remains to verify the strict comparison uniformly, rather than by sampling ranks. For fixed y the last fraction defining $\mathcal{A}$ is fractional linear in A_0. It lies between its value at a_* and its limit at infinity. For each of these two bounds, $\mathcal{L} - \mathcal{A}$ is a concave quadratic in y. Its minimum on $[0, 1/4]$ is therefore at an endpoint: subtracting the line joining the endpoint values gives a nonnegative multiple of $y(1/4 - y)$. Thus four exact rational comparisons per case prove the assertion for every rank and every admissible A_0. Substituting the tables gives these strict lower bounds for the gap:

Case	$\mathcal{L} - \mathcal{A}$ greater than
I.1	$27/500$
I.2	$41/1000$
II	$3/10$
III.1	$93/1000$
III.2	$93/1000$
IV	$91/250$
V	$4/3$

All exceed 1/25. The exact rational calculation records the four differences in each case. We have therefore bounded the same hypothetical nonzero value strictly above and strictly below incompatible quantities. It must be zero. Additive orders exceeding D_{a} vanish by degree, so the conclusion covers the whole stated cutoff. □

The theorem supplies the first enlarged integer block. Further integer extensions, fractional values and support contraction retain their own field and denominator costs.

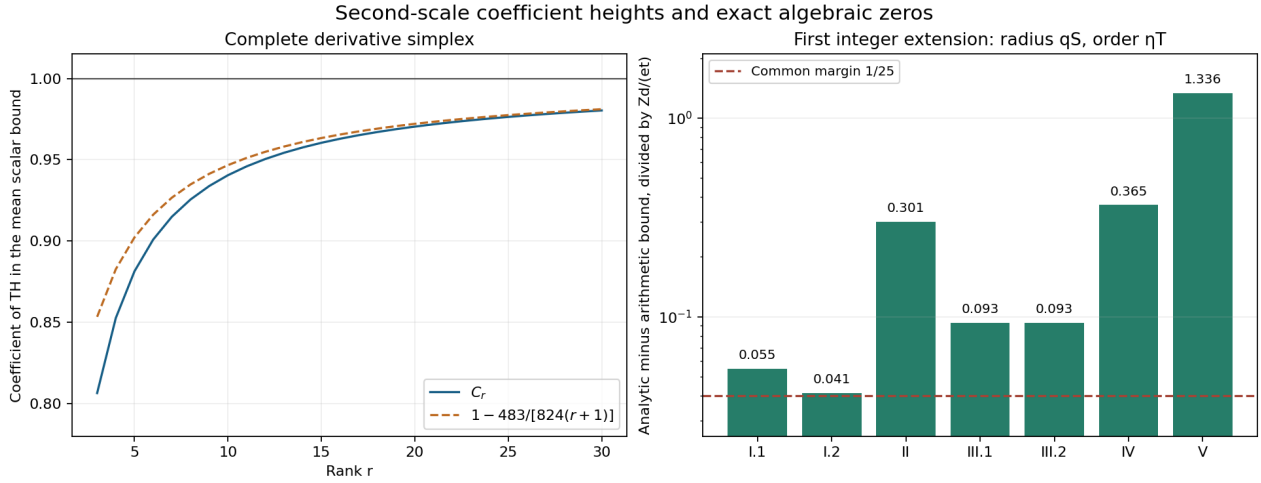


Figure 10.57.

The left panel shows the complete-simplex coefficient C_r and its proved upper envelope; the displayed ranks are samples of a bound proved for all ranks. The right panel shows the smaller of the four exact endpoint gaps in each field case. Fractional linearity in A_0 and concavity in $1/(r+1)$ make these endpoint checks uniform. The dashed line is the common proved margin 1/25. Theorem 10.197 gives exact zeros at integer nodes through qS, at total order at most ηT; it does not assert fractional zeros. Reproducible calculation and figure. Human-source context: the selected-support induction in Section 11 of Yu's freely accessible 1999 paper and the second constants in Section 1.1 of his freely accessible 2013 paper.

65. The complete second-scale integer zero block

The first enlargement can be repeated without changing the coefficient vector or its field. The real interval grows by q at each step, while the available derivative order is multiplied by η. We prove the comparison for every rank, including the accumulated integer rounding and ordinary-jet costs.

Theorem 10.198 (all integer stages at the second scale). Retain all hypotheses and notation of Section 64, and choose the coefficient vector of Proposition 10.196. Under $v_p(\Xi-1)\geq U_r$,

$$\Phi(s;\boldsymbol{t})=0 \quad \text{if } s\in\mathbb{Z}, \quad |s|\leq q^jS, \quad |\boldsymbol{t}|\leq \eta^jT, \qquad 0\leq j\leq r. \tag{10.835}$$

These are the original prepared values at contraction depth zero. In particular the conclusion through q^rS retains the actual global field K and every original polynomial column.

Proof. Put $y=1/(r+1)$. The initial stage is Theorem 10.184, and the step from zero to one is Theorem 10.197. We establish the input and arithmetic comparisons for the remaining steps $1\leq j\leq r-1$.

At step j set

$$R_j=q^jS, \qquad s_j=\lfloor R_j\rfloor, \qquad \mu_j=\left\lfloor \frac{c_5\eta^jT}{r+1}\right\rfloor+1. \tag{10.836}$$

Bernoulli's inequality, proved by induction after multiplying by $1 - c_5/(r+1)$, gives $\eta^j > 1 - c_5$ for $j \le r$. Equation (10.709) therefore gives

$$\frac{r+1}{c_5\eta^j T} < \frac{29}{12c_5(1-c_5)12^r} \le \frac{725}{82944} < \frac{1}{100}, \qquad \mu_j < \frac{101}{100}\frac{c_5\eta^j T}{r+1}. \tag{10.837}$$

Here both choices of c_5 satisfy $c_5(1-c_5) \ge 4/25$. The exact floor inequality gives

$$\lfloor \eta^{j+1}T \rfloor + \mu_j - 1 \le \lfloor \eta^j T \rfloor. \tag{10.838}$$

Thus the preceding block supplies every ordinary input jet.

We next control its separation cost. The proof of (10.756) gives $\log(2S) < H/2 + 3 + r + \log d$. Since $j \le r$, $q \le 3$, and $\log 3 < 11/10$,

$$\log(2R_j) < H/2 + 3 + (21/10)r + \log d < H. \tag{10.839}$$

For the last inequality use (10.755). After discarding the nonnegative $(r-1)\log d/2$, the difference between its lower bound for $H/2$ and $3 + (21/10)r + \log d$ is at least $(15r^2 - r - 25)/10 > 0$. Consequently the full interval separation bound of Theorem 10.34 and the original coefficient and lcm costs obey $M_j \le H/\log p < SN/3$, exactly as in (10.757)–(10.758).

Using $S > 16$, $j \le r-1$, and $U_r = c_1 q^r STN/(r+1)$, we obtain

$$\frac{\mu_j[(2s_j+1)N + M_j]}{U_r} < \frac{101}{100}\frac{c_5}{c_1 q}\left(2 + \frac{1}{16} + \frac{1}{3}\right) < \frac{15}{16}. \tag{10.840}$$

The two upper bounds, using only $c_1 \ge 7/10$, are $20907/22400$ for q = 2 and $2323/2520$ for q = 3. Strict normality is unchanged from Theorem 10.190. Theorem 10.182 therefore gives at every target integer node

$$v_p(\Phi(s;\boldsymbol{t})) - \beta \ge (2s_j+1)\mu_j N - D_{\mathrm{a}}(N + 1/(p-1)). \tag{10.841}$$

It remains to force exact zeros by the global comparison. Use the same scale $\mathcal{Z} = Zd/(et)$, the same case data $a_*, d_*, \gamma = c_3 q d_*$, and the same normalizing upper bound $\mathcal{N}$ as in Theorem 10.197. The positive analytic term is strictly greater than $\mathcal{Z}\mathcal{L}_j(y)$, where

$$\mathcal{L}_j(y) = \frac{2c_5 q^{j+1}}{c_1}(1-c_5 y)^j - \frac{c_5 q}{c_1\gamma}y^2(1-c_5 y)^j. \tag{10.842}$$

This retains $2\lfloor q^j S \rfloor + 1 > 2q^j S - 1$. If a target value is nonzero, it is in K by Theorem 10.55. There is no q-denominator at these integer nodes and depth zero. Theorem 10.92, Lemma 10.194 and Proposition 10.196 then bound its arithmetic height, with the normalizing loss added, by $\mathcal{Z}\mathcal{A}_j(y, A_0)$, as follows:

$$
\begin{aligned}
\ell_q &= \begin{cases} 7/50, & q=2,\\ 11/50, & q=3, \end{cases}\\
B_j &= \lambda(1+(j+1)\ell_q+10/17),\\
K_j(y) &= \left(\frac{107}{103}+\frac{3}{8}y\right)(1-c_5y)^{j+1}+\frac{10}{17}\left(1-\frac{483}{824}y\right),\\
\mathcal{A}_j(y,A_0) &= \frac{q^{j+1}}{c_1c_2}+\frac{10}{17}\frac{1}{2c_1c_2}\left(1+\frac{y^2}{2\gamma}\right)\\
&\quad+\frac{y^2}{2}+\frac{22}{17}\frac{13}{18000}+\frac{27}{17}e_0+\mathcal{N}\\
&\quad+\frac{101}{100c_1}\frac{B_jA_0/c_4+K_j(y)/c_3}{1+A_0}.
\end{aligned}
\tag{10.843}
$$

Indeed $q^{j+1}S+2k \le q^{j+1}(S+2k)$, and $\log q/A_r < \ell_q$ because $A_r > 5$. The pointwise derivative term is bounded using $a/H \le 107/103+3y/8$ at order $\eta^{j+1}T$. The initial mean term uses $C_r \le 1-483y/824$. Every other contribution is exactly the corresponding bound in Theorem 10.197, with its actual d,e,f. In particular this comparison does not discard the coefficient-field or logarithmic column term.

Here is a finite exact verification for steps 1 through 4, followed by a uniform argument for all later steps. For fixed y, the last fraction is fractional linear in A_0, so it lies between its value at a_* and its limit at infinity. At either endpoint $\mathcal{L}_j-\mathcal{A}_j$ is a polynomial P of degree at most $n=j+2$. Its required interval is

$$
0 \le y \le b_j := \frac{1}{\max\{4,j+2\}},
\tag{10.844}
$$

since $r \ge 3$ and $j \le r-1$. Write $P(y)=\sum_{k=0}^{n} a_k y^k$ and define

$$
v_i = \sum_{k=0}^{i} a_k b_j^k \frac{\binom{i}{k}}{\binom{n}{k}}, \qquad P(y)=\sum_{i=0}^{n} v_i \binom{n}{i}(y/b_j)^i(1-y/b_j)^{n-i}.
\tag{10.845}
$$

To prove the identity, substitute the first formula into the second and use $\binom{n}{i}\binom{i}{k}=\binom{n}{k}\binom{n-k}{i-k}$. The inner binomial sum is one after taking out $(y/b_j)^k$. The displayed weights are nonnegative and sum to one. Thus P is at least the smallest of the rational numbers v_i. Substitution of the seven case rows gives the strict lower bounds in the second column of the table below. This checks 56 polynomials with 308 coefficients; the identity proves that the checks cover whole intervals, rather than a sample of ranks.

For the tail first observe

$$
\eta^j \ge \eta^r > \exp(-c_5) > z_q, \qquad z_2=29/50, \quad z_3=4/9.
\tag{10.846}
$$

Integration gives $\log(1-x) > -x/(1-x)$ for $0<x<1$. Taking $x=c_5/(r+1)$ yields $r\log\eta > -rc_5/(r+1-c_5) > -c_5$. For the rational exponential bounds, at c = 27/50 and c = 4/5, respectively, the positive series gives

$$
\exp(c) \le \sum_{k=0}^{6}\frac{c^k}{k!}+\frac{c^7}{7!}\frac{1}{1-c/8} < \begin{cases} 50/29, & c=27/50,\\ 9/4, & c=4/5. \end{cases}
\tag{10.847}
$$

The tail majorant follows because every ratio after the seventh term is at most c/8; both final comparisons are rational.

For $j \ge 5$, put $K = 107/103 + 10/17$, $\xi = 101/100$, $\varepsilon = 13/18000$, and define

$$\begin{aligned} \mathfrak{a} &= \frac{2c_5 z_q - 1/c_2}{c_1}, & \mathfrak{c} &= \frac{\xi \lambda \ell_q}{c_1 c_4}, \\ \mathfrak{b} &= \frac{c_5 q}{16 c_1 \gamma} + \frac{10}{17}\frac{1}{2c_1 c_2}\left(1 + \frac{1}{32\gamma}\right) + \frac{1}{32} + \frac{22}{17}\varepsilon + \frac{27}{17}e_0 + \mathcal{N} \\ &\quad + \frac{\xi}{c_1}\left[\frac{\lambda(1 + 10/17)}{c_4} + \frac{K}{c_3(1 + a_*)}\right]. \end{aligned} \tag{10.848}$$

The bound in Theorem 10.197 implies $(107/103 + 3y/8)\eta^{j+1} \le 107/103$, so $K_j(y) \le K$. Use $y \le 1/4$, $A_0/(1 + A_0) \le 1$, and $1/(1 + A_0) \le 1/(1 + a_*)$. The preceding estimates give

$$\mathcal{L}_j - \mathcal{A}_j > F_j := \mathfrak{a} q^{j+1} - \mathfrak{b} - \mathfrak{c}(j + 1). \tag{10.849}$$

In every case $\mathfrak{a} > 0$, $\mathfrak{a}(q - 1)q^6 - \mathfrak{c} > 0$, and the third column below is a strict lower bound for F_5. Consequently $F_{j+1} - F_j > 0$ for every $j \ge 5$.

Case	Steps 1–4: every coefficient greater than	Tail: F_5 greater than
I.1	7/10	12/5
I.2	69/100	119/50
II	93/100	67/25
III.1	73/100	67/50
III.2	18/25	33/25
IV	93/100	41/50
V	9/2	57/10

The exact calculation records all rational coefficients, both exponential bounds, and each tail increment. Every table entry is greater than 1/2. We have therefore proved $\mathcal{L}_j - \mathcal{A}_j > 1/2$ for every remaining integer step and every admissible rank and field. A nonzero target value would violate the two bounds for its valuation, so it is zero. Together with the exact jet inequality, induction completes all stages through r. Above-degree additive orders vanish identically, as in Theorem 10.197. □

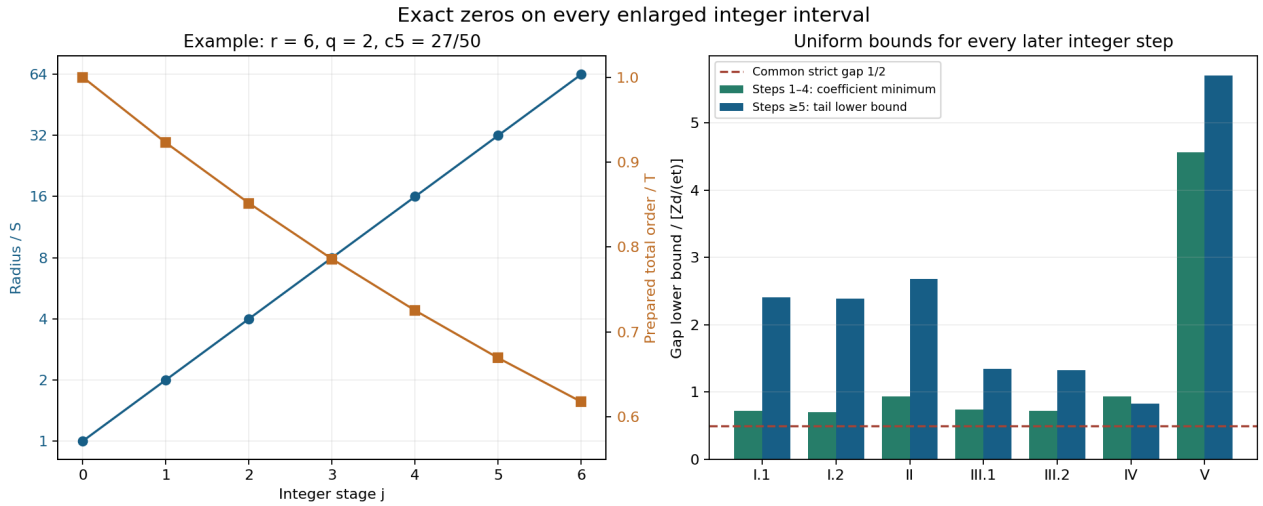


Figure 10.58.

Left: the radii $q^j S$ and orders $\eta^j T$ of Theorem 10.198 at r = 6 and q = 2, for j from zero to six. The points show the real cutoffs; the proof retains their integer floors. Right: the smallest Bernstein

coefficient over steps 1–4 and the tail lower bound F_5 in each field case. Both give a strict gap greater than $1/2$ in units of $Zd/(et)$; the first step has the separate bound in Theorem 10.197. The conclusion is an integer zero block at depth zero. Fractional values and support contractions require their additional field and denominator estimates. Reproducible calculation and figure. Human-source context: Section 11 of Yu's freely accessible 1999 paper and the second constants in Section 1.1 of his freely accessible 2013 paper.

66. The first fractional block at the second scale

The integer intervals contain a nested collection of derivative data. Their smaller intervals retain higher orders, and all these orders can be used together. At a fractional node the global root field still has to be paid for. Two quantitative features of the selected construction supply the comparison: its actual coefficient surplus and the strict room between its auxiliary scale and the stated second constant.

Retain Sections 61–65, including the original coefficient conditions (10.806), the saturated phase data and compatible principal roots of Theorems 10.51 and 10.56. Write

$$\mathcal{Z} = \frac{Zd}{et}, \qquad y = \frac{1}{r+1}, \qquad \eta = 1 - c_5 y, \qquad \mathfrak{h} = \eta^{r+1}, \qquad S_1 = \mathfrak{h}^{-1} S, \quad T_1 = \mathfrak{h} T. \tag{10.853}$$

Thus H continues to mean $h + \log J$; it is distinct from the order fraction $\mathfrak{h}$. The constants c_5 are those of Theorem 10.197.

The coefficient surplus and the scale comparison

Lemma 10.199 (retaining the full selected-column surplus). Let M be the full number of initial equations in Theorem 10.184. Its selected columns satisfy

$$\frac{N_c}{M} > c_0 p^{f A_0} \frac{\delta_r \mathcal{R}_r}{p^f - 1} > c_0 p^{f A_0}. \tag{10.854}$$

In particular the kernel may be chosen with the bound (10.823) in which the factor $10/17$ is replaced by w from this table:

Case	w	χ
I.1	2/25	11/10
I.2	2/25	51/50
II	1/134	21/20
III.1	10/71	109/100
III.2	10/71	6/5
IV	4/23	13/10
V	5/211	7/6

Every such choice still satisfies Proposition 10.196 and Theorem 10.198.

Proof. Use the column and row bounds in the proof of Theorem 10.184, retaining the factor $p^{-f A_0}$ in (10.668). The ratio of the actual column lower bound to the full row upper bound is strictly greater than

$$c_0 \ell_d \, \frac{2 + 1/g_S}{2 + 1/S} \, \frac{\delta_r \mathcal{R}_r}{p^f - 1} \, p^{f A_0}. \tag{10.855}$$

This follows by substituting (10.702) into the sufficient column inequality (10.713): the factors ζ_2^r, $(1+\tau)^r$, the powers of t and all factorials cancel. Here $S \ge g_S$ and $\ell_d \ge 1$. Also $\delta_r \mathcal{R}_r \ge p^f$. These inequalities give the two asserted strict bounds.

The weighted integral Siegel proof in Proposition 10.183 uses the row-to-surplus ratio $M/(N_c - M)$. It is now smaller than $(c_0 p^{fA_0} - 1)^{-1}$. In the seven cases, respectively, the following are lower bounds for p^{fA_0}:

$$5, \quad 5, \quad 50, \quad 3, \quad 3, \quad 5/2, \quad 16. \tag{10.856}$$

For I use $p = 3, A_0 \ge 3/2$ and $3^3 > 5^2$. For II use $p = 5, A_0 \ge 5/2$ and $5^5 > 50^2$. For III use $p \ge 5, A_0 \ge 3/4$ and $5^3 > 3^4$. For IV use $p \ge 7, A_0 \ge 1/2$ and $7 > (5/2)^2$. In V, $p^f \ge 4, A_0 \ge 2$, so 16 is allowed with equality. Substitution of these bounds and $c_0 = 27/10$ gives exactly w. The strict column inequality makes the final ratio strict also in V.

Apply the same complete-simplex mean estimate of Lemma 10.194 to the same full list of equations. The coefficient-field generator term and the leading logarithmic column term are unchanged; only the row-height multiplier decreases to w. This proves the claimed height bound. All its bracket terms are nonnegative. Since every w is less than $10/17$, the former height bound and all its integer-zero consequences remain valid. □

Lemma 10.200 (room in the second constant). With χ in the table, the same parameters satisfy

$$\chi e U_r < C_2(r, d, p, \boldsymbol{\alpha})\, h \Omega_r. \tag{10.857}$$

The rank-transfer conclusion of Lemma 10.187 remains valid with χU_r in place of U_r.

Proof. In the seven rows of Proposition 10.186, multiply its upper prefactors 627, 631, 627, 452, 454, 1763, 343 by the corresponding χ. The results are strictly below 1438, 648, 690, 495, 557, 2418, 406, respectively. All these are rational comparisons; for example $631(51/50) = 32181/50 < 648$. In III the common $1/\vartheta$ remains cancelled on both sides. Every other factor in that proposition is unchanged. The rank transfer uses exactly the same cancellation of $\mathcal{R}_r$ and the same strict factorial ratio as Lemma 10.187; χ depends on the field case, not the rank. □

Nested interpolation with the selected monomials

Theorem 10.201 (the first second-scale fractional zeros). Choose the kernel of Lemma 10.199. Under $v_p(\Xi - 1) \ge \chi U_r$, its original prepared values satisfy

$$\Phi(s/q; \boldsymbol{t}) = 0 \quad \text{for all } s \in \mathbb{Z}, \quad |s| \le q(\lfloor S_1 \rfloor + 1), \quad |\boldsymbol{t}| \le T_1. \tag{10.858}$$

The global root field may have degree as large as $q^r d$. The assertion retains that degree and the complete rational denominator at these fractional nodes.

Proof. Since $\chi > 1$, Theorem 10.198 supplies every integer block. Set

$$\begin{aligned} R_j &= \lfloor q^j S \rfloor, & \mu_j &= \lfloor \eta^j T \rfloor - \lfloor \mathfrak{h} T \rfloor + 1, \quad 0 \le j \le r, \\ N_* &= (2R_0 + 1)\mu_0 + 2\sum_{j=1}^{r} (R_j - R_{j-1})\mu_j, & B &= \lfloor \log_p(2R_r) \rfloor, \\ M_* &= \max\{B, v_p(V(k)), v_p(b_n)\}, & Q_D &= D_{\mathrm{a}}(N + 1/(p-1)). \end{aligned} \tag{10.859}$$

The radii are strictly increasing, and the multiplicities are positive and nonincreasing. Every required ordinary jet is available, since an output index has order at most $\lfloor \mathfrak{h} T \rfloor$. The proofs of (10.839) and (10.757) give $M_* \le H/\log p$.

We use the normal-series interpolation of Lemma 10.94 with the selected combined frequencies. Theorem 10.180 normalizes the projected function by a scalar of valuation $Q_D - \beta$, where β is the minimum valuation of its nonzero auxiliary coefficients. Its strict disc conditions follow from Theorem 10.190, since the counterhypothesis is stronger. The prepared projection error has valuation at least $\beta + \kappa + \chi U_r - v_p(b_n)$. Lemma 10.38 gives the ordinary-jet loss lC_0, with $C_0 = \max\{v_p(V(k)), v_p(b_n)\}$, exactly as in Theorem 10.182.

In Lemma 10.94 the complete cardinal polynomial at a node of multiplicity $\mu(s)$ loses at most $B(\mu_0 - \mu(s))$; its Taylor inverse of degree a loses at most aB. In a term using jet l, one has $a + l \le \mu(s) - 1$. Its total loss is consequently at most $(\mu_0 - 1)M_*$. The monic quotient contributes $N_* N$. Subtracting the normalization and then adding back the projection error proves, for every $x \in \mathbb{Z}_p$,

$$\begin{aligned} v_p(\Phi(x;\boldsymbol{t})) - \beta \ge \min\{&N_* N - Q_D, \\ &\kappa + \chi U_r - v_p(b_n) - (\mu_0 - 1)M_*\}. \end{aligned} \tag{10.860}$$

This proves the needed selected-support nested bound directly from the general interpolation lemma. It makes no assertion about the unverified last uniform-multiplicity input at scale U_r.

The two analytic lower bounds

Let d_* and a_* have their case values in Theorem 10.197, and put

$$\begin{aligned} S_- &= c_3 q d_* (r+1)^2, & T_- &= \frac{12}{29}(r+1)12^r, \\ \Gamma_r &= \frac{2c_5 q}{q^r}\sum_{j=0}^{r}(q\eta)^j - \frac{q(r+1)}{q^r S_-}\left[1 - \mathfrak{h} + 2\sum_{j=1}^{r}(\eta^j - \mathfrak{h})\right], \\ \mathcal{I}_r &= c_1 q\chi - \frac{101}{100}\frac{1 - \mathfrak{h} + 2/T_-}{q^r c_3(1 + a_*)}. \end{aligned} \tag{10.861}$$

Counting the annuli, with $\mu_j > (\eta^j - \mathfrak{h})T$, gives

$$\frac{c_1 N_* N}{q^r \mathcal{Z}} > \Gamma_r. \tag{10.862}$$

For completeness, the inner node count exceeds $2S - 1$, and the j-th annulus count exceeds $2(q^j - q^{j-1})S - 2 > 0$. Their principal real mass is $2c_5 ST\sum_{j=0}^{r}(q\eta)^j/(r+1)$. Their removed floor mass is at most $T[1 - \mathfrak{h} + 2\sum_{j=1}^{r}(\eta^j - \mathfrak{h})]$. Now $c_1 STN/\mathcal{Z} = q(r+1)$ and $S \ge S_-$. This proves the bound, including the differences of radius floors.

For the second entry, retain the original integer coefficient: $v_p(b_n) \le H/\log p$. Also $\mu_0 - 1 \le (1 - \mathfrak{h})T + 1$. Since $d \ge ef$,

$$\frac{c_1 TH}{\mathcal{Z}\log p} = \frac{f}{dc_3 N} \le \frac{1}{c_3 eN} < \frac{101}{100c_3(1 + A_0)}. \tag{10.863}$$

Discarding only the nonnegative κ, and using $T > T_-$, shows that the second entry multiplied by $c_1/(q^r\mathcal{Z})$ exceeds $\mathcal{I}_r$. Write $\mathcal{N}$ for the casewise normalizing upper bound in Theorem 10.197. The first entry in this same normalization exceeds $\Gamma_r - c_1\mathcal{N}/q^r$.

The arithmetic upper bound in the actual field

Theorem 10.56 applies to the actual selected support, which is still in its original phase class. Its normalized fractional value lies in E_Λ, of degree $q^{h_\Lambda}d$, where $h_\Lambda \le r$. The chosen local degrees are e,f. The common principal-unit monomial removed in normalizing the value has valuation zero. Thus it suffices to compare the normalized algebraic value V with the same local lower bound.

Use ε_r, e_0 from Lemma 10.195, taking the sharper $e_r = 1/79 + 1/(71928b_r)$ in place of its rank-three upper bound when convenient. Set $\lambda = 59/58$, $\xi = 101/100$, and $\ell_2 = 7/50, \ell_3 = 11/50$. Define

$$\begin{aligned} J_r &= \frac{\mathfrak{h}^{-1} + S_-^{-1}}{c_2} + \frac{w}{2c_2}\left(1 + \frac{1}{2S_-}\right) + \frac{y^2}{c_2c_3q} + (1 + w/2)c_1\varepsilon_r + (1 + w)c_1e_r, \\ B_r &= \lambda\left(1 + \ell_q + w + \frac{q}{q-1}\ell_q\right), \\ K_r &= \left(\frac{107}{103} + \frac{3}{8}y\right)\mathfrak{h} + w\left(1 - \frac{483}{824}y\right), \\ \mathcal{B}_r(A_0) &= J_r + \xi\frac{B_rA_0/c_4 + K_r/c_3}{1 + A_0}. \end{aligned} \tag{10.864}$$

If V is nonzero, the global product-formula proof in Theorems 10.57 and 10.92, with Lemma 10.199 for its coefficients, gives

$$\frac{c_1}{q^r\mathcal{Z}}(v_p(V) - \beta) \le \mathcal{B}_r(A_0). \tag{10.865}$$

Here is each contribution. The torus radius obeys $X/S \le \mathfrak{h}^{-1} + S_-^{-1} < q$, giving the first term of J_r. The mean torus term in the initial coefficient height gives its second. The coefficient-field generator gives its third. The leading coefficient count, the row count in that height bound, and the further $(\log N_c)/2$ give $(1 + w/2)c_1\varepsilon_r$. The two scalar remainders give $(1 + w)c_1e_r$.

The pointwise additive term at $X < qS$ is bounded as in Theorem 10.197 by $\lambda(1 + \ell_q)$, and the mean additive term has multiplier $w\lambda$. The fractional denominator is bounded by $D_{\mathrm{a}}(1 + 1/(q-1))$, using (10.243) at depth zero and root denominator q. Corollary 10.185 and $\log q/A_r < \ell_q$ therefore add precisely $\lambda q\ell_q/(q-1)$ to B_r. The individual derivative bound at order $\mathfrak{h}T$ and the initial complete-simplex mean give K_r, by Lemma 10.194. Finally $eN > (1 + A_0)/\xi$ gives the last displayed fraction. In this argument the entire nonnegative global height majorant is multiplied by $q^{h_\Lambda} \le q^r$. No multiplicity of primes or equality of conjugate valuations is assumed.

To verify the radius inequality just used, the function $(1 - c/v)^v$ increases for $v > c$: its logarithmic derivative is $\log(1 - z) + z/(1 - z) > 0$, where $z = c/v$. The latter inequality follows by integrating $1/(1-u) < 1/(1-z)$ from zero to z. It follows that $\mathfrak{h}^{-1} + S_-^{-1}$ decreases with r. At odd primes check r = 3 and the smallest c_3d_* in its row; in V check r = 4. The seven resulting rational numbers are less than q. Rank three is excluded in V by the positive-optimizer argument in Proposition 10.186.

Uniform strict comparisons

The required two inequalities are

$$\Gamma_r - \frac{c_1\mathcal{N}}{q^r} - \mathcal{B}_r(A_0) > \frac{1}{2}, \qquad \mathcal{I}_r - \mathcal{B}_r(A_0) > \frac{1}{1000}. \tag{10.866}$$

We give finite rational reductions that cover all ranks. At fixed r the dependence on A_0 is fractional linear. Its maximum is therefore at $A_0 = a_*$ or its limit at infinity. For r = 3 through 7, omitting the impossible r = 3 row in V, all displayed quantities are rational and can be substituted exactly.

For every r at least eight, put

$$H_- = (1 - c_5/9)^9,$$
$$E_- = \sum_{j=0}^{10} \frac{c_5^j}{j!}, \qquad E_+ = E_- + \frac{c_5^{11}}{11!(1 - c_5/12)}, \quad H_+ = E_-^{-1}. \tag{10.867}$$

The positive exponential series gives $E_- < \exp(c_5) < E_+$. The monotonicity just proved and $\log(1-z) < -z$ give $H_- \le \mathfrak{h} < \exp(-c_5) < H_+$. Equation (10.846) also gives $\eta^r > \exp(-c_5) > 1/E_+$. In the reversed geometric sum defining Γ_r, retain its last four terms and use $q\eta < q$. This proves

$$\Gamma_r > \frac{2c_5 q}{E_+} \sum_{j=0}^{3} q^{-j} - \frac{153q}{q^8 S_-(8)}. \tag{10.868}$$

Indeed the removed floor term is at most $q(r+1)(2r+1)/(q^r S_-(r))$. Since $S_-(r) = c_3 q d_*(r+1)^2$, this bound decreases for $r \ge 8$: its consecutive ratio is $(2r+3)(r+1)/(q(2r+1)(r+2)) < 1$.

For the arithmetic upper bound replace y by $1/9$, $S_-, b_r, \varepsilon_r, e_r$ by their conservative rank-eight values, $\mathfrak{h}^{-1}$ by H_-^{-1}, and the factor $\mathfrak{h}$ in K_r by H_+. Replace $1 - 483y/824$ there by one. All these substitutions increase the bound. For the input lower bound, use H_-, $T_-(8)$ and q^8; these substitutions decrease it. Use q^8 also for the upper normalizing loss. This reduces every remaining rank to two more rational comparisons per case.

There are 34 finite-rank rows and seven uniform rows, each with two endpoints, for 82 comparisons. The following are strict lower bounds for their smaller differences:

Case	Mass minus normalization and arithmetic	Input minus arithmetic
I.1	99/100	3/25
I.2	99/100	3/2000
II	109/100	3/20
III.1	93/100	1/20
III.2	93/100	19/100
IV	51/50	11/25
V	27/25	69/100

The exact calculation records every rational endpoint and difference, the seven radius bounds and the seven comparisons with the second constant. The uniform reductions above prove why these finite computations cover all allowed ranks and fields. Each column exceeds its asserted common lower bound. Thus both entries of the analytic minimum strictly exceed the arithmetic upper bound for a nonzero V. This contradiction proves $\mathrm{V} = 0$, hence the original prepared value is zero. Indices whose additive order exceeds D_{a} vanish by degree. This completes the whole stated fractional block. □

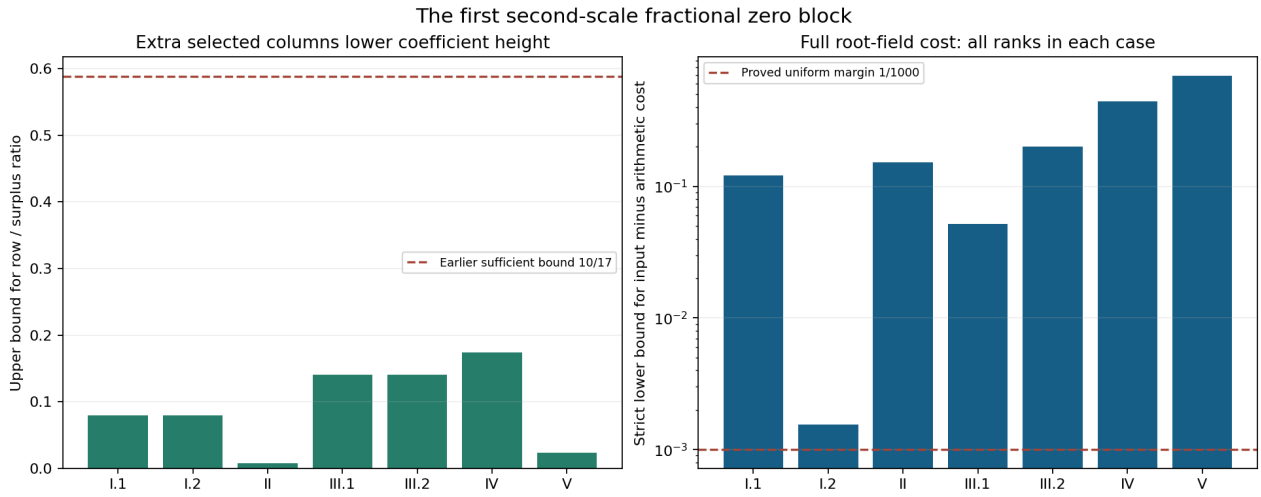


Figure 10.59.

Left: the row-to-surplus upper bounds of Lemma 10.199, compared with the sufficient 10/17 bound of Proposition 10.196. Right: the smaller exact input margin among all finite-rank and uniform-tail endpoint comparisons in Theorem 10.201, in units $q^r Zd/(c_1 et)$. Both the full root-field degree and the q-denominator are included. The theorem concerns the first fractional block; subsequent support contractions retain their separate depth costs. Reproducible calculation and figure. Human-source context: the selected-support induction in Section 11 of Yu's freely accessible 1999 paper and the second constants in Section 1.1 of his freely accessible 2013 paper.

67. The first support contraction at the second scale

The fractional zeros separate the exponent classes. This produces a smaller nonzero polynomial whose coefficients are selected from the original kernel. The new integer values lie in the original number field. Their rational denominators still contribute to the height bound, and filling the missing multiples of q uses both losses in the interpolation formula for a q-deleted interval.

Retain the parameters, kernel and counterhypothesis of Theorem 10.201. In particular, $c = c_5$, $\mathfrak{h} = \eta^{r+1}$, $S_1 = S/\mathfrak{h}$, $T_1 = \mathfrak{h}T$, and $v_p(\Xi - 1) \geq \chi U_r$. Let β be the minimum valuation of the nonzero initial coefficients. The meanings of H, N and $\mathcal{Z}$ remain unchanged.

Theorem 10.202 (the first full contracted block at the second scale). There is a nonzero contracted polynomial with an integral coefficient subvector of the kernel in Lemma 10.199. Its coefficient minimum is β, its coefficient height does not increase, its torus widths are at most D_i/q, and its additive argument is x/q. Its phase family Φ_1 has every integer value in K. Put

$$R = q(\lfloor S_1 \rfloor + 1). \tag{10.872}$$

The family satisfies both collections of zeros

$$\begin{aligned} &\Phi_1(s;\boldsymbol{t}) = 0 \quad (|s| \leq R,\ q \nmid s,\ |\boldsymbol{t}| \leq \lfloor T_1 \rfloor), \\ &\Phi_1(s;\boldsymbol{t}) = 0 \quad (|s| \leq R,\ |\boldsymbol{t}| \leq \lfloor \eta T_1 \rfloor). \end{aligned} \tag{10.873}$$

Its selected combined-frequency congruence retains the depth of Theorem 10.176. The stronger first collection remains available for subsequent interpolation.

Proof. Apply the exact phase and support passage of Theorem 10.51 to the fractional zeros of Theorem 10.201. Include the second phase partition when q divides G_0. The saturated root-monomial basis separates the exponent classes at every integer s prime to q. Choose a class containing a coefficient attaining β. The resulting subvector is nonzero and integral, preserves its minimum

valuation, and has no larger projective or Euclidean coefficient height, by the place-by-place subset inequalities in Theorem 10.48.

The affine prepared-jet transfer in that theorem preserves the whole total-order cutoff: its triangular leading factors are powers of q, which are units at p. The new widths and additive argument are exactly those of (10.141)–(10.142). All additive polynomial indices remain in each selected exponent fibre. Their independence in Lemma 10.29 therefore makes the contracted polynomial nonzero. The phase identity in Theorem 10.55 puts its integer prepared values in K. These arguments prove the first collection of zeros.

Restriction to a q-class followed by division by q preserves the selected congruence, by the final assertion of Theorem 10.176. Consequently Theorem 10.180 and Corollary 10.179 still give the normalization with the same N. The new coefficient minimum is still β, so the normalizing valuation is $Q_D - \beta$, where $Q_D = D_{\mathrm{a}}(N + 1/(p-1))$. The projection estimate retains the original coefficient b_n and the original Euler directions. The strict disc conditions follow from Theorem 10.190, as in Theorem 10.201.

Filling the missing nodes

Set

$$\begin{aligned} M &= \left\lfloor \frac{cT_1}{r+1} \right\rfloor + 1, \quad n_E = 2(q-1)(\lfloor S_1 \rfloor + 1), \\ B &= \lfloor \log_p(2R) \rfloor, \qquad M_* = \max\{B, v_p(V(k)), v_p(b_n)\}. \end{aligned} \tag{10.874}$$

The q-deleted interval has exactly n_E nodes, since q divides R. The elementary inequality $\lfloor a \rfloor + \lfloor b \rfloor \le \lfloor a + b \rfloor$, applied to $a = \eta T_1$ and $b = cT_1/(r+1)$, gives

$$\lfloor \eta T_1 \rfloor + M - 1 \le \lfloor T_1 \rfloor. \tag{10.875}$$

Thus all required ordinary jets are present. The radius inequality in Theorem 10.201 gives $R < q^2 S \le q^r S$. The proof of (10.839), including the original coefficient and least-common-multiple bounds, therefore gives $M_* \le H/\log p$.

Apply the q-deleted form (10.690) of Theorem 10.182, before imposing its sufficient one-term budget. For every output order in the second collection and every $x \in \mathbb{Z}_p$, it gives

$$\begin{aligned} v_p(\Phi_1(x;\boldsymbol{t})) - \beta \ge \min\{&n_E MN - Q_D, \\ &\kappa + \chi U_r - v_p(b_n) - MB - (M-1)M_*\}. \end{aligned} \tag{10.876}$$

Here MB is the additional deleted-node loss; the inverse-polynomial and ordinary-jet loss is the separate term $(M-1)M_*$. Together with the original coefficient loss they are at most $2MM_*$, since both B and $v_p(b_n)$ are at most M_*.

Write $\xi = 101/100$ and retain a_*, T_-, $\mathcal{N}$, and $\mathcal{B}_r(A_0)$ from Theorem 10.201. Define

$$\mathcal{I}_r^{\mathrm{c}} = c_1 \chi q^{r+1} - \frac{\xi(2c\mathfrak{h}/(r+1) + 2/T_-)}{c_3(1+a_*)}. \tag{10.877}$$

Since $M > cT_1/(r+1)$, $\lfloor S_1 \rfloor + 1 > S_1$, and $S_1 T_1 = ST$, the identity in the proof of (10.862) yields

$$\frac{c_1 n_E MN}{\mathcal{Z}} > 2cq(q-1), \qquad \frac{c_1 Q_D}{\mathcal{Z}} < c_1 \mathcal{N}. \tag{10.878}$$

For the other entry use $M \le cT_1/(r+1)+1$, $T > T_-$, and the complete original-coefficient identity (10.863). Discarding only the nonnegative κ gives

$$\frac{c_1}{\mathcal{Z}}(\kappa + \chi U_r - v_p(b_n) - MB - (M-1)M_*) > \mathcal{I}_r^{\mathrm{c}}. \tag{10.879}$$

The original-field arithmetic comparison

At a target integer, the value lies in K by the proved phase identity. If it were nonzero, the same product-formula derivation as (10.865) would give

$$\frac{c_1}{\mathcal{Z}}(v_p(\Phi_1(s;\boldsymbol{t})) - \beta) \le \mathcal{B}_r(A_0). \tag{10.880}$$

We justify each changed argument in that bound. The coefficient height and minimum have their original bounds, and the number of remaining columns is no larger. The torus contribution has radius $q^{-1}R/S \le \mathfrak{h}^{-1} + S_-^{-1}$. The additive scalar is evaluated at an argument of absolute value at most $q^{-1}R = \lfloor S_1 \rfloor + 1 < qS$. The derivative order ηT_1 is smaller than T_1. For the Euler remainder, the depth-one bound (10.808) gives

$$\frac{E_1}{Z} < \frac{1}{79q} + \frac{r-1}{999Z} < e_r. \tag{10.881}$$

Only this common upper envelope is required; no comparison of the exact numerical remainders at successive depths is assumed. Finally, (10.243) gives the full rational clearing cost $\Xi_{1,0,u} \le D_{\mathrm{a}}(1 + 1/(q-1))$. Thus every term in (10.864) remains an upper bound. The normalization here is $c_1/\mathcal{Z}$: the actual global field is K, of degree d, with its original local degrees e and f. The factor q^r paid at fractional nodes is absent, while the q-denominator remains included.

Strict inequalities at every permitted rank

The two comparisons are

$$2cq(q-1) - c_1\mathcal{N} - \mathcal{B}_r(A_0) > \frac{3}{5}, \qquad \mathcal{I}_r^{\mathrm{c}} - \mathcal{B}_r(A_0) > 9. \tag{10.882}$$

Here is a finite rational reduction covering every rank. Use the 34 finite-rank rows and seven uniform rows of Theorem 10.201, with the two endpoints $A_0 = a_*$ and $A_0 \to \infty$. The arithmetic expression is the same (10.864), so its proved fractional-linear endpoint reduction and uniform upper envelope apply unchanged. The normalizing loss is now $c_1\mathcal{N} = \lambda\nu/c_4$, with the same casewise factor ν used there, and has no dependence on r.

For the input comparison at every $r \ge 8$, use $q^{r+1} \ge q^9$, $\mathfrak{h} < H_+$, $1/(r+1) \le 1/9$, and $T_- \ge T_-(8)$. Each substitution decreases the displayed input lower bound. Thus the same 82 rational endpoint comparisons prove the result for all ranks. Strict lower bounds for their smallest differences are as follows:

Case	Mass minus normalization and arithmetic	Input minus arithmetic
I.1	61/100	10
I.2	3/5	9
II	37/50	10
III.1	63/100	10
III.2	31/50	11
IV	69/100	13

Case	Mass minus normalization and arithmetic	Input minus arithmetic
V	71/10	230

The exact calculation lists every rational endpoint and verifies these strict bounds. The finite computations certify the rational inequalities after the proved reductions; the uniform rows cover the infinite tail. Both entries of the analytic minimum therefore exceed the arithmetic upper bound for a nonzero target value. Each such value must be zero. Additive orders above D_{a} vanish by degree. This proves the full second collection, while retaining the first. □

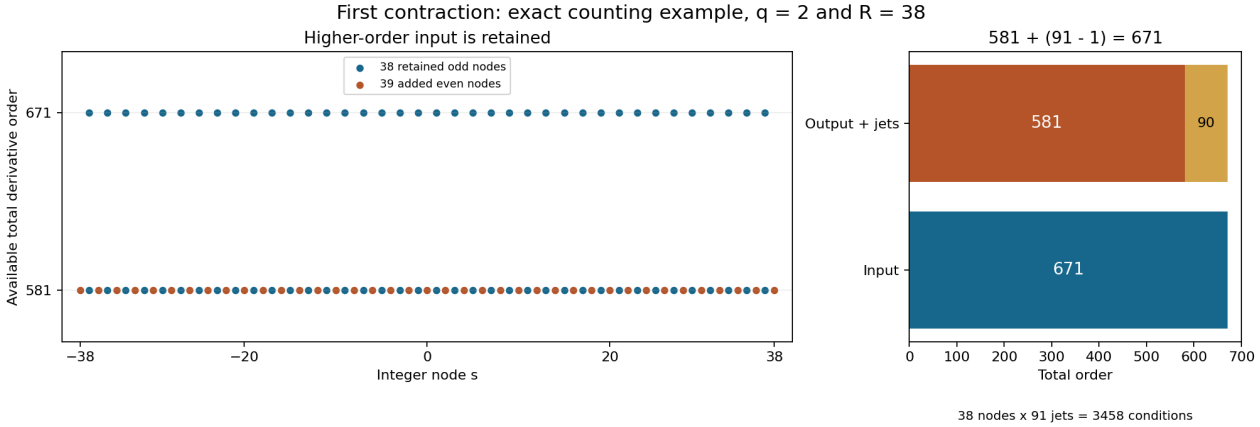


Figure 10.60.

The exact counting example of Solution 81 has $q = 2$, R = 38, input order 671 and output order 581. The 38 odd nodes retain order 671; interpolation of multiplicity 91 adds the 39 even nodes at order 581. Its cardinal polynomial has degree 3458. The displayed orders describe available vanishing conditions, not function values. The example illustrates the exact floors and node counts, without claiming that these small numbers are an instance of the second-scale parameter construction. The field and precision statements are proved in Theorem 10.202. Reproducible calculation and figure.

68. Saturation and the cost of contraction depth

The q-primary saturation index contributes both a factor to the lattice count and a logarithmic cost to repeated contraction. These two effects can be balanced in the polynomial scale. We give the construction, its geometric stopping bounds and the arithmetic estimate for integer nodes.

A scale that includes the saturation index

Retain the field and height assumptions of Sections 61–67, including (10.806). Write $A = A_r$, $x = \log J$, $B = A + x$, $H = h + x$, and $g = B/A$. Define

$$\widehat{\mathcal{D}} = g\mathcal{D}, \qquad \widehat{T} = gT, \qquad \widehat{D}_i = gD_i, \qquad \widehat{Z} = \frac{S\widehat{\mathcal{D}}}{d}, \qquad \widehat{\mathcal{Z}} = \frac{S\widehat{\mathcal{D}}}{et}. \tag{10.884}$$

Keep S, k, N, c_i, η and the field and residue data fixed. In the polynomial parameter (10.703), use

$$\widehat{X} = \frac{S\widehat{\mathcal{D}}}{c_1c_4(N/\vartheta)dBk} = X, \qquad \widehat{L} = L, \qquad \widehat{D}_{\mathrm{a}} = D_{\mathrm{a}}, \qquad \widehat{U}_r = q^{r+1}\widehat{\mathcal{Z}}. \tag{10.885}$$

Theorem 10.203 (the kernel with a saturation-dependent scale). There is an integral kernel for these parameters with the full selected-column surplus of Lemma 10.199. The coefficient-height bounds of Lemmas 10.194–10.195 and Proposition 10.196 hold with hats on the scales and with the

same rational envelopes. Under $v_p(\Xi-1)\ge \chi\widehat{U}_r$, its integer, fractional and first contracted zero blocks satisfy Theorems 10.198, 10.201 and 10.202 with the hatted parameters. Moreover,

$$\chi e\widehat{U}_r < C_2(r,d,p,\boldsymbol{\alpha})\,h\Omega_r. \tag{10.886}$$

The coefficients are chosen for the hatted system of equations. The construction retains its actual row field and a coefficient of minimum valuation throughout phase extraction.

Proof. The integer polynomial parameters are unchanged because $\widehat{\mathcal{D}}/B=\mathcal{D}/A$. Both the width volume and the real derivative count acquire a factor g^r. Equivalently, substitute $\widehat{\mathcal{D}},B$ in (10.713): both sides acquire the factor g. The lattice, phase and unit-congruence selections of Proposition 10.183 apply to this box. The floor estimates (10.711)–(10.712) hold because $\widehat{T}\ge T>101r^2$ and $\widehat{X}=X>58$. Keeping the factors in (10.855) gives the same strict column-to-row surplus, and hence the same w in Lemma 10.199. Choose its integral coefficient vector for the complete hatted initial row list.

We check the estimates used in the stated consequences. Corollary 10.185 gives the degree bound with $\widehat{\mathcal{D}},B$. The normalizing estimate (10.832) uses only $B\ge A\ge t$. The generating-series proof of Lemma 10.194 applies at arbitrary widths and orders: (10.812), divided by $\widehat{Z}$, has the same leading term and a smaller $(r-1)/(999\widehat{Z})$. The derivative coefficient still satisfies $a/H\le 107/103+3/[8(r+1)]$, and the complete-simplex mean (10.810) is unchanged. Estimate (10.824) uses only S,k,H, so its logarithm is less than $A\le B$. Also

$$\frac{\widehat{T}H}{\widehat{Z}}=\frac{1}{c_1c_3eN},\qquad P\sum_i \widehat{D}_i\sigma_i=\frac{\widehat{\mathcal{D}}}{c_1c_2d}. \tag{10.887}$$

Thus the normalized mean derivative and coefficient-generator bounds remain valid. For Lemma 10.195 use $\widehat{\mathcal{D}}/A\ge \mathcal{D}/A>b_r$. The actual lattice count still gives (10.821) with $\widehat{\mathcal{D}}$. The functions $\log z/z$ decrease on the same ranges, and $rx/\widehat{Z}$ decreases. Consequently the same ε_r,e_r bound all column and scalar remainders. This proves the asserted coefficient bound.

The radius S and its lower bound S_- are unchanged, while $\widehat{T}>T_-$. Identity (10.863) is unchanged with hats. The proof of (10.839), including the bounds for the original $v_p(b_n)$ and $V(k)$, still gives $\log(2q^rS)<H$. Strict normality follows from Theorems 10.180 and 10.190 with $\widehat{U}_r\ge U_r$. Every normalized positive interpolation mass uses the unchanged identity $c_1S\widehat{T}N/\widehat{\mathcal{Z}}=q(r+1)$. Thus the floor, ordinary-jet, normalization and arithmetic arguments in Theorems 10.198, 10.201 and 10.202 apply to this kernel. Their denominator terms use $\log q/B\le\log q/A$, so all their rational endpoint comparisons remain valid.

For the final constant, substitute the hatted parameters in (10.719)–(10.720), retaining the factor H/J. The resulting saturation factor is HB/J. Since $h>40, A>5, x\ge 0$,

$$\left(1+\frac{x}{h}\right)\left(1+\frac{x}{A}\right)\le 1+\frac{9}{40}x+\frac{x^2}{200}\le 1+x+\frac{x^2}{2}\le \exp x. \tag{10.888}$$

The last inequality is the positive exponential series. Hence $HB/J\le hA$, with equality only at $x=0$. After this substitution the prefactors of Lemma 10.200 are identical, proving the strict claimed bound. The factorial and residue cancellation in Lemma 10.187 uses this same final bound, so its rank-transfer conclusion also applies. □

The geometric stop and its rank product

Write $\mathfrak{h}=\eta^{r+1}$, $Q=q\mathfrak{h}$. Let I be a nonnegative integer with

$$E = Q^I > \exp(3B), \qquad \bar{S} = q\mathfrak{h}^{-I}S, \qquad \bar{T} = \eta\mathfrak{h}^I\widehat{T}. \tag{10.889}$$

Use $\mathsf{D}_0 = D_{\mathrm{a}}$, $\mathsf{D}_j = 2Jq^{-I}\widehat{D}_j$, and the integer allocations (10.769) with these $\bar{S}, \bar{T}$.

Theorem 10.204 (degree and rank bounds at the logarithmic stop). All the allocation and degree hypotheses of Theorem 10.25 hold. More precisely,

$$\begin{aligned} &\mathsf{D}_0 > 20\mathsf{D}_j \ (1 \le j \le r), \qquad \mathsf{D}_0 > 4\bar{T}, \qquad \mathsf{D}_0 > 4r, \\ &\mathcal{R}_0 > \frac{32}{11}, \qquad \mathcal{R}_{m,J'} > \frac{2^{10}}{29} \ (1 \le m < r), \qquad \mathcal{R}_r > 31. \end{aligned} \tag{10.890}$$

If the nonzero contracted family has the full prepared integer zeros through $\bar{S}$ and order $\bar{T}$, with the independent bases and hyperplane hypotheses of Corollary 10.192, its rank conclusion holds with the strict factor 10^{-6} in (10.787), including the actual residue maxima of both families.

Proof. As in (10.773), $Q > 1$, while $q\mathfrak{h}^2 < 1$. Thus $\mathfrak{h}^{-I} > E$ and $q^I/J > E/J > \exp(3A + 2x)$. The derivative of $\exp(2x)/(A + x)$ is positive for $x \ge 0$, since $2 - 1/(A + x) > 0$. Consequently

$$\frac{q^I}{JB} > \frac{\exp(3A)}{A}, \qquad \frac{\exp(3B)}{B} \ge \frac{\exp(3A)}{A}. \tag{10.891}$$

The cancellation (10.775), with B in the denominator, gives $\mathsf{D}_0/\mathsf{D}_j > (9/392)(r+1)^2 d\sigma_j \exp(3A)/A$. Use $d\sigma_j > 1/(87d^2)$, $\exp(2A) > 256(r+1)^2d^2$, $\exp(A)/A > 32/5$, and $r + 1 \ge 4$. The resulting rational lower bound is $9 \cdot 256^2 \cdot 32/(392 \cdot 87 \cdot 5) > 20$. Similarly, the cancellation in (10.777) gives $\mathsf{D}_0/\bar{T} > (3/35)\exp(3A)/A > (3/35)2^{15}/5 > 4$. Finally $kL > 58(r + 1) > 4r$. The allocation floors still obey (10.778). Formula (10.779) has B in place of A, so $B \ge t$ gives $\mathcal{R}_0 > 32/11$.

For an m-coordinate subset, (10.780) holds with this value of B_0. Its bracket exceeds $(6/7)d\exp(3A)$. Apply the height-product estimate (10.781), $\mathrm{e} < 3$, and

$$\frac{2}{7}\exp(3A) > \frac{2}{7}\,2^{13}(r+1)^2d^2 > 2^{11}(r+1)^2d^2. \tag{10.892}$$

Here the two factors arise from $2A$ and A separately, using $A \ge 4 + \log((r + 1)d) > 5$. The factorial and field-degree argument of (10.782) now yields

$$\mathcal{R}_{m,J'} > \frac{4}{29}\frac{(2^{11})^m}{(m+1)^3} \ge \frac{2^{10}}{29}. \tag{10.893}$$

The last expression increases with m: its consecutive ratio is at least $2^{11}(2/3)^3 > 1$. In the full-rank cancellation (10.783), B cancels with the same factor in $\widehat{\mathcal{D}}$. Also $J^{1-r}E^r > \exp(3rA)J^{2r+1} \ge \exp(3rA)$. Using (10.784)–(10.785) gives

$$\begin{aligned} \mathcal{R}_r &> \frac{\exp((3r-1)A)}{6r(r+1)![12(r+1)]^r\ell_d A} \\ &> \frac{16^{2r}}{6r(r+1)12^r} > 31. \end{aligned} \tag{10.894}$$

For the middle inequality use $\exp((r-1)A) > A$, $\exp(2rA) > [16(r+1)d]^{2r}$, $(r+1)! \le (r+1)^{r+1}$, and $d^{2r} \ge \ell_d$. The final expression has consecutive ratio $(256/12)r/(r+2) > 1$ for $r \ge 3$, and its value at three is $32768/243 > 31$. This proves the degree and allocation claims.

It remains to check the rank-product constant. Under the stipulated zeros, Theorem 10.25 and Corollary 10.26 supply the rank output exactly as in Corollary 10.192. The cancellation (10.789) is unchanged: the explicit B in $\widehat{\mathcal{D}}$ cancels its polynomial denominator. Its last depth factor satisfies

$$J^{m-1}\frac{\mathfrak{h}^I}{E^m} < \exp(-3(m+1)A)J^{-2m-4} \le \exp(-3(m+1)A). \tag{10.895}$$

Let $\mathcal{Q}$ denote the normalized product quotient in the proof of Corollary 10.192. Keeping all its factorials and its bound $\mathcal{R}_m \le \exp(A)A$, we obtain

$$\mathcal{Q} < 6r(m+1)!m!\ell_d A(35r)^m \exp(-(3m+2)A). \tag{10.896}$$

Set $V = (r+1)d$. The function $z\exp(-(3m+2)z)$ decreases for $z \ge 5$. Since $A \ge 4 + \log V$, $4 + \log V \le 5V$, and $\ell_d \le d$,

$$\mathcal{Q} < 30 \cdot 35^m \exp(-12m-8)F_{r,m,d}, \qquad F_{r,m,d} = \frac{r^{m+1}(m+1)!m!}{(r+1)^{3m+1}d^{3m}}. \tag{10.897}$$

For $m = 1, r \ge 3$, differentiation gives $F_{r,1,d} \le 2r^2/(r+1)^4 \le 9/128$. For $m \ge 2$, the function $r^{m+1}/(r+1)^{3m+1}$ decreases for $r \ge m+1$: the numerator of its logarithmic derivative is $m+1-2mr < 0$. Use the already proved factorial inequality $n! \le n^n/2^{n-1}$ at $n = m, m+1$, followed by $r = m+1$. This gives $F_{r,m,d} < 2(m+1)/4^m \le 1$. Now $\mathrm{e} > 27/10$ gives the exact rational bounds

$$\begin{aligned} m = 1: &\quad \mathcal{Q} < \frac{9450}{128}\left(\frac{10}{27}\right)^{20} < 10^{-6}, \\ m \ge 2: &\quad \mathcal{Q} < 30 \cdot 35^2\left(\frac{10}{27}\right)^{32} < 10^{-6}. \end{aligned} \tag{10.898}$$

The second bound decreases as m increases, since $35(10/27)^{12} < 1$. These are integer comparisons after clearing denominators. They prove the claimed product estimate, with all field and independence assertions provided by Corollary 10.26. □

The full denominator at an integer stage

Lemma 10.205 (scalar costs before the logarithmic stop). Let a selected family be at integer depth $I \ge 0$, with widths $q^{-I}\widehat{D}_i$ and additive argument $q^{-I}x$. Take $j \ge 1$, $S_I = \mathfrak{h}^{-I}S$, and target radius $R = q^{j+1}S_I$. Suppose $I\log Q \le 3B$. For every additive order $u \le D_{\mathrm{a}}$,

$$\begin{aligned} &\frac{c_1}{\widehat{Z}}\left[\Xi_{I,0,u}\log q + PRq^{-I}\sum_i \widehat{D}_i\sigma_i\right] \le \frac{q^{j+1}}{c_2} + \frac{\lambda\vartheta\log q}{c_4NB(q-1)}, \\ &\frac{D_{\mathrm{a}}\log(q^{-I}R+2k) - L\log(k!)}{\widehat{Z}} < \frac{\lambda\vartheta}{c_1c_4N}\left(1 + \frac{(j+1)\log q}{B}\right). \end{aligned} \tag{10.899}$$

Here $\lambda = 59/58$, and $\Xi_{0,0,u} = 0$. The exponent $\Xi_{I,0,u}$ is the full denominator exponent from (10.243).

Proof. That formula bounds $\Xi_{I,0,u} \le D_{\mathrm{a}}(I + 1/(q-1))$. The hatted degree estimate gives $c_1D_{\mathrm{a}}/\widehat{Z} < \lambda\vartheta/(c_4NB)$. The torus contribution is exactly $q^{j+1}Q^{-I}/c_2$. Put $a_j = q^{j+1}/c_2$, $b = \lambda\vartheta\log q/(c_4NB)$. The second derivative of $a_jQ^{-z} + bz$ is positive. On $0 \le z \le 3B/\log Q$, convexity therefore bounds this function by the larger endpoint value. At zero it is a_j. At the other endpoint its ratio to a_j is

$$\exp(-3B) + \frac{3\lambda(\vartheta/N)c_2\log q}{c_4q^{j+1}\log Q}. \tag{10.900}$$

We have $\vartheta/N < 1+\tau < 101/100 = \xi$. At $q=2$, use $Q \geq 2(173/200)^4$. At $q=3$, rank three is excluded by the optimizer condition, so $Q \geq 3(21/25)^5$. These follow from the increasing function $(1-c/v)^v$ proved in Theorem 10.201. Also $\log Q > 2(Q-1)/(Q+1)$: integrate $2/(1-z^2) > 2$ from zero to $(Q-1)/(Q+1)$. With $\log 2 < 7/10$, $\log 3 < 11/10$, and the largest c_2 in each q-case, exact substitution bounds the second term by $3/5$ at two and $2/5$ at three. The rational calculation records every field case. The first term is less than $1/1000$, since $B > 5$ and $\exp(15) > 2^{15} > 1000$. Thus the depth-proportional cost is absorbed. The remaining $1/(q-1)$ term gives precisely the displayed remainder.

For the additive scalar use $Q^{-I} \leq 1$ and $q^{-I}R + 2k \leq q^{j+1}(S+2k)$. The factorial argument of (10.824) bounds $\log[\mathrm{e}(S+2k)/k]$ by $A \leq B$. Substitution of the hatted degree bound proves the second inequality. □

The lemma charges all denominator primes in the global height. Later zero-production comparisons must combine this arithmetic bound with the selected-series interpolation bound; fractional nodes also require their actual global root-field degree.

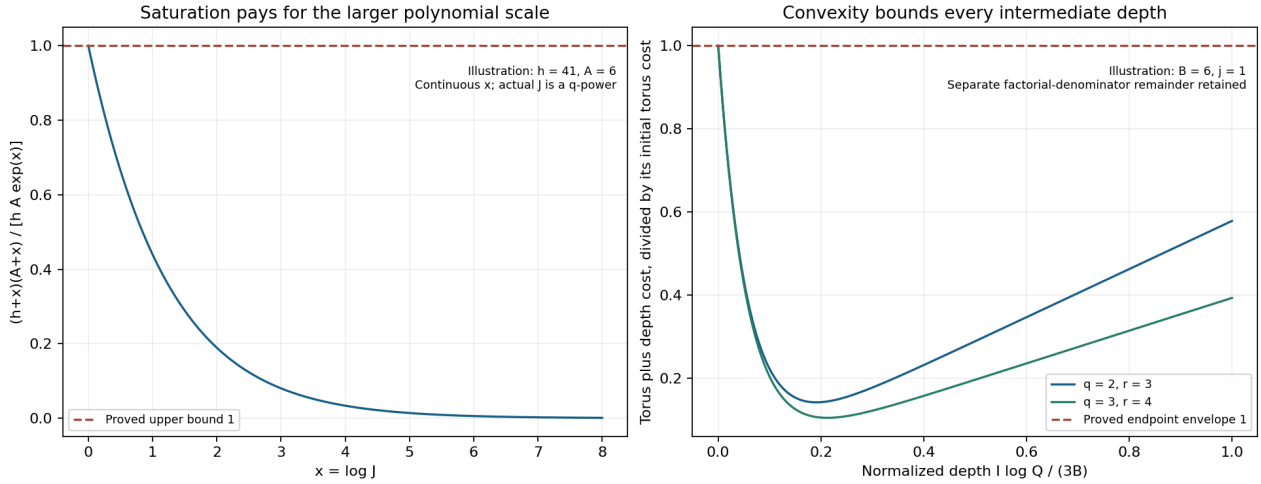


Figure 10.61.

Left: the ratio $(h+x)(A+x)/(hA\exp x)$ at $h=41, A=6$; Theorem 10.203 proves its upper bound for every admissible height. The curve uses continuous x; actual saturation indices have $x = \nu \log q$. Right: the convex endpoint argument of Lemma 10.205 at $B=6, j=1$, with $(q,r)=(2,3),(3,4)$ and ϑ/N replaced by its upper bound $101/100$. The separate factorial-denominator remainder remains in the lemma. These parameter illustrations explain the inequalities; they are not full arithmetic instances. Reproducible figure and exact comparisons. Human-source context: the selected-support and contraction methods in Sections 10–11 of Yu's freely accessible 1999 paper; the parameter, depth and product inequalities used here are proved above.

69. Integer zeros at a contracted depth

Contraction increases the integer radius while decreasing the available derivative order. Their product remains fixed. The scalar denominator grows with depth, but Lemma 10.205 absorbs its depth-dependent part into the torus-height budget. We now combine that arithmetic estimate with the selected-function interpolation theorem.

Theorem 10.206 (the integer extension after contraction). Use the hatted kernel of Theorem 10.203 and assume $v_p(\Xi - 1) \geq \chi \widehat{U}_r$. Put $c = c_5$, $\mathfrak{h} = \eta^{r+1}$, and $Q = q\mathfrak{h}$. Let $I \geq 1$ be an integer with $I \log Q \leq 3B$, and set

$$S_I = \mathfrak{h}^{-I} S, \qquad T_I = \mathfrak{h}^I \widehat{T}. \tag{10.901}$$

Suppose a nonzero phase family Φ_I is obtained by the prime-to-p support and phase restrictions of Theorems 10.48, 10.51 and 10.176, retaining a coefficient of the original minimum valuation β. Its widths are at most $q^{-I}\widehat{D}_i$, its additive argument is $q^{-I}x$, and its integer values belong to K. If

$$\Phi_I(s;\boldsymbol{t}) = 0 \quad (s \in \mathbb{Z},\ |s| \le qS_I,\ |\boldsymbol{t}| \le \eta T_I), \tag{10.902}$$

then, for every $1 \le j \le r$,

$$\Phi_I(s;\boldsymbol{t}) = 0 \quad (s \in \mathbb{Z},\ |s| \le q^j S_I,\ |\boldsymbol{t}| \le \eta^j T_I). \tag{10.903}$$

All cutoffs mean their integer floors. The coefficient vector, its height bound and its minimum valuation remain unchanged in these extensions. In particular, Theorems 10.202–10.203 provide the required starting block at $I = 1$; its higher-order q-deleted block remains available. At later depths this theorem applies once the stated starting block has been obtained.

Proof. Let $1 \le j \le r-1$, and assume the j-th block. Use the full node interval $|s| \le s_j$, where

$$s_j = \lfloor q^j S_I \rfloor, \qquad \mu_j = \left\lfloor \frac{c\eta^j T_I}{r+1} \right\rfloor + 1, \qquad \lfloor \eta^{j+1} T_I \rfloor + \mu_j - 1 \le \lfloor \eta^j T_I \rfloor. \tag{10.904}$$

The last inequality follows from $\eta^{j+1}T_I + c\eta^j T_I/(r+1) = \eta^j T_I$ and the elementary floor inequality. It holds even when $\mu_j = 1$. Thus all ordinary jets required by Theorem 10.182 are supplied by the assumed prepared zeros.

The two interpolation bounds

Write M_* for the maximum of the full-interval separation cost, $v_p(V(k))$, and the original $v_p(b_n)$. Equation (10.839) and $H > 40 > 1$ imply

$$\begin{aligned} \log(2q^j S_I) < H + I\log(\mathfrak{h}^{-1}) \le \mathfrak{h}^{-I}H, \\ M_* \le \frac{\mathfrak{h}^{-I}H}{\log p}, \qquad v_p(b_n) \le \frac{H}{\log p}. \end{aligned} \tag{10.905}$$

Indeed $1 + z \le \exp z$ for $z \ge 0$, so $H + z \le H\exp z$. The separation bound is $\lfloor \log_p(2q^j S_I) \rfloor$, and the other two costs have the bounds already proved in (10.757). The last displayed bound is kept separately; it is the original integer coefficient, not a coefficient enlarged by contraction.

All phase restrictions retain the selected combined-frequency congruence by Theorem 10.176. The strict disc conditions of Theorem 10.180 consequently persist, and its perturbation still uses the original b_n. Theorem 10.190 and $\chi\widehat{U}_r \ge U_r$ give the strict perturbation condition. Multiplication of the additive argument by the p-adic unit q^{-I} changes neither its coefficient integrality nor its degree D_{a}. Thus Corollary 10.179 supplies the normalizing valuation $Q_D - \beta$, where $Q_D = D_{\mathrm{a}}(N + 1/(p-1))$.

The prepared projection error of Lemma 10.181 at every integer node has valuation at least $\beta + \kappa + \chi\widehat{U}_r - v_p(b_n)$. Its ordinary divided jets are those of the original selected directions, as in the proof of Theorem 10.182; their l-th cost is at most $l\max\{v_p(V(k)), v_p(b_n)\}$. The full-cardinal interpolation bound (10.690) therefore gives, for every target integer and every permitted output order,

$$v_p(\Phi_I(s;\boldsymbol{t})) - \beta \ge \min\left\{(2s_j+1)\mu_j N - Q_D,\quad \kappa + \chi\widehat{U}_r - v_p(b_n) - (\mu_j - 1)M_*\right\}. \tag{10.906}$$

There is no deleted-node loss in this full interval. In particular, the second entry is used as it stands, without replacing it by the stronger sufficient one-term budget (10.691).

Put $y = 1/(r+1)$, $\lambda = 59/58$, and $\xi = 101/100$. Retain the case constants a_*, d_* of Theorem 10.197, $\gamma = c_3 q d_*$, and its normalizing factor ν, so $c_1 Q_D/\widehat{\mathcal{Z}} < \lambda\nu/c_4$. The lower bound $S \geq \gamma/y^2$, the identity $S_I T_I = S\widehat{T}$, and $c_1 S\widehat{T}N/\widehat{\mathcal{Z}} = q(r+1)$ give

$$\begin{aligned} \frac{c_1((2s_j+1)\mu_j N - Q_D)}{q^j \widehat{\mathcal{Z}}} &> \mathcal{L}_j(y), \\ \mathcal{L}_j(y) &= 2cq(1-cy)^j \left(1 - \frac{y^2}{2q^j\gamma}\right) - \frac{\lambda\nu}{c_4 q^j}. \end{aligned} \tag{10.907}$$

Here $2\lfloor q^j S_I \rfloor + 1 > 2q^j S_I - 1$ and $\mu_j > c\eta^j T_I/(r+1)$; no multiplicity rounding is discarded. Also $\mu_j - 1 \leq c\eta^j T_I/(r+1)$. Applying the hatted identity (10.863), and discarding only $\kappa \geq 0$, gives the following strict lower bound for the other normalized entry:

$$\mathcal{I}_j = c_1 \chi q^{r+1-j} - \frac{\xi}{q^j c_3 (1+a_*)} \left(\frac{1}{\widehat{T}} + \frac{c\eta^j}{r+1}\right). \tag{10.908}$$

That is, the entry is greater than $\mathcal{I}_j$. The term $1/\widehat{T}$ accounts for the original b_n, and the second term accounts for the ordinary jets and separation.

The arithmetic upper bound

The phase identity (10.151), retained at each restriction, puts every target integer value in K. Its global degree and local degrees are therefore d,e,f. Restricting the coefficient vector does not increase $h_2(c)$ or its column count, and the retained minimum is β. Theorem 10.92 applies with its full q-denominator.

For an explicit envelope, put $\varepsilon = 13/18000$, $e_0 = 1/79 + 1/(71928 \cdot 1500)$, and $\ell_q = 7/50$ at two and $11/50$ at three. Let $E_-(c), E_+(c)$ be the degree-ten rational exponential bounds in Theorem 10.201, and write $H_+ = 1/E_-(c)$. In particular $\exp(-c) < H_+$. Define

$$\begin{aligned} W(y) &= \frac{w}{2c_2}\left(1 + \frac{y^2}{2\gamma}\right) + \frac{y^2}{c_2 c_3 q} + (1 + w/2)c_1\varepsilon + (1+w)c_1 e_0, \\ b_j &= \lambda\left(1 + (j+1)\ell_q + w + \frac{\ell_q}{q-1}\right), \\ k_j(y) &= \left(\frac{107}{103} + \frac{3}{8}y\right)(1-cy)^{j+1} H_+ + w\left(1 - \frac{483}{824}y\right), \\ \mathcal{A}_j(y, A_0) &= \frac{q}{c_2} + \frac{W(y)}{q^j} + \frac{\xi}{q^j}\frac{b_j A_0/c_4 + k_j(y)/c_3}{1 + A_0}. \end{aligned} \tag{10.909}$$

If a target value is nonzero, we claim

$$\frac{c_1}{q^j \widehat{\mathcal{Z}}}(v_p(\Phi_I(s; \boldsymbol{t})) - \beta) \leq \mathcal{A}_j(y, A_0). \tag{10.910}$$

To check every term, apply (10.244) with $R = q^{j+1} S_I$ and output order $O \leq \eta^{j+1} T_I$. Since $t = f \log p$, its field factor converts the height normalization into $c_1/(q^j \widehat{Z})$. Lemma 10.205 bounds the combined torus and depth-dependent scalar cost by q/c_2, with the remaining $\lambda\ell_q/(q-1)$ contribution to b_j. Its additive-factorial bound gives $\lambda(1 + (j+1)\ell_q)$, since $\log q/B < \ell_q$. The initial mean additive cost adds $w\lambda$. Use $\vartheta/N < \xi A_0/(1+A_0)$ for these terms.

The initial mean torus cost, coefficient-field generator, logarithmic column counts and scalar remainders give the four terms of W, respectively, exactly as in (10.864). We used $\varepsilon_r \leq \varepsilon$, $e_r \leq e_0$, and the

arbitrary-depth bound $E_I/\widehat{Z} < e_0$ from Lemmas 10.194–10.195. For the pointwise derivative cost, $a/H \le 107/103 + 3y/8$ and $\mathfrak{h}^I \le \mathfrak{h} < \exp(-c) < H_+$: the strict exponential inequality follows by integrating $1/(1-z) > 1$ to obtain $\log(1-z) < -z$. The mean derivative cost has the multiplier $wC_r \le w(1 - 483y/824)$. Finally $1/(eN) < \xi/(1 + A_0)$ gives exactly k_j. This proves the complete arithmetic bound in the actual field.

Comparisons for every rank and integer step

We prove that both analytic entries are strictly larger than this arithmetic bound. For fixed y, its dependence on $A_0 \ge a_*$ is fractional linear, so its maximum is at a_* or at its limit at infinity. For $1 \le j \le 4$, at either endpoint $P(y) = \mathcal{L}_j(y) - \mathcal{A}_j(y, A_0)$ is a polynomial of degree at most $j + 2$. Its required interval is

$$0 \le y \le \frac{1}{\max\{r_0 + 1, j + 2\}}, \qquad r_0 = \begin{cases} 3, & q = 2, \\ 4, & q = 3. \end{cases} \tag{10.911}$$

The q = 3 rank restriction is the positive-optimizer consequence in Proposition 10.186. Use the Bernstein identity (10.845) on this interval. Substitution of the seven case rows gives 56 polynomials, with 308 rational Bernstein coefficients, all greater than 1/4. The exact calculation constructs the coefficients from the displayed formulas and reconstructs each polynomial exactly. The nonnegative weights in (10.845) sum to one, so the comparison holds on the whole interval.

For $j \ge 5$, one has $r \ge j + 1 \ge 6$, hence $y \le 1/7$. The inequality $\log(1-z) > -z/(1-z)$ used in (10.846) gives $\eta^j \ge \eta^{r-1} > \exp(-c) > 1/E_+(c)$. Both q^{-j} and $(j+1)q^{-j}$ decrease with j. Replace y by 1/7 in W, replace k_j by $(107/103+3/56)H_+ + w$, and replace j by five in the remaining decreasing costs. This bounds the mass from below and the arithmetic expression from above by

$$\begin{aligned} L_* &= \frac{2cq}{E_+(c)}\left(1 - \frac{1}{98q^5\gamma}\right) - \frac{\lambda\nu}{c_4 q^5}, \\ A_*(A_0) &= \frac{q}{c_2} + \frac{W(1/7)}{q^5} + \frac{\xi}{q^5}\frac{b_5 A_0/c_4 + [(107/103 + 3/56)H_+ + w]/c_3}{1 + A_0}. \end{aligned} \tag{10.912}$$

The same two endpoints reduce this tail to 14 rational comparisons. Each gives $L_* - A_* > 3/100$.

For the input entry put $y_0 = 1/(r_0 + 1)$ and $T_0 = (12/29)(r_0 + 1)12^{r_0}$. Since $\widehat{T} > T_0$, $r + 1 - j \ge 2$, and $q^j \ge q$, the input is greater than

$$I_* = c_1\chi q^2 - \frac{\xi(1/T_0 + cy_0)}{qc_3(1 + a_*)}. \tag{10.913}$$

Meanwhile $(j+1)/q^j \le 2/q$, $W(y) \le W(y_0)$, and $\eta^{j+1} \le 1$ give the uniform arithmetic upper bound

$$U_* = \frac{q}{c_2} + \frac{W(y_0)}{q} + \frac{\xi}{q}\left[\frac{\lambda(1 + w + \ell_q/(q-1) + 2\ell_q)}{c_4} + \frac{(107/103 + 3y_0/8)H_+ + w}{c_3(1 + a_*)}\right]. \tag{10.914}$$

Each of the seven rational differences $I_* - U_*$ exceeds 7/5. Thus the finite-step mass gap exceeds 1/4, the remaining mass gap exceeds 3/100, and the input gap exceeds 7/5, all in the normalization $c_1/(q^j\widehat{\mathcal{Z}})$. These are exact rational comparisons after the analytic reductions; no rank or contraction depth is left to numerical sampling.

A nonzero target value would have its normalized valuation at most $\mathcal{A}_j$ and strictly greater than that quantity by both entries of the interpolation minimum. It must therefore be zero. Additive orders greater than D_{a} vanish identically. Induction proves every asserted block through j = r. The

phase family is unchanged during this induction, so all previously known zeros, including a stronger q-deleted block, are retained.

At depth one, Theorem 10.202 supplies a full radius $q(\lfloor S_1 \rfloor + 1) > qS_1$ with order $\lfloor \eta T_1 \rfloor$. Also $\log Q < \log 3 < 3B$. This verifies the stated unconditional application at that depth. □

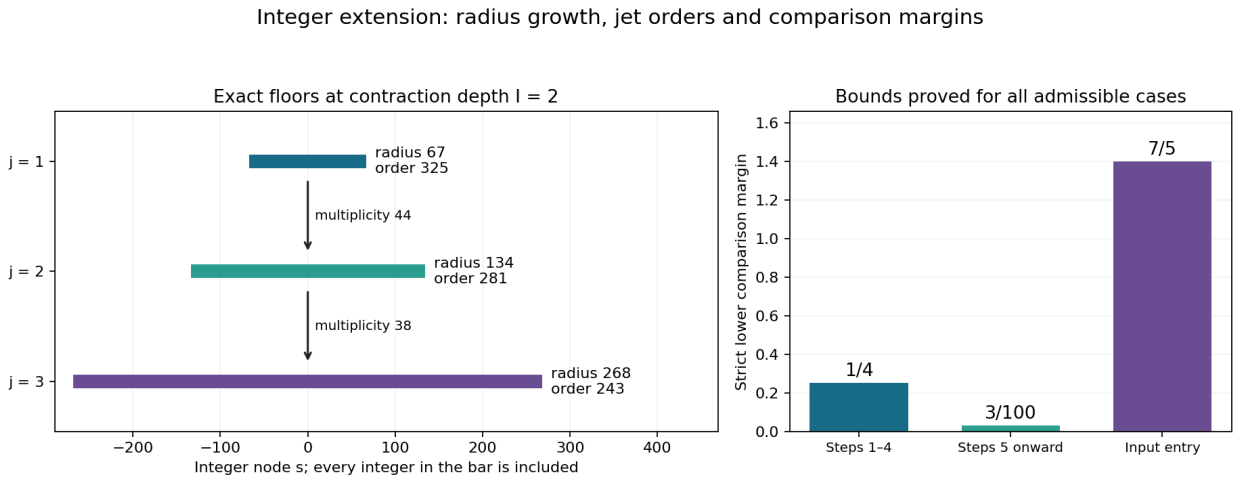


Figure 10.62.

Left: the counting example of Solution 83 at $q = 2, r = 3, I = 2, S = 21/2, \widehat{T} = 1200$. Each full integer interval carries the displayed derivative order; the arrows give the ordinary interpolation multiplicities. This is an illustration of exact floor arithmetic, not a number-field instance of the parameter construction. Right: exact strict lower margins from Theorem 10.206, in units of $c_1/(q^j \widehat{\mathcal{Z}})$: 1/4 for steps 1–4, 3/100 for the infinite tail, and 7/5 for the input entry. The proof connects these comparisons to the actual selected functions. Reproducible calculation and figure. Human-source context: the extrapolation and contraction methods of Section 11 of Yu's freely accessible 1999 paper; the selected-function and height estimates used here are proved in this lesson.

70. Fractional zeros and successive contractions

At a contracted depth, the inner interval retains two different orders: a higher order away from multiples of q, and a lower order at every integer. The outer integer intervals supplied by Theorem 10.206 add further layers. We use all these layers, while retaining the denominator of the fractional target and the degree of its actual root field.

Use the hatted kernel of Theorem 10.203 throughout. Put $a = r + 1$, $c = c_5$, $\mathfrak{h} = \eta^a$, $Q = q\mathfrak{h}$, and

$$S_I = \mathfrak{h}^{-I} S, \qquad T_I = \mathfrak{h}^I \widehat{T}, \qquad \widehat{\mathcal{Z}} = \frac{S\widehat{\mathcal{D}}}{et}. \tag{10.917}$$

The logarithmic scale is $B = A_r + \log J$; it is distinct from the integer separation cost used below. Retain the counterhypothesis $v_p(\Xi - 1) \geq \chi \widehat{U}_r$, including the casewise χ and coefficient-height multiplier w of Lemma 10.199.

The complete mixed fractional step

Theorem 10.207 (fractional zeros after a contraction). Let $I \geq 1$ be an integer satisfying

$$I \log Q \leq 3B, \qquad \frac{cT_{I+1}}{a} \geq 1. \tag{10.918}$$

Suppose a nonzero selected family Φ_I has been obtained by the support and phase restrictions of Theorems 10.48, 10.51 and 10.176. It retains the original coefficient minimum β, an integral coefficient subvector, the complete polynomial-index fibres, widths $q^{-I}\widehat{D}_i$, and additive argument $q^{-I}x$. Put $R_1 = q(\lfloor S_I \rfloor + 1)$ and $R_j = \lfloor q^j S_I \rfloor$ for $2 \le j \le r$. Suppose its prepared zeros comprise:

- the q-deleted integers through R_1, to order $\lfloor T_I \rfloor$;
- every integer through R_1, to order $\lfloor \eta T_I \rfloor$;
- every integer through R_j, to order $\lfloor \eta^j T_I \rfloor$, for $2 \le j \le r$.

Then

$$\Phi_I(s/q; \boldsymbol{t}) = 0 \quad (s \in \mathbb{Z}, \quad |s| \le q(\lfloor S_{I+1} \rfloor + 1), \quad |\boldsymbol{t}| \le \lfloor T_{I+1} \rfloor)\,. \tag{10.919}$$

Proof. Set $O = \lfloor T_{I+1} \rfloor$ and define

$$\begin{aligned}
\mu_j &= \lfloor \eta^j T_I \rfloor - O + 1 && (0 \le j \le r),\\
E &= \mu_0 - \mu_1, \qquad M = \mu_0,\\
N_I &= (2R_r + 1)\mu_r + \sum_{j=1}^{r-1} (2R_j + 1)(\mu_j - \mu_{j+1}) + 2(q-1)(\lfloor S_I \rfloor + 1)E,\\
B_I &= \lfloor \log_p(2R_r) \rfloor, \qquad M_* = \max\{B_I, v_p(V(k)), v_p(b_n)\},\\
Q_D &= D_{\mathrm{a}}(N + 1/(p-1)).
\end{aligned} \tag{10.920}$$

These are legitimate nested radii and multiplicities. Indeed $S_I \ge S > 16$ gives $q(q-1)S_I - q - 1 > 0$, hence $R_2 > R_1$, and all subsequent radii increase. The last real order difference is $(\eta^r - \mathfrak{h})T_I = c\eta^r T_I/a \ge 1/\eta > 1$. Thus every μ_j is positive, and they are nonincreasing. At a node in layer j, an output order at most O plus its largest ordinary-jet order is at most $O + \mu_j - 1$, exactly the provided input floor. The deleted increment E is a nonnegative integer.

Apply the normal-series assertion (10.351) of Theorem 10.132, with the deleted radius equal to R_1. Theorem 10.180 gives the normalizing scalar valuation $Q_D - \beta$. Theorem 10.182 and Lemma 10.38 give the ordinary-jet loss $l \max\{v_p(V(k)), v_p(b_n)\}$, with original baseline $\beta + \kappa + \chi\widehat{U}_r - v_p(b_n)$. Theorem 10.176 preserves the combined-frequency congruence under each stipulated restriction; the factor q^{-I} is a p-adic unit. The normal-disc conditions are those of Theorem 10.190 for the hatted scale. Subtract the normalizing valuation after applying (10.351), and add back the projection error. This gives

$$\begin{aligned}
v_p(\Phi_I(x; \boldsymbol{t})) - \beta \ge \min\{&N_I N - Q_D,\\
&\kappa + \chi\widehat{U}_r - v_p(b_n) - EB_I - (\mu_0 - 1)M_*\} \qquad (x \in \mathbb{Z}_p).
\end{aligned} \tag{10.921}$$

In particular, both the deleted-cardinal cost and the inverse/jet cost remain. No prepared-family constants from (10.352) or (10.353) are substituted for this selected-family normalization.

The proof of (10.839), together with $H > 1$, gives

$$\log(2q^r S_I) < H + I\log(\mathfrak{h}^{-1}) \le \mathfrak{h}^{-I}H, \qquad M_* \le \frac{\mathfrak{h}^{-I}H}{\log p}, \qquad v_p(b_n) \le \frac{H}{\log p}. \tag{10.922}$$

For the middle inequality use $H + z \le H\exp z$ for $z \ge 0$. Keep the original coefficient bound separately. The floor estimates are $E \le cT_I/a + 1$ and $\mu_0 - 1 \le (1 - \mathfrak{h})T_I + 1$. Using (10.863), $T_I = \mathfrak{h}^I\widehat{T}$, and $2/T_I \le 2c\mathfrak{h}/a$, the second analytic entry, multiplied by $c_1/(q^r\widehat{\mathcal{Z}})$, is strictly greater than

$$\mathcal{I}_r = c_1 q\chi - \frac{\xi}{q^r c_3(1+a_*)}\left[1-\mathfrak{h}+\frac{c(1+2\mathfrak{h})}{a}+\frac{1}{T_-}\right], \quad \xi=\frac{101}{100}, \quad T_- = \frac{12}{29}a12^r. \tag{10.923}$$

Here a_* is the casewise lower bound for A_0 in Theorem 10.197. The term $1/T_-$ pays for the original b_n; it is additional to the two rounded interpolation losses.

To count the first entry, separate the inner deleted nodes, the remaining inner multiples of q, and the outer annuli. Their counts are $2(q-1)(\lfloor S_I\rfloor+1)$, $2(\lfloor S_I\rfloor+1)+1$, and $2(R_j-R_{j-1})$. The first two exceed $2(q-1)S_I$ and $2S_I$. The first annulus exceeds $2(q^2-q)S_I-2(q+1)>0$; later annuli exceed $2(q^j-q^{j-1})S_I-2>0$. Also $\mu_j>(\eta^j-\mathfrak{h})T_I$. Telescoping gives

$$\begin{aligned}(q-1)(1-\mathfrak{h})+(\eta-\mathfrak{h})+\sum_{j=2}^{r}(q^j-q^{j-1})(\eta^j-\mathfrak{h})\\ =\frac{c}{a}\left[\sum_{j=0}^{r}(q\eta)^j+q-2\right].\end{aligned} \tag{10.924}$$

Use $S_IT_I=S\widehat{T}$, $c_1S\widehat{T}N/\widehat{\mathcal{Z}}=qa$, and $S_I\ge S_-=c_3qd_*a^2$. The normalized mass is strictly greater than

$$\begin{aligned}\Gamma_r=\frac{2cq}{q^r}\left[\sum_{j=0}^{r}(q\eta)^j+q-2\right]\\ -\frac{qa}{q^rS_-}\left[2(q+1)(\eta^2-\mathfrak{h})+2\sum_{j=3}^{r}(\eta^j-\mathfrak{h})\right].\end{aligned} \tag{10.925}$$

The first analytic entry therefore exceeds $\Gamma_r-\lambda\nu/(c_4q^r)$, where $\lambda=59/58$ and ν is the case factor in Theorem 10.197, so that $c_1\mathcal{N}=\lambda\nu/c_4$.

The full fractional denominator over the permitted depths

Let $X=\lfloor S_{I+1}\rfloor+1$, the absolute target radius for $x=s/q$. By Theorem 10.56 its normalized value V lies in the actual root field of degree at most q^rd, with chosen local degrees e,f. Removing its common principal-unit monomial does not change the valuation. The original minimum β and the coefficient-height bound of Lemma 10.199 remain valid for the subvector.

Put $y=1/a$, let ε_r,e_r be the bounds of Lemma 10.195, and use $\ell_2=7/50,\ell_3=11/50$. Define

$$\begin{aligned}u_r&=\frac{\mathfrak{h}^{-1}+S_-^{-1}}{c_2},\\ W_r&=\frac{w}{2c_2}\left(1+\frac{1}{2S_-}\right)+\frac{y^2}{c_2c_3q}+(1+w/2)c_1\varepsilon_r+(1+w)c_1e_r,\\ b'_r&=\lambda\left(1+\ell_q+w+\frac{q}{q-1}\ell_q\right),\\ k'_r&=\left(\frac{107}{103}+\frac{3}{8}y\right)\mathfrak{h}^2+w\left(1-\frac{483}{824}y\right),\\ L_Q&=\frac{2(Q-1)}{Q+1},\qquad v_r=\frac{3\lambda l_q}{c_4L_Q},\qquad l_2=\frac{7}{10},\quad l_3=\frac{11}{10},\\ \mathcal{B}_{r,0}(A_0)&=u_r+W_r+\xi\frac{b'_rA_0/c_4+k'_r/c_3}{1+A_0},\\ \mathcal{B}_{r,1}(A_0)&=\frac{u_r}{32768}+W_r+\xi\frac{(b'_r/c_4+v_r)A_0+k'_r/c_3}{1+A_0}.\end{aligned} \tag{10.926}$$

If V is nonzero, Theorem 10.92 and the same height contributions as in Theorem 10.201 give

$$\frac{c_1}{q^r\widehat{\mathcal{Z}}}(v_p(V)-\beta)\le\max\{\mathcal{B}_{r,0}(A_0),\mathcal{B}_{r,1}(A_0)\}. \tag{10.927}$$

Here is the depth estimate and a check of the other contributions. Since $q^{-I}\le Q^{-I}$, the torus cost is at most u_rQ^{-I}. Formula (10.243) bounds the full denominator by $D_{\mathrm{a}}(I+1+1/(q-1))$. Its part proportional to I has normalized height at most $\lambda(\vartheta/N)I\log q/(c_4B)$. The sum of this linear term and u_rQ^{-I} is convex as a function of real I on $[0,3B/\log Q]$. At zero its value is u_r. At the other endpoint it is bounded by

$$u_r\exp(-3B)+\frac{3\lambda(\vartheta/N)\log q}{c_4\log Q}<\frac{u_r}{32768}+\xi\frac{A_0}{1+A_0}v_r. \tag{10.928}$$

Indeed $B>5$, $\exp(15)>2^{15}$, $\vartheta/N=(1+\tau)A_0/(m+A_0)<\xi A_0/(1+A_0)$, $\log q<l_q$, and $\log Q>L_Q>0$, as proved in Lemma 10.205. Convexity bounds every intermediate integer depth. The remainder $1+1/(q-1)$ gives exactly the term $\lambda q\ell_q/(q-1)$ in b'_r.

The mean torus, coefficient generator, coefficient count and scalar remainders give W_r, with the same terms as in (10.864). The scalar radius obeys $q^{-I}X\le\mathfrak{h}^{-1}S+1<qS$, by Theorem 10.201. Thus its additive term is bounded by $\lambda(1+\ell_q)$, while the initial mean contributes $w\lambda$. The target derivative order is at most $T_{I+1}\le\mathfrak{h}^2\widehat{T}$, because $I\ge1$. Together with the unchanged initial complete-simplex mean, Lemma 10.194 gives k'_r. At depth I its Euler remainder is bounded by $1/(q^I79)+(r-1)/(999\widehat{Z})<e_r$. Finally use $eN>(1+A_0)/\xi$, retaining the full factor q^r on every nonnegative height contribution. No equality between valuations at different primes is assumed.

Exact comparisons covering every rank

For the finite ranks 3 through 7, or 4 through 7 in case V, use the optimizer hypothesis from Theorem 10.184:

$$\frac{r}{t}>A_0+\frac{3}{5},\qquad t>1,\qquad a_*\le A_0<r. \tag{10.929}$$

Each $\mathcal{B}_{r,i}$ is fractional linear in A_0. Its maximum on this interval is bounded by its values at a_* and r. This uses the actual rank restriction even when the upper endpoint r itself is not attainable.

For all ranks at least eight, use the exponential bounds $H_-=(1-c/9)^9$, $H_+=E_-^{-1}$, and E_+ from Theorem 10.201. They give $H_-\le\mathfrak{h}<\exp(-c)<H_+$ and $\eta^r>1/E_+$. Keeping the last four geometric terms and bounding every removed order fraction by one gives

$$\Gamma_r>\frac{2cq}{E_+}\sum_{j=0}^{3}q^{-j}-\frac{18q(q+7)}{q^8S_-(8)}. \tag{10.930}$$

The discarded floor term is at most $2q(r+1)(q+r-1)/(q^rS_-(r))$. It decreases with r: its consecutive ratio is $(q+r)(r+1)/(q(q+r-1)(r+2))<1$.

For both arithmetic endpoints, replace y and all decreasing size remainders by their rank-eight bounds, replace $\mathfrak{h}^{-1}$ by H_-^{-1}, the derivative factor $\mathfrak{h}^2$ by H_+^2, and $1-483y/824$ by one. The function $2(Q-1)/(Q+1)$ increases with Q, so using $Q=qH_-$ increases v_r. All these changes increase the arithmetic bounds. In the input loss, use $1-H_-+c(1+2H_+)/9+1/T_-(8)$, and use q^8 in its denominator and in the normalizing loss. These changes decrease the two analytic lower bounds. The resulting arithmetic expressions are again fractional linear; their extrema on $[a_*,\infty)$ are bounded by the value at a_* and the limit at infinity.

There are 34 finite-rank rows and seven uniform rows, with two depth endpoints and two parameter endpoints each. Exact rational substitution in these 164 comparisons gives the following strict lower bounds:

Case	Mass minus normalization and arithmetic	Input minus arithmetic
I.1	107/100	1/5
I.2	107/100	2/25
II	11/10	21/100
III.1	103/100	13/100
III.2	51/50	7/25
IV	26/25	49/100
V	18/25	39/100

The exact calculation records every rational difference. The reductions above prove why these comparisons cover all permitted depths, ranks and field parameters. In particular both analytic entries strictly exceed the nonzero-value arithmetic bound, by more than 7/10 and 2/25, respectively. Thus V, and hence the original prepared value, is zero. Additive orders above D_{a} vanish by degree. This proves the entire stated fractional block. □

The following integer block

Theorem 10.208 (the next contracted starting profile). Under the hypotheses of Theorem 10.207, a support and phase restriction produces a nonzero family Φ_{I+1} of the same kind at depth $I+1$, with the same minimum β. At $R = q(\lfloor S_{I+1} \rfloor + 1)$, it has q-deleted zeros through order $\lfloor T_{I+1} \rfloor$ and full integer zeros through order $\lfloor \eta T_{I+1} \rfloor$.

Proof. Apply the exact phase/coset passage of Theorems 10.48 and 10.51 to the whole fractional block, including the extra phase partition when q divides G_0. Keep the class containing an original coefficient of minimum valuation. The Kummer monomial basis separates the classes for $q \nmid s$. The affine jet change is triangular with diagonal powers of q, so it retains every provided index. Theorem 10.55 identifies the new integer values in K itself. Theorems 10.29 and 10.176 preserve nonzero support, the polynomial-index fibres, and the combined-frequency congruence. Thus the same argument as Theorem 10.202 gives the stipulated q-deleted zeros, widths $q^{-I-1}\widehat{D}_i$, and additive argument $q^{-I-1}x$.

To fill the multiples of q, put $T' = T_{I+1}$ and $M' = \lfloor cT'/a \rfloor + 1$. Then $\lfloor \eta T' \rfloor + M' - 1 \le \lfloor T' \rfloor$. There are exactly $n_E = 2(q-1)(\lfloor S_{I+1} \rfloor + 1)$ input nodes. Their separation maximum is at most $\mathfrak{h}^{-I-1} H / \log p$, since $R < q^2 S_{I+1} \le q^r S_{I+1}$. The q-deleted form of Theorem 10.182 consequently gives analytic entries

$$n_E M' N - Q_D, \qquad \kappa + \chi \widehat{U}_r - v_p(b_n) - M' \lfloor \log_p(2R) \rfloor - (M'-1) M_*. \tag{10.931}$$

In normalization $c_1/\widehat{\mathcal{Z}}$, their strict lower bounds are

$$2cq(q-1) - \frac{\lambda \nu}{c_4}, \qquad \mathcal{I}_r^{\mathrm{c}} = c_1 \chi q^{r+1} - \frac{\xi}{c_3(1+a_*)} \left(\frac{3c}{a} + \frac{1}{T_-} \right). \tag{10.932}$$

For the first use $n_E > 2(q-1)S_{I+1}$, $M' > cT'/a$, and $S_{I+1} T' = S\widehat{T}$. For the second, the two separation/jet terms have total multiplier $2M' - 1 \le 2cT'/a + 1$. Since $cT'/a \ge 1$, their normalized loss is at most $3c/a$; the original coefficient again contributes the separate $1/T_-$.

The arithmetic upper bound is the same maximum of $\mathcal{B}_{r,0}, \mathcal{B}_{r,1}$, now in this integer normalization. Indeed the actual field is K; the torus radius is $q^{-I-1}R = q^{-I}(\lfloor S_{I+1} \rfloor + 1)$, precisely the preceding fractional radius after scaling. Formula (10.243) has the same total depth $I+1$. The derivative order has decreased, and the coefficient height has not increased. These facts preserve each contribution to the preceding height bound. The convex estimate still uses the previous I, so it remains valid even if $(I+1)\log Q > 3B$.

For the same 164 endpoint comparisons the two new differences exceed 7/10 and 9, respectively. In the uniform rank-eight row, q^{r+1} is replaced by q^9 in the positive term, and the loss uses $a = 9, T_-(8)$. All substitutions have the required direction. Thus a nonzero integer value would contradict the product-formula bound. Every missing multiple of q is filled, and the stronger q-deleted block remains available. □

Corollary 10.209 (iteration up to the geometric or order threshold). Define

$$I_G = \min\{I \geq 1 : Q^I > \exp(3B)\}, \qquad I_M = \max\{I \geq 1 : cT_I/a \geq 1\}. \tag{10.933}$$

These are finite integers. At every $1 \leq I \leq \min(I_G, I_M)$, an actual nonzero selected family exists with the two rounded starting blocks of Theorem 10.208. At every such I with $I < I_G$, it also has all the outer integer blocks required by Theorem 10.207. If $I_G \leq I_M$, the actual family at I_G supplies the zeros in Theorem 10.204 and hence its rank conclusion whenever the remaining independence and hyperplane hypotheses there hold.

Proof. The inequalities $Q > 1$ and $0 < \mathfrak{h} < 1$ give finiteness. The set defining I_M is nonempty: $\mathfrak{h} > 1/3$, checked at the least rank in each q-case using its monotonicity, and $cT_1/a > (27/50)(1/3)(12/29)12^3 > 1$. Theorems 10.202–10.203 construct the rounded depth-one profile. If $I < I_G$, Theorem 10.206 supplies its outer integer blocks, since $I \log Q \leq 3B$. If also $I < I_M$, then $cT_{I+1}/a \geq 1$, so Theorems 10.207–10.208 construct the next profile. Induction proves the whole interval. At I_G, its rounded full interval contains $|s| \leq qS_{I_G}$, with the required order ηT_{I_G}; these are exactly the zeros in Theorem 10.204. □

When $I_M < I_G$, the constructed family has reached the order threshold first. The corollary does not extend the mixed step past its multiplicity hypothesis. That endpoint requires a separate argument with a fixed derivative order.

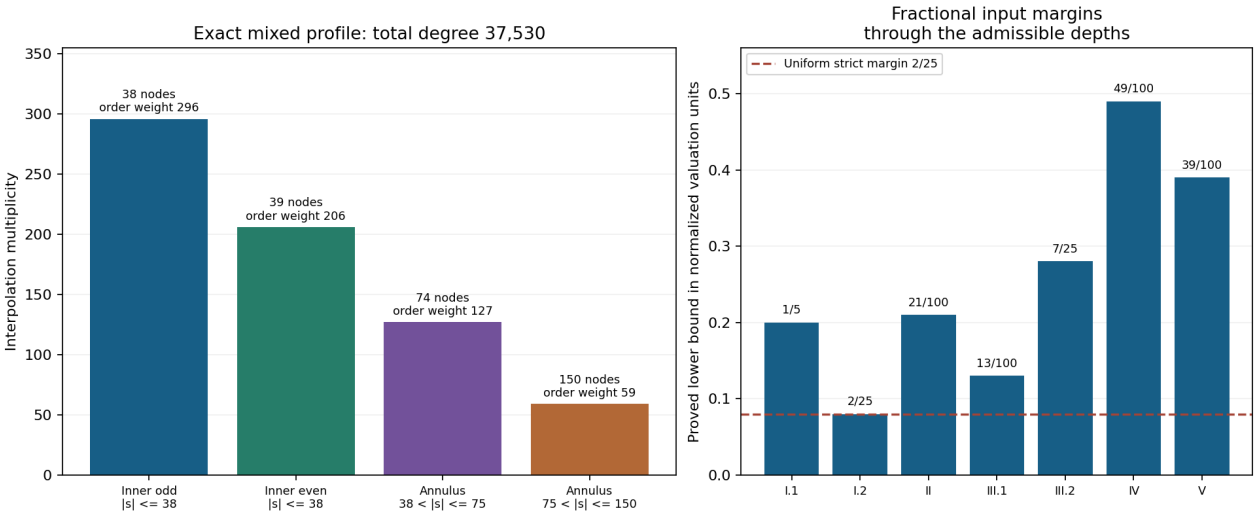


Figure 10.63.

Left: the exact node types and multiplicities for $q = 2, r = 3, I = 1, S = 21/2, \widehat{T} = 1200, c = 27/50$, with output order $\lfloor T_2 \rfloor$. This is a counting illustration, not a complete number-field parameter instance. Right: strict rational lower bounds for the input-minus-arithmetic margins in Theorem 10.207; each includes both depth endpoints and every rank in its field case. Theorem 10.208 supplies

the next starting block, and Corollary 10.209 states the resulting finite induction. Reproducible exact calculation and figure. Human-source context: Section 11 of Yu's freely accessible 1999 paper and the second constants in Section 1.1 of his freely accessible 2013 paper.

71. The last integer extension and the fixed-order endpoint

The derivative-order threshold leaves one more full integer extension available. This extension identifies the actual starting interval for a continuation at fixed order. We then prove the contradiction that such a continuation must reach; the intermediate fractional steps are a separate requirement.

Theorem 10.210 (the last integer extension). Under the hypotheses of Theorem 10.206, the same selected family satisfies

$$\Phi_I(s;\boldsymbol{t}) = 0 \quad (s \in \mathbb{Z},\ |s| \le q^{r+1}S_I, |\boldsymbol{t}| \le \mathfrak{h}T_I). \tag{10.936}$$

All cutoffs mean their integer floors. This assertion does not require $cT_{I+1}/(r+1) \ge 1$.

Proof. Write $a = r+1$. Theorem 10.206 supplies the full integer interval of radius $s_r = \lfloor q^r S_I \rfloor$ through order $\lfloor \eta^r T_I \rfloor$. Set

$$\mu = \left\lfloor \frac{c\eta^r T_I}{a} \right\rfloor + 1, \qquad \lfloor \mathfrak{h}T_I \rfloor + \mu - 1 \le \lfloor \eta^r T_I \rfloor. \tag{10.937}$$

The second inequality follows from $\mathfrak{h}T_I + c\eta^r T_I/a = \eta^r T_I$. It holds when $\mu = 1$ as well. Thus every ordinary divided jet required by Theorem 10.182 is available for each asserted target order.

Retain the original minimum coefficient valuation β, the normalizing cost $Q_D = D_{\mathrm{a}}(N + 1/(p-1))$, and the original integer b_n. The input nodes form a complete integer interval. The interpolation bound is therefore

$$v_p(\Phi_I(s;\boldsymbol{t})) - \beta \ge \min\left\{(2s_r+1)\mu N - Q_D, \quad \kappa + \chi\widehat{U}_r - v_p(b_n) - (\mu-1)M_*\right\}. \tag{10.938}$$

There is no deleted-node loss. The separation and jet estimate $M_* \le \mathfrak{h}^{-I}H/\log p$ remains valid at $j = r$: equation (10.839) bounds $\log(2q^r S)$, and the proof of (10.905) then applies unchanged. The original coefficient separately satisfies $v_p(b_n) \le H/\log p$. The strict disc, perturbation, phase and congruence conditions are those verified in Theorem 10.206. The integer target lies in K, and the family has not changed.

Use $y = 1/a$, $\lambda = 59/58$, $\xi = 101/100$, and the case data a_*, d_*, w, ν, χ of Theorem 10.197 and Lemmas 10.199–10.200. Define

$$S_- = c_3 q d_* a^2, \qquad T_- = (12/29)a12^r. \tag{10.939}$$

In units $c_1/(q^r\widehat{\mathcal{Z}})$, the two interpolation entries are strictly greater than

$$\begin{aligned} L_r &= 2cq\eta^r\left(1 - \frac{1}{2q^r S_-}\right) - \frac{\lambda\nu}{c_4 q^r}, \\ J_r &= c_1\chi q - \frac{\xi(1/T_- + c\eta^r/a)}{q^r c_3(1+a_*)}. \end{aligned} \tag{10.940}$$

For the first entry use $\mu > c\eta^r T_I/a$, $2s_r + 1 > 2q^r S_I - 1$, and $c_1 S_I T_I N/\widehat{\mathcal{Z}} = qa$. For the second use $\mu - 1 \le c\eta^r T_I/a$, the separate original coefficient bound, and (10.863). The strict lower bound $\widehat{T} > T_-$ preserves the strict inequality even at a floor endpoint.

Here is the arithmetic estimate with all terms retained. Put $\ell_q = 7/50$ for $q = 2$, and $\ell_q = 11/50$ for $q = 3$. Let $E_-(c) < \exp c < E_+(c)$ be the rational bounds from Theorem 10.201 and set $H_+ = 1/E_-(c)$. Define

$$\begin{aligned}
\epsilon &= \frac{13}{18000}, \qquad e_0 = \frac{1}{79} + \frac{1}{71928 \cdot 1500}, \\
W_r &= \frac{w}{2c_2}\left(1 + \frac{1}{2S_-}\right) + \frac{y^2}{c_2 c_3 q} + (1 + w/2)c_1\epsilon + (1 + w)c_1 e_0, \\
b_r &= \lambda\left(1 + (r+1)\ell_q + w + \frac{\ell_q}{q-1}\right), \\
k_r &= \left(\frac{107}{103} + \frac{3y}{8}\right)\eta^{r+1}H_+ + w\left(1 - \frac{483y}{824}\right), \\
\mathcal{A}_r(z) &= \frac{q}{c_2} + \frac{W_r}{q^r} + \frac{\xi}{q^r}\frac{b_r z/c_4 + k_r/c_3}{1+z}.
\end{aligned} \tag{10.941}$$

For a nonzero target, $c_1(v_p(\Phi_I) - \beta)/(q^r\widehat{\mathcal{Z}}) \le \mathcal{A}_r(A_0)$. To verify this endpoint of the arithmetic argument in Theorem 10.206, first apply Lemma 10.205 at $j = r$. Its statement covers every $j \ge 1$, and absorbs the complete depth-dependent denominator and torus cost. The remaining constant denominator gives $\ell_q/(q-1)$. The physical radius is at most $q^{r+1}S$, giving the additive factorial term $\lambda(1+(r+1)\ell_q)$; the initial mean contributes $w\lambda$. These terms are multiplied by $\vartheta/N < \xi A_0/(1+A_0)$.

The derivative order is at most $\eta^{r+1}\mathfrak{h}^I\widehat{T}$. Since $I \ge 1$ and $\mathfrak{h} < \exp(-c) < H_+$, its pointwise contribution is bounded by the first part of k_r. The mean derivative contribution is the second part. The remaining coefficient-generator, mean torus, column-count and scalar-remainder terms are exactly W_r, with the uniform remainder bounds used in Theorem 10.206. Thus the arithmetic argument applies at $j = r$; its numerical comparison is supplied below.

For ranks 3 through 7, or 4 through 7 in case V, evaluate $\mathcal{A}_r(z)$ at $z = a_*$ and at infinity. A function $(bz + k)/(1 + z)$ has derivative $(b - k)/(1 + z)^2$, so its maximum on $[a_*, \infty)$ occurs at one of these endpoints.

For every $r \ge 8$ use $y_8 = 1/9$, $S_8 = c_3 q d_* 9^2$, and $T_8 = (12/29)9 \cdot 12^8$. The integral inequality $\log(1-u) > -u/(1-u)$ gives $\eta^r > \exp(-c) > 1/E_+(c)$, because $r < a - c$. Consequently uniform lower bounds for the two entries are

$$\begin{aligned}
L_{\ge 8} &= \frac{2cq}{E_+(c)}\left(1 - \frac{1}{2q^8 S_8}\right) - \frac{\lambda\nu}{c_4 q^8}, \\
J_{\ge 8} &= c_1\chi q - \frac{\xi(1/T_8 + c/9)}{q^8 c_3(1 + a_*)}.
\end{aligned} \tag{10.942}$$

For an upper bound on $\mathcal{A}_r$, use y_8, S_8 in W_r, bound $\eta^{r+1}H_+$ by H_+^2, and bound $1 - 483y/824$ by 1. Each replacement increases its nonnegative term. Finally use $r = 8$ in q^{-r} and $(r+1)q^{-r}$; both decrease for $r \ge 8$. These operations give a uniform upper bound for all remaining ranks.

The resulting exact rational differences exceed the following values.

Case	Mass minus arithmetic	Input minus arithmetic
I.1	91/1000	33/100
I.2	91/1000	11/50
II	91/1000	27/100
III.1	66/1000	29/100
III.2	66/1000	11/25
IV	49/1000	29/50

Case	Mass minus arithmetic	Input minus arithmetic
V	1/20	3/4

There are 34 finite-rank rows and seven uniform tail rows, each with two parameter endpoints. The displayed formulas therefore reduce the proof to 82 comparisons of each interpolation entry with its arithmetic bound. The exact rational calculations, including the tail substitutions, are given in the reproducible source `yu_second_terminal_parameters.py`. They yield the strict margins in the table. Hence each entry exceeds the arithmetic upper bound, contradicting a nonzero target. Additive orders above D_{a} vanish by degree. □

Lemma 10.211 (the actual fixed-order starting interval). Suppose $I_M < I_G$ in Corollary 10.209. Put $I_1 = I_M$, $S_1 = \mathfrak{h}^{-I_1} S$, and $T_1 = \mathfrak{h}^{I_1} \widehat{T}$. There is a unique integer $0 \le j_0 \le r$ for which

$$1 \le \frac{c\eta^{j_0} T_1}{a} < \frac{1}{\eta}. \tag{10.943}$$

Define

$$\begin{aligned} &T_{\mathrm{f}} = \eta^{j_0+1} T_1, \qquad O = \lfloor T_{\mathrm{f}} \rfloor, \qquad R = q \lfloor q^{j_0} S_1 \rfloor, \\ &I_2 = \left\lfloor \frac{3B - I_1 \log Q}{\log q} \right\rfloor + 1, \qquad I_3 = I_1 + I_2. \end{aligned} \tag{10.944}$$

Then $\eta \le cT_{\mathrm{f}}/a < 1$, $I_2 \ge 1$, and

$$q^{I_3} \mathfrak{h}^{I_1} > \exp(3B). \tag{10.945}$$

The actual family at I_1 from Corollary 10.209 has all integer zeros $|s| \le R$ through order O. Its original minimum coefficient, phase, integral coefficient subvector and complete polynomial fibres are retained.

Proof. The definition of I_M gives $cT_1/a \ge 1$ and $c\mathfrak{h}T_1/a < 1$. The sequence $c\eta^j T_1/a$ decreases strictly. Its last index with value at least 1 is therefore a unique $j_0 \in \{0, \dots, r\}$. Multiplication by η gives the asserted interval for cT_{f}/a.

Since $I_1 < I_G$, we have $I_1 \log Q \le 3B$, hence $I_2 \ge 1$. The floor plus one is strictly larger than its argument, including when that argument is an integer. Thus $Q^{I_1} q^{I_2} > \exp(3B)$, as asserted.

If $j_0 = 0$, the rounded starting interval from Corollary 10.209 contains $|s| \le q\lfloor S_1 \rfloor$ through order $\lfloor \eta T_1 \rfloor$. If $1 \le j_0 < r$, Theorem 10.206 gives radius $\lfloor q^{j_0+1} S_1 \rfloor$ and order $\lfloor \eta^{j_0+1} T_1 \rfloor$. If $j_0 = r$, Theorem 10.210 gives the final integer level with the same order. In every case $q\lfloor q^{j_0} S_1 \rfloor \le \lfloor q^{j_0+1} S_1 \rfloor$, so the required interval is present. None of these extensions changes the family. □

Proposition 10.212 (the final additive-polynomial contradiction). Suppose a stage-I_3 family retains a nonzero integral coefficient subvector of the hatted kernel of Theorem 10.203, its complete polynomial fibres, and the integral normalized support of Proposition 10.31, with widths at most $q^{-I_3} \widehat{D}_i$. Suppose its prepared values vanish at every integer $|s| \le R$ through order O, where R, O, I_3 are defined in Lemma 10.211. Then such a family cannot exist. Each torus degree is strictly less than $609/163840$, whereas the number of integer roots exceeds $(1305/236) D_{\mathrm{a}}$.

Proof. The coordinate-degree bound from Proposition 10.31 and Theorem 10.191 is $2Jq^{-I_3} \widehat{D}_i$. The strict stopping inequality of Lemma 10.211 and $T_1 < a/(c\mathfrak{h})$ give

$$\begin{aligned} 2Jq^{-I_3}\widehat{D}_i &< 2JT_1\frac{\widehat{D}_i}{\widehat{T}}\exp(-3B) \\ &< \frac{2eNt}{cc_2qrP\mathfrak{h}d\sigma_i}\exp(-3A). \end{aligned} \tag{10.946}$$

Equation (10.702) and Theorem 10.203 give the ratio $\widehat{D}_i/\widehat{T} = eNt/(c_2qraPd\sigma_i)$. The remaining exponential factor satisfies $J\exp(-3B) = J^{-2}\exp(-3A) \le \exp(-3A)$. The optimizer gives $eNt < \zeta_1 r < 7r/5$. Also $c > 1/2$, $c_2q > 3$, $P \ge 1$, and $\mathfrak{h} > 1/3$. The one-base height bound in Theorem 10.18 gives $1/(d\sigma_i) < 87d^2$. Therefore the degree is strictly less than $(2436/5)d^2\exp(-3A)$.

Since $A \ge 4 + \log(ad) > 5$, $a \ge 4$, and $\exp(1) > 2$,

$$d^2\exp(-2A) \le \frac{\exp(-8)}{a^2}, \qquad \frac{2436}{5}d^2\exp(-3A) < \frac{2436}{5\cdot 16\cdot 2^{13}} = \frac{609}{163840} < 1. \tag{10.947}$$

The nonnegative integer torus degree is thus zero. The remaining ordinary polynomial depends only on the additive coordinate and has degree at most D_{a}. It is nonzero: its retained coefficient vector is nonzero, and the complete polynomial fibres are expressed in the independent polynomial basis proved in Lemma 10.29 and Proposition 10.31.

Finally $S_1 \ge S > 16$ and $j_0 \ge 0$ give $2R + 1 > 2q(S_1 - 1) > (15/8)qS_1$. The hatted degree bound from Corollary 10.185 is

$$D_{\mathrm{a}} < \frac{\lambda S\widehat{D}}{c_1c_4(N/\vartheta)dB}. \tag{10.948}$$

Using $S_1/S = \widehat{T}/T_1$ and $\widehat{T} = qa\widehat{D}/(c_1eNt)$, we obtain

$$\begin{aligned} \frac{2R+1}{D_{\mathrm{a}}} &> \frac{15q^2ac_4dB}{8\lambda e\vartheta tT_1} \\ &> \frac{15q^2cc_4\mathfrak{h}}{8\lambda\vartheta}\frac{d}{e}\frac{B}{t} > \frac{45}{8\lambda} = \frac{1305}{236} > 1. \end{aligned} \tag{10.949}$$

For the last step use $q\mathfrak{h} > 1$, $q \ge 2$, $c > 1/2$, $c_4 \ge 6$, $d/e \ge 1$, $B/t \ge 1$, and $\vartheta \le 2$. Order zero is available since $O \ge 0$. By Proposition 10.31 these zeros yield $2R + 1$ distinct additive roots. For each distinct root, division by its linear factor gives zero remainder. Induction on the number of distinct roots therefore makes their product divide the polynomial. Its degree cannot exceed that of the nonzero polynomial, contradicting the displayed strict inequality. □

Proposition 10.212 gives the contradiction once the final family and its zeros have been constructed. Lemma 10.211 provides the starting family at I_1; Theorems 10.216–10.217 propagate its fractional zeros and restore the full integer intervals through the same order O, retaining the complete root-field and rational-denominator costs.

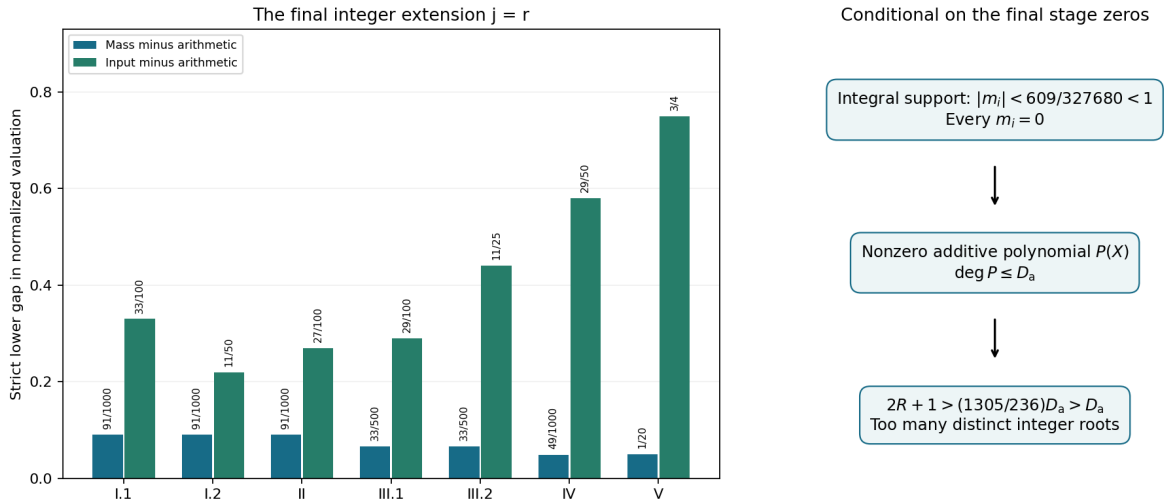


Figure 10.64. Left: the strict rational lower margins proved in Theorem 10.210, in units $c_1/(q^r\widehat{\mathcal{Z}})$. The proof includes every rank through its finite rows and uniform tail. Right: Proposition 10.212, conditional on obtaining the final-stage zeros. Integral support collapses to the additive coordinate; the nonzero polynomial then has more roots than its degree. Its normalized support bound is half the coordinate-degree bound. The reproducible figure source is ***yu_second_terminal_parameters.py****. The stopping alternatives in the freely accessible Yu (2013), Section 7, provide context; every mathematical input and bound used here is proved in this programme.*

72. Keeping the derivative order through the final contractions

The interval constructed in Lemma 10.211 has reached the smallest derivative scale used by the geometric induction. We now keep its order fixed. Before each fractional step, two integer extensions supply enough interpolation mass to pay for the entire root-field degree. After restriction to a phase class, interpolation on the q-deleted interval restores the missing integers at that same order.

Use the hatted kernel of Theorem 10.203 and the counterhypothesis $v_p(\Xi - 1) \ge \chi\widehat{U}_r$. Assume $I_M < I_G$, and retain $I_1, S_1, T_1, T_{\mathrm{f}}, O, R, I_3$ from Lemma 10.211. Put $a = r + 1$, $c = c_5$, $\eta = 1 - c/a$, $\mathfrak{h} = \eta^a$, and $Q = q\mathfrak{h} > 1$. The constants a_*, d_*, w, ν, χ have their case values in Theorem 10.197 and Lemma 10.199. Here $c_1 Q_D/\widehat{\mathcal{Z}} < \lambda\nu/c_4$, where $Q_D = D_{\mathrm{a}}(N + 1/(p-1))$ and $\lambda = 59/58$. Also set $\xi = 101/100$ and $\zeta = 99/100$. The symbol ζ in this section is distinct from the optimizer constant ζ_2.

The lower bound for the actual stopping depth

Lemma 10.213 (size and depth bounds). Define

$$\begin{aligned} P_0 &= \max\{1, a_*\}, & F_r &= \frac{r^r (r+1)^r}{(r!)^2}, \\ L_r &= \frac{2c_0c_4qa}{58}(c_2q)^r F_r P_0^{r-1} \max\left\{1, \frac{5(a_* + 3/5)}{r}\right\}, & & \\ m_r &= \max\{m \in \mathbb{Z}_{\ge 0} : cL_r\mathfrak{h}^m/a \ge 1\}, & & \\ \mu_r &= q^r Q^{-m_r}, & j_r &= \min\{j \in \mathbb{Z}_{\ge 0} : \mu_r \le q^j\}, \\ S_- &= c_3 q d_* a^2, & \sigma_r &= \frac{qa}{S_- L_r}. \end{aligned} \tag{10.952}$$

Then $\widehat{T} > L_r$, $I_1 \ge m_r \ge 1$, and $c_1 N/\widehat{\mathcal{Z}} < \sigma_r$. For $\alpha = -\log\mathfrak{h}$ and every integer $I_1 \le I < I_3$, the pair (I_1, I) belongs to the triangle

$$m_r \le x \le \frac{3B}{\log Q}, \qquad x \le z \le \frac{3B + \alpha x}{\log q}. \tag{10.953}$$

Proof. Substitute (10.707), with the hatted scales, into $\widehat{T} = qa\widehat{\mathcal{D}}/(c_1eNt)$. Since $A_0 = e\vartheta$, this gives

$$\widehat{T} > \frac{2c_0c_4qa}{\rho A_0}(c_2qP)^r F_r \frac{B}{t}. \tag{10.954}$$

Use $\rho \le 58$, $P \ge A_0$, $P \ge P_0$, and $B/t \ge 1$. The optimizer inequality $r/t > A_0 + 3/5$ and $B > 5$ also give $B/t > 5(A_0 + 3/5)/r \ge 5(a_* + 3/5)/r$. This proves $\widehat{T} > L_r$. For existence without a logarithmic rounding assumption, (10.708), $c_0 \ge 2, c_4 \ge 6, q \ge 2$ and $c_2q > 3$ give $L_r/a > (12/29)12^r$. Since $c > 1/2$, $\mathfrak{h} > 1/3$, and $r \ge 3$, this makes $cL_r\mathfrak{h}/a > 1$. Since $0 < \mathfrak{h} < 1$, the maximum defining m_r exists. The integer $I_1 = I_M$ is the last depth with $c\widehat{T}\mathfrak{h}^{I_1}/a \ge 1$, so $I_1 \ge m_r$. The identity $c_1S\widehat{T}N/\widehat{\mathcal{Z}} = qa$ and $S \ge S_-$ prove the bound for σ_r.

Finally $I_1 \log Q \le 3B$. The exact floor in (10.944) gives $I \le I_3 - 1 \le I_1 + (3B - I_1 \log Q)/\log q = (3B + \alpha I_1)/\log q$. These are exactly the triangle inequalities. Its vertices are (m_r, m_r), $(m_r, (3B + \alpha m_r)/\log q)$, and $(3B/\log Q, 3B/\log Q)$. □

A joint bound for the radius and the rational denominator

Write $\ell_2 = 7/50$, $\ell_3 = 11/50$, and $l_q = 5\ell_q$. Define the positive rational logarithm bound

$$v_Q = \frac{Q-1}{Q+1}, \qquad L_Q = 2\sum_{k=0}^{4} \frac{v_Q^{2k+1}}{2k+1} < \log Q. \tag{10.955}$$

The strict inequality is the positive logarithm series proved in Solution 34. In particular no numerical logarithm determines m_r or j_r: their definitions use rational powers and integer comparisons.

For $z \ge a_*$, put

$$\begin{aligned}
b_r &= 1500\,12^{r-3}, \qquad \varepsilon_r = \frac{8r+15}{9ab_r}, \qquad e_r = \frac{1}{79} + \frac{1}{71928b_r},\\
W_r &= \frac{w}{2c_2}\left(1 + \frac{1}{2S_-}\right) + \frac{1}{a^2c_2c_3q}\\
&\quad + (1 + w/2)c_1\varepsilon_r + (1+w)c_1e_r,\\
K_j(z) &= W_r + \frac{\xi}{1+z}\left\{\frac{\lambda[1 + w + q\ell_q/(q-1) + j\ell_q]z}{c_4}\right.\\
&\quad \left. + \frac{(107/103 + 3/(8a))a/(cL_r) + w(1 - 483/(824a))}{c_3}\right\},\\
V_\theta(z) &= \max\left\{\frac{\theta\mu_r}{c_2} + \frac{\xi z}{1+z}\frac{\lambda l_q}{c_4}\min\left(\frac{m_r}{5}, \frac{3}{L_Q}\right),\right.\\
&\quad \left.\frac{\theta}{c_2}\min\left(\frac{q^r}{32768}, \mu_r\right) + \frac{\xi z}{1+z}\frac{3\lambda l_q}{c_4L_Q}\right\},\\
\mathcal{H}_{\mathrm{s}}(z) &= K_{j_r}(z) + V_1(z),\\
\mathcal{H}_{\mathrm{l}}(z) &= K_{j_r+1}(z) + V_q(z).
\end{aligned} \tag{10.956}$$

Lemma 10.214 (complete fixed-order arithmetic bounds). Let $\theta \in \{1, q\}$. At a depth $I_1 \le I < I_3$, consider a nonzero prepared value of an admissible selected subfamily, at order at most O. If its physical radius X obeys

$$\frac{X}{S} \le \theta q^r \exp(\alpha I_1 - I\log q), \tag{10.957}$$

then the following bounds hold, provided $X \le q^{j}S$. For an integer value in K at depth I, its normalized valuation $c_1(v_p(V)-\beta)/\widehat{\mathcal{Z}}$ is at most $K_j(A_0)+V_\theta(A_0)$. The same bound holds for an integer value at depth $I+1$ with the same physical radius. For a fractional value at depth I, divide the normalized valuation by q^{r}; its upper bound is again $K_j(A_0)+V_\theta(A_0)$. The fractional assertion retains the full global field degree and the chosen local degrees e,f.

Proof. The global height argument is Theorem 10.92 with the coefficient bound of Lemma 10.199 and the hatted estimates of Theorem 10.203. Its pointwise torus contribution is at most $X/(c_2S)$. The part of its denominator proportional to I is at most $\lambda(\vartheta/N)I\log q/(c_4B)$. To bound their sum, on the triangle in Lemma 10.213 consider

$$F(x,z)=\frac{\theta q^{r}}{c_2}e^{\alpha x-z\log q}+\frac{\lambda\vartheta}{c_4NB}z\log q. \tag{10.958}$$

The Hessian of the exponential term is its positive value times the outer product of $(\alpha,-\log q)$ with itself; the other term is linear. Thus F is convex. Expressing a point in the triangle as a convex combination of its vertices bounds F by its largest vertex value. At both vertices on the upper edge, the exponential is e^{-3B}, and the vertex with $x=z=3B/\log Q$ has the larger linear term. Only that vertex and (m_r,m_r) need to be retained.

At the first vertex the torus term is $\theta\mu_r/c_2$. Use $B>5$, $m_r\log Q\le 3B$, $\log q<l_q$, and $\log Q>L_Q$ to bound its denominator part by the first linear term in V_θ. At the other vertex, $e^{-3B}<2^{-15}$ and $e^{-3B}\le Q^{-m_r}$. Its torus term is therefore bounded by the minimum in the second entry. The denominator term is at most $3\lambda(\vartheta/N)\log q/(c_4\log Q)$. In both entries use $\vartheta/N<\xi A_0/(1+A_0)$. This proves exactly the bound $V_\theta(A_0)$.

Here are all the remaining contributions. The initial mean torus height, coefficient-field generator, logarithmic column counts and two scalar remainders give the four terms of W_r, exactly as in (10.864), with the sharper e_r permitted there. The factorial estimate in (10.824) and $X\le q^{j}S$ give

$$\log\frac{\mathrm{e}(X+2k)}{k}\le j\log q+\log\frac{\mathrm{e}(S+2k)}{k}<B+j\log q. \tag{10.959}$$

The pointwise additive contribution is consequently bounded by $\lambda(1+j\ell_q)$, and its initial mean has multiplier $w\lambda$. The fractional denominator in (10.243) is $D_{\mathrm{a}}(I+1+1/(q-1))$. The constant part contributes $\lambda q\ell_q/(q-1)$. An integer at depth I has a smaller denominator; an integer at depth $I+1$ has the same total denominator as the fractional target just considered. Thus the same bound covers all three cases without discarding a power of q.

For the derivative term, $O\le T_{\mathrm{f}}<a/c$ and $\widehat{T}>L_r$, so $O/\widehat{T}<a/(cL_r)$. Lemma 10.194 supplies the multiplier $107/103+3/(8a)$. The unchanged initial mean is bounded by $w(1-483/(824a))$. Finally $1/(eN)<\xi/(1+A_0)$. These terms are exactly the second fraction in K_j; the additive terms use $\vartheta/N<\xi A_0/(1+A_0)$.

An integer value is in K by Theorem 10.55. A fractional normalized value is in the field of Theorem 10.56, of degree at most $q^{r}d$ and with the specified local degrees e,f. Its removed common principal-unit monomial has valuation zero. Multiply the entire nonnegative height majorant by q^{r} in this case. The coefficient minimum is still β, and restriction has not increased the coefficient height. This proves all the stated bounds without an equality assumption for valuations at different primes. □

Rational comparisons in every rank

Put $C=\lambda\nu/c_4$ and

$$D_r=\frac{103\xi c}{100c_3(1+a_*)a}. \tag{10.960}$$

Lemma 10.215 (the fixed-order comparison margins). In every admissible case and rank, $j_r \le 1$, and each of the following quantities is strictly positive for $a_* \le z \le r$:

$$\begin{aligned}
P_1(z) &= 2cq^2\eta - 2q\sigma_r - C - \mathcal{H}_{\mathrm{s}}(z),\\
P_2(z) &= \zeta\chi c_1 q^r - 2\sigma_r - C - \mathcal{H}_{\mathrm{l}}(z),\\
P_3(z) &= \zeta\chi c_1 q - \frac{C + 2\sigma_r}{q^r} - \mathcal{H}_{\mathrm{s}}(z),\\
P_4(z) &= 2cq(q-1)\eta - 2(q-1)\sigma_r - C - D_r - \mathcal{H}_{\mathrm{s}}(z),\\
P_5(z) &= c_1\chi q^{r+1} - \frac{\xi}{c_3(1+a_*)L_r} - D_r - \mathcal{H}_{\mathrm{s}}(z),\\
P_6 &= (1-\zeta)c_1\chi q^{r+1} - \frac{\xi}{c_3(1+a_*)L_r},\\
P_7 &= 1 - \frac{\zeta\chi c_1}{2cq\mathfrak{h}}.
\end{aligned} \tag{10.961}$$

The strict lower bounds for P_1, P_2, P_3, P_4, in units of 10^{-6}, are as follows.

Case	P_1	P_2	P_3	P_4
I.1	2422817	4303236	315715	507605
I.2	2422663	3859563	204821	507452
II	2506006	4370042	312956	604168
III.1	2408114	3728458	138904	489560
III.2	2407902	4338090	291349	489352
IV	2369299	4839514	416027	469693
V	9589027	72053237	338353	5464556

Moreover $P_5 > 10$, $P_6 > 11/100$, and $P_7 > 1/4$.

Proof. We give the finite reduction and its unbounded-rank argument. For each fixed rank the arithmetic bounds are maxima of two fractional-linear functions of z. Under $u = 1/(1+z)$, each of these is affine. Its maximum on an interval is attained at an endpoint. It is therefore sufficient to substitute $z = a_*$ and $z = r$, since the optimizer gives $a_* \le A_0 < r$.

For the tail, the factorial computation in Lemma 10.148 proves $F_{r+1}/F_r > 13/2$ for $r \ge 8$. The ratio of $\max(1, K/(r+2))$ to $\max(1, K/r)$ is at least $r/(r+2)$: check separately whether K lies below r, between r and $r+2$, or above $r+2$. Consequently

$$\frac{L_{r+2}/(r+3)}{L_r/(r+1)} > (c_2 q P_0)^2 (13/2)^2 \frac{r}{r+2} =: G_r \qquad (r \ge 32). \tag{10.962}$$

Both $\mathfrak{h}$ and Q increase with rank by the logarithmic derivative argument in Theorem 10.201. For each case, direct rational substitution at ranks 32 and 33 gives

$$G_r \mathfrak{h}_r^{d_0} > 1, \qquad Q_r^{d_0} > q^2, \qquad d_0 = \begin{cases} 10, & q = 2,\\ 9, & q = 3. \end{cases} \tag{10.963}$$

These inequalities continue to hold at larger ranks, since their left sides increase. Multiplication of $cL_r\mathfrak{h}_r^{m_r}/(r+1) \ge 1$ by the first growth factor, and use of $\mathfrak{h}_{r+2} \ge \mathfrak{h}_r$, give $m_{r+2} \ge m_r + d_0$. Thus $\mu_{r+2} \le \mu_r q^2/Q_r^{d_0} < \mu_r$, and j_r is nonincreasing on each parity sequence. In particular it is not permissible to hold m_r fixed while bounding q^r.

Here is a uniform envelope for each parity tail starting at $b = 32$ or 33. In K_j, replace $1 - 483/(824a)$ by one, and use the base-rank values for W_r, $(107/103 + 3/(8a))a/L_r$, and j_r. In V_θ, use the larger common bound

$$\frac{\theta\mu_b}{c_2} + \frac{\xi z}{1+z}\frac{3\lambda l_q}{c_4 L_{Q_b}}. \tag{10.964}$$

Every replacement increases the arithmetic bound. Indeed S_- increases, a/L_r decreases by the growth inequality, ε_r, e_r decrease from their displayed formulas, μ_r, j_r decrease along the parity sequence, and L_Q increases with Q. The initial mean factor was replaced by one because it increases, rather than decreases, with rank. For the positive mass terms use the base values of η and q^r; for $\sigma_r, D_r, 1/L_r$ and negative q^{-r} terms use their base upper bounds. These substitutions decrease each comparison. For P_7, the base value of $\mathfrak{h}$ also decreases the comparison. The resulting tail bounds are again maxima of affine functions of $1/(1+z)$, so use $z = a_*$ and the limit $z \to \infty$.

The exact calculation substitutes ranks 3 through 33 in the six odd-prime cases and ranks 4 through 33 in V, and then the two parity tail envelopes in every case. It checks seven inequalities at each of the two parameter endpoints: 3,220 rational inequalities in total. The 14 parity steps also check both displayed growth inequalities. The integers m_r, j_r are obtained by multiplication until their defining rational inequalities change sign. These same finite and tail checks give $m_r \geq 1$, $j_r \leq 1$, and the listed strict margins. Together with the tail argument this proves the claims for every admissible rank. □

Two integer extensions and the fractional zeros

Theorem 10.216 (the second-scale fixed-order fractional block). Suppose an admissible selected family at depth $I_1 \leq I < I_3$ has every prepared zero on the integer interval $|s| \leq R$ through order O. It then has every prepared zero at s/q, for integers $|s| \leq R$, through the same order O. Its coefficients, minimum valuation, phase and support are unchanged.

Proof. All restrictions retain the combined-frequency congruence by Theorem 10.176, at every finite depth. The strict disc and perturbation conditions follow from Theorems 10.180, 10.190 and 10.203. The original distinguished coefficient b_n is used in the perturbation. Since q is prime to p, scaling the additive argument by q^{-I} preserves its local integrality and degree. For a full interval with n known integer nodes, Theorem 10.182 with $\mu = 1$ gives, at every $x \in \mathbb{Z}_p$,

$$v_p(\Phi_I(x;\boldsymbol{t})) - \beta \geq \min\{nN - Q_D,\ \kappa + \chi\widehat{U}_r - v_p(b_n)\} \qquad (|\boldsymbol{t}| \leq O). \tag{10.965}$$

There is no ordinary-jet loss because $\mu - 1 = 0$. Equation (10.863), with hats, separately bounds the original loss:

$$\frac{c_1 v_p(b_n)}{\widehat{\mathcal{Z}}} < \frac{\xi}{c_3(1+a_*)\widehat{T}} < \frac{\xi}{c_3(1+a_*)L_r}. \tag{10.966}$$

In particular $P_6 > 0$ gives $v_p(b_n) < (1-\zeta)\chi\widehat{U}_r$. Define two integer radii

$$R_{\mathrm{s}} = \left\lfloor \frac{\zeta\chi\widehat{U}_r/(qN) - 1}{2} \right\rfloor, \qquad R_{\mathrm{l}} = \left\lfloor \frac{\zeta\chi\widehat{U}_r/N - 1}{2} \right\rfloor. \tag{10.967}$$

They are positive. For example $\widehat{U}_r/(qN) = c_1 q^{r-1} S\widehat{T}/a$, while $c_1 \geq 7/10$, $q^{r-1} \geq 4$, $S > 16$, and $\widehat{T}/a > (12/29)12^r > 700$. Even the smaller radius therefore exceeds one. With $n_{\mathrm{s}} = 2R_{\mathrm{s}} + 1$ and $n_{\mathrm{l}} = 2R_{\mathrm{l}} + 1$, the exact floor inequalities are

$$\begin{aligned}\zeta\chi\widehat{U}_r/q - 2N < n_{\mathrm{s}}N \le \zeta\chi\widehat{U}_r/q,\\ \zeta\chi\widehat{U}_r - 2N < n_{\mathrm{l}}N \le \zeta\chi\widehat{U}_r,\\ qR_{\mathrm{s}} \le R_{\mathrm{l}}.\end{aligned} \tag{10.968}$$

The last follows because $qR_{\mathrm{s}} \le (\zeta\chi\widehat{U}_r/N - q)/2 \le (\zeta\chi\widehat{U}_r/N - 1)/2$ and its left side is integral. No comparison between R and either boosted radius is required.

First extend to R_{s}. The starting integer interval has $2R + 1 > 2q^{j_0+1}S_1 - 2q$ nodes. Since $a/T_1 > c\eta^{j_0+1}$, $q\eta > 1$, and $S_1T_1 = S\widehat{T}$, its first analytic entry, multiplied by $c_1/\widehat{\mathcal{Z}}$, is greater than

$$2cq^2\eta(q\eta)^{j_0} - 2q\sigma_r - C \ge 2cq^2\eta - 2q\sigma_r - C. \tag{10.969}$$

The second analytic entry is greater than $c_1\chi q^{r+1} - \xi/[c_3(1 + a_*)L_r]$. We retain this second entry separately; we do not assume that the original interval's node mass is smaller than $\chi\widehat{U}_r$.

The physical target radius satisfies

$$\begin{aligned}q^{-I}R_{\mathrm{s}} &< \frac{\zeta\chi c_1}{2cq\mathfrak{h}}q^r e^{\alpha I_1 - I\log q}S\\ &< q^r e^{\alpha I_1 - I\log q}S \le \mu_r S \le q^{j_r}S.\end{aligned} \tag{10.970}$$

Here $T_1 < a/(c\mathfrak{h})$, $S\widehat{T} = S_1T_1$, $I \ge I_1 \ge m_r$, and $P_7 > 0$ were used. The arithmetic bound is $\mathcal{H}_{\mathrm{s}}(A_0)$, by Lemma 10.214. The first analytic entry exceeds it by $P_1 > 0$. The second exceeds it by $P_5 + D_r > 0$. Thus each proposed nonzero integer target is impossible. Orders exceeding the additive degree already vanish identically. This gives all zeros through O on $|s| \le R_{\mathrm{s}}$, without changing the family.

Use this smaller boosted interval as the interpolation input for the extension to R_{l}. Its node mass is at most $\zeta\chi\widehat{U}_r/q$, while the second analytic entry is greater than $\zeta\chi\widehat{U}_r$, by $P_6 > 0$. Hence the minimum is at least $n_{\mathrm{s}}N - Q_D$. Its normalized lower bound is $\zeta\chi c_1q^r - 2\sigma_r - C$. The physical large radius is less than q times the real upper bound just used for the small radius. Thus Lemma 10.214 applies with $\theta = q, j = j_r + 1$, giving $\mathcal{H}_{\mathrm{l}}(A_0)$. The strict comparison $P_2 > 0$ forces every integer target to zero. All these extensions retain the original R-block as well.

Finally interpolate from the full R_{l}-interval. Again the second entry is greater than its node mass, so the analytic lower bound is $n_{\mathrm{l}}N - Q_D$. Its normalization is greater than $\zeta\chi c_1q^{r+1} - 2\sigma_r - C$. For the prescribed fractional targets,

$$q^{-I}R/q \le q^r e^{\alpha I_1 - I\log q}S \le q^{j_r}S. \tag{10.971}$$

They belong locally to $\mathbb{Z}_p$. Their global field degree is at most $q^r d$; keep this full factor. Lemma 10.214 bounds their normalized valuation divided by q^r by $\mathcal{H}_{\mathrm{s}}(A_0)$. Dividing the analytic bound by the same q^r, the strict difference is $P_3 > 0$. Thus every such value is zero. All target orders are still at most O, and every interpolation used multiplicity one. □

Restoring every integer and reaching the endpoint

Theorem 10.217 (fixed-order successor and terminal contradiction). Under the same hypotheses, a nonzero admissible stage-I family with its full R-block through O produces such a family at depth $I + 1$, for every $I_1 \le I < I_3$. Consequently the alternative $I_M < I_G$ is impossible.

Proof. Theorem 10.216 provides the entire fractional block. Apply the exact support and phase extraction of Theorems 10.48 and 10.51, including the extra phase partition when q divides G_0. Keep

the class containing a coefficient of minimum valuation β. The Kummer monomial basis separates the classes for $q \nmid s$. The triangular affine change in the prepared indices preserves every total order through O, with diagonal powers of q. The resulting family has q-deleted zeros at $|s| \le R$ through O, widths at most $q^{-I-1}\widehat{D}_i$, additive argument $q^{-I-1}s$, and integer values in K. Theorems 10.29, 10.55 and 10.176 preserve the nonzero polynomial, complete polynomial fibres, original coefficient minimum, phase and selected combined-frequency congruence. The coefficient height does not increase.

It remains to fill the multiples of q. Since R is divisible by q, the deleted interval contains exactly $n_E = 2(q-1)R/q$ nodes. Write $B_E = \lfloor \log_p(2R) \rfloor$. Theorem 10.182 with $\mu = 1$ gives the two analytic entries

$$n_E N - Q_D, \qquad \kappa + \chi \widehat{U}_r - v_p(b_n) - B_E. \tag{10.972}$$

There is exactly one deleted-node separation loss in the second entry and no inverse-jet loss. For an even smaller common lower bound we may subtract B_E from the first entry as well. The normalization is still $Q_D - \beta$; it has not been discarded in either integer extension or in this closure.

We bound the separation loss using the fixed starting depth, rather than the later stage. Since $j_0 \le r$, $R \le q^{r+1} S_1$, while Theorem 10.203 gives $\log(2q^r S) < H$. Hence

$$\log(2R) < H + \log q + \alpha I_1 < \frac{103}{100} H e^{\alpha I_1}. \tag{10.973}$$

Indeed $H > 40$, $\log q < 11/10 < 3H/100$, and $1 + u \le e^u$ for $u \ge 0$. Applying (10.863) with hats and $T_1 = \mathfrak{h}^{I_1} \widehat{T} \ge a/c$ gives

$$\frac{c_1 B_E}{\widehat{\mathcal{Z}}} < \frac{103 \xi}{100 c_3 (1 + a_*) T_1} \le D_r. \tag{10.974}$$

The exact count satisfies $n_E > 2(q-1) q^{j_0} S_1 - 2(q-1)$. As in the preceding proof, $a/T_1 > c\eta^{j_0+1}$ and $q\eta > 1$ show that the first, conservatively lowered analytic entry has normalized lower bound $2cq(q-1)\eta - 2(q-1)\sigma_r - C - D_r$. The second has lower bound $c_1 \chi q^{r+1} - \xi/[c_3(1 + a_*)L_r] - D_r$.

For the successor integer target, the physical radius is $q^{-I-1}R$, exactly the preceding fractional radius. Its denominator has total depth $I + 1$, also exactly the one just charged. Its field is K, so there is no factor q^r on its arithmetic bound. Its order is unchanged and its coefficient height has not increased. Lemma 10.214 therefore gives $\mathcal{H}_{\mathrm{s}}(A_0)$, using the previous depth I in the triangle. This remains valid at $I + 1 = I_3$; no extension of the triangle beyond its proved range has been made. The two strict differences are $P_4 > 0$ and $P_5 > 0$. Every integer target is consequently zero, through the same order O.

Lemma 10.211 supplies the actual initial family at I_1. Finite induction with the successor just proved reaches I_3, retaining all hypotheses of Proposition 10.212. That proposition forces the torus degree to be zero and gives more integer roots than the degree of its nonzero additive polynomial, a contradiction. Thus $I_M < I_G$ is impossible. □

The argument retains the fixed derivative order, the entire original integer interval, the original coefficient loss and the complete root-field factor. Its intermediate integer radii are used only to gain interpolation mass; the final contradiction concerns the same R-block that was constructed in Lemma 10.211.

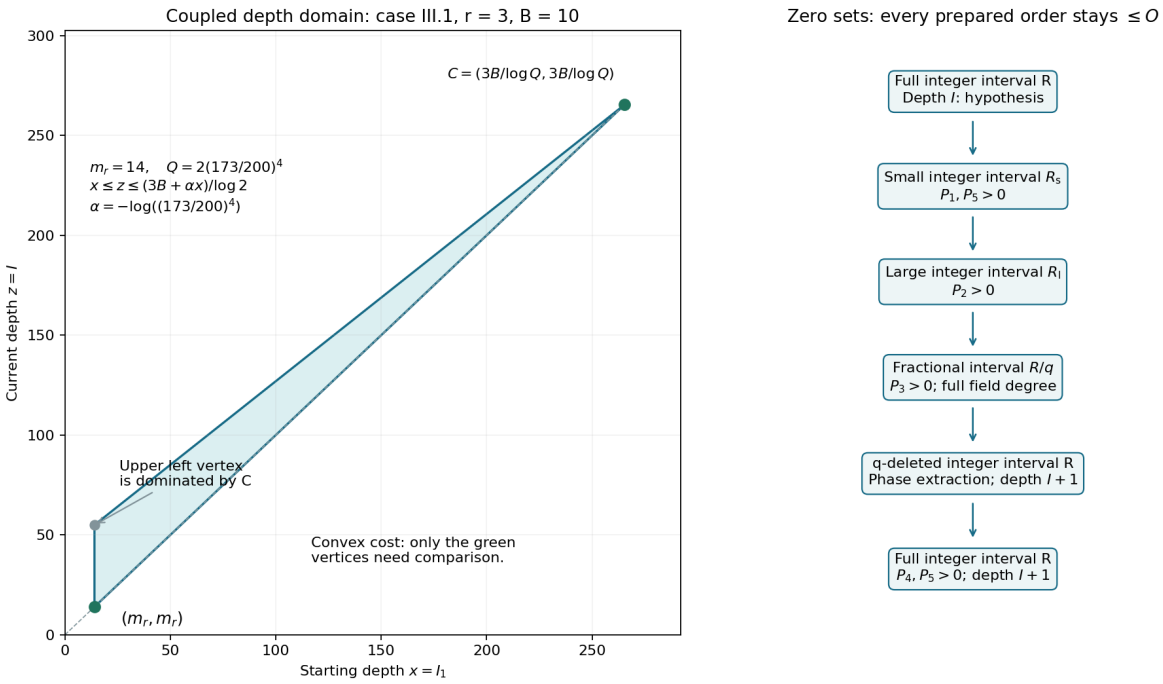


Figure 10.65. Left: the real depth enclosure of Lemma 10.213 for the illustrative parameters of case III.1, rank three and $B = 10$. The two green vertices bound the convex sum of torus-radius and denominator costs; the upper left vertex has the same exponential term as C and a smaller linear term. The plotted domain is a parameter enclosure, not a list of realized number-field families. Right: the zero sets constructed in Theorems 10.216–10.217, with their exact comparison labels. Every step retains order O. The fractional comparison uses the complete global degree bound $q^r d$; the successor integer comparison uses K itself. Exact calculation and figure program. Human-source context: the auxiliary-function extrapolation and contraction method in Sections 10–11 of Yu's freely accessible 1999 paper. The bounds and induction used in this figure are proved in Lemmas 10.213–10.215 and Theorems 10.216–10.217; Solution 86 checks the rational depth example and the node counts.

73. Closing the two-base case at an unramified prime above five

The comparison in Theorem 10.189 left one range: two bases, a local field equal to $\mathbb{Q}_5$, global degree greater than one, and a small coefficient parameter. A two-logarithm determinant handles this range. Its usual height floor is unnecessarily large here. We first lower that floor when the interpolation parameter is large, and retain every rounding and common-coefficient term in the proof.

A smaller height floor at a large interpolation parameter

Lemma 10.218 (the determinant estimate with a tenth-height floor). Use the algebraic, local-unit and independence hypotheses of Theorem 10.170, with nonzero integer coefficients. Its global degree may also be the degree of a specified number field containing both bases; use that field's local residue degree in $D = d/f$. Choose

$$\begin{aligned} L_i &\geq \max\{h(\alpha_i), \log p/(10D)\}, \\ B' &= \frac{|b_1|}{DL_2} + \frac{|b_2|}{DL_1}, \\ H &\geq \max\{\log B' + \log\log p + 2/5, 10\log p/D, 10\}. \end{aligned} \tag{10.979}$$

If $B = DH/\log p \geq 1200$, then

$$v_p(\alpha_1^{b_1} - \alpha_2^{b_2}) < 24\frac{pgD^4}{(p-1)(\log p)^4} L_1 L_2 H^2, \tag{10.980}$$

where g is the order of the subgroup generated by the residues. The ordinary floor $L_i \ge \log p/D$ gives the same weak inequality without the extra condition $B \ge 1200$, by Theorem 10.170.

Proof. The height calculation in Lemma 10.6 and Theorem 10.169 uses the product formula in a specified field containing the bases. For a place of local degree ef its weight is ef/d, giving exactly $D = d/f$. Neither the polynomial zero lemma nor the local power-law argument requires the two bases to generate that field. Thus the proof of Theorem 10.170, including its ordinary-floor conclusion, applies to these ambient degrees. In particular $D \ge e \ge 1$. This observation does not identify global and local degrees.

Invert bases and multiply by a local unit to reduce to positive coefficients, as in that theorem. Put $a_i = DL_i/\log p$, $z = ga_1a_2$, $k = 3, \ell = 13/5$, and take the same integers $L = \lfloor \ell B \rfloor + 2$, $K = \lfloor 3zL \rfloor + 1$ and the four rounded square-root sides of Theorem 10.165. Those side-product and side-sum inequalities use only $a_i > 0$. Here

$$a_i \ge 1/10, \qquad z \ge 1/100, \qquad zB \ge 12, \qquad K > 3(1/100)(13/5)1200 > 85. \tag{10.981}$$

The two residue-class cardinalities therefore have the same proof as in Theorem 10.170, unless linear labels collide. First suppose there is no such collision. The factorial argument (10.611)–(10.613) uses $K \ge 85$, not a lower bound on either a_i. Consequently $\log \beta < \log(B' \log p) + 2/5 \le H$.

The compressed height calculation (10.614)–(10.616) is also unchanged. Its final use of $a_i \ge 1$ must now be replaced: since $g(a_1 + a_2) = z(1/a_1 + 1/a_2) \le 20z$, it gives

$$\gamma_1 R a_1 + \gamma_2 S a_2 \le \frac{2}{3}\sqrt{3}\, zL + \frac{4}{3}\sqrt{zL} + \frac{40}{3}z. \tag{10.982}$$

Put $\lambda = 14/5$; then $L \le \lambda B$. The sufficient determinant inequality (10.618), divided by zB^2, has left side minus the coefficient term greater than $3(13/5)^2 - 3(13/5) - 6/B$. The three height terms are at most

$$\frac{2}{3}\sqrt{3}\lambda^2, \qquad \frac{4}{3}\frac{\lambda^{3/2}}{\sqrt{zB}}, \qquad \frac{40\lambda}{3B}. \tag{10.983}$$

For the remaining logarithm use $D/\log p = B/H \le B/10$ and $N \le 3z\lambda^2 B^2 + \lambda B$. The function $\log(az + b)/z$ decreases when $az + b >$ e: its derivative has numerator $az/(az + b) - \log(az + b) < 0$. At $z = 1/100$, the function $\log(3\lambda^2 B^2/100 + \lambda B)/B$ decreases for $B \ge 1200$, because its logarithmic derivative times B is less than two and its logarithm is greater than two. It follows that

$$\frac{3(D/\log p)\log N}{zB^2} \le \frac{\log 342048}{40} < \frac{13}{40}. \tag{10.984}$$

The remaining comparisons are entirely rational. Use $\sqrt{3} < 26/15$, $\lambda^{3/2}/\sqrt{12} < 34/25$, and $B \ge 1200$. The strict sufficient margin is

$$\frac{3848}{1125} - \left(\frac{1}{200} + \frac{136}{75} + \frac{7}{225} + \frac{13}{40}\right) > 0. \tag{10.985}$$

Thus Theorem 10.169 applies. Its common coefficient p-part is not discarded. Since $\log(b_1/a_2) < H$, $\log b_1 \le \max\{1, a_2\}H$: for $a_2 \ge 1$ use $\log a_2 \le a_2 - 1$ and $H \ge 1$; otherwise use $\log a_2 \le 0$. Also $\max\{1, a_2\} \le 100z$. Hence $u \le 100zB/D \le 100zB$. With $C = p/(p-1)$, the final bound $CN + u$, divided by CzB^2, is strictly below

$$3\lambda^2+\frac{\lambda}{12}+\frac{100}{1200}=\frac{7151}{300}<24. \tag{10.986}$$

If linear labels collide, the exact residue-order and power-law argument (10.625)–(10.627) applies with these same positive heights. Its bound is $u+2C\sqrt{3}\,zL+(D/\log p)\log 2$. Dividing by CzB^2, using $\sqrt{3}<7/4$ and $\log 2<7/10$, gives the strict upper bound

$$\frac{1}{12}+\frac{49}{6000}+\frac{7}{1200}=\frac{73}{750}<1. \tag{10.987}$$

Finally $zB^2=gD^4L_1L_2H^2/(\log p)^4$, as before. This proves the assertion in both cardinality alternatives. □

Comparing the determinant with the second constant

Set $t=\log 5$, and retain the independent original bases and coefficient conventions of Theorem 10.152. In this subsection $n=2, e=f=1, d>1$, $A=4+\log(3d)$, $\Omega=h(a_1)h(a_2)$, and h_2 is (10.643).

Lemma 10.219 (three degree ranges). In this setting, $C_1\mathsf{h}<C_2\mathsf{h}_2$ for $d\geq 2048$. For $2\leq d\leq 2047$, suppose $\mathsf{h}_2<G_1(2,d)$ and both coefficients are nonzero. Then

$$\frac{v_5(\Xi-1)}{C_2\mathsf{h}_2\Omega}<\begin{cases}1/2, & 2\leq d\leq 64,\\ 3/5, & 65\leq d\leq 2047.\end{cases} \tag{10.988}$$

Proof. Write $a_0=2+\log 7$. The constants independent of d in $G_1/3$ and $G_2/3$ are respectively

$$g_1=2a_0+471/100+\log(2a_0+471/100+\log 2),\qquad g_2=2a_0+199/100+\log 3. \tag{10.989}$$

They satisfy $g_1<76/5$, $g_2>549/50$. The exact quotient of the first and second constants is $C_2/C_1=(495/1794)(3/8)\mathrm{e}^2t>123/100$. All three strict numerical comparisons are proved by the rational logarithm calculation described below. Since $\log 2>69/100$, at $d\geq 2048=2^{11}$ we have

$$\frac{123}{100}(549/50+\log d)-(76/5+\log d)\geq\frac{511}{10000}>0. \tag{10.990}$$

Thus $G_1<(123/100)G_2$. Every other entry in the two coefficient maxima agrees, so $\mathsf{h}\leq(123/100)\mathsf{h}_2$. The strict quotient proves the large-degree assertion.

For the bounded-degree ranges let $w=w_K$. Reduction embeds the roots of unity of K into $\mathbb{F}_5^*$: a root reducing to one lies in $1+5\mathbb{Z}_5$, where the deep-unit power laws of Lesson 9 rule out nontrivial torsion. Therefore $w\mid 4$, and $w=q^u$. The singleton and two-element cases of Theorem 10.18 give, with $\ell_d=\max\{1,\log d\}$,

$$h(a_i)\geq\frac{w}{58\mathrm{e}d^2\ell_d},\qquad \Omega\geq\frac{w}{29\mathrm{e}^2d^3\ell_d}. \tag{10.991}$$

Use $\epsilon=1$ for $d\leq 64$, and $\epsilon=1/10$ otherwise. Set $L_i=\max\{h(a_i),\epsilon t/d\}$. Separating the cases where neither, one, or both heights are raised, (10.991) gives

$$\begin{aligned}\frac{L_1L_2}{\Omega} &\le \max\left\{1, \frac{58\mathrm{e}\epsilon t d\ell_d}{w}, \frac{29\mathrm{e}^2\epsilon^2t^2d\ell_d}{w}\right\}\\ &\le \begin{cases}29\mathrm{e}^2t^2d\ell_d/w, & \epsilon = 1,\\ (58/10)\mathrm{e}td\ell_d/w, & \epsilon = 1/10.\end{cases}\end{aligned} \tag{10.992}$$

For the last comparisons use $2 < \mathrm{e}t < 6$, $d \ge 2$, $w \le 4$ and $\ell_d \ge 1$. Each selected term exceeds one; the ratio of the third term to the second is $\mathrm{e}\epsilon t/2$.

To apply the two-logarithm estimate to $\Xi - 1$, use $(\alpha_1, \alpha_2) = (a_1, a_2^{-1})$ and (b_1, b_2) as its exponents. The resulting difference differs from $\Xi - 1$ by a local unit. Its residue subgroup has order at most four. Put

$$c(d) = \log t + 3\log\log(3d) + 2/5, \qquad H = \mathsf{h}_2 + c(d). \tag{10.993}$$

Since the padded reciprocal heights decrease, $B' \le R_{\max}/d \le W(d)\exp(\mathsf{h}_2)$. Thus H satisfies the logarithmic requirement of Lemma 10.218. We also have $\mathsf{h}_2 > 33$, $c(d) > 0$, and $10t/d < 10$, since $d \ge 2, t < 13/8$. For $d \ge 65$, $B = dH/t > 65 \cdot 33/(13/8) = 1320 > 1200$. The ordinary estimate handles the lower range; Lemma 10.218 handles the upper one.

The residue maximum in C2 is exactly its second entry: $\mathrm{e}^2t^3/4 > 5 \ge 5/\delta(\boldsymbol{a})$. The complete formula (10.642) therefore reduces to

$$C_2 = 218295\mathrm{e}^4d^4\ell_dA/w. \tag{10.994}$$

Combining (10.992), the bound with coefficient 24, and $g \le 4$, bounds the desired ratio above by

$$\begin{cases}\dfrac{3480}{218295\mathrm{e}^2t^2}, & d \le 64,\\ \dfrac{696}{218295\mathrm{e}^3t^3}, & 65 \le d \le 2047\end{cases} \qquad \frac{d}{A}\frac{(\mathsf{h}_2 + c(d))^2}{\mathsf{h}_2}. \tag{10.995}$$

Here $d/A(d)$ increases, since its derivative has the sign of $A - 1 > 0$. Write $\bar{G}(d) = 3(76/5 + \log d)$. On $2 \le d \le 2048$ we have $0 < c(d) < 15/2 < 33 < \mathsf{h}_2 < G_1 < \bar{G}$. The function $(x + c)^2/x$ increases in x when $x > c > 0$, and increases in c. Both $\bar{G}(d)$ and c(d) increase. Consequently it suffices to bound the two expressions at d=64 and d=2048, respectively. At the first endpoint use $A > 46/5, \bar{G} < 291/5, c < 6$; at the second use $A > 12, \bar{G} < 687/10, c < 15/2$. Together with $\mathrm{e} > 27/10, t > 8/5$, these give

$$\frac{3320210000}{7889661549} < \frac{1}{2}, \qquad \frac{37419280000}{65596294071} < \frac{3}{5}. \tag{10.996}$$

For completeness, all logarithmic endpoints used here admit one short exact verification. For rational $1 \le x \le 2$, set $y = (x - 1)/(x + 1)$. Integration of the geometric series gives

$$0 \le \log x - 2\sum_{j=0}^{N-1}\frac{y^{2j+1}}{2j+1} \le \frac{2y^{2N+1}}{(2N+1)(1-y^2)}. \tag{10.997}$$

Reduce any larger rational argument by a power of two, and use N=64 in both it and $\log 2$. Monotonicity permits rational interval endpoints inside the one nested logarithm in (10.989). The positive exponential series through degree ten supplies the lower bound for e. Substitution checks (10.989)–(10.990), $8/5 < t < 13/8$, $\log t < 1/2$, $26/5 < \log 192 < 53/10$, $\log(53/10) < 17/10$,

$\log 64 < 21/5$, $8 < \log 6144 < 44/5$, and $\log(44/5) < 11/5$. These are exactly the endpoint bounds above. The same calculation verifies $\log 342048 < 13$ for Lemma 10.218. The editable rational calculation checks these and all remaining displayed rational margins; the monotonicity argument, not a sample over degrees, supplies the quantified ranges. □

Theorem 10.220 (the second estimate for every two-base field case). Under the hypotheses of Theorem 10.152 with n=2,

$$\operatorname{ord}_{\mathfrak{p}}(\Xi - 1) < C_2 \mathsf{h}_2 h(a_1) h(a_2). \tag{10.998}$$

Proof. Theorem 10.189 covers every case except $p = 5, e = f = 1, d > 1, \mathsf{h}_2 < G_1(2, d)$. When both coefficients are nonzero, Lemma 10.219 covers that range; for d at least 2048 use Theorem 10.152 after its constant comparison. Since e=1, its p-adic valuation is the ideal valuation in (10.998).

If exactly one coefficient b is nonzero, put $a = a_i$ for its base. Its residue order is at most four. The elementary one-base estimate (10.60), $\log|b| \le \mathsf{h}_2$, and $\mathsf{h}_2 > 33$ give $v_5(a^b - 1) < (34/33)d\mathsf{h}_2/t + 8dh(a)/t$. Divide by the target, use the product lower bound in (10.991) for the first term and the singleton bound for the unused base in the second, and substitute (10.994). The ratio is strictly less than

$$\frac{(34/33)29}{218295\mathrm{e}^2 At} + \frac{464}{218295\mathrm{e}^3 dA\mathsf{h}_2 t} < \frac{1}{1000} < 1. \tag{10.999}$$

The last rational comparison follows already from $\mathrm{e} > 2, A > 5, t > 1, d \ge 2, \mathsf{h}_2 > 33$. This includes the zero-coefficient alternative without feeding it into a determinant requiring two nonzero coefficients. □

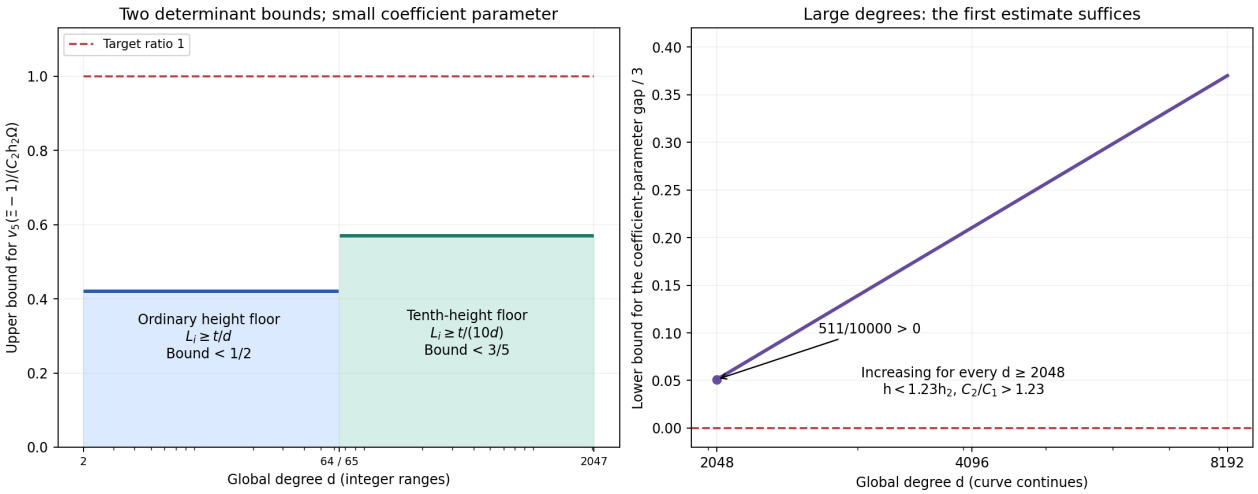


Figure 10.66. Left: the two rational upper bounds from (10.996), valid throughout their indicated integer-degree ranges when both coefficients are nonzero and $\mathsf{h}_2 < G_1$. The auxiliary height floors differ by a factor of ten; Lemma 10.218 proves the second determinant estimate, and Lemma 10.219 absorbs its actual padding costs. Right: the lower bound $511/10000 + (23/100)\log(d/2048)$ for the normalized coefficient-parameter gap is positive for every $d \ge 2048$, although only the range through 8192 is drawn. Theorem 10.220 includes the zero-coefficient case by its separate one-base proof. These are proved bounds, not samples of algebraic inputs. Exact rational calculation and figure program. Human-source context: the two-logarithm determinant in Bugeaud–Laurent's freely accessible 1996 paper and the second constant in Yu's freely accessible 2013 paper. The lower floor and degree comparisons are proved above.

74. The second explicit estimate for an arbitrary number of bases

The second estimate follows by combining the two stopping arguments with a minimum-rank representation. The coefficients of that representation are rational. Its coefficient bounds must

therefore continue to refer to the original integers. We first establish the precise weighted form of the first estimate that is needed at a small intermediate rank.

The first estimate for integral forms with prescribed weights

Retain the original field, independent units $a_1, \dots, a_n$, integers b_j, distinguished coefficient b_n, and $\mathcal{L} = \sum_j b_j z_j$ of Theorem 10.152. In particular, $b_n \neq 0$ has minimum p-adic valuation. Let a_0 generate the q-primary roots of unity. Suppose $L_0 = z_0, L_1, \dots, L_r$ are independent integral forms, $2 \leq r \leq n$, and

$$\mathcal{L} = B_0 L_0 + \sum_{i=1}^{r} B_i L_i, \qquad L_i = \sum_{k=0}^{n} a_{ik} z_k, \qquad B_i \in \mathbb{Q}. \tag{10.1002}$$

Set $\alpha_i = a_0^{a_{i0}} \prod_{k=1}^{n} a_k^{a_{ik}}$. Choose positive weights $\sigma_i \geq \sum_{k=1}^{n} |a_{ik}| h(a_k)$, and write $\Omega_r = \prod_i \sigma_i$. The bases α_i are independent, by the argument following (10.277). In constants bearing a rank argument, use that family's actual saturated residue index.

Lemma 10.221 (a weighted first estimate with the original coefficients). If

$$H \geq \max\left\{ \log \frac{R_{\max}}{dW(d)}, \log B_{\dagger}, G_1(r,d), (r+1)t \right\}, \tag{10.1003}$$

where $R_{\max}, B_{\dagger}$ still concern the original a_k, b_k, then

$$\operatorname{ord}_{\mathfrak{p}}(\Xi - 1) < C_1(r, d, p, \boldsymbol{\alpha}) H \Omega_r. \tag{10.1004}$$

No size or integrality hypothesis on the rational B_i is added.

Proof. We give the minimum-rank argument with the stated starting forms. This identifies the coefficient and height data at each step. Choose $\tau = 10^{-26}/(2n)$, and let H_* be the first-scale quantity in (10.272), so $qH_* = \mathfrak{a}(1+\tau)$. In this proof only, put $\Gamma = \mathrm{e} H_* (r+1) d$. Consider integral forms

$$F_j = u_{j0} L_0 + \sum_{i=1}^{r} u_{ji} L_i, \qquad s_j \geq \sum_{i=1}^{r} |u_{ji}| \sigma_i, \tag{10.1005}$$

independent with L_0, whose rational span with L_0 contains $\mathcal{L}$. For a family of length s require

$$\prod_{j=1}^{s} s_j \leq \Gamma^{r-s} \frac{M_r(\boldsymbol{\alpha})}{M_s(\boldsymbol{\beta})} \Omega_r, \qquad \beta_j = a_0^{u_{j0}} \prod_i \alpha_i^{u_{ji}}. \tag{10.1006}$$

Here M_j is (10.272), so $t^j M_j = \mathcal{R}_j$. The initial family $F_i = L_i$ with weights σ_i is admissible. There is a least positive admissible length s.

For every such family, the coefficient triangle inequality gives

$$\sum_{k=1}^{n} \left| \sum_i u_{ji} a_{ik} \right| h(a_k) \leq \sum_i |u_{ji}| \sigma_i \leq s_j. \tag{10.1007}$$

Thus (10.218) holds with the original a_k, b_k. Lemma 10.84 bounds each integral Euler argument by the original $R_{\max}$, and Lemma 10.23 uses the same original b_n to choose the derivations. The scalar bounds and the common coefficient loss therefore use (10.1003). Since $G_1(s,d) \leq G_1(r,d)$

and $(s+1)t \le (r+1)t$, all their rank-s height hypotheses hold. The evaluation of the original form remains $P \log \Xi$; no logarithm of a rational coordinate coefficient is substituted.

Suppose first $s \ge 2$. Form the first-scale parameters (10.183)–(10.186), using these bases and weights and height H. Under $v_p(\Xi - 1) \ge U_s$, the kernel and initial zero arguments in the proof of Theorem 10.152 apply: the principal-unit depth is supplied by (10.280), the integer coefficient vector by Theorem 10.71, and the complete subsequent zero blocks by Corollaries 10.140 and 10.150. At the geometric stop, Theorem 10.102 applies with the fixed reference rank r. Its proof uses this reference rank only through $\Gamma = \mathrm{e}H_*(r+1)d$ and the inequality $s \le r$. It produces a shorter family with product strictly smaller than $\Gamma^{s-m}(M_s/M_m)\prod_j s_j/700$. Composing its integral rows with (10.1005) preserves the stronger weight condition there, by the triangle inequality. Its product is strictly below the right side of (10.1006) at length m. This contradicts the chosen minimum. At the other stop, Corollary 10.150 gives the nonzero additive-polynomial contradiction. The alternatives exhaust the two finite stopping indices. Hence $v_p(\Xi - 1) < U_s$.

The exact cancellation (10.452)–(10.453) now has r as its fixed reference rank and Ω_r as the initial product. The residue factor M_s cancels through (10.1006). The remaining ratio of rank factors is at most

$$\left(\frac{\mathrm{e}}{q}\right)^{r-s} \frac{s^s/s!}{r^r/r!} \left(\frac{s+1}{r+1}\right)^{s+1} \le 1. \tag{10.1008}$$

Consecutive ratios of $j^j/j!$ are $(1+1/j)^j \ge 2$; also $\mathrm{e} < 3$ and $q \ge 2$. These prove the displayed inequality. We have $A_s \le A_r$, and $(1+\tau)^s < 1 + 10^{-26}$ because $s \le n$. Lemma 10.151 therefore gives (10.1004).

If the least length is one, write its nonzero rational coefficient as v/w in lowest terms, $w > 0$. The form F_1 is integral in the original variables. Comparing their coefficients shows $v \mid b_k$ for every k, hence $|v| \le B_\dagger$. As in (10.454), raising to wq^u kills the torsion factor and gives $\Xi^{wq^u} = \beta_1^{vq^u} \ne 1$. The elementary residue-order bound yields

$$\operatorname{ord}_{\mathfrak{p}}(\Xi - 1) \le \frac{d}{t}(2H + 2egs_1), \tag{10.1009}$$

where g is the actual residue order of β_1. Here $\log(2q^u B_\dagger) < 2H$ follows from $q^u < p^f$, $t > 1$, $H \ge 3t$, and $H \ge \log B_\dagger$. The weight product of the initial independent bases satisfies (10.55), since $\sigma_i \ge h(\alpha_i)$. Consequently the calculation (10.455)–(10.456), with r in place of its reference rank, bounds the two terms in (10.1009) by respectively $1/400$ and $1/4000$ of the target in (10.1004). Its second calculation uses (10.1006), $M_1 > g/t$, and $eq^u \le 2d$; all are available here. The sum $11/4000 < 1$ completes this case. This proves the lemma. □

Closing the small intermediate ranks

Use the second rank allowance (10.725), and retain the integral-form and individual weighted conditions (10.1002). Let $h = \mathsf{h}_2$ for the original n-base form, and put

$$\mathcal{T} = C_2(n, d, p, \boldsymbol{a}) h \prod_{k=1}^{n} h(a_k), \qquad \Gamma_2 = a^{(2)}\mathrm{e}^2 Pd/q. \tag{10.1010}$$

Lemma 10.222 (small-rank bounds with the second allowance). For $n \ge 3$, a representation of rank two satisfying (10.725) gives

$$\operatorname{ord}_{\mathfrak{p}}(\Xi - 1) < \frac{25}{41}\mathcal{T}. \tag{10.1011}$$

For $3 \le r \le n$, the second estimate holds whenever $r/t - A_0 \le 3/5$. In the latter assertion r is the rank of the representation, rather than the original number of bases.

Proof. Put $H = \max\{h, G_1(r,d)\}$. The function $G_2(j,d)$ increases with j: each positive factor in (10.643) increases. Lemma 10.172 gives $G_1(r,d) < 3G_2(r,d)/2 \le 3h/2$. Every other entry in (10.1003) is at most h. Thus Lemma 10.221 applies and $H \le 3h/2$.

After multiplication by the allowance (10.725), the exact quotient between the rank-r second constant and the rank-n target is at most

$$\frac{C_2(r,\boldsymbol{\alpha})\Omega_r}{C_2(n,\boldsymbol{a})\prod_k h(a_k)} \le \left(\frac{\mathrm{e}}{q}\right)^{n-r} \frac{F(r)}{F(n)}\frac{A_r}{A_n} \le 1. \tag{10.1012}$$

The actual $\mathcal{R}_r$ cancels, even if its residue index differs from the original one. Lemma 10.187 proves the purely algebraic ratio in this display for every $r \ge 1$: its proof that $F(j+1)/F(j) > \mathrm{e}$ does not use an optimizer condition. For $r < n$, the right side is strictly less than q^{r-n}.

At r=2, the exact quotient (10.648) gives uniformly

$$\frac{C_2(2,\boldsymbol{\alpha})}{C_1(2,\boldsymbol{\alpha})} > \frac{123}{100}. \tag{10.1013}$$

Indeed $2 \le Pt$ gives the stronger 3/2 bound in Theorem 10.173. If $2 > Pt$, the field-case proof of Theorem 10.189 leaves only p=5 or 7, with f=1 and P=1. At seven its two rational comparisons give 3/2. At five and d=1, (10.744) gives $11/8 > 123/100$. At five and d>1, the exact quotient is $(495/1794)(3/8)\mathrm{e}^2 \log 5 > 123/100$, proved by the rational series calculation in Lemma 10.219. These comparisons depend only on the field data; the common residue maximum cancels. Since $n \ge 3$, (10.1012) now gives

$$\frac{\mathrm{ord}_{\mathfrak{p}}(\Xi - 1)}{\mathcal{T}} < \frac{3}{2}\frac{100}{123}q^{2-n} \le \frac{25}{41} < 1. \tag{10.1014}$$

For r at least three in the stated range, the constant comparison in Theorem 10.188 gives $C_2(r)/C_1(r) > 3/2$. Its argument uses only the rank and field constants, so it applies to this r. Combining that comparison with Lemma 10.221, $H \le 3h/2$, and (10.1012) proves the second assertion. This argument has never treated the rational B_i as original integer exponents. □

The complete second estimate

Theorem 10.223 (Yu's second explicit estimate, with proof). Under every hypothesis of Theorem 10.152, with $n \ge 2$, the second constant and height parameter (10.642)–(10.643) satisfy

$$\mathrm{ord}_{\mathfrak{p}}(\Xi - 1) < C_2 \mathsf{h}_2 \prod_{k=1}^{n} h(a_k). \tag{10.1015}$$

Proof. Theorem 10.220 gives n=2, including the exceptional unramified-five field case. Suppose n is at least three. Among the integral-form representations in (10.277), replace its product allowance by (10.725) and choose one of least positive rank r. This set is nonempty: $L_i = z_i$ and $\sigma_i = h(a_i)$ give the original rank-n representation with equality in the allowance. The original weighted conditions make its bases independent local units in K, and its nonzero form gives a nonzero rational coefficient vector.

Rank one is settled by Proposition 10.193. Rank two is settled by Lemma 10.222, which also handles every $r \ge 3$ with $r/t - A_0 \le 3/5$. We may therefore assume $r \ge 3$ and $r/t > A_0 + 3/5$. Put

$h = \mathsf{h}_2$. Monotonicity of G_2 gives all of (10.806), with the original $R_{\max}$, $B_\dagger$, and minimum-valuation coefficient b_n still in place.

Use the hatted construction of Theorem 10.203, with $\tau = 10^{-26}/(2n)$. Its scale comparison and the cancellation in (10.1012) give

$$\chi e\widehat{U}_r < C_2(r, \boldsymbol{\alpha}) h \Omega_r \le \mathcal{T}. \tag{10.1016}$$

Assume for contradiction that $\operatorname{ord}_{\mathfrak{p}}(\Xi - 1) \ge \mathcal{T}$. Then $v_p(\Xi - 1) \ge \chi\widehat{U}_r$. Theorem 10.203 constructs an actual nonzero selected kernel with its initial integer, fractional and contracted zero blocks. The original integral forms provide (10.218); Lemma 10.23 chooses the Euler directions using b_n. Their coefficient vector is nonzero and their hyperplane is the one used in Theorem 10.25. The phase and support extraction preserves those directions, independence of the torus bases, a nonzero coefficient of the original minimum valuation, and complete polynomial fibres. Thus these are the families and the hyperplane hypotheses in Theorems 10.204 and 10.217.

Let I_G, I_M be (10.933). They are finite and their defining sets are nonempty, as proved in Corollary 10.209. If $I_G \le I_M$, that corollary produces the complete prepared integer zeros required by Theorem 10.204. The latter produces $1 \le m < r$, integral vectors u_j, and weights $R_j = \sum_i |u_{ji}| \sigma_i$ such that

$$\prod_{j=1}^{m} R_j < 10^{-6} \Gamma_2^{r-m} \frac{\mathcal{R}_r(\boldsymbol{\alpha})}{\mathcal{R}_m(\boldsymbol{\beta})} \Omega_r \le 10^{-6} \Gamma_2^{n-m} \frac{\mathcal{R}_n(\boldsymbol{a})}{\mathcal{R}_m(\boldsymbol{\beta})} \prod_k h(a_k). \tag{10.1017}$$

Compose the original forms with these integral vectors. The rational span conclusion of Theorem 10.25 preserves the representation of $\mathcal{L}$. Corollary 10.26 gives independence and bases in K. The coefficient triangle inequality (10.1007) gives each original weighted condition, and (10.1017) is stronger than its required product allowance. Hence m is an admissible smaller rank, a contradiction.

If $I_M < I_G$, Lemma 10.211 supplies the actual starting interval at its fixed order. Theorems 10.216–10.217 produce every fractional and integer successor through I_3, retaining that order, the full root-field factor in fractional comparisons and the original coefficient loss. Proposition 10.212 then gives a nonzero additive polynomial with more distinct integer roots than its degree. This is again a contradiction. Equality of the stopping indices belongs to the first case, so the two cases exhaust the possibilities. The counterhypothesis is impossible, proving (10.1015). □

Both explicit constants now bound the original form with their own coefficient parameters. The second estimate retains the field-case constants, the saturated residue index of the original bases, and the minimum-valuation choice of the original integer coefficient.

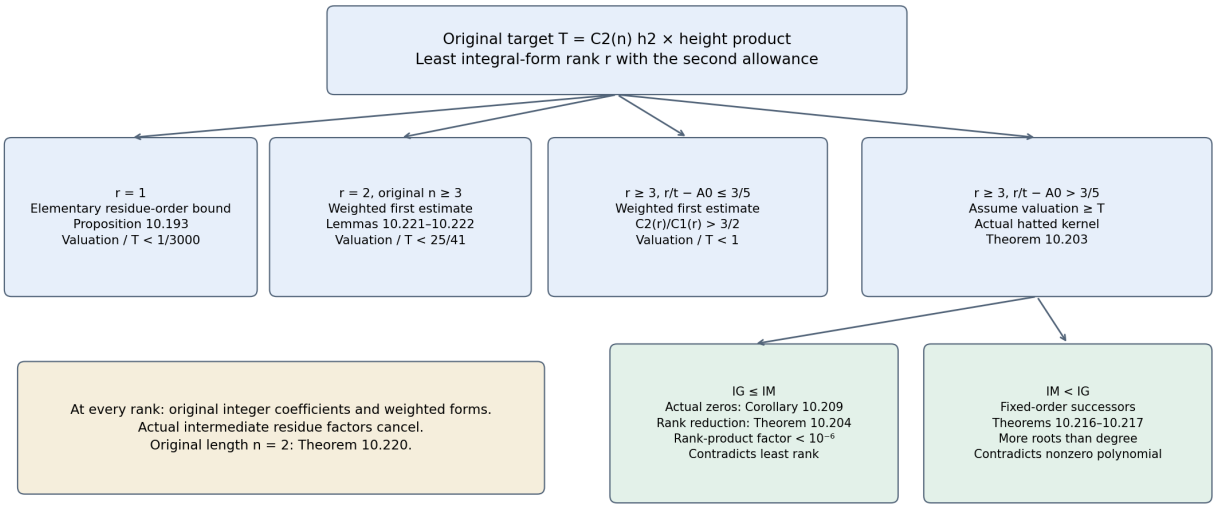


Figure 10.67. Theorem 10.223 separates the least admissible rank into four cases. The one-base ratio is less than $1/3000$*, and the two-base intermediate ratio is less than* $25/41$ *for original length* n *at least three. In the small-optimizer case the first constant and its coefficient parameter combine through Lemma 10.222. The positive-optimizer case has an actual induction to one of two exhaustive stops; its geometric branch decreases rank with factor* 10^{-6}*, and its order branch contradicts the degree of a nonzero polynomial. Every ratio is relative to the exact original target* $\mathcal{T}$*, except the explicitly marked rank-product factor. The diagram is a proof map, not a numerical sample of algebraic inputs. Exact comparison and figure program. Human-source context: the two estimates of Theorem I in Yu's freely accessible 2013 paper and the extrapolation method in Sections 10–11 of his freely accessible 1999 paper. The weighted extension and the assembly of the proof are established in Lemmas 10.221–10.222 and Theorem 10.223 above.*

75. A coefficient refinement for independent rational local-unit parts

A separate bound for the last coefficient is useful when the other coefficients can be much larger. We obtain the numerical bound appearing in Theorem 2.3 of Bugeaud's freely accessible author paper, under the explicit independence hypothesis below. The proof uses the second estimate proved in Theorem 10.223. In particular, the cited statement is not used as a proof.

Enlarging the height bounds

Write $\omega(d) = 1/(dW(d))$, as in (10.222). The constants C_1, C_2 concern the actual bases and their saturated residue index; they do not change when upper bounds for their heights are enlarged.

Lemma 10.224 (monotonicity with padded heights). Retain the hypotheses of Theorem 10.152. For $i = 1, 2$, put

$$H_i = \max\{\log B_\dagger, G_i(n,d), (n+1)t\},$$
$$F_i(\boldsymbol{w}) = \left(\prod_{j=1}^{n} w_j\right) \max\left\{H_i, \max_{j<n} \log\left[\omega(d)\left(\frac{|b_n|}{w_j} + \frac{|b_j|}{w_n}\right)\right]\right\}. \tag{10.1021}$$

The function F_i is nondecreasing in each positive coordinate. Consequently, for arbitrary real $w_j \geq h(a_j)$,

$$\operatorname{ord}_{\mathfrak{p}}(\Xi - 1) < C_i F_i(\boldsymbol{w}). \tag{10.1022}$$

Proof. We have $H_i \geq (n+1)t > 1$. Fix all coordinates except one, denoted by u. Apart from a positive constant factor, the function to consider is the maximum of finitely many functions

$$u \max\{H_i, \log(A + D/u)\}, \qquad A, D \geq 0, \quad A + D > 0. \tag{10.1023}$$

The constant logarithmic entries are included by taking D=0. Below the threshold where the logarithm exceeds H_i, the function is uH_i, hence increasing. Above that threshold its derivative is $\log(A + D/u) - D/(Au + D) \geq H_i - 1 > 0$. At the threshold the two formulas agree. This proves monotonicity of each function and of their finite maximum. Apply it successively to the n height coordinates. Theorem 10.152, for i=1, and Theorem 10.223, for i=2, give the strict inequality at the actual heights. Their constants and bases are unchanged, so the inequality persists at the larger weights. □

A uniform comparison of the second constant

Let $a_1, \ldots, a_n$ be independent rational p-adic units, $n \geq 2$. At an odd prime use $K = \mathbb{Q}$. At two use $K = \mathbb{Q}(\zeta_3)$. In this last field, $d = 2, e = 1, f = 2, q^u = 3$. Indeed the prime-to-two root of unity reduces injectively, by the proof before (10.183), so its order three forces the residue degree to be even. Since $ef \leq d = 2$, we get f=2 and e=1. The same argument gives $q^u \mid 2^2 - 1$, and the field contains ζ_3, so $q^u = 3$. Heights of the rational bases do not change, and $\mathrm{ord}_{\mathfrak{p}} = v_2$ on rational numbers.

Define

$$\mathcal{C}_n(p) = (16\mathrm{e})^{2(n+1)} n^{3/2} (\log(2n))^2 \frac{p}{(\log p)^2}. \tag{10.1024}$$

Lemma 10.225 (the rational coefficient constant). For the fields just specified, the constant in (10.642) satisfies

$$2C_2 < \mathcal{C}_n(p) \qquad (n \geq 2). \tag{10.1025}$$

Proof. All comparisons below hold at every rank. We record the elementary estimates used in them. The positive exponential series gives $8/3 < \mathrm{e} < 11/4$: the sum through degree three is 8/3, and the remaining terms are at most $(1/24)/(1 - 1/5) = 5/96$. Integration of the geometric series at x=1/3 gives $\log 2 = 2\sum_{k\geq 0}(1/3)^{2k+1}/(2k+1)$. Its first term and the geometric upper bound for its tail show $2/3 < \log 2 < 25/36 < 7/10$. Thus $\log 3 > 1$, $\log 5 > 3/2$, and $\log(2n) > 4/3$ for n at least two. For the middle assertion, $\mathrm{e}^3 < (11/4)^3 < 25$ suffices.

Integrating the increasing function $\log x$ gives $(n-1)! \geq ((n-1)/\mathrm{e})^{n-1}$. Also $(1+2/(n-1))^{n-1} < \mathrm{e}^2$, by $\log(1+x) < x$. Hence

$$\frac{(n+1)^{n+2}}{(n-1)!} < \mathrm{e}^{n+1}(n+1)^3, \qquad \frac{(n+1)^{n+1}}{(n-1)!} < \mathrm{e}^{n+1}(n+1)^2. \tag{10.1026}$$

For every t>0 the maximum of $t^{n+1}\exp(-t)$ is $(n+1)^{n+1}\exp(-n-1)$. Therefore

$$\frac{\mathrm{e}^n t^{n+1}}{n^n} < (n+1)\exp(t). \tag{10.1027}$$

First let $p \geq 5$. Here $P = 1, d = f = 1, q^u = 2$, $c^{(2)} \leq 2228/3$, and $a^{(2)} = 7$. The residue index is at least one. Put $A = \max\{4 + \log(n+1), t\}$, where t=log p. Since $\log(n+1) \leq n$ and $t > 3/2$, $A/t \leq n + 2$. Substitution into (10.642), followed by (10.1026) and (10.1027), gives

$$\frac{2C_2}{\mathcal{C}_n(p)} < \frac{2228}{768\mathrm{e}} \left(\frac{7}{256}\right)^n \frac{(n+1)^3(n+2)}{n^{3/2}(\log(2n))^2} < 10n^3 \left(\frac{7}{256}\right)^n. \tag{10.1028}$$

For the last inequality use $\mathrm{e} > 2$, $\log(2n) > 1$, $n+1 \le 3n/2$, $n+2 \le 2n$, and $n^{5/2} \le n^3$. The resulting scalar coefficient is $20052/2048 < 10$. The final expression is $245/4096$ at n=2, and its successive ratio is at most $189/2048$. It is always below one.

At p=3, $P = 3, d = f = 1, q^u = 2, c^{(2)} = 648$. Now $A/t \le 2(n+2)$. The same calculation yields

$$\frac{2C_2}{\mathcal{C}_n(3)} < 6n^3 \left(\frac{21}{256}\right)^n. \tag{10.1029}$$

Here the scalar coefficient before rounding to 6 is $729/128$. The right side starts at $1323/4096$ and has successive ratio at most $567/2048$. Again it is below one.

At p=2 use the quadratic field above. We have $P = 4, t = 2\log 2, c^{(2)} = 406, a^{(2)} = 13$ and $A = 4 + \log(2(n+1))$. Since t<7/5, the function $(\mathrm{e}t/n)^n$ decreases for real n at least two: the derivative of its logarithm is $\log(t/n) < 0$. Thus

$$\max\left\{4, \frac{\mathrm{e}^n t^{n+1}}{n^n}\right\} < 6, \qquad \frac{(11/4)^2(7/5)^3}{4} = \frac{41503}{8000} < 6. \tag{10.1030}$$

Substitute $d = 2, q^u = 3$ and $(\log 2)^2 = t^2/4$ exactly; do not replace the residue-field cardinality 4 by 2. This gives

$$\begin{aligned} \frac{2C_2}{\mathcal{C}_n(2)} &< \frac{203}{256\mathrm{e}} \left(\frac{13}{32}\right)^n \frac{(n+1)^2 A}{t n^{3/2}(\log(2n))^2} \\ &< \frac{345303}{262144} n^{3/2} \left(\frac{13}{32}\right)^n < \frac{4}{3} n^2 \left(\frac{13}{32}\right)^n. \end{aligned} \tag{10.1031}$$

In the second line we used $\mathrm{e} > 8/3$, t>1, $\log(2n) > 4/3$, $n+1 \le 3n/2$, and $A < n+5 \le 7n/2$. The rational comparison with 4/3 is strict. The last expression starts at $169/192$ and has successive ratio at most $117/128$. This completes all primes and all n. □

Inflating just the last height

For a nonzero rational $a = x/y$ in lowest terms, with y>0, define its p-unit part by $a^\circ = ap^{-v_p(a)}$. Removing the p-power from either x or y proves $h(a^\circ) \le h(a)$. For integers b_j, put $s = \sum_j b_j v_p(a_j)$. If s>0, then $v_p(\prod_j a_j^{b_j} - 1) = 0$; if s<0, that valuation is s. These follow from the unequal-valuation case of the ultrametric inequality, proved in Lesson 9. If s=0, the product is exactly $\prod_j (a_j^\circ)^{b_j}$.

Theorem 10.226 (the coefficient refinement with independent unit parts). Let p be prime and n at least two. Let $a_j = x_j/y_j$ be nonzero rational numbers, with coprime integers x_j, y_j and $y_j > 0$. Assume that $a_1^\circ, \dots, a_n^\circ$ are multiplicatively independent. Choose

$$\begin{gathered} A_j \ge \max\{|x_j|, y_j, \mathrm{e}\}, \qquad w_j = \log A_j, \\ b_j \in \mathbb{Z} \setminus \{0\}, \qquad v_p(b_n) \le v_p(b_j) \quad (1 \le j \le n), \\ B \ge \max\{3, |b_1|, \dots, |b_n|\}, \qquad B \ge B_n \ge |b_n|, \qquad 0 < \varepsilon \le \tfrac{1}{2}. \end{gathered} \tag{10.1032}$$

Write $\Xi = \prod_j a_j^{b_j}$, and set

$$V=\prod_{j<n} w_j, \qquad \Omega=\prod_{j=1}^{n} w_j, \qquad r=B/B_n,$$
$$D_n=(n+1)(6n+5), \quad T=2B_n\varepsilon^{-1}\exp(D_n)p^{n+1}V, \quad L=\log T. \tag{10.1033}$$

Then

$$v_p(\Xi-1)<\mathcal{C}_n(p)\max\{\Omega\log T,\varepsilon B/B_n\}. \tag{10.1034}$$

The parameter ε is the auxiliary parameter called delta in the cited rational statement. It is distinct from the saturated residue index in C_2.

Proof. If s is nonzero, the preceding valuation calculation proves the conclusion, whose right side is positive. Suppose s=0. Replace the bases by their p-unit parts. Their heights are bounded by w_j, every w_j is at least one, and the product and coefficient conditions are unchanged. Use the fields of Lemma 10.225. Their ramification indices are one, so their prime-ideal valuations of $\Xi-1$ equal $v_p(\Xi-1)$.

Enlarge only the final weight, taking

$$w_n^{*}=\max\left\{w_n,\frac{\varepsilon r}{VL}\right\}. \tag{10.1035}$$

We check the entire maximum in (10.1021) against 2L. In both fields $\omega(d)<1$: at d=1 this follows from $\omega(1)=\log 2\log 3/\log 6$, and at d=2 from $\omega(2)=1/(2\log^3 6)$. For every j<n,

$$\omega(d)\left(\frac{|b_n|}{w_j}+\frac{|b_j|}{w_n^{*}}\right)\le B_n\left(1+\frac{VL}{\varepsilon}\right)\le\frac{B_nV}{\varepsilon}(L+\varepsilon)<\exp(2L). \tag{10.1036}$$

For the last inequality, both $B_nV/\varepsilon<\exp L$ and $L+\varepsilon<\exp L$ hold. The latter follows from $L>1$, $\varepsilon\le 1/2$ and the positive exponential series. Also $\log B_{\dagger}\le\log B_n<L$.

The relevant rows of (10.643) satisfy $a_0^{(2)}<7, a_1^{(2)}<4$ and $\log d<1$. For example $\log 52<5$ follows from $\mathrm{e}^5>(8/3)^5>52$; the other rows have smaller logarithmic arguments. Hence $G_2(n,d)<(n+1)(8n+5)<2D_n<2L$. Finally f is at most two, so $(n+1)t\le 2(n+1)\log p<2L$. All entries of the coefficient maximum are therefore at most 2L.

Lemma 10.224 now gives

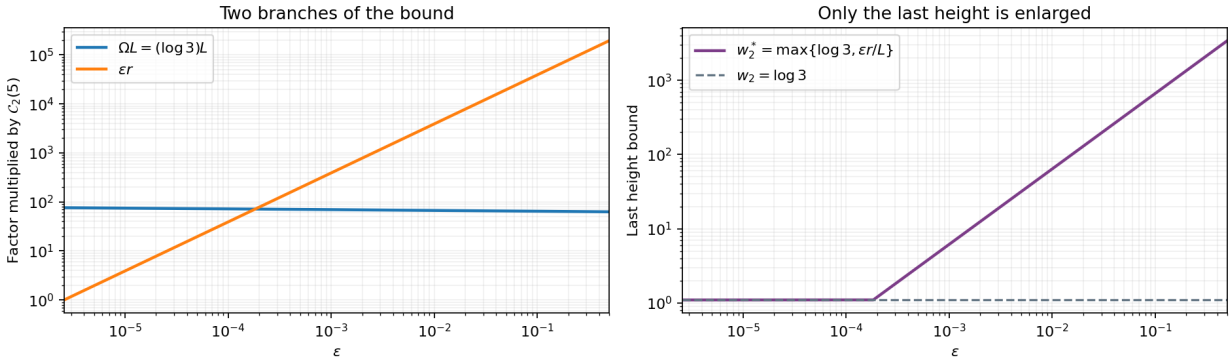
$$v_p(\Xi-1)<2C_2VLw_n^{*}=2C_2\max\{\Omega L,\varepsilon r\}. \tag{10.1037}$$

Apply Lemma 10.225. The maximum is positive, so strictness is preserved. □

For p=5, the bases (2,3) and coefficients $(5^k,1)$, $k\ge 1$, meet the hypotheses. Their independence follows by taking the valuations at two and three. Here $B=5^k, B_n=1$ and the exact valuation is one: $5^k\equiv 5 \pmod{20}$, $2^{20}\equiv 1 \pmod{25}$, and $3\cdot 2^5-1\equiv 20 \pmod{25}$. With $w_1=1, w_2=\log 3$, the two entries in (10.1034) are $(\log 3)\log(250\exp(51)/\varepsilon)$ and $\varepsilon 5^k$. The parameter can be chosen to balance them.

The independence condition in Theorem 10.226 is essential to its proof: removing p-powers can create multiplicative dependencies or roots of unity. For instance (2,6,3) at two has unit parts (1,3,3). Theorem 10.232 treats arbitrary rational lists by controlling their relations while retaining the distinguished coefficient bound.

$p = 5,\ (a_1, a_2) = (2, 3),\ (b_1, b_2) = (5^8, 1),\ B_n = 1,\ r = 390625$



$L = 51 + \log(250/\varepsilon)$; plotted values are samples of the explicit formulas. The exact valuation is 1.

Figure 10.68. For the exact inputs of Solution 89, the left panel samples the two factors in the bound; their maximum is multiplied by $\mathcal{C}_2(5)$. The right panel samples the inflated last height from (10.1035). The switch occurs exactly when $\varepsilon r = \Omega L$. These curves are bounds and height choices, not sampled valuations: the actual valuation is one. Lemma 10.224 proves the monotonicity, Lemma 10.225 proves the constant comparison at every rank, and Theorem 10.226 supplies the coefficient estimate. Reproducible figure and exact rational comparisons. Human-source context: Theorem 2.3 of Bugeaud's free author paper.

76. Rational dependencies and the two-base coefficient refinement

Multiplicative dependence can simplify the coefficient refinement when the bases are rational. In rank one, all their prime-valuation vectors lie on an integral line. We first construct its primitive generator; then the elementary power identity from Lesson 9 replaces the many-logarithm estimate. This proves the coefficient bound for every rational two-base input, including dependent bases. The target constant is the one in Theorem 2.3 of Bugeaud's freely accessible author paper.

Lemma 10.227 (a rational generator with exact height weights). Let $c_1, \dots, c_n \in \mathbb{Q}^*$ have multiplicative rank one modulo signs. There are a positive rational $\beta \neq 1$, integers m_i, and signs $\sigma_i \in \{1, -1\}$ such that

$$c_i = \sigma_i \beta^{m_i}, \qquad h(c_i) = |m_i| h(\beta), \qquad h(\beta) \geq \log 2. \tag{10.1038}$$

If all the c_i are p-adic units, so is β. For integers b_i, write $M = \sum_i m_i b_i$. If $w_i \geq \max\{h(c_i), 1\}$, $\Omega = \prod_i w_i$, and $B \geq \max\{1, |b_1|, \dots, |b_n|\}$, then

$$\prod_i c_i^{b_i} = \sigma \beta^M, \qquad \sigma \in \{1, -1\}, \qquad |M| < 2nB\Omega. \tag{10.1039}$$

Proof. Use the finite set of rational primes occurring in the numerators or denominators of the c_i. The prime-valuation map has kernel $\{1, -1\}$. Its vectors span a one-dimensional rational subspace. Choose a nonzero integral vector on this line and divide its coordinates by their greatest common divisor, obtaining a primitive integral vector v.

Every integral vector on the line is an integral multiple of v. Indeed, there are integers u_ℓ with $\sum_\ell u_\ell v_\ell = 1$; if the vector is av, this identity gives $a = \sum_\ell u_\ell (a v_\ell) \in \mathbb{Z}$. For completeness, the integer linear combinations of the coordinates form an ideal in $\mathbb{Z}$. Its least positive element divides each coordinate by division with remainder, so is their greatest common divisor. This proves the integer-combination assertion just used.

Set $\beta = \prod_\ell \ell^{v_\ell} > 0$. Unique prime factorization proves the representation in (10.1038). If $\beta = x/y$ is reduced, its positive and negative powers have reduced numerator and denominator $x^{|m|}, y^{|m|}$, in one order or the other. Thus $h(\beta^m) = |m| h(\beta)$. Since $\beta \neq 1$, at least one of x,y is at least two; this

proves the height lower bound. The p-coordinate is zero when all original vectors have p-coordinate zero, proving the unit assertion.

The sign in (10.1039) is $\prod_i \sigma_i^{b_i}$. Moreover $\log 2 > 1/2$, as proved in Lemma 10.225, and $w_i \le \Omega$ because every weight is at least one. Hence

$$|M| \le B \sum_i |m_i| \le \frac{B}{h(\beta)} \sum_i w_i \le \frac{nB\Omega}{h(\beta)} < 2nB\Omega. \tag{10.1040}$$

At least one m_i is nonzero, so (10.1038) also gives $h(\beta) \le \max_i w_i \le \Omega$. □

Theorem 10.228 (the coefficient refinement in rational rank at most one). Let $n \ge 2$, and retain the rational inputs, height bounds, coefficient bounds and parameter ε of (10.1032), except that the independence condition is replaced by the condition that the p-unit parts have rank at most one modulo signs. Assume $\Xi \ne 1$. Use V, Ω, r, T, L of (10.1033), and put

$$U = \max\{\Omega L, \varepsilon r\}. \tag{10.1041}$$

Then

$$v_p(\Xi - 1) < \frac{5p}{(\log p)^2} U < \mathcal{C}_n(p) U. \tag{10.1042}$$

In fact this rank-at-most-one argument does not require the minimum p-valuation condition on the distinguished coefficient.

Proof. The valuation reduction before Theorem 10.226 applies unchanged. If $\sum_i b_i v_p(a_i) \ne 0$, then $v_p(\Xi - 1) \le 0$, and the conclusion follows. Otherwise replace the bases by their p-unit parts c_i, without increasing their heights or changing the product. All weights are at least one.

In rank zero all c_i are signs. If their product with the given coefficients is not one, it is minus one, so its distance from one has valuation at most one. The same conclusion holds in rank one when the exponent M in Lemma 10.227 is zero. These cases satisfy (10.1042): since $U \ge L > 3 \log p$ and $\log p < p$, its first upper bound is greater than 15.

We may therefore assume rank one and $M \ne 0$. Take $N = 2(p-1)$. For odd p the residue of β^{p-1} is one: multiplication by a nonzero residue permutes the nonzero residues, and comparison of their products proves this assertion. At two an odd rational unit has square congruent to one modulo eight, since each odd integer has that property. Thus the power identity of Lemma 9.5 applies to β^N at every prime. Raising a unit to a positive integer power cannot decrease the valuation of its difference from one, by the factorization of $X^N - 1$. Since N is even, the sign disappears, and

$$\begin{aligned} v_p(\Xi - 1) &\le v_p(\Xi^N - 1) = v_p(\beta^N - 1) + v_p(M) \\ &\le \frac{2(p-1)h(\beta) + \log 2 + \log |M|}{\log p}. \end{aligned} \tag{10.1043}$$

The last line uses the rational local-height inequality proved within Theorem 10.21 for $\beta^N - 1$ and the elementary inequality $p^{v_p(M)} \le |M|$ for a nonzero integer. Negative M causes no change, since $\beta^{-N|M|} - 1$ is a p-adic unit times $\beta^{N|M|} - 1$.

It remains to retain the coefficient bound B_n in this elementary estimate. Lemma 10.227, $B = B_n r$, and $\varepsilon r \le U$ give

$$\begin{aligned} \log |M| &< \log(2n) + \log B_n + \log \Omega + \log r \\ &\le \log(2n) + \log B_n + \log(\varepsilon^{-1}) + \log \Omega + \log U < 3U. \end{aligned} \tag{10.1044}$$

Here $L > \log(2n) + \log B_n + \log(\varepsilon^{-1})$, because $D_n > \log n$ and $V \geq 1$. Also $U \geq \Omega L \geq L > 1$, $\log \Omega < \Omega \leq U$, and $\log U < U$. These prove every inequality in (10.1044). We have $\log 2 < U$ and $h(\beta) \leq \Omega < U/((n+1)\log p)$. Substitution in (10.1043) gives

$$\begin{aligned} v_p(\Xi - 1) &< \frac{U}{(\log p)^2}\left(\frac{2(p-1)}{n+1} + 4\log p\right) \\ &< \frac{14p}{3(\log p)^2}U < \frac{5p}{(\log p)^2}U. \end{aligned} \tag{10.1045}$$

We used $n \geq 2$ and $\log p < p - 1$, which follows by integrating $1/x < 1$ from one to p. Finally the factor $(16\mathrm{e})^{2(n+1)} n^{3/2} \log^2(2n)$ in $\mathcal{C}_n(p)$ exceeds 5: $n \geq 2$, $\log(2n) > 1$, and $16^6 > 5$ suffice. This proves the second strict inequality of (10.1042). □

Corollary 10.229 (arbitrary two-base rational inputs). Take $n = 2$ and all the data and inequalities in (10.1032), with no independence condition. If $\Xi \neq 1$, then the coefficient bound (10.1034) holds.

Proof. The two p-unit parts have rank zero, one, or two modulo signs. Apply Theorem 10.228 in the first two cases. In rank two they are multiplicatively independent, so Theorem 10.226 applies. A relation with product minus one would square to a relation with product one, so no torsion relation is overlooked. □

For an exact dependent example, take p=5, the bases (8,4,2), and coefficients $(5^k, 2, 1)$, with $k \geq 1$. The primitive generator is 2, the exponent vector is (3,2,1), and

$$M = 3 \cdot 5^k + 5, \qquad \Xi = 2^M, \qquad v_5(\Xi - 1) = 2. \tag{10.1046}$$

Indeed M is divisible by four and has 5-adic valuation one. For k=1 this is immediate from M=20; for k greater than one write $M = 5(3 \cdot 5^{k-1} + 1)$. Since $v_5(2^4 - 1) = 1$, Lemma 9.5 proves (10.1046). Valid weights are $(\log 8, \log 4, 1)$, and $B = 5^k, B_n = 1$. Thus

$$V = \Omega = (\log 8)(\log 4), \qquad L = 92 + \log(1250\Omega/\varepsilon), \qquad r = 5^k. \tag{10.1047}$$

The distinguished coefficient is one. Its p-valuation is minimal, as required for the common statement, even though the rank-one proof does not need that condition.

These results complete the rational coefficient refinement for two bases and for lists of any length whose unit parts have rank at most one. Theorem 10.232 treats the intermediate dependent ranks for three or more rational bases. The general-number-field coefficient refinement is a separate result.

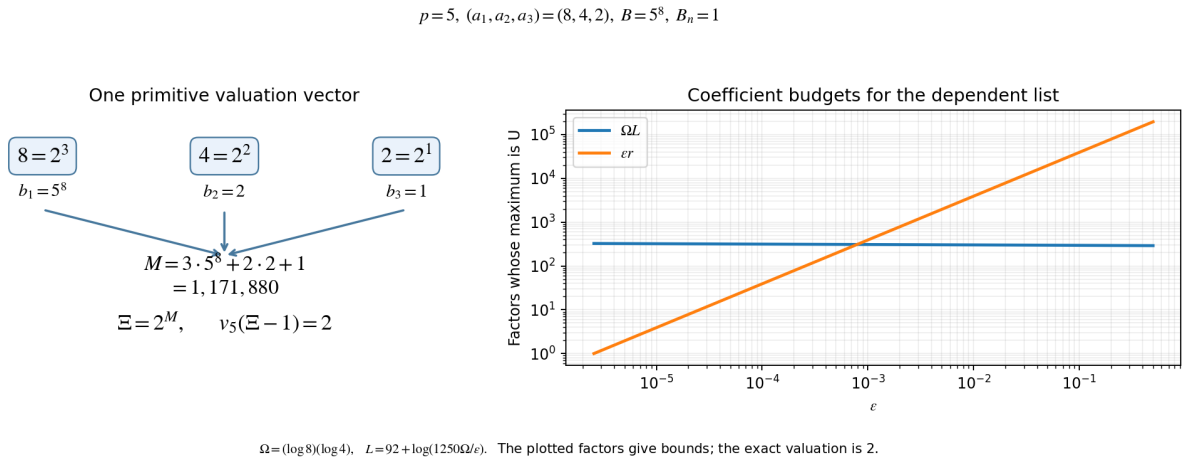


Figure 10.69. At p=5, the valuation vectors of (8,4,2) lie on the integral line generated by the valuation vector of 2. The left panel shows the exact exponent transfer for $k = 8$. The right panel samples the two factors defining U in (10.1041), using the exact inputs (10.1047). Their maximum is multiplied by $5p/(\log p)^2$ in the proved rank-one bound; the actual valuation is 2. The plotted curves are bound factors, not observed valuations. Lemma 10.227 supplies the generator and Theorem 10.228 supplies the inequality. Reproducible figure and exact example checks. Human-source context: Theorem 2.3 of Bugeaud's free author paper.

77. The coefficient refinement for arbitrary rational inputs

We now remove the independence restriction at every rational rank. The distinguished base determines the proof. If its valuation vector lies in the span of the others, we eliminate that base and use the height factor thereby saved. Otherwise a dependence among the other bases preserves the minimum-valuation condition on the distinguished coefficient. Both operations retain the exact constant in (10.1024).

Throughout this section a rank or a span means the rank or rational span of the prime-valuation vectors, equivalently multiplicative rank modulo signs. Zero coefficients are allowed; the distinguished coefficient is nonzero and we set $v_p(0) = +\infty$.

Lemma 10.230 (rational determinants and primitive relations). Let $n \geq 2$, let $c_1, \dots, c_n$ be nonzero rational numbers, and let $w_i \geq \max\{h(c_i), 1\}$. Put

$$K_n = 2^n n!, \qquad V = \prod_{i<n} w_i, \qquad \Omega = V w_n. \tag{10.1048}$$

(a) Suppose the first $n-1$ vectors span all the vectors, with rank $s \geq 2$. Choose a basis sublist of s of those first vectors, indexed by J, and put $W = \prod_{j\in J} w_j$. For integers b_i with $|b_i| \leq B$, there are a positive integer D and integers M_j such that

$$\left(\prod_i c_i^{b_i}\right)^D = \prod_{j\in J} |c_j|^{M_j}, \qquad |M_j| \leq K_n B \Omega V, \qquad W \leq V. \tag{10.1049}$$

(b) If the first $n-1$ vectors are dependent, there is a relation, supported on those indices,

$$\prod_{i<n} c_i^{k_i} = \eta \in \{1, -1\}, \qquad \gcd(k_1, \dots, k_{n-1}) = 1, \qquad |k_i| \leq 2^{n-2}(n-2)!V. \tag{10.1050}$$

In particular, for every prime p at least one nonzero k_i is prime to p.

Proof. Use all rational primes appearing in a numerator or denominator. Each coordinate of the valuation vector of c_i has absolute value at most $h(c_i)/\log 2 < 2w_i$. Indeed a prime power dividing

the reduced numerator or denominator is at most their maximum. This proves the coordinate bound also for negative valuations. The kernel of this vector map is exactly $\{1, -1\}$.

For (a), choose s rows on which the selected s columns give a nonsingular integer matrix A. Such rows exist by elementary elimination: each independent column increases the pivot count, giving s pivot rows. Write $d_0 = |\det A| \geq 1$. Cramer's formula expresses every column in their rational span with coefficients d_{ji}/d_0, where the signed integers d_{ji} are the determinants obtained by replacing column j, with the sign of $\det A$ absorbed. The coefficients agree on every row, since the chosen rows determine a vector in the selected column span. Cramer's identity itself follows by multiplying by the adjugate: expansion of a determinant along a row gives $A\,\mathrm{adj}(A) = (\det A)I$.

The determinant expansion has $s!$ terms. The coordinate bound therefore gives

$$|d_{ji}| \leq 2^s s! w_i \frac{W}{w_j}. \tag{10.1051}$$

Take $D = 2d_0$ and $M_j = 2\sum_i d_{ji} b_i$. Unique prime factorization gives (10.1049); the even power removes all signs. Since $w_i \leq \Omega$, $W/w_j \leq V$, and $s \leq n-1$,

$$|M_j| \leq 2^{s+1} s! B \frac{W}{w_j} \sum_i w_i \leq 2^{s+1} s! n B \Omega V \leq K_n B \Omega V. \tag{10.1052}$$

For (b), choose a nonempty dependent set of smallest size m among the first $n-1$ vectors. If m=1, its vector is zero and its single coefficient can be one. Otherwise this set has rank m-1. Choose m-1 independent rows. Their signed maximal minors form a nonzero kernel vector, by determinant expansion with a repeated row. Every entry is nonzero: a zero entry would give a dependence on a proper subset. The selected and full matrices both have one-dimensional kernels, so this is a relation among the full vectors. Divide the entries by their greatest common divisor. The same determinant bound gives $2^{m-1}(m-1)!$ times the product of the other column weights, at most the bound in (10.1050), since $m \leq n-1$ and all weights are at least one. The singleton case satisfies that bound too. The product has zero valuation at every rational prime, so is a sign. Primitivity supplies an entry prime to any specified p. □

Proposition 10.231 (a redundant distinguished base). Retain the rational data and bounds of (10.1032), allowing zero coefficients other than b_n, and assume $\Xi \neq 1$. If the p-unit part of a_n lies in the span of the other p-unit parts, then (10.1034) holds. This case does not require the minimum-valuation condition on b_n.

Proof. Put $U = \max\{\Omega L, \varepsilon r\}$, with the notation of (10.1033). A nonzero total p-valuation gives $v_p(\Xi - 1) \leq 0$, so reduce to the p-unit parts c_i. If their rank is at most one, the proof of Theorem 10.228 applies: it permits zero coefficients and uses only the nonzero product condition. Hence assume the rank s is at least two. It is at most n-1. If $\Xi = -1$, the valuation is at most one and the same elementary bound from Theorem 10.228 suffices. Thus $\Xi \notin \{1, -1\}$.

Use Lemma 10.230(a). The independent positive bases $|c_j|$ are p-adic units. Their exponent vector (M_j) is nonzero, because a non-torsion rational number has no positive power equal to one. Put

$$B^* = K_n B \Omega V, \qquad W = \prod_{j \in J} w_j \leq V. \tag{10.1053}$$

Choose a nonzero M_j of minimum p-valuation as the last coefficient. Theorem 10.223 permits the other coefficients to be zero. Use $\mathbb{Q}$ at odd p and $\mathbb{Q}(\zeta_3)$ at two, as in Lemma 10.225. Lemma 10.224 pads the heights to the selected weights. Since these are at least one and $\omega(d) < 1$, its logarithmic coefficient entry is at most $\log(2B^*)$. Also $\log B_\dagger \leq \log B^*$, $G_2(s,d) < 2D_s \leq 2D_n < 2L$, and

$(s+1)f\log p \le 2n\log p < 2L$. The parameter bound for G_2 is the one proved in Theorem 10.226. Consequently, with $C_2^{[s]}$ denoting the second constant for this selected list,

$$v_p(\Xi-1) \le v_p(\Xi^D-1) < C_2^{[s]} W \max\{\log(2B^*), 2L\}. \tag{10.1054}$$

The first inequality follows by factoring a positive integral power of a unit. It needs no assumption on $v_p(D)$.

We now keep the original distinguished bound inside this estimate. For every $n \ge 2$,

$$\log K_n = n\log 2 + \log(n!) < n + n^2 < D_n. \tag{10.1055}$$

Here $n! \le n^n$, $\log 2 < 1$, and $\log n < n$. Since $r \le U/\varepsilon$, expanding $B^* = K_n B_n r\Omega V$ gives

$$\begin{aligned}\log(2B^*) &\le \log(2K_n) + \log B_n + \log(\varepsilon^{-1}) + 3\log V + \log w_n + \log(U/V)\\ &< L + 2\log V + \log w_n + \log(U/V).\end{aligned} \tag{10.1056}$$

Multiply by W. We have $\log V < L$, $WL \le U$, $W\log w_n < Ww_n \le \Omega \le U/L < U$, and $W\log(U/V) < WU/V \le U$. The logarithmic inequalities $\log x < x$ hold for every positive x, and $L > 1$. Thus

$$W\log(2B^*) < 5U, \qquad 2WL \le 2U. \tag{10.1057}$$

Lemma 10.225 gives $2C_2^{[s]} < \mathcal{C}_s(p)$. Moreover (10.1024) directly gives

$$\frac{\mathcal{C}_{j+1}(p)}{\mathcal{C}_j(p)} > (16\mathrm{e})^2 > 256 \qquad (j \ge 2). \tag{10.1058}$$

Since $2 \le s < n$, (10.1054) is less than $(5/2)\mathcal{C}_s(p)U < \mathcal{C}_n(p)U$. This is (10.1034). □

Theorem 10.232 (the rational coefficient refinement, all ranks). Let p be prime, $n \ge 2$, and take all the rational inputs, height bounds and coefficient bounds in (10.1032), with no independence condition. The coefficients may be zero except for $b_n \ne 0$, with $v_p(0) = +\infty$ in the minimum-valuation condition. Assume $\Xi \ne 1$. Then the estimate (10.1034) holds with exactly $\mathcal{C}_n(p), T, \Omega$ and ε specified in (10.1024) and (10.1033).

Proof. As before, reduce to rational p-unit bases c_i, unless the total valuation already proves the result. Their heights are at most w_i. Signs cause no height change. We argue by induction on n. Independent lists are covered by the proof of Theorem 10.226, including zero coefficients: its use of Theorem 10.223 and all inequalities (10.1035)–(10.1037) require only that the distinguished coefficient be nonzero. Lists of rank at most one are covered by the proof of Theorem 10.228, again including zero coefficients. These two cases establish n=2. The torsion product $\Xi = -1$ is included in the elementary argument in that proof, even when the list has larger rank. Hence assume n at least three, intermediate rank, and $\Xi \notin \{1, -1\}$.

If the distinguished vector lies in the span of the others, apply Proposition 10.231. Otherwise the first $n-1$ vectors are dependent, since adjoining the distinguished vector increases their rank by one but the whole list is dependent. Take a primitive relation (k_i) from Lemma 10.230(b). Choose an index $x < n$ for which $p \nmid k_x$, and reverse the signs of all the k_i if necessary so that $k_x > 0$. At every prime put $\kappa_i = 2k_i$. Thus κ_x is positive, $v_p(\kappa_x) = v_p(2)$, and every κ_i has valuation at least $v_p(2)$. Set $\kappa_n = 0$, and put

$$K = K_n V, \qquad b'_i = \kappa_x b_i - \kappa_i b_x \quad (i \neq x), \qquad b'_n = \kappa_x b_n. \tag{10.1059}$$

Lemma 10.230 implies $|\kappa_i| \le K$, so $|b'_i| \le 2KB$. Every b'_i has p-valuation at least $v_p(b_n) + v_p(2)$, and the distinguished coefficient has exactly that valuation. It remains nonzero. This retains the minimum-valuation condition also at two: the common extra factor two occurs in every transferred coefficient.

The doubled relation has product one, regardless of its original sign. Eliminating c_x therefore gives

$$\Xi' = \prod_{i \neq x} c_i^{b'_i} = \Xi^{\kappa_x} \neq 1, \qquad v_p(b'_n) = v_p(b_n) + v_p(2) = \min_{i \neq x} v_p(b'_i). \tag{10.1060}$$

The nonvanishing follows because $\Xi \notin \{1, -1\}$ is rational. Keep the surviving weights and the distinguished last index, and choose

$$\begin{aligned} &B' = 2KB, \quad B'_n = \kappa_x B_n, && \varepsilon' = \frac{\kappa_x}{2K}\varepsilon, \\ &V' = V/w_x, \quad \Omega' = \Omega/w_x, && \varepsilon' B'/B'_n = \varepsilon B/B_n. \end{aligned} \tag{10.1061}$$

These are valid bounds: $B' \ge 3$, $B' \ge B'_n \ge |b'_n|$, and $0 < \varepsilon' \le \varepsilon/2 \le 1/4$. All hypotheses of the induction statement hold for this list of length n-1, including possible zero coefficients.

Let L' be its parameter (10.1033). Exact cancellation of κ_x gives

$$L' = L + \log(2K/w_x) - (D_n - D_{n-1}) - \log p < 2L. \tag{10.1062}$$

Indeed $D_n - D_{n-1} = 12n + 5 > 0$, $w_x \ge 1$, and $\log(2K) = \log 2 + \log K_n + \log V < L$ by (10.1055). Thus $U' = \max\{\Omega' L', \varepsilon' B'/B'_n\} \le 2U$. Induction, the factored-power inequality for local units, and (10.1058) now give

$$v_p(\Xi - 1) \le v_p(\Xi' - 1) < \mathcal{C}_{n-1}(p) U' \le 2\mathcal{C}_{n-1}(p) U < \mathcal{C}_n(p) U. \tag{10.1063}$$

This completes the induction and all rational ranks. □

Theorem 10.232 proves the full rational statement in Theorem 2.3 of Bugeaud's free author paper, and also permits zero non-distinguished coefficients. Its proof uses the second independent-base estimate already proved in Theorem 10.223; no citation substitutes for the reduction of dependent inputs. The general-number-field refinement has additional arithmetic data and is not asserted by this rational theorem.

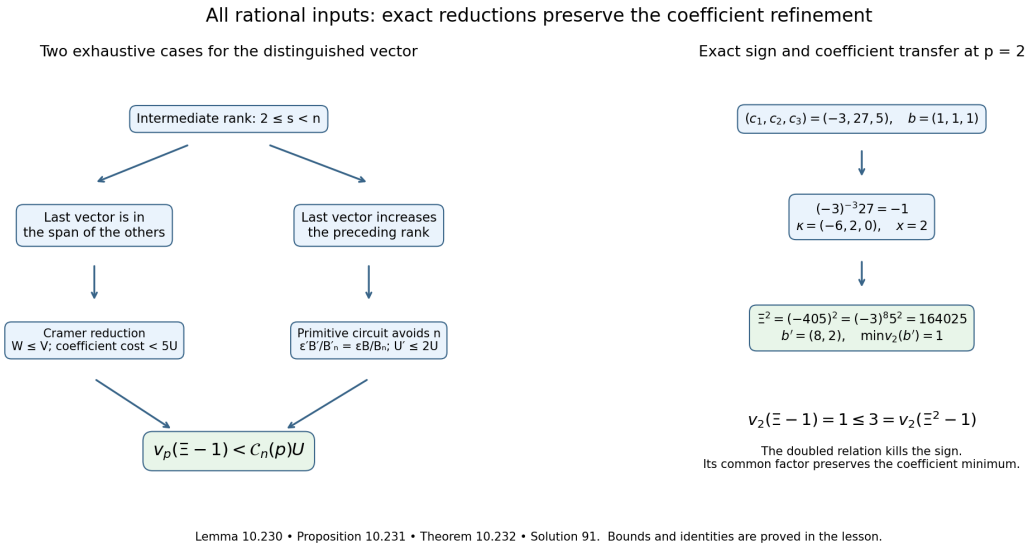


Figure 10.70. The left panel displays the two cases for the distinguished valuation vector used in Theorem 10.232. The redundant-vector case uses Proposition 10.231 and the saved height product W; the other case uses a primitive relation supported on the preceding indices. The right panel records the exact p=2 example of Solution 91, including the common factor in all transferred coefficients. The displayed valuations are exact; the inequalities involving U are proved bounds. Reproducible figure and exact reduction checks. Human-source target: Theorem 2.3 of Bugeaud's free author paper.

78. The second application estimate for every multiplicative rank

The second estimate also admits a bound using the largest coefficient, even when the bases are dependent. Its height floor differs from the first estimate's floor, so we prove the coefficient conversion and the rank reduction with that floor retained.

Keep the field and prime conventions of Sections 52 and 58. Thus the bases are local units in a number field of degree d; at two the field contains a primitive cube root of unity. Write $t = f \log p$, $P = p^{\kappa}$, and $a = a^{(2)}, c = c^{(2)}$, with the constants of (10.641)–(10.643). Here $a = 7$ at odd primes and $a = 13$ at two. Let ρ and ℓ_d have the meaning of (10.55), and let w_K be the number of roots of unity in K. The already proved bounds include $\rho \le 58$, $q^u \le 2d$, and $w_K \ge q^u$. At two, $w_K/q^u \ge 2$, since the orders two and q^u are coprime.

Sort an indexed list $a_1, \dots, a_n$ by nondecreasing height and select its greedy basis $\boldsymbol{\beta}$ modulo roots of unity, of rank $r \ge 1$. At equal heights use any fixed ordering. Let δ_β be the saturated residue index of this basis if $r \ge 2$, and the original residue index if $r = 1$. With $\varkappa = 25$ at odd primes and $\varkappa = 48$ at two, define

$$
\begin{aligned}
X_n &= \max\{n, t\}, \qquad L_n = \frac{X_n}{\varkappa P dt},\\
H_n(a_i) &= \max\{h(a_i), L_n\},\\
\Omega_n &= \prod_{a_i \in \beta} h(a_i) \prod_{a_i \notin \beta} H_n(a_i),\\
M_n &= \max\left\{\frac{p^f}{\delta_\beta}, \frac{\mathrm{e}^n t^{n+1}}{n^n}\right\},\\
A_n &= \max\{4 + \log((n+1)d), e, t\},\\
C_n &= \frac{c(a\mathrm{e}P)^n}{P} \frac{(n+1)^{n+2} d^{n+2} \ell_d}{(n-1)! q^u t^3} M_n A_n,\\
H &= \max\{\log B, t\}, \qquad \mathcal{T}_n = C_n \Omega_n H,\\
\lambda &= \frac{2099}{2100}.
\end{aligned}
\tag{10.1064}
$$

Here e without roman type is the ramification index, $B \geq \max\{3, |b_1|, \dots, |b_n|\}$, and $\Xi = \prod_i a_i^{b_i} \neq 1$. Repeated bases remain distinct indices. The greedy argument of Lemma 10.155 applies to this floor as well: the chosen basis minimizes the weighted product, and is unchanged when a discarded base is removed. The constant C_n uses the full list length n and the residue index of this fixed basis.

The conversion for an independent list

We first record elementary envelopes used in both parts of the proof. Since $\delta_\beta \leq p^f - 1$, we have $M_n > 1$. For $n \geq 2$ and every integer $1 \leq j \leq n+1$,

$$
\frac{t^j}{M_n} \leq \left(\frac{n}{\mathrm{e}}\right)^{nj/(n+1)} < \frac{n^j}{\mathrm{e}^{j-1}}, \qquad \frac{M_n A_n}{t^2} > \frac{1}{n}. \tag{10.1065}
$$

Indeed, replace M_n by $\max\{1, \mathrm{e}^n t^{n+1}/n^n\}$. The quotient with numerator t^j increases up to $t_0 = (n/\mathrm{e})^{n/(n+1)}$ and is nonincreasing after it. Its maximum is t_0^j. The ratio of this value to n^j/e^{j-1} is $\mathrm{e}^{j/(n+1)-1} n^{-j/(n+1)} < 1$. For the last assertion use $A_n \geq t$, the first bound at j=1, and $t_0 < n$.

The factorial argument in Lemma 10.153 also gives, for $n \geq 2$,

$$
R_n := \frac{(n+1)^{n+2}(n-1)^{n-1}}{(n!)^2} > \mathrm{e}^{2n-2}. \tag{10.1066}
$$

In detail, $n! \leq \mathrm{e}\sqrt{n}(n/\mathrm{e})^n$, and $(1+1/n)^{n+2}(1-1/n)^{n-1} > 1$, by the integral inequalities proved there. Substitution proves (10.1066).

Lemma 10.233 (second coefficient-height conversion). Suppose the list is independent and $n \geq 2$. Put $\Omega = \prod_i h(a_i)$ and $W = \lambda t C_n \Omega / (nd \max_i h(a_i))$. If $B/\log B > W$, every entry of the parameter h_2 is at most $(n+1)H$, after moving a nonzero coefficient of minimum p-valuation to the last position.

Proof. Delete a base of maximal height and apply (10.55) to the remaining $n-1$ independent bases. Exact cancellation gives

$$
W \geq \lambda \frac{c\mathrm{e}}{\rho} \frac{w_K}{q^u} \frac{(aP)^n}{P} n R_n \frac{M_n A_n}{t^2} d. \tag{10.1067}
$$

Let P_0 be respectively $3, 3, 5, 1, 1, 1, 4$ in the seven field cases. Then $P \geq P_0$: the defining strict inequality for P supplies these minima, including $P \geq 4$ at two. Let c_0 be the lower bound for

c in the table below, and $\sigma = 2$ in V and one otherwise. Set $K_0 = 4\lambda c_0\sigma/(11\rho P_0)$. Equations (10.1065)–(10.1067), together with $\mathrm{e} < 11/4$, imply

$$W > K_0 d \exp(a_0^{(2)} n), \qquad a_0^{(2)} = 2 + \log(aP_0). \tag{10.1068}$$

The actual degree-one value is $\rho = 17$; the other rows use the admissible upper bound58. The three numerical comparisons needed below are finite, one for each row:

$$\begin{aligned} K_0 a_0^{(2)} &> \exp(a_1^{(2)}), \\ K_0(2a_0^{(2)} + \log d_0 + \log K_0) &> 3\exp(a_1^{(2)}), \\ 4a_0^{(2)} + 2\log K_0 &> 6. \end{aligned} \tag{10.1069}$$

Case	c_0	ρ	P_0	d_0	σ	$a_1^{(2)}$
I.1	1438	58	3	2	1	1.94
I.2	648	17	3	1	1	2.76
II	690	58	5	2	1	0.71
III.1	495	58	1	2	1	1.99
III.2	557	17	1	1	1	3.32
IV	2418	58	1	2	1	3.58
V	406	58	4	2	2	1.48

Here is a reproducible rational verification, with no rounded decimal test used as an inference. For each logarithm reduce its positive argument to $[1, 2]$, evaluate the first32 terms of $2\sum_{k\ge 0} z^{2k+1}/(2k+1)$, and enclose the tail by $2z^{65}/(65(1-z^2))$, where $z = (x-1)/(x+1)$. Bound the exponential at $a_1^{(2)}$ from above by the sum through degree64 plus its geometric tail. These are the enclosures proved in Lemma10.153. The exact calculation records positive rational lower margins for all21 comparisons. It also verifies $2a_0^{(2)} + \log d_0 + \log K_0 > 1$.

The first two inequalities of (10.1069) give, by a positive linear slope in n and monotonicity in d, $K_0(a_0^{(2)} n + \log d + \log K_0) > (n+1)\exp(a_1^{(2)})$ for every $n \ge 2, d \ge d_0$. Thus (10.1068) implies $\log W + \log\log W > G_2(n,d)/(n+1)$. The large-coefficient assumption gives $B > W\log B > W\log W$, because $\log B > 1$, $W > 1$, and the logarithm increases. It follows that $(n+1)\log B > G_2(n,d)$.

The singleton height bound used in (10.466) gives $R_{\max}/(dW(d)) \le \rho e d^2 B$. Also $\log\rho + 1 < 6$. The third inequality in (10.1069) implies $n\log W > 2\log d + 6$: at n=2 it is precisely that inequality, and $a_0^{(2)} n^2 + n\log K_0 + (n-2)\log d - 6$ increases for $n \ge 2, d \ge 1$, since $a_0^{(2)} > 3$ and $\log K_0 > -1$. Hence the logarithmic ratio entry is below $(n+1)\log B$. The entries $\log B_\dagger$ and $(n+1)t$ are bounded by $(n+1)H$ directly. This proves the lemma. □

Theorem 10.234 (independent second application estimate). For every independent local-unit list of length $n \ge 2$, $\mathrm{ord}_{\mathfrak{p}}(\Xi - 1) < \mathcal{T}_n$. The singleton case has the stronger bound $\mathcal{T}_1/4000$.

Proof. At length one this is Theorem10.161. For $n \ge 2$, the full height-product bound (10.55) and (10.1065) give

$$\frac{t\mathcal{T}_n}{dH} > \frac{c}{\rho}\frac{w_K}{q^u} a^n P^{n-1} \frac{n^n (n+1)^{n+2}}{(n!)^2} > 7900. \tag{10.1070}$$

For the last strict inequality the rank factor is at least72, as proved in (10.464); use $c \ge 200, a \ge 7, \rho \le 58$. In particular $\mathcal{T}_n > 2100(d/t)\log 2$. The local height inequality gives $\mathrm{ord}_{\mathfrak{p}}(\Xi - 1) \le$

$(d/t)(nB\max_i h(a_i)+\log 2)$. If $B/\log B\le W$, its first term is at most $\lambda\mathcal{T}_n$ and its second is strictly less than $\mathcal{T}_n/2100$. Otherwise Lemma10.233 supplies every coefficient condition, and Theorem10.223 gives the asserted bound, since $C_n=(n+1)C_2$. □

Uniform budgets for removing a dependent base

Lemma 10.235 (second weighted rank budgets). Let $2\le r<n$, and define $W_n=\lambda tC_n\Omega_n/(ndh(a_n))$. Then

$$\mathcal{T}_n>7900\frac{d}{t}\log 2,\qquad W_n>200\frac{w_K}{q^u}d\mathrm{e}^n>\max\{d\mathrm{e}^n,w_K\}. \tag{10.1071}$$

If $B/\log B>W_n$ and the first base is not torsion, let x be the first discarded index of the greedy procedure. There is an exact circuit relation, with positive coefficient at x, whose coefficients have absolute value at most K, where

$$K<\begin{cases}B/8, & x<n,\\ BdH_n(a_x)/8, & x=n.\end{cases} \tag{10.1072}$$

If the list after removal has length $n-1$ and modified height product Ω', with the same greedy basis, then

$$\frac{C_n\Omega_n}{C_{n-1}\Omega'}>a\mathrm{e}^2Pt\frac{dH_n(a_x)}{X_n}. \tag{10.1073}$$

Proof. The height-product bound (10.55), including the empty-list case proved in Lemma10.155, is

$$\prod_{i=1}^{j}h(\gamma_i)\ge\frac{w_K}{\rho d^{j+1}\ell_d}\frac{j^j}{j!\mathrm{e}^j} \tag{10.1074}$$

for an independent list of length j, with the last ratio one at j=0. Flooring the other heights and applying (10.1065) with $j=n-r+1$, using $A_n\ge t$, gives

$$\frac{t\mathcal{T}_n}{dH}>\frac{c}{\rho}\frac{w_K}{q^u}a^r\left(\frac{a\mathrm{e}^2}{\varkappa}\right)^{n-r}P^{r-1}\frac{(n+1)^{n+2}}{n!}\frac{r^r}{r!}. \tag{10.1075}$$

The positive exponential series gives $\mathrm{e}>1957/720$, its sum through degree six. Exact rational squaring shows $a\mathrm{e}^2/\varkappa>2$ in both prime cases. Also $(n+1)^{n+2}/n!$ increases with n and is at least$81/2$ for $n\ge2$, and $r^r/r!\ge2$ for $r\ge2$. The right side of (10.1075) is therefore greater than $200\cdot7^2\cdot81/58=396900/29>7900$. Since $H>1>\log 2$, this proves the first assertion.

For W, remove a_n from its product. If a_n is selected, put k=r and apply (10.1074) to the remaining k-1 selected bases. Otherwise put k=r+1, use $H_n(a_n)/h(a_n)\ge1$, and apply it to the same k-1 selected bases. In both cases $1\le k\le n$, and direct substitution gives

$$\begin{aligned}W_n\ge\lambda\frac{c}{\rho}\frac{w_K}{q^u}dA_n\frac{a^n\mathrm{e}^{n-k+1}(n+1)^{n+2}}{n!}\frac{(k-1)^{k-1}}{(k-1)!}\\ \times\frac{P^{k-1}X_n^{n-k}}{\varkappa^{n-k}}\frac{M_n}{t^{n-k+2}}.\end{aligned} \tag{10.1076}$$

Use (10.1065), $X_n\ge n$, and $n!\le n^{n+1}\mathrm{e}^{1-n}$, the latter obtained by bounding $\sum_{j<n}\log j$ by $\int_1^n\log x\,dx$ and then multiplying by n. Drop the factors $P^{k-1}\ge1$ and $(k-1)^{k-1}/(k-1)!\ge1$. The result is

$$\frac{W_n}{d\mathrm{e}^n} > \lambda \frac{c\mathrm{e}}{\rho} \frac{w_K}{q^u} \frac{A_n}{n} a^k \left(\frac{a\mathrm{e}^2}{\varkappa} \right)^{n-k} > \frac{67168}{261} \frac{w_K}{q^u}. \tag{10.1077}$$

Here $2^{n-1} \geq n$, $a \geq 7$, $A_n \geq 4$, $c \geq 200$, $\rho \leq 58$, and $\mathrm{e} > 8/3$. The last rational number exceeds200. Since $q^u \leq 2d$, both remaining assertions of (10.1071) follow.

For the circuit, choose v earlier selected bases forming a minimal dependent set with a_x, where $1 \leq v \leq r$. Theorem10.20 supplies an exact relation. All earlier heights are at most $h(a_x)$, and its coefficient bound is

$$K \leq \rho d^{v+1} \ell_d \frac{v! \mathrm{e}^v}{v^v} \frac{h(a_x) \prod_{a_i \in S} h(a_i)}{\min_{a_i \in S} h(a_i)}. \tag{10.1078}$$

Set $j = r - v + 1$ and $s = n - r - 1 \geq 0$. The selected bases outside S, together with its smallest-height member, form an independent list of length j. Apply (10.1074). Then

$$\frac{K}{\Omega_n} \leq \frac{\rho^2 d^{r+3} \ell_d^2}{w_K} \mathrm{e}^{r+1} \frac{v! j!}{v^v j^j} \frac{h(a_x)}{H_n(a_x)} L_n^{-s}. \tag{10.1079}$$

We give the full cancellation needed to normalize this bound. At x=n, divide by $BdH_n(a_x)/8$, use $B > W_n \log B$, $\log B > 1$, and substitute (10.1064). Use $q^u/w_K \leq 1$, $\ell_d/A_n \leq 1$, $h(a_x)/H_n(a_x) \leq 1$, and $v!j!/(v^v j^j) \leq 1$. The normalized ratio is at most

$$\frac{8n\rho^2}{\lambda c a^{r+1}} \frac{(n-1)!}{(n+1)^{n+2}} \left(\frac{\varkappa}{a\mathrm{e}} \right)^s \frac{t^{s+2}}{P^r X_n^s M_n}. \tag{10.1080}$$

The degree powers cancel because $r + 3 + s = n + 2$. If x<n and a_n is selected, cancel its height before applying (10.1074), using length j-1 instead of j. The same envelope has an extra factor 1/e, including j=1 by its empty-list case. If x<n and a_n is not selected, cancel its height against $H_n(a_n)$; now s is at least one. The same expression has $\varkappa^s/P^r$ replaced by $\varkappa^{s-1}/P^{r+1}$, and t^{s+2}/X_n^s by t^{s+1}/X_n^{s-1}. These cancellations cover every position of the maximal-height base.

In (10.1080), (10.1065) and $X_n \geq n$ give $t^{s+2}/(X_n^s M_n) < n^2$. They give the same bound for the altered expression, with exponent s+1. Use $(n-1)! \leq n^n \mathrm{e}^{1-n}$, and $\varkappa/(a\mathrm{e}^2) < 1/2$. All three cases are bounded by

$$\frac{8\rho^2 n}{\lambda c a^{r+1} \mathrm{e}^r} 2^{-s} \leq \frac{24 \cdot 58^2}{\lambda \cdot 200 \cdot 7^3 (8/3)^2} = \frac{68121}{411404} < \frac{1}{5} < 1. \tag{10.1081}$$

For the middle inequality the maximum occurs at r=2,s=0,n=3: increasing s multiplies $n2^{-s}$ by $(n+1)/(2n) < 1$, and increasing r multiplies $n/(a^{r+1}\mathrm{e}^r)$ by $(n+1)/(na\mathrm{e}) < 1$. This proves (10.1072) uniformly, not just at the sampled ranks.

Finally the two entries of M_n have ratios one and $\mathrm{e}t(n-1)^{n-1}/n^n > t/n$ compared with M_{n-1}. Their maxima therefore have ratio at least t/X_n. Also $A_n \geq A_{n-1}$, giving

$$\frac{C_n}{C_{n-1}} \geq a\mathrm{e}Pd \frac{t}{X_n} \frac{(n+1)^{n+2}}{(n-1)n^{n+1}}. \tag{10.1082}$$

The last factor exceeds e, since its logarithm is $(n+2)\log(1+1/n) - \log(1-1/n) > 1 + 5/(2n) - 1/n^2 > 1$. This uses $\log(1+y) \geq y - y^2/2$ and $-\log(1-y) > y$, proved by integration. The floor L_n is nondecreasing in n, so $\Omega_n/\Omega' \geq H_n(a_x)$. These prove (10.1073). □

Theorem 10.236 (the full second application estimate). For every indexed local-unit list with positive multiplicative rank,

$$\operatorname{ord}_{\mathfrak{p}}(\Xi-1)<C_n\Omega_nH. \tag{10.1083}$$

At rank one, the right side may be divided by4000. The constant, floor and greedy basis are exactly those in (10.1064).

Proof. Rank one, including all torsion patterns, is Theorem10.163. Full rank is Theorem10.234. Induct on n in the remaining cases $2\le r<n$. The small-coefficient alternative $B/\log B\le W_n$ follows from the local height inequality and (10.1071), with its two contributions respectively at most $\lambda\mathcal{T}_n$ and less than $\mathcal{T}_n/2100$.

Otherwise $B>W_n\log B>W_n$, so $\log B>n+\log d$, $B>w_K$, and $H\ge X_n$. If the first base is not torsion, use the exact circuit of Lemma10.235. Raising Ξ to its positive x-th coefficient removes a_x and leaves every selected base, hence the same basis and residue index. Its new integer coefficients are bounded by2BK. If this power is one, Ξ is a nontrivial root of unity and Theorem10.21 bounds its valuation by $(d/t)\log 2<\mathcal{T}_n$.

Otherwise put $z=dH_n(a_x)$. The bounds (10.1072) give coefficients at most $B^2/4$ if x<n, and at most $B^2z/4$ if x=n. As $\log u\le u/\mathrm{e}$ for positive u, a valid new bound is $B'=B^2\exp(z/(4\mathrm{e}))$. Consequently

$$H'\le H\left(2+\frac{z}{4\mathrm{e}X_n}\right). \tag{10.1084}$$

The ratio (10.1073) is strictly larger than this increase. Indeed $z/X_n\ge 1/(\varkappa Pt)$, and

$$a\mathrm{e}^2-\frac{1}{4\mathrm{e}Pt}>2\varkappa. \tag{10.1085}$$

At an odd prime, $Pt>1$ and the strict margin is at least $7(1957/720)^2-50-3/32=840343/518400>0$. At two, the primitive cube root forces $f\ge 2$; hence $Pt\ge 4\log 4>16/3$. The strict margin there is at least $13(1957/720)^2-96-9/512=25049/1036800>0$. Thus $(a\mathrm{e}^2Pt-1/(4\mathrm{e}))z/X_n>2$. Apply the inductive bound to the shorter list and the elementary factored-power inequality to return to Ξ. Equations (10.1073)–(10.1085) prove (10.1083).

If the first base is torsion, raise Ξ to its order $w\le w_K<B$, deleting that base. Again a resulting power equal to one is handled by the torsion bound. Otherwise its coefficients are below B^2, so $H'\le 2H$. The greedy basis is unchanged. In (10.1073) its deleted modified height is L_n, so the ratio is greater than $a\mathrm{e}^2/\varkappa>2$. Induction and the same factored-power inequality finish this case. Every list of positive rank is now covered. □

This proves the second application estimate of Theorem2 in Yu's freely accessible 2013 paper, with its original modified heights and residue index. The proof retains the stronger rank-one factor already established in Theorem10.163. Rank zero is separate: a nontrivial torsion product has valuation at most $(d/t)\log 2$, by Theorem10.21.

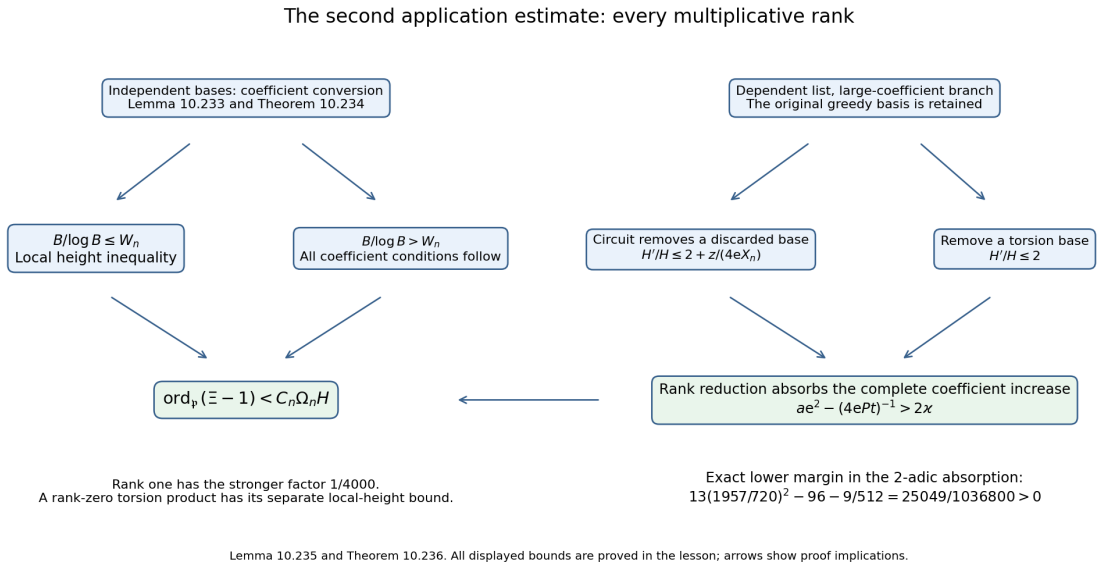


Figure 10.71. The left panel shows the independent-list coefficient conversion. Its local-height alternative also applies to dependent lists. The right panel shows the remaining large-coefficient branch for dependent lists: a circuit or a torsion relation removes one unselected base while preserving the chosen basis and its residue index. Lemma10.235 proves that the ratio of successive full budgets exceeds the complete increase in the coefficient parameter; Theorem10.236 returns the bound to the original product. The displayed 2-adic margin is an exact positive rational number, not a numerical sample. Rank one retains its separate factor1/4000. Reproducible figure and exact checks. Human-source context: Theorem2 of Yu's freely accessible2013 paper.

79. The number-field coefficient refinement for independent normalized units

This section uses the absolute logarithmic height and the prime-ideal valuation. Write $d = [K : \mathbb{Q}]$, $e = e_{\mathfrak{p}}$, $f = f_{\mathfrak{p}}$, $N = p^f$, and $t = \log N$. The symbol e denotes the exponential constant, not the ramification index. At two, assume in this section that $\zeta_3 \in K$. The remaining two-adic fields and dependent normalized lists are not covered by the theorem below.

For $n \geq 2$, define

$$\begin{aligned} Q &= 16\mathrm{e}^2 d^2, & D_n &= (n+1)(6n+5), \\ \mathcal{A}_n &= (16\mathrm{e}d)^{2(n+1)} n^{3/2} \log(2nd) \log(2d) \frac{e^n N}{t^2}, & & \\ c_3(n,d) &= (2d)^{2n+1} \log(2d) \log^3(3d), & & \\ c_4(n,d) &= 2 \exp(D_n) d^{3n} \log(2d), & J &= e + d/f. \end{aligned} \tag{10.1086}$$

The target is the number-field estimate stated in Theorem B of the free author paper by Bérczes, Evertse and Győry. Here it is proved for the explicitly specified class of inputs by reducing to Theorem 10.223. The external statement supplies no proof step.

Lemma 10.237 (normalization inside the original number field). For $\alpha_i \in K^*$, put

$$v_i = \mathrm{ord}_{\mathfrak{p}}(\alpha_i), \qquad \beta_i = \alpha_i^e p^{-v_i}. \tag{10.1087}$$

Each β_i is a local unit in K and $h(\beta_i) \leq J h(\alpha_i)$. For integers b_i, put $\Xi = \prod_i \alpha_i^{b_i}$ and $s = \sum_i b_i v_i$. If $s \neq 0$, then $\mathrm{ord}_{\mathfrak{p}}(\Xi - 1) = \min\{s, 0\}$. If $s = 0$, then

$$\prod_i \beta_i^{b_i} = \Xi^e. \tag{10.1088}$$

If this last product is not one, its valuation after subtracting one is at least $\operatorname{ord}_{\mathfrak{p}}(\Xi-1)$.

Proof. The normalization $\operatorname{ord}_{\mathfrak{p}}(p)=e$ gives $\operatorname{ord}_{\mathfrak{p}}(\beta_i)=ev_i-ev_i=0$. The local contribution to the height of α_i or its inverse gives $|v_i|f\log p\le dh(\alpha_i)$, as in the proof of Theorem 10.21. Hence the product and power inequalities for height give

$$h(\beta_i)\le eh(\alpha_i)+|v_i|\log p\le (e+d/f)h(\alpha_i).$$

The unequal-valuation rule gives the assertion for $s\neq 0$. Multiplication proves (10.1088) when s vanishes. In that case Ξ is a unit. The identity $\Xi^e-1=(\Xi-1)(1+\Xi+\cdots+\Xi^{e-1})$ has an integral second factor, proving the final assertion. No root extraction, field extension, or change of valuation normalization is involved. □

Lemma 10.238 (absorbing the normalization cost). For every independent list of local units in the field cases of Theorem 10.152, its second constant satisfies

$$2C_2J^n<\mathcal{A}_n\qquad (n\ge 2). \tag{10.1089}$$

Proof. Retain the actual saturated residue index in C2. It is at least one. Formula (10.1027) gives

$$\max\{N/\delta(\boldsymbol{\beta}),\mathrm{e}^n t^{n+1}/n^n\}<(n+1)N.$$

The factorial bound (10.1026), and $J\le 2d/f$, therefore give, with $a=a^{(2)}$, $c=c^{(2)}$, and $P=p^{\kappa}$,

$$\frac{2C_2J^n}{\mathcal{A}_n}<\frac{c}{128\mathrm{e}q^u}\left(\frac{a}{128f}\right)^n\left(\frac{P}{e}\right)^{n-1}\frac{(n+1)^3}{n^{3/2}}\frac{\ell_dA_n}{et\log(2nd)\log(2d)}. \tag{10.1090}$$

Suppose first that $d\ge 2$. Then $\ell_d\le\log(2d)$, and $\log(2nd)\ge\log 8>2$. Also $\log((n+1)d)\le\log(2nd)$. The three entries of $A_n=\max\{4+\log((n+1)d),e,t\}$ consequently show that

$$\frac{A_n}{et\log(2nd)}<\begin{cases}3, & p \text{ odd},\\ 9/4, & p=2,\ \zeta_3\in K.\end{cases} \tag{10.1091}$$

For the first entry use $(4+\log(2nd))/\log(2nd)<3$. For the other two entries use respectively $1/(t\log(2nd))$ and $1/(e\log(2nd))$. At odd primes $t>1$. At two, injectivity of reduction on the prime-to-two roots of unity gives $3\mid 2^f-1$, hence $f\ge 2$ and $t\ge\log 4>4/3$. These verify both rows.

At odd primes $P/e\le 3$, $q^u\ge 2$, $c\le 2418$, and a=7. At two $P/e\le 4$, $q^u\ge 3$, c=406, and a=13. The bounds on P/e follow directly from (10.641). Using e $>8/3$, (10.1090) is therefore bounded by the first two rows below. For $d=1$, the admissible fields here have odd p; e=f=1 and J=2. At three we have P=3,c=648; at $p\ge 5$ we have P=1 and $c\le 2228/3$. Since $\log(2n)>4/3$, $\log 2>2/3$, and $A_n/t\le n+4$ at three and $A_n/t\le 2(n+4)/3$ at primes at least five, (10.1090) gives the last two rows. Here $F_n=(n+1)^3/n^{3/2}$.

Field range	Upper bound for $2C_2J^n/\mathcal{A}_n$
$d\ge 2$, odd p	$(3627/1024)(21/128)^nF_n$
$d\ge 2$, p=2	$(1827/8192)(13/64)^nF_n$
d=1, p=3	$(729/2048)(21/128)^n(n+4)F_n$
d=1, $p\ge 5$	$(1671/2048)(7/128)^n(n+4)F_n$

These bounds hold for every integer n at least two. To verify that they are below one uniformly, use $F_2=27/(2\sqrt{2})<135/14$, since $\sqrt{2}>7/5$. The four resulting rational endpoints are below one. Their successive ratios are at most, respectively,

$$\frac{7}{18}, \qquad \frac{13}{27}, \qquad \frac{49}{108}, \qquad \frac{49}{324}. \tag{10.1092}$$

Indeed $F_{n+1}/F_n < ((n+2)/(n+1))^3 \le 64/27$, and the extra factor in the last two rows has ratio $(n+5)/(n+4) \le 7/6$. Thus every envelope decreases. The exact endpoint fractions are recorded by the accompanying calculation; the argument here covers the infinite rank range. □

Theorem 10.239 (the number-field coefficient refinement for independent normalized units). Let $n \ge 2$, $\alpha_i \in K^*$, and choose real weights and coefficient bounds such that

$$\begin{aligned} w_i \ge \max\{h(\alpha_i), Q^{-1}\}, \qquad b_i \in \mathbb{Z}, \quad b_n \ne 0, \quad v_p(b_n) = \min_i v_p(b_i), \\ B \ge \max_i |b_i|, \qquad B \ge B_n \ge |b_n|, \qquad 0 < \varepsilon \le \tfrac{1}{2}. \end{aligned} \tag{10.1093}$$

Use $v_p(0) = +\infty$. Assume that the units β_i of (10.1087) are multiplicatively independent. Put

$$V = \prod_{i<n} w_i, \qquad \Omega = V w_n, \qquad r = B/B_n, \qquad L = \log\left(B_n c_4(n,d) N^{n+1} V/\varepsilon\right). \tag{10.1094}$$

Then

$$\operatorname{ord}_{\mathfrak{p}}(\Xi - 1) < \mathcal{A}_n \max\left\{\Omega L, \frac{\varepsilon r}{c_3(n,d)}\right\}. \tag{10.1095}$$

In particular, choosing the smallest allowed weights gives the exact height floor and constants in the stated number-field target, on this class of inputs.

Proof. We first verify all scalar comparisons, including the small height floor. The exponential series gives $\log 16 < 3$ and $\log 52 < 4$. For the second assertion one may use $\mathrm{e} > 1957/720 > 19/7$ and $(19/7)^4 > 52$. Since $\log Q < 5 + 2\log d$ and $2\log(2d) > 1$,

$$\log(c_4/Q^n) > D_n - 5n + n\log d > 0. \tag{10.1096}$$

For every x>0, maximizing $(\log x)/x^{1/3}$ gives $\log x \le 3x^{1/3}/\mathrm{e}$. At x=3d this implies $\log^3(3d) \le 81d/\mathrm{e}^3 < 5d$. Consequently

$$\frac{c_3}{c_4} = 4^n \exp(-D_n) d^{1-n} \log^3(3d) < 5\, 4^n \exp(-D_n) < 1.$$

The last step uses $\mathrm{e} > 2$ and $D_n > 2n + 3$. Using $V \ge Q^{1-n}$, $B_n \ge 1$, and $\varepsilon^{-1} \ge 2$, we also obtain

$$L > \log B_n + (n+1)t + D_n - 5(n-1) + (n+2)\log d > 1. \tag{10.1097}$$

Here the discarded constant is $\log(4\log(2d)) > 0$. All quantities in the claimed bound are therefore positive.

If $\sum b_i v_i \ne 0$, Lemma 10.237 proves the conclusion immediately. Otherwise Ξ is a unit and $\prod \beta_i^{b_i} = \Xi^e \ne 1$, by independence and $b_n \ne 0$. Enlarge only the last unnormalized weight:

$$w_n^* = \max\left\{w_n, \frac{\varepsilon r}{c_3 V L}\right\}. \tag{10.1098}$$

Use weights $J w_j$ for $j < n$ and $J w_n^*$ for the independent units. Lemma 10.237 verifies these height bounds. Put $\omega(d) = 1/(dW(d))$. Formula (10.222) implies $\omega(d) < 1$: at d=1, $\omega(1) =$

$\log 2 \log 3 / \log 6 < 1$; at d at least two, $d \log^3(3d) > 1$. Since $J \geq 1$, every coefficient entry in Lemma 10.224 is at most the logarithm of

$$B_n \left(Q + \frac{c_3 V L}{\varepsilon} \right) < \exp(L)(1 + L) < \exp(2L). \tag{10.1099}$$

For the first strict inequality, (10.1096) and $Q^{n-1}V \geq 1$ give $\exp L > B_n Q$; $c_4 > c_3$ gives $\exp L > B_n c_3 V/\varepsilon$. The second is the positive-series inequality $1 + L < \exp L$.

Also $\log B_\dagger \leq \log B_n < L$, and $(n+1)t < L$. Every row of (10.643) has $a_0^{(2)} < 6$ and $a_1^{(2)} < 4$, so

$$G_2(n, d) < (n + 1)(7n + 4 + \log d) < 2L.$$

To check the final inequality, subtract this upper bound from twice the non-t term of (10.1097). The difference is $5n^2 + n + 16 + (n + 3) \log d > 0$. Thus the entire padded coefficient maximum is less than 2L.

Lemma 10.224, Theorem 10.223, and Lemma 10.237 now give

$$\operatorname{ord}_{\mathfrak{p}}(\Xi - 1) \leq \operatorname{ord}_{\mathfrak{p}}(\Xi^e - 1) < 2C_2 J^n V L w_n^* = 2C_2 J^n \max\{\Omega L, \varepsilon r/c_3\}. \tag{10.1100}$$

Lemma 10.238 proves (10.1095). The original distinguished coefficient, field degree, residue degree and prime-ideal valuation have all been retained. □

An exact quadratic example

Let $K = \mathbb{Q}(\sqrt{2})$, p=7, and choose the place with $\sqrt{2} \equiv 3 \pmod 7$. The simple root 3 of $X^2 - 2$ modulo seven lifts to $\mathbb{Q}_7$, by the proved lifting lemma, so this place has e=f=1. Take

$$\alpha_1 = 7(1 + \sqrt{2}), \qquad \alpha_2 = (6 + \sqrt{2})/7, \qquad b_1 = b_2 = 1.$$

Their valuations are 1 and -1. The normalized units are $\beta_1 = 1 + \sqrt{2}$, $\beta_2 = 6 + \sqrt{2}$, and J=3. Multiplication gives $\Xi = 8 + 7\sqrt{2}$ and $\Xi - 1 = 7(1 + \sqrt{2})$, whose valuation is exactly one. They are independent: conjugation sends their products to norms -1 and 34. A torsion relation would force the exponent of the second base to vanish; the real value $1 + \sqrt{2} > 1$ then forces the first exponent to vanish.

The primitive minimal polynomials of the original inputs are $X^2 - 14X - 49$ and $49X^2 - 84X + 34$. The height formula therefore gives

$$h(\alpha_1) = \log 7, \qquad h(\alpha_2) = \tfrac{1}{2} \log(7(6 + \sqrt{2})), \qquad h(\beta_1) = \tfrac{1}{2} \log(1 + \sqrt{2}), \qquad h(\beta_2) = \tfrac{1}{2} \log 34.$$

For the first polynomial both roots have absolute value greater than one; for the second only $(6 + \sqrt{2})/7$ does. These facts verify the displayed heights directly. With B=Bn=1 and $\varepsilon = 1/2$, the theorem applies although neither original input is a local unit.

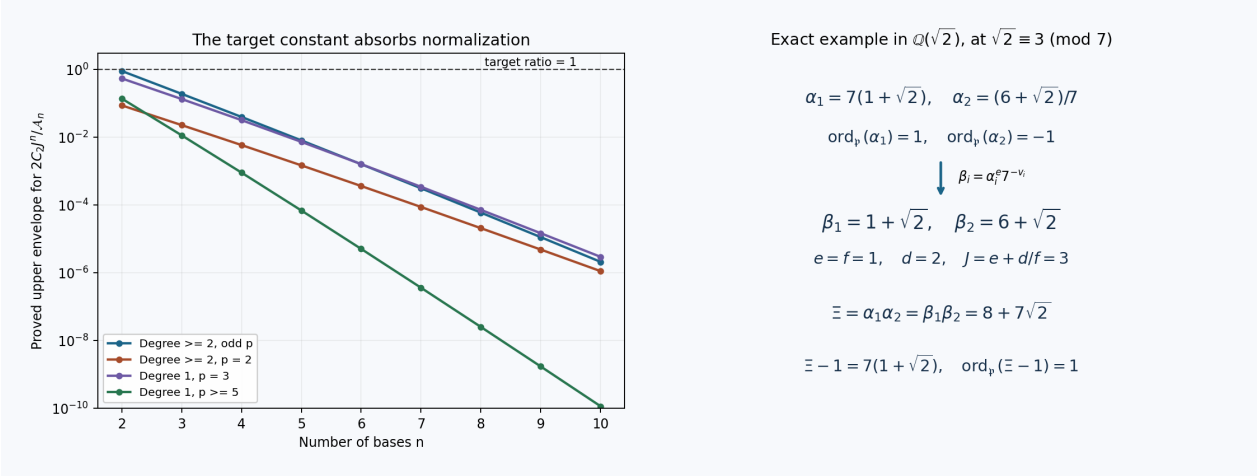


Figure 10.72. The left panel samples the four upper envelopes proved in Lemma 10.238; the endpoint and successive-ratio argument covers every rank, beyond the displayed samples. The two-adic row assumes that the field contains a primitive cube root of unity. The right panel shows the exact original-field normalization of Lemma 10.237 and the quadratic example: both original inputs are nonunits, their valuations cancel, and the product minus one has valuation one. Theorem 10.239 retains the distinguished coefficient and absorbs the height cost J. Reproducible diagram and exact rational checks. Human-source target: Theorem B of Bérczes, Evertse and Győry.

80. The number-field coefficient refinement for dependent inputs

We retain $K, d, e, f, N, t, Q, J, \mathcal{A}_n, c_3, c_4, D_n$ from Section 79. Thus p is odd, or p=2 and $\zeta_3 \in K$. We first work with arbitrary local units β_i, with positive weights satisfying $h(\beta_i) \le Jw_i$ and $w_i \ge Q^{-1}$. The units may be dependent or torsion. Define

$$u_i = Qw_i \ge 1, \qquad R = \prod_{i<n} u_i = Q^{n-1}V, \qquad S = Ru_n = Q^n\Omega, \qquad A = 58d\ell_d. \tag{10.1101}$$

The scalar $J = e + d/f$ remains fixed when an index is deleted. It is distinct from the saturation index used below. The bounds B, B_n, ε and the quantities V, Ω, r, L have their meanings in (10.1093)–(10.1094). Put

$$U = \max\{\Omega L, \varepsilon r/c_3(n,d)\}, \qquad \widehat{U} = Q^n U. \tag{10.1102}$$

In particular $\widehat{U} \ge SL > 1$. All inequalities for L in Theorem 10.239 depend only on the weights and coefficient bounds, so remain valid here without independence.

Lemma 10.240 (weighted relations and common denominators). The following assertions hold for these local units and weights.

(a) If the first n-1 bases are dependent, they admit a nonzero exact integer relation $\prod_{i<n} \beta_i^{k_i} = 1$, with $|k_i| \le \kappa := AR$.

(b) Suppose that all bases lie, modulo roots of unity, in the rational span of an independent sublist of the first n-1 bases. Let I be the indices of this sublist, of length $s \ge 1$, and put $W_u = \prod_{j\in I} u_j$, $W = Q^{-s}W_u$. For $Y = \prod_i \beta_i^{b_i}$, there are a positive integer D and integers M_j, j in I, such that

$$Y^D = \prod_{j\in I} \beta_j^{M_j}, \qquad |M_j| \le B^* := nA^2BSR^2, \qquad W_u \le R. \tag{10.1103}$$

Proof. The empty-list height bound in Lemma 10.155 gives $w_K \le 58d\ell_d = A$. For a minimal circuit of rank a at least one, Corollary 10.20 bounds each exact relation coefficient by

$$58d^{a+1}\ell_d \frac{a!\mathrm{e}^a}{a^a}\left(\frac{J}{Q}\right)^a \prod u_j \le A\left(\frac{\mathrm{e}dJ}{Q}\right)^a \prod u_j \le A\prod u_j.$$

The product omits the coefficient's own index. We used $a! \le a^a$ and $\mathrm{e}dJ/Q \le 1/(8\mathrm{e}f) < 1$. Every missing u-factor can be inserted since it is at least one. A singleton torsion circuit instead uses its order, at most w_K. Applying these alternatives to a minimal dependence among the first n-1 indices proves (a).

For (b), write $\gamma_j = \beta_j$ for j in I. Each base has unique rational coordinates x_{ji} in this basis modulo torsion. Concretely, an integral multiple of its coordinate vector expresses an integral power of β_i as a product of powers of the γ_j, up to a root of unity. Raising once more by w_K removes that root. Thus the coordinate vector belongs to the lattice $M_K(\boldsymbol{\gamma})$ of Theorem 10.18. Let $m = [M_K : \mathbb{Z}^s]$. The order of each class divides m, by partitioning the finite group into cosets of its cyclic subgroup. Hence $mx_{ji} \in \mathbb{Z}$. The saturation bound (10.55), with the same calculation as above, gives

$$w_K m \le AW_u. \tag{10.1104}$$

The quotient $\beta_i^m / \prod_j \gamma_j^{mx_{ji}}$ lies in K and has a positive power equal to one. Its order therefore divides w_K. Taking $D = w_K m$ and $M_j = D\sum_i x_{ji}b_i$ gives the exact identity in (10.1103).

We also need an individual coordinate bound. If β_i is torsion its coordinates vanish. Otherwise its nonzero coordinates select a minimal circuit consisting of that base and the corresponding basis members. The coefficient at β_i is a nonzero integer, so has absolute value at least one. Dividing the simultaneous bounds of Corollary 10.20 by this coefficient gives

$$|x_{ji}| \le Au_iW_u/u_j. \tag{10.1105}$$

For a base already in I its coordinates are Kronecker symbols and the same bound holds. Consequently

$$|M_j| \le A^2B(W_u^2/u_j)\sum_i u_i \le nA^2BSR^2.$$

Here $u_i \le S$, $u_j \ge 1$, and $W_u \le R$. All denominators and roots of unity have been removed in the original field. □

Lemma 10.241 (logarithmic budgets and the elementary cases). For n at least two the preceding data satisfy

$$\log R < L, \quad \log(2Q) < L, \quad \log(2AB_n) < L, \quad \log\kappa < L - (D_n - 5n) - n\log d. \tag{10.1106}$$

For the basis in Lemma 10.240(b), put $F = Q^{n-s}U$. Then

$$W\log(2QB^*) < 8F, \qquad 2WL \le 2F. \tag{10.1107}$$

Finally, suppose Y is not one. The desired estimate $\mathrm{ord}_{\mathfrak{p}}(Y-1) < \mathcal{A}_nU$ holds if Y is torsion; if the list has rank one and its distinguished last base lies in the span of the others; or if all preceding bases are torsion.

Proof. We have $\ell_d \le d$, $\log 58 < 5$, and $\log Q < 5 + 2\log d$. Hence $\log\kappa < 5n + 2n\log d + \log V$. On the other hand, the definition gives $L > D_n + 3n\log d + \log V$, since $B_n/\varepsilon \ge 2$, $2\log(2d) > 1$, and N>1. This proves the last inequality of (10.1106), and the first follows since A>1. The explicit lower bound (10.1097), together with $\log(2Q) < 6 + 2\log d$ and $\log(2A) < 6 + 2\log d$, proves the other two.

We first bound a scalar coefficient needed for (10.1107):

$$\frac{nA^2c_3(n,d)}{Qc_4(n,d)} < 1. \tag{10.1108}$$

Indeed, $A^2/Q < 30\ell_d^2$, and exact cancellation gives $c_3/c_4 = 4^n \exp(-D_n) d^{1-n} \log^3(3d)$. Since $\ell_d \le \log(3d)$, maximizing $(\log x)^5/x$ for x at least one yields

$$d^{1-n}\ell_d^2 \log^3(3d) \le \frac{\log^5(3d)}{d} \le 3(5/\mathrm{e})^5 < 3(15/8)^5 < 70.$$

Thus the left side of (10.1108) is less than $2100n4^n \exp(-D_n)$. Use $2100 < 2^{12}$, $n \le 2^{n-1}$, e > 2, and $D_n > 3n + 11$ for n at least two to prove (10.1108) for every n. Because $Q^nV \ge Q$, it follows that

$$\log\left(\frac{nA^2c_3B_n}{\varepsilon Q^n}\right) < L. \tag{10.1109}$$

The definition of U implies $r \le c_3U/\varepsilon$. Inserting this into (10.1103) and writing S=Ru_n gives

$$\log B^* \le \log\left(\frac{nA^2c_3B_n}{\varepsilon Q^n}\right) + 4\log R + \log u_n + \log(\widehat{U}/R). \tag{10.1110}$$

Now $WL \le F$, since $W_u \le R \le S$ and $\widehat{U} \ge SL$. The first and the next four logarithmic contributions after multiplication by W are therefore less than 5F. Also $W \log u_n < W u_n \le S/Q^s \le F/L < F$, and $W \log(\widehat{U}/R) < W\widehat{U}/R \le F$. These use $\log x < x$ for x>0. We obtain $W \log B^* < 7F$; (10.1106) adds the final F needed in (10.1107).

For the elementary cases note the uniform lower bound

$$\frac{\mathcal{A}_n}{Q^n} > 16^4 d^2 e^n N/t^2. \tag{10.1111}$$

It follows directly from (10.1086): the exact quotient has factor $16^{n+2}\mathrm{e}^2d^2n^{3/2}\log(2nd)\log(2d)$, and $\mathrm{e}^2n^{3/2}\log(2nd)\log(2d) > 1$. For example $n \ge 2$, $\log(2nd) > 4/3$, and $\log(2d) > 2/3$ suffice.

If Y is a nonidentity root of unity, the local height inequality gives $\mathrm{ord}_{\mathfrak{p}}(Y-1) \le (d/t)\log 2$. It is below the target by (10.1111) and $Q^nU \ge L > 1$, using t<N.

Suppose next that the list has rank one, that its distinguished base is redundant, and that Y is not torsion. Lemma 10.240(b) gives $Y^D = \gamma^M \ne 1$ for one selected preceding base γ, with $|M| \le B^*$ and $h(\gamma) \le JW$. Theorem 10.21, its residue-order bound $g < N$, and the positive power factorization imply

$$\mathrm{ord}_{\mathfrak{p}}(Y-1) \le \frac{d}{t}\{\log(2B^*) + 2eNJW\}.$$

By (10.1107), $W \ge Q^{-1}$, and $W \le Q^{n-1}U/L$, this is strictly below

$$\frac{d}{t}\left\{8Q^n + \frac{2eNJQ^{n-1}}{(n+1)t}\right\}U. \tag{10.1112}$$

We retained $L > (n+1)t$ in the second term; dropping that factor would lose the required logarithm in the target. Dividing (10.1112) by $\mathcal{A}_nU$, and using (10.1111), bounds its two terms respectively by $8/16^4$ and $4/16^4$. The first uses t<N; the second uses $J \le 2d/f$, Q>1, e,f at least one. Their sum is $3/16384 < 1$.

Finally suppose that all preceding bases are torsion and Y is not. Raising to w_K gives $Y^{w_K} = \beta_n^{w_K b_n} \neq 1$. Use Theorem 10.21, $w_K \le A$, $\log(2AB_n) < L \le Q^n U$, and $w_n \le Q^{n-1}U/L$. This gives (10.1112), with its first coefficient 8 replaced by 1, and proves the same conclusion. These alternatives also cover rank zero. □

Theorem 10.242 (all normalized local-unit lists). Let n be at least two and let $\beta_i \in K^*$ be arbitrary local units with $h(\beta_i) \le Jw_i$, $w_i \ge Q^{-1}$. Use all the coefficient hypotheses (10.1093). If $Y = \prod_i \beta_i^{b_i} \neq 1$, then

$$\operatorname{ord}_{\mathfrak{p}}(Y-1) < \mathcal{A}_n U. \tag{10.1113}$$

Proof. Independent lists satisfy the result directly by the padded height proof of Theorem 10.239: use the weights Jw_i and Jw_n^* in Lemma 10.224, apply the scalar inequalities (10.1096)–(10.1099), and conclude with (10.1089). This argument uses only the displayed height upper bounds for the local units. Lemma 10.241 handles torsion products and the specified rank-one cases. These cases establish the statement at n=2.

Induct on n. Suppose n is at least three, the list is dependent, and Y is not torsion. If the distinguished base lies in the span of the preceding ones, choose a basis I among those indices. Its length $s < n$ is positive; s=1 was handled in Lemma 10.241, so take $2 \le s < n$. Lemma 10.240(b) gives $Y^D = \prod_{j\in I} \beta_j^{M_j} \neq 1$. Renumber the selected bases so that a nonzero M_j of minimum p-adic valuation is last. The other coefficients may vanish.

Apply the independent estimate 10.223 to this selected list, with padded weights Jw_j. Its logarithmic coefficient entry is at most $\log(2QB^*)$, because $w_j \ge Q^{-1}$, J is at least one, and $\omega(d) < 1$. Its minimum absolute nonzero coefficient is at most B*. The other two entries satisfy $G_2(s,d) \le G_2(n,d) < 2L$ and $(s+1)t < 2L$. Consequently (10.1107) yields

$$\operatorname{ord}_{\mathfrak{p}}(Y-1) < 8C_2^{[s]} J^s Q^{n-s} U < 4\mathcal{A}_s Q^{n-s} U. \tag{10.1114}$$

The first step also uses the factorization of $Y^D - 1$, and the second is Lemma 10.238 at rank s. Directly from (10.1086),

$$\mathcal{A}_j / \mathcal{A}_{j-1} > 16Qe \qquad (j \ge 3). \tag{10.1115}$$

Since n-s is positive, (10.1114) is less than $\mathcal{A}_n U$. This resolves the redundant distinguished-base case without using induction or assuming any coordinate relation is prime to p.

Otherwise the distinguished base raises the rank. If every preceding base is torsion, Lemma 10.241 applies. In the remaining case, the first n-1 bases are dependent: their rank is one less than the rank of the whole dependent list. Choose a relation from Lemma 10.240(a), and choose a nonzero coefficient k_x, x<n, with minimum p-adic valuation. Reverse all its signs if needed to make $k_x > 0$. Set $k_n = 0$ and eliminate index x by defining

$$b_i' = k_x b_i - k_i b_x \quad (i \neq x), \qquad b_n' = k_x b_n. \tag{10.1116}$$

Then $\prod_{i\neq x} \beta_i^{b_i'} = Y^{k_x} \neq 1$, and $|b_i'| \le 2\kappa B$, where $\kappa = AR$. Every new coefficient has p-adic valuation at least $v_p(k_x) + v_p(b_n)$, and the distinguished coefficient has exactly that valuation and is nonzero. This follows from the minimum-valuation choices of both vectors and the ultrametric inequality; no coefficient is assumed to be prime to p.

Keep the surviving weights and choose

$$B' = 2\kappa B, \qquad B'_n = k_x B_n,$$
$$\varepsilon' = \frac{k_x}{2\kappa}\varepsilon\frac{c_3(n-1,d)}{c_3(n,d)}, \quad \frac{\varepsilon' B'}{B'_n c_3(n-1,d)} = \frac{\varepsilon B}{B_n c_3(n,d)}. \tag{10.1117}$$

These satisfy $B' \geq B'_n \geq |b'_n|$, all coefficient bounds, and $0 < \varepsilon' \leq 1/2$. Exact cancellation gives

$$L' = L + \log(8\kappa d^2) - (12n+5) - 3\log d - t - \log w_x < 2L. \tag{10.1118}$$

Here $c_3(n,d)/c_3(n-1,d) = 4d^2$, $D_n - D_{n-1} = 12n+5$, and $c_4(n-1,d)/c_4(n,d) = \exp(-12n-5)d^{-3}$. For the final inequality use $w_x \geq Q^{-1}$, $\log(8Q) < 7 + 2\log d$, and (10.1106). The excess above 2L is strictly less than $-D_n - 7n + 2 - (n-1)\log d - t < 0$.

The new height product is $\Omega' = \Omega/w_x$, so $U' \leq 2QU$. The induction hypothesis, the power factorization for local units, and (10.1115) give

$$\mathrm{ord}_{\mathfrak{p}}(Y-1) \leq \mathrm{ord}_{\mathfrak{p}}(Y^{k_x}-1) < \mathcal{A}_{n-1}U' \leq 2Q\mathcal{A}_{n-1}U < \mathcal{A}_n U. \tag{10.1119}$$

All dependence, torsion and zero-coefficient alternatives have now been covered. □

Corollary 10.243 (arbitrary original inputs in the stated fields). Let $\alpha_i \in K^*$ be arbitrary and take every coefficient and height hypothesis in (10.1093). Assume $\Xi = \prod_i \alpha_i^{b_i} \neq 1$. Then (10.1095) holds without an independence or local-unit assumption on the original bases. The field condition remains: p is odd, or p=2 and $\zeta_3 \in K$.

Proof. Normalize by Lemma 10.237. A nonzero total valuation makes the left side nonpositive. At total valuation zero, the normalized product is Ξ^e. If Ξ is torsion, the elementary torsion estimate in Lemma 10.241 applies to the original product, using the same positive target and weights. Otherwise $\Xi^e \neq 1$, and Theorem 10.242 applies to the normalized units with $h(\beta_i) \leq Jw_i$. Factoring $\Xi^e - 1$ gives the required inequality for $\Xi - 1$. Every field datum, original weight and original distinguished coefficient is retained. □

An internal relation in a quadratic field

Use $K = \mathbb{Q}(\sqrt{2})$ and the place above seven from Section 79. Write $a = 1+\sqrt{2}$, $c = 6+\sqrt{2}$, and take the four local units $(a, c, ac, 3)$, with coefficients $(1,1,1,6)$. Their exact relation $a^{-1}c^{-1}(ac) = 1$ deletes the third base and gives coefficients $(2,2,6)$ on $(a,c,3)$. This remaining list is independent: applying conjugation and multiplication gives norms -1,34,9, so the valuations at the rational primes seventeen and three force the second and third relation coefficients to vanish; the real value a>1 forces the first to vanish. Thus the distinguished base raises the rank as required in the internal case.

Take $B = B_n = 6$. Using the actual relation bound $\kappa = 1$, the parameter transfer (10.1117) gives $B' = 12$, $B'_n = 6$, and $\varepsilon' = \varepsilon/32$, since d=2. The second entry of the maximum is exactly unchanged. Lifting the chosen root of $X^2 - 2$ to 108 modulo 343 gives

$$3^6(ac)^2 - 1 \equiv 245 \pmod{343}.$$

It has valuation exactly two. Both the original and the reduced list represent that same product. This example verifies a coefficient transfer and its exact valuation; the proof for all inputs is the induction above.

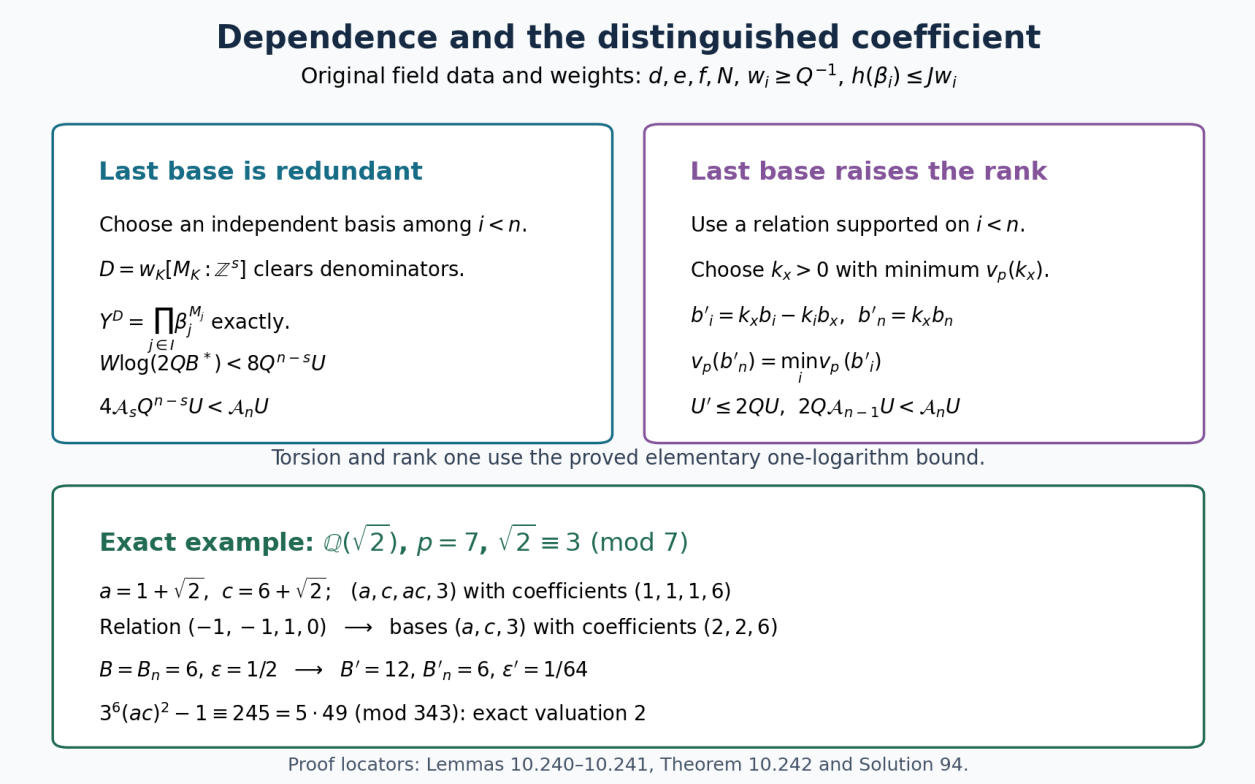


Figure 10.73. The upper paths distinguish whether the last base belongs to the rational span of the preceding bases modulo roots of unity. The common-denominator path uses Lemmas 10.240–10.241; the internal-relation path uses (10.1116)–(10.1119). The lower panel gives an exact instance of the latter path in the quadratic field above seven. The parameter transfer keeps the second term of the maximum unchanged, and both expressions have valuation two. Theorem 10.242 proves the bounds uniformly, including torsion and zero coefficients. Reproducible diagram and exact arithmetic checks. Human-source target: Theorem B of Bérczes, Evertse and Győry; the proof here uses the programme results identified above.

81. The coefficient refinement at two over every number field

Adjoining a cube root of unity doubles the global degree. At a prime above two it can also square the residue-field cardinality. Applying a bound with these larger parameters would not yet give the estimate with the original field data. The saturated residue index supplies the needed compensation. We retain it until the roots-of-unity factor has cancelled.

Throughout this section, K does not contain ζ_3, and

$$L = K(\zeta_3), \qquad d = [K : \mathbb{Q}], \qquad e = e_{\mathfrak{p}}, \quad f = f_{\mathfrak{p}}, \quad N = 2^f, \quad t = \log N.$$

The exponential constant is e. All the constants $Q, J, \mathcal{A}_n, c_3, c_4$ retain the original degree and local data of (10.1086).

Cubic saturation and the residue group

Lemma 10.244 (descent of cubic saturation). Let $\gamma_1, \dots, \gamma_s$ be independent local units in K. Define the two lattices M_3^K and M_3^L by (10.44), using roots in the indicated field. Then $M_3^K = M_3^L$. A saturated basis for the calculation over L can consequently be represented by exact roots θ_j in K.

At any prime $\mathfrak{P}$ of L above $\mathfrak{p}$, the relative ramification index is one. Write $f' = \sigma f$ for the residue degree over $\mathbb{F}_2$. Then $\sigma = 1$ for even f, and $\sigma = 2$ for odd f. If f is odd and $m = 3^u$ is the order of the group of 3-primary roots of unity in L, there is a divisor g of $N - 1$ such that

$$\delta_L = \frac{N^2 - 1}{mg}, \qquad \frac{N^2}{m\delta_L} = \frac{N^2 g}{N^2 - 1} \leq \frac{N^2}{N + 1} < N.$$

Proof. The global extension has degree two, since $X^2 + X + 1$ is irreducible over K. Suppose $\boldsymbol{v} \in M_3^L$. Choose $T = 3^a$ such that $T\boldsymbol{v}$ is integral and $\beta^T = A = \prod_i \gamma_i^{Tv_i}$ for some $\beta \in L$. Let $z = \beta\overline{\beta} \in K$, where the bar is the nonidentity K-automorphism of L. Then $z^T = A^2$. Because T is odd, the element

$$\theta = z^{(T+1)/2}A^{-1} \in K, \qquad \theta^T = A^{T+1}A^{-T} = A \tag{10.1120}$$

proves that $\boldsymbol{v} \in M_3^K$. The reverse inclusion is immediate. Lemma 10.12 supplies a finite lattice basis, and this argument supplies its roots in K. By Lemma 10.15, root choices and integral changes of basis do not change the residue group over L: their extra factors are powers of its 3-primary root of unity.

Reduction is injective on roots of odd order, as proved in Section 1. If f is even, the cyclic group $\mathbb{F}_N^*$ contains an element of order three. Its unique prime-to-two root-of-unity lift from Section 1 has order three: its cube has residue one, so injectivity makes that cube one. Thus the two roots of $X^2 + X + 1$ already lie in $K_{\mathfrak{p}}$, and every completion $L_{\mathfrak{P}}$ equals $K_{\mathfrak{p}}$.

If f is odd, $3 \nmid N - 1$, so the reduction of this polynomial has no root in $\mathbb{F}_N$. Its root in the residue field of $L_{\mathfrak{P}}$ gives a residue extension of degree at least two. The local extension degree is at most two. The elementary inequality between local degree and ramification times residue degree therefore forces residue degree two and ramification one. To recall the inequality, take lifts of a residue-field basis and multiply them by powers of a uniformizer in distinct valuation classes modulo the base value group. A linear relation over the base field, after scaling its coefficients to their least valuation, has a unique least valuation class; reducing within that class contradicts independence of the residue basis. These products are linearly independent, proving the inequality. We have established

$$[L : \mathbb{Q}] = 2d, \qquad e_{\mathfrak{P}} = e, \qquad f' = \begin{cases} f, & f \text{ even}, \\ 2f, & f \text{ odd}, \end{cases} \qquad \operatorname{ord}_{\mathfrak{P}}|_K = \operatorname{ord}_{\mathfrak{p}}. \tag{10.1121}$$

Now assume f is odd. The reductions of the chosen θ_j belong to $\mathbb{F}_N^*$; let their generated group have order $g \mid N - 1$. Reduction embeds the group of m-th roots of unity into $\mathbb{F}_{N^2}^*$. Thus $m \mid N^2 - 1$, and $3 \nmid N - 1$ gives $m \mid N + 1$. The two residue groups have coprime orders, so their generated group has exactly mg elements. The definition (10.49) now gives

$$m\delta_L = \frac{N^2 - 1}{g} \geq N + 1, \qquad \frac{N^2}{m\delta_L} \leq \frac{N^2}{N+1} < N. \tag{10.1122}$$

This proves the assertions. Only 3-primary saturation has been used; no claim about full saturation in a quadratic extension is needed. □

The constant with the original degree and residue size

Lemma 10.245 (the quadratic transfer constant). For an independent list of $n \geq 2$ local units belonging to K, let $C_2(L)$ be (10.642) calculated over L, with the actual saturated residue index of that list. Then

$$2C_2(L)J^n < \mathcal{A}_n(K). \tag{10.1123}$$

Proof. Put $\ell' = \max\{1, \log(2d)\}$, $A' = \max\{4 + \log(2(n+1)d), e, \sigma t\}$, and

$$H = \max\left\{\frac{N^\sigma}{m\delta_L}, \frac{\mathrm{e}^n(\sigma t)^{n+1}}{mn^n}\right\}, \qquad T_n = \frac{(n+1)^{n+1}}{(n-1)!\,n^{3/2}}, \qquad Z = \frac{\ell' A'}{et\log(2nd)\log(2d)}. \tag{10.1124}$$

Here $m \ge 3$. Formula (10.641) at two gives $P/e \le 4$. Also $ef \le d$, so $J = e + d/f \le 2d/f$. Substituting $2d, \sigma t$ in (10.642), and cancelling the powers of d, t, and e, gives

$$\frac{2C_2(L)J^n}{\mathcal{A}_n(K)} \le \frac{406}{128\sigma^3 \mathrm{e}^{n+2}} \left(\frac{13}{16f}\right)^n T_n Z \frac{H}{N}. \tag{10.1125}$$

In particular, the residue contribution here is $N^\sigma/(m\delta_L)$, not N^σ/δ_L. To check the cancellation directly, before using $P/e \le 4$ the power of two in the denominator is 2^{6n+5}, and the remaining ramification factor is $(P/e)^{n-1}/e$. Multiplying by 4^{n-1} changes that denominator to 2^{4n+7}, as in (10.1125); the $1/e$ is included in Z.

We use the elementary inequalities from Lemma 10.225: $8/3 < \mathrm{e} < 11/4$, $2/3 < \log 2 < 7/10$, and $\log(3/2) < 1/2$. If $d \ge 2$, then $\ell' = \log(2d)$ and $x = \log(2nd) > 2$. Consequently $4 + \log(2(n+1)d) < x + 9/2$. Bounding the three entries of A' separately gives

$$Z < \begin{cases} 39/16, & d \ge 2,\ f \text{ even}, \\ 13/8, & d \ge 2,\ f \ge 3 \text{ odd}, \\ 39/8, & d \ge 2,\ f = 1, \\ 81/8, & d = 1. \end{cases} \tag{10.1126}$$

For the first three rows the entries are bounded by $39/(8ef)$, $1/(tx)$, and $\sigma/(ex)$, respectively. Use $e \ge 1$, $t > 2f/3$, $x > 2$, and $\sigma \le 2$. In the last row $e = f = 1$, $\sigma = 2$, and $\ell' = 1$. The first entry of A' is the largest. Now $x = \log(2n) > 4/3$ and $A'/x < 1 + (9/2)/(4/3) = 35/8 < 9/2$. Since $(\log 2)^{-2} < 9/4$, the last bound follows.

For even f, (10.1027) and $\delta_L \ge 1$ give $H/N < (n+1)/3$. For odd $f \ge 3$, combine Lemma 10.244 with (10.1027) to get $H/N < 2^{n+1}(n+1)/3$. If $f = 1$, the residue entry is less than two. The other entry is also less than two for every $n \ge 2$: the derivative of $n \log(\mathrm{e}(2t)/n)$ is $\log(2t/n) < 0$, and at $n = 2$ an upper bound is $(11/4)^2(7/5)^3/12 = 41503/24000 < 2$. Thus

$$\frac{H}{N} < W_n, \qquad W_n = \begin{cases} (n+1)/3, & f \text{ even}, \\ 2^{n+1}(n+1)/3, & f \ge 3 \text{ odd}, \\ 1, & f = 1. \end{cases} \tag{10.1127}$$

Retaining the factorial makes one endpoint calculation sufficient in each row. Indeed,

$$\frac{T_{n+1}}{T_n} < \mathrm{e}\,\frac{(n+2)\sqrt{n}}{(n+1)^{3/2}} \le \mathrm{e}\sqrt{\frac{32}{27}} < \frac{11\mathrm{e}}{10}. \tag{10.1128}$$

The first inequality is $(1 + 1/(n+1))^{n+1} < \mathrm{e}$. The logarithmic derivative of the remaining factor is $(2-n)/(2n(n+1)(n+2))$, which is nonpositive for $n \ge 2$; its maximum is therefore the stated value at two. Finally $32/27 < 121/100$.

Let E_n be the right side of (10.1125) with Z and H/N replaced by their row bounds. Its successive ratio is at most $143/240$ for even $f \ge 2$, at most $143/180$ for odd $f \ge 3$, and at most $143/160$ for $f = 1$. These follow by multiplying $143/(160f)$ by the successive ratio of W_n, which is at most $4/3$, $8/3$, or one. All are less than one. At $n = 2$, use $T_2 = 27/(2\sqrt{2}) < 135/14$ and $\mathrm{e}^{-4} < (3/8)^4$. The resulting exact bounds are

field range	endpoint bound	ratio bound
$d \ge 2,\ f$ even	2090104965/8589934592	143/240
$d \ge 2,\ f \ge 3$ odd	77411295/1073741824	143/180
$d \ge 2,\ f = 1$	2090104965/8589934592	143/160
$d = 1$	4340987235/8589934592	143/160

(10.1129)

Every endpoint is below one. The decreasing envelopes prove the comparison at all ranks, including the rational field. □

The estimate for arbitrary inputs

Theorem 10.246 (the coefficient refinement over every number field). Let K be any number field, $\mathfrak{p}$ a prime above any rational prime p, and $n \geq 2$. Let $\alpha_i \in K^*$, $\Xi = \prod_i \alpha_i^{b_i} \neq 1$, and use the original field data and constants (10.1086). Assume all the weight and coefficient conditions (10.1093), including $w_i \geq \max\{h(\alpha_i), Q^{-1}\}$, $b_n \neq 0$, $v_p(b_n) = \min_i v_p(b_i)$, $B \geq B_n \geq |b_n|$, $B \geq \max_i |b_i|$, and $0 < \varepsilon \leq 1/2$. With V, Ω, r, L defined in (10.1094), one has

$$\operatorname{ord}_{\mathfrak{p}}(\Xi - 1) < \mathcal{A}_n \max\left\{\Omega L, \frac{\varepsilon r}{c_3(n,d)}\right\}. \tag{10.1130}$$

There is no independence, local-unit, or root-of-unity hypothesis on the original list. Zero nondistinguished coefficients are allowed. Taking the smallest allowed w_i gives the original height floor and constants of Theorem B in the free paper of Bérczes, Evertse and Győry. The proof here uses the preceding programme proofs.

Proof. Corollary 10.243 covers odd p and the fields containing ζ_3 at two. It remains to prove the result when $p = 2$ and $\zeta_3 \notin K$. In the rest of this proof the extension field is denoted by $K' = K(\zeta_3)$; L continues to denote the real logarithmic expression (10.1094).

First consider independent local units $\beta_i \in K$ with $h(\beta_i) \leq Jw_i$, $w_i \geq Q^{-1}$. Enlarge only w_n to w_n^* as in (10.1098), and apply Lemma 10.224 and Theorem 10.223 over K' with padded weights Jw_i, Jw_n^*. Independence is unchanged by a field extension. Since $\omega(2d) < 1$, the coefficient logarithms are still less than $2L$ by (10.1096)–(10.1099), using the original Q, c_3, c_4. The two remaining comparisons are

$$\begin{aligned} (n+1)f' \log 2 &\leq 2(n+1)t < 2L, \\ G_2(n, 2d) &< (n+1)(7n + 4 + \log d + \log 2) < 2L. \end{aligned} \tag{10.1131}$$

For the last step the lower bound (10.1097), after subtraction, leaves $5n^2 + n + 16 + (n+3)\log d - (n+1)\log 2 > 0$. This is positive since $\log 2 < 1$, $n \geq 2$, and $d \geq 1$. Also $\log B_\dagger \leq \log B_n < L$. The valuation is unchanged by Lemma 10.244. Lemma 10.245 therefore gives $\operatorname{ord}_{\mathfrak{p}}(\prod_i \beta_i^{b_i} - 1) < 2C_2(K')J^n U < \mathcal{A}_n U$, where $U = \max\{\Omega L, \varepsilon r/c_3\}$.

We spell out why the dependence argument uses the same original parameters. Lemmas 10.240–10.241 use height counts, exact multiplicative relations in K, and the elementary one-logarithm bound; none uses $\zeta_3 \in K$. Torsion products, rank-one products, and the two-base cases are handled by their same estimates, in K. For a nontorsion product of larger rank, proceed by induction on the list length, as in Theorem 10.242.

If the distinguished last base is redundant, choose an independent basis among the preceding bases. Its members still belong to K. Lemma 10.240 supplies the exact common power Y^D, integer coefficients M_j, and B^*. Renumber this selected list so that a nonzero M_j of least two-adic valuation is last. If its rank is $s \geq 2$, apply the second estimate over K'. Every coefficient logarithm is at most $\log(2QB^*)$, since the original weights are at least Q^{-1} and $\omega(2d) < 1$. Also $\log B_\dagger \leq \log B^*$. The noncoefficient terms are less than $2L$, by (10.1131) and monotonicity in $s \leq n$. Writing $W = \prod_{j \in I} w_j$, Lemma 10.241 gives $W \log(2QB^*) < 8Q^{n-s}U$ and $WL \leq Q^{n-s}U$. Hence the padded coefficient maximum times W is less than $8Q^{n-s}U$. The power identity and Lemma 10.245 give

$$\operatorname{ord}_{\mathfrak{p}}(Y - 1) \leq \operatorname{ord}_{\mathfrak{p}}(Y^D - 1) < 8C_2(K')J^s Q^{n-s}U < 4\mathcal{A}_s Q^{n-s}U < \mathcal{A}_n U. \tag{10.1132}$$

The last inequality is the rank absorption proved in Theorem 10.242.

If the last base raises the rank, use the relation supported on the preceding bases from Lemma 10.240. The exact transfer $b_i' = k_x b_i - k_i b_x$, $b_n' = k_x b_n$, with $k_x > 0$ of minimum two-adic valuation among its nonzero entries, preserves the distinguished minimum valuation. Formula (10.1117) preserves the second target entry exactly, and (10.1119) gives $U' \leq 2QU$. The shortened bases still belong to K. Induction and $2Q\mathcal{A}_{n-1} < \mathcal{A}_n$ prove the bound. These two branches exhaust every dependent list, and use no relation chosen outside the original field.

Finally apply Lemma 10.237 to the original α_i. If their total valuation is nonzero, the required valuation is nonpositive. If it is zero, their normalized product is Ξ^e. For nontorsion Ξ, apply the unit-list estimate just proved and the integral power factor. For torsion $\Xi \neq 1$, use the torsion estimate of Lemma 10.241 directly in K, including when $\Xi^e = 1$. This completes all cases. $\square$

A residue field of eight elements

Let a be the real root of $X^3 - X - 1$ between one and two, and put $K = \mathbb{Q}(a)$. The polynomial has no rational root, so $d = 3$ and K cannot contain a quadratic subfield. Its reduction $X^3 + X + 1$ has no root over $\mathbb{F}_2$, hence is irreducible. At a prime above two the residue degree is three and the ramification index is one, by the degree inequality used in Lemma 10.244. Thus $N = 8$, whereas $K(\zeta_3)$ has residue cardinality 64.

The units $a, 3$ are independent. Their norms are one and 27, so a relation forces the exponent of three to vanish; the real inequality $a > 1$ then forces the other exponent to vanish. The reduction of a is a nonidentity element of $\mathbb{F}_8^\ast$, so has order seven. The descended saturated residue group therefore has $g = 7$. Indeed the integral vector of a is an integer combination of the saturated lattice basis. The corresponding product of the θ_j equals a up to a 3-primary root of unity in K, which is necessarily one. Thus this residue group contains the original residue of a, and is contained in a group of order seven. Consequently

$$\begin{aligned} m\delta_L = 9, \qquad \frac{64}{m\delta_L} = \frac{64}{9} < 8, \\ 3a^7 - 1 = 2(3a^2 + 3a + 1), \qquad \operatorname{ord}_{\mathfrak{p}}(3a^7 - 1) = 1. \end{aligned} \tag{10.1133}$$

For the valuation, use $a^3 = a + 1$, hence $a^7 = 2a^2 + 2a + 1$. The second factor has nonzero reduction, since a polynomial of degree two cannot vanish at the residue of a, whose minimal polynomial has degree three. This gives a concrete application with $n = 2$, coefficients $(7, 1)$, $B = 7$, $B_n = 1$, and $\varepsilon = 1/2$.

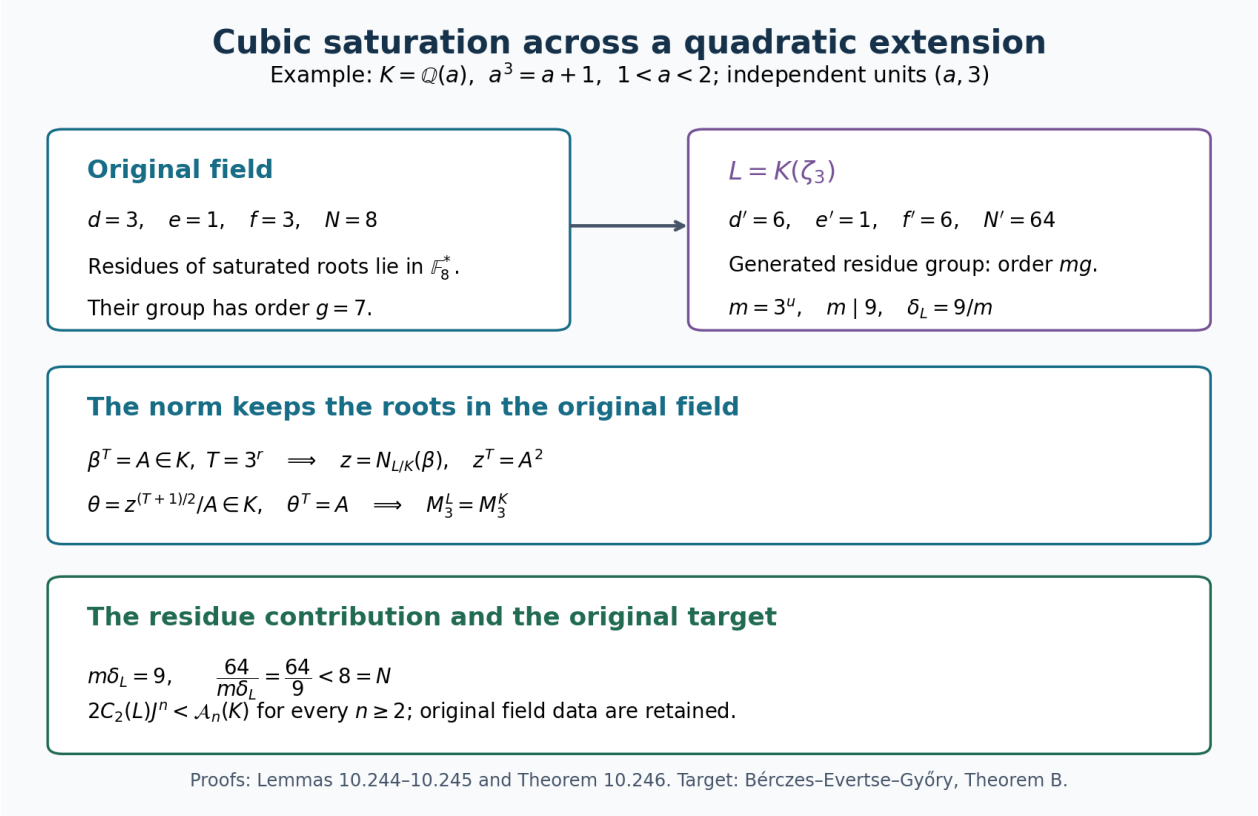


Figure 10.74. Cubic saturation is represented by roots in the original field. In the example, the residue field grows from eight to sixty-four elements, while the product $m\delta_L=9$ reduces its contribution to $64/9<8$. Lemmas 10.244–10.245 prove the general transfer; Theorem 10.246 retains the original field constants. Reproducible figure and exact arithmetic.

A bound using only the largest coefficient

The coefficient parameter in the second explicit estimate contains fixed field terms as well as logarithms of the integer coefficients. A local height estimate treats the small-coefficient range. In the complementary range, the coefficient itself is large enough to absorb every fixed term. An exact relation then removes a dependent base. Choosing a base of largest weight in that relation makes its coefficient cost fit inside the decrease in the height-product constant. The resulting bound uses the original field degree, for arbitrary nonzero algebraic bases.

Write $d=[K:\mathbb{Q}]$, $e=e_{\mathfrak{p}}$, $f=f_{\mathfrak{p}}$, $t=f\log p$, and $Q=16\mathrm{e}^2d^2$. The roman e is the exponential constant. Set $J=e+d/f$, $A=58d\max\{1,\log d\}$. For $n\geq 1$, put

$$\begin{aligned} x&=\log(2nd), \qquad y=\log(2d),\\ \mathcal{G}_n&=(16\mathrm{e}d)^{2(n+1)}n^{5/2}x^2d^n\frac{p^d-1}{(\log p)^2},\\ Z_n&=\frac{et\mathcal{G}_n}{2dJn}Q^{1-n}=\frac{16^{n+3}\mathrm{e}^4ef}{2J}n^{3/2}x^2d^{n+3}\frac{p^d-1}{\log p}. \end{aligned} \tag{10.1134}$$

For $n\geq 2$, also put

$$R_n=\frac{e\mathcal{G}_n}{\mathcal{A}_n}=\frac{nd^nf^2}{e^{n-1}}\frac{p^d-1}{p^f}\frac{x}{y},$$

where $\mathcal{A}_n$ is the original-field constant (10.1086).

Lemma 10.247 (the coefficient threshold). For every such field, prime and integer $n \geq 2$,

$$\begin{gathered} Z_n > 2Q, \qquad R_n \geq nd, \\ \log Z_n > \max\{n, \log(2A)\}, \\ (n+1)(7n+5+\log d) < 2R_n \log Z_n, \\ (n+1)t < R_n \log Z_n. \end{gathered} \tag{10.1135}$$

Proof. The positive logarithm series $\log 2 = 2\sum_{j\geq 0} 1/((2j+1)3^{2j+1})$ gives $11/16 < \log 2 < 3/4$. For the lower bound its first three terms sum to $842/1215 > 11/16$. For the upper bound, replace every denominator $2j+1$, $j \geq 1$, by three; the resulting sum is $25/36 < 3/4$. In particular $11/4 < \log 16 < 3$, and $\log 2 > 2/3$.

Since $p^d - 1 \geq d \log p$, the formula for Z_n, $ef \geq 1$, $J \leq 2d$, $x \geq \log 4 > 1$, and $(3/2)\log n \geq (3/2)\log 2 > 1$ give

$$\log Z_n > \frac{11n}{4} + \frac{47}{4} + (n+3)\log d. \tag{10.1136}$$

This exceeds $\log(2Q) = \log 32 + 2 + 2\log d$, because $\log 32 < 15/4$. Thus $Z_n > 2Q$. It also exceeds n and $\log(2A)$. For the latter, use $\max\{1, \log d\} \leq d$ and $\log 116 < 5$, so $\log(2A) < 5 + 2\log d$. The exponential series gives $\mathrm{e} > 8/3$, and $(8/3)^5 > 116$, proving the stated logarithmic comparison.

Use $ef \leq d$ to bound the exact ratio in (10.1134):

$$R_n \geq ndf^{n+1}\frac{p^d - 1}{p^f}. \tag{10.1137}$$

If $d \geq 2$ and $f = 1$, the last quotient is $p^{d-1} - p^{-1} \geq 3/2$, so $R_n \geq 3n$. If $f \geq 2$, that quotient is at least $1 - p^{-d} \geq 3/4$, and again $R_n \geq 3n$. Both estimates also imply $R_n \geq nd$. If $d = 1$, then $e = f = 1$, and the exact expression is

$$R_n = n(1 - p^{-1})\frac{\log(2n)}{\log 2} \geq n. \tag{10.1138}$$

It is at least $2n$ for $n \geq 8$.

Whenever $d \geq 2$ or $n \geq 8$, we therefore have

$$\frac{(n+1)(7n+5+\log d)}{2R_n} \leq \frac{3}{8}(7n+5+\log d) < \log Z_n$$

by (10.1136). In the remaining cases $d = 1$, $2 \leq n \leq 7$, use $R_n \geq n$. The required comparison with (10.1136) reduces to

$$-3n^2 + 23n - 10 > 0.$$

This concave polynomial is at least the smaller of its endpoint values 24 and 4 on $[2, 7]$. The coefficient-parameter assertion follows.

For the last assertion, if $p = 2$ or 3, then $t < 2d$ and $R_n \geq nd$, so $(n+1)t/R_n < 3 < \log Z_n$. If $p \geq 5$, formulas (10.1137)–(10.1138) give $R_n \geq 8n/5$. Consequently

$$\frac{(n+1)t}{R_n} \leq \frac{15}{16}d\log p.$$

For every $l > 0$, maximization of $\log l - l/16$ gives $\log l \leq l/16 + \log 16 - 1 < l/16 + 2$. The factor preceding $(p^d - 1)/\log p$ in the formula for Z_n is greater than e^4. Also $p^d - 1 \geq p^d/2$. Taking $l = \log p$ yields

$$\log Z_n > 4 + dl - \log 2 - \log l > \frac{15}{16}dl.$$

This proves the last assertion. □

Lemma 10.248 (one base and the height floor). For every $n \ge 1$,

$$\mathcal{G}_n Q^{-n} > \frac{1000d}{\log p}. \tag{10.1139}$$

If β is a local unit, $h(\beta) \le Jw$, $w \ge Q^{-1}$, $B \ge \max\{3, |b|\}$, and $\beta^b \ne 1$, then

$$v_p(\beta^b - 1) < \mathcal{G}_1 w \log B. \tag{10.1140}$$

Proof. Exact cancellation gives

$$\frac{\mathcal{G}_n}{Q\mathcal{G}_{n-1}} = 16d\left(\frac{n}{n-1}\right)^{5/2}\left(\frac{\log(2nd)}{\log(2(n-1)d)}\right)^2 > 16d \qquad (n \ge 2). \tag{10.1141}$$

At n=1, writing $l = \log p$ and $y = \log(2d) > 2/3$,

$$\frac{l\mathcal{G}_1}{Q} = 4096\mathrm{e}^2 d^3 y^2 \frac{p^d - 1}{l} \ge 4096\mathrm{e}^2 d^4 y^2 > 1000d.$$

This and (10.1141) prove (10.1139).

For n=1, the prefactor of $(p^d - 1)/l$ in Z_1 is at least $16^4\mathrm{e}^4 d^3 y^2/4 > 2$. Consequently $Z_1 > 2Q$: divide its lower bound, using $(p^d - 1)/l \ge d$, by $2Q$ to obtain $512\mathrm{e}^2 d^2 y^2 > 1$. Also $p^d - 1 \ge p/2$, so $\log Z_1 > l - \log l > l/2$. The last inequality follows by maximizing $\log l - l/2$, whose maximum is $\log 2 - 1 < 0$.

Put $T = \mathcal{G}_1 w \log B$. If $B \le Z_1 \log B$, the local height bound gives

$$v_p(\beta^b - 1) \le \frac{d}{et}(\log 2 + JBw) < T.$$

Indeed the variable term is at most $T/2$ by the definition of Z_1, while $T > 1000d \log B/l$ makes the constant term less than $T/2$.

Otherwise $\log B > \log Z_1 > l/2$. Let g be the order of the residue of β, so $g \le p^f - 1$. Theorem 10.21 yields

$$v_p(\beta^b - 1) \le \frac{\log B}{l} + \frac{dJgp}{fl(p-1)}w + \frac{d\log 2}{et}.$$

The first and third terms are each less than $T/3$, by (10.1139). The middle term is at most $4d^2(p^d - 1)w/l$, since $J \le 2d/f$ and $p/(p-1) \le 2$. It too is less than $T/3$, because $\log B > l/2$ and $16^4\mathrm{e}^4 d^3 y^2 > 24$. This proves (10.1140), including torsion bases allowed in Theorem 10.21. □

Theorem 10.249 (the maximum-coefficient bound for local units). Let $\beta_1, \dots, \beta_n \in K^*$ be local units, $n \ge 1$, with no independence assumption. Suppose

$$h(\beta_i) \le Jw_i, \quad w_i \ge Q^{-1}, \quad B \ge \max\{3, |b_1|, \dots, |b_n|\}, \quad Y = \prod_i \beta_i^{b_i} \ne 1, \quad \Omega = \prod_i w_i.$$

Then, with $v_p(p) = 1$,

$$v_p(Y - 1) < \mathcal{G}_n \Omega \log B. \tag{10.1142}$$

Proof. We induct on n. Lemma 10.248 proves n=1. For every n, a torsion product has height zero, so $v_p(Y - 1) \le d\log 2/(et)$; (10.1139) places this strictly below the target. Hence assume n at least two and Y nontorsion. Put $w = \max_i w_i$. The local height inequality and the sum and power inequalities for height give

$$v_p(Y - 1) \le \frac{d}{et}\left(\log 2 + B\sum_i h(\beta_i)\right) \le \frac{d}{et}(\log 2 + JnBw). \tag{10.1143}$$

The inequality is immediate if the valuation is nonpositive; otherwise it follows from the local contribution to the height of $(Y-1)^{-1}$, just as in Theorem 10.21.

Suppose first that $B \le Z_n \log B$. Since $\Omega \ge wQ^{1-n}$, the definition of Z_n gives

$$\frac{d}{et} JnBw \le \frac{1}{2}\mathcal{G}_n \Omega \log B.$$

Moreover $\Omega \ge Q^{-n}$ and $Z_n > 2Q$, so

$$\frac{et}{2d}\mathcal{G}_n \Omega \log B \ge \frac{JnZ_n}{Q}\log B > \log 2.$$

The constant term in (10.1143) is strictly below the other half of the target. This proves (10.1142) in the first case.

Now let $B > Z_n \log B$. As $B \ge 3$, we have $\log B > \log Z_n$.

First suppose the bases are independent. Apply the padded-weight form of Theorem 10.223 supplied by Lemma 10.224, with weights Jw_i. Renumber so a nonzero coefficient of minimum p-adic valuation is last. No coefficient or height bound changes. Use the same field unless $p = 2$ and $\zeta_3 \notin K$; in that case use $K' = K(\zeta_3)$. Its degree is $2d$, its local ramification index remains e, its residue degree is at most $2f$, and its prime-ideal valuation restricts to the original one, by Lemma 10.244.

In either case Lemma 10.238 or 10.245 gives $2C_2 J^n < \mathcal{A}_n$. The coefficient parameter satisfies

$$\mathsf{h}_2 \le \max\left\{\log(2BQ),\ (n+1)(7n+5+\log d),\ 2(n+1)t\right\}. \tag{10.1144}$$

Indeed $1/(d'W(d')) < 1$, $J \ge 1$, and $w_i \ge Q^{-1}$ give the first entry. Also $\log B_\dagger \le \log B < \log(2BQ)$. In every row of (10.643), $a_0^{(2)} < 6$ and $a_1^{(2)} < 4$. Using $d' \le 2d$, $\log(n+1) \le n$, and $\log 2 < 1$ gives the second entry. The residue-degree bound gives the third entry.

Lemma 10.247 bounds the last two entries strictly by $2R_n \log B$. For the first, $B > Z_n > 2Q$ implies $\log(2BQ) < 2\log B \le 2R_n \log B$. Therefore

$$v_p(Y-1) < \frac{\mathcal{A}_n}{2e}\Omega \mathsf{h}_2 < \frac{\mathcal{A}_n}{e} R_n \Omega \log B = \mathcal{G}_n \Omega \log B. \tag{10.1145}$$

This proves the independent case.

If the bases are dependent, choose a minimally dependent sublist C. Put $u_i = Qw_i \ge 1$, and choose $x \in C$ maximizing u_x on C. There is an exact integer relation, supported on C,

$$\prod_i \beta_i^{k_i} = 1, \qquad k_x > 0, \qquad |k_i| \le A u_x^{n-1}. \tag{10.1146}$$

For a singleton torsion sublist, use its order, at most A by Lemma 10.155. Otherwise put $a = |C| - 1$. The simultaneous bounds of Corollary 10.20, as calculated in the proof of Lemma 10.240, give

$$|k_i| \le A\left(\frac{\mathrm{e}dJ}{Q}\right)^a \prod_{j \in C\setminus\{i\}} u_j \le A u_x^a \le A u_x^{n-1}.$$

Every coefficient on a minimal circuit is nonzero; reversing their signs makes $k_x > 0$. The relation is exact, since Corollary 10.20 has already removed the root of unity.

Delete the x-th base by raising the original product to k_x:

$$Y^{k_x} = \prod_{i \ne x} \beta_i^{b_i'}, \qquad b_i' = k_x b_i - k_i b_x, \qquad |b_i'| \le B' := 2ABu_x^{n-1}. \tag{10.1147}$$

Here $B' \geq 3$, and $Y^{k_x} \neq 1$ because Y is nontorsion. The field, e, f, Q and J are unchanged. Since $\log B > \log Z_n > \max\{n, \log(2A)\}$, we have

$$\frac{\log B'}{\log B} = 1 + \frac{\log(2A) + (n-1)\log u_x}{\log B} < 2 + \log u_x \leq 2u_x. \tag{10.1148}$$

The last inequality follows from $\log u \leq u - 1$ for $u \geq 1$. Induction and the integral geometric-sum factor therefore give

$$\begin{aligned} v_p(Y-1) &\leq v_p(Y^{k_x} - 1) \\ &< \mathcal{G}_{n-1} \frac{\Omega}{w_x} \log B' < 2Q\mathcal{G}_{n-1}\Omega \log B < \mathcal{G}_n \Omega \log B, \end{aligned}$$

where (10.1141) supplies the last comparison. Each deletion reduces n, so the induction includes every dependence and torsion pattern. □

Corollary 10.250 (arbitrary algebraic bases and the original degree). Let K be any number field of degree d, $\mathfrak{p}$ a prime above any rational prime p, and $\alpha_1, \ldots, \alpha_n \in K^*$, $n \geq 1$. For integers b_i, suppose $\Xi = \prod_i \alpha_i^{b_i} \neq 1$. If

$$w_i \geq \max\{h(\alpha_i), (16\mathrm{e}^2 d^2)^{-1}\}, \qquad B \geq \max\{3, |b_1|, \ldots, |b_n|\},$$

then

$$v_p(\Xi - 1) < (16\mathrm{e}d)^{2(n+1)} n^{5/2} (\log(2nd))^2 d^n \frac{p^d - 1}{(\log p)^2} \left(\prod_i w_i \right) \log B. \tag{10.1149}$$

The original bases may be dependent, torsion, or nonunits at $\mathfrak{p}$; some coefficients may vanish.

Proof. A nontrivial torsion product is handled by the height-zero argument in Theorem 10.249. Otherwise apply Lemma 10.237, putting $v_i = \mathrm{ord}_{\mathfrak{p}}(\alpha_i)$ and $\beta_i = \alpha_i^e p^{-v_i}$. If $\sum_i b_i v_i \neq 0$, the required valuation is nonpositive. If the sum is zero, the β_i are local units satisfying $h(\beta_i) \leq J w_i$, and their product is $\Xi^e \neq 1$. Lemma 10.237 and Theorem 10.249 give $v_p(\Xi - 1) \leq v_p(\Xi^e - 1) < \mathcal{G}_n \Omega \log B$. All constants use the original field. □

The free BIRS notes of Bugeaud, Theorem G give this maximum-coefficient constant with p^d in place of $p^d - 1$. Here the bound $B \geq 3$ is explicit: a positive logarithmic coefficient factor is necessary even when every nonzero coefficient has absolute value one. Corollary 10.250 has the same height floor and requires no independence. Its proof uses the programme results above, with the second explicit constant of Yu's freely accessible 2013 paper.

Example. In $K = \mathbb{Q}$ at $p = 5$, take bases 2 and 3, weights $\log 2, \log 3$, and coefficients $a, -b$. Independence follows by comparing prime exponents. With $B = \max\{3, |a|, |b|\}$ and $2^a 3^{-b} \neq 1$, the theorem gives

$$v_5(2^a 3^{-b} - 1) < (16\mathrm{e})^6 \, 2^{5/2} (\log 4)^2 \frac{4}{(\log 5)^2} \log 2 \log 3 \log B. \tag{10.1150}$$

For $(a, b) = (3, 1)$, the argument of the valuation is $5/3$, so the exact valuation is one.

Exercise. Apply Corollary 10.250 over $\mathbb{Q}$ at two to $(6, 10)$ with coefficients $(1, -1)$, and then to $(6, 10, 15)$ with coefficients $(1, -1, 1)$. Compute the normalized units and exact valuations. For the second list, exhibit the relation and the two-base product after deleting the normalized part of 15.

Solution. Here $d = e = f = 1$. The normalized units in the first list are 3 and 5; the product is $3/5$, giving $v_2(3/5 - 1) = 1$. In the second list the valuations are $(1, 1, 0)$ and the normalized units are $(3, 5, 15)$. Their exact relation is $3^{-1}5^{-1}15 = 1$. Choosing weights $(\log 6, \log 10, \log 15)$ makes the third weight largest. Formula (10.1147), with $k = (-1, -1, 1)$, replaces the coefficient vector by $(2, 0)$. Both products equal 9, and $v_2(9 - 1) = 3$. In both cases take B=3; the original valuation sums vanish. The original bases include nonunits, the second normalized list is dependent, and

the original field has no primitive cube root of unity. Lemmas 10.244–10.245 and Corollary 10.250 nevertheless retain $d = 1$ in the bound.

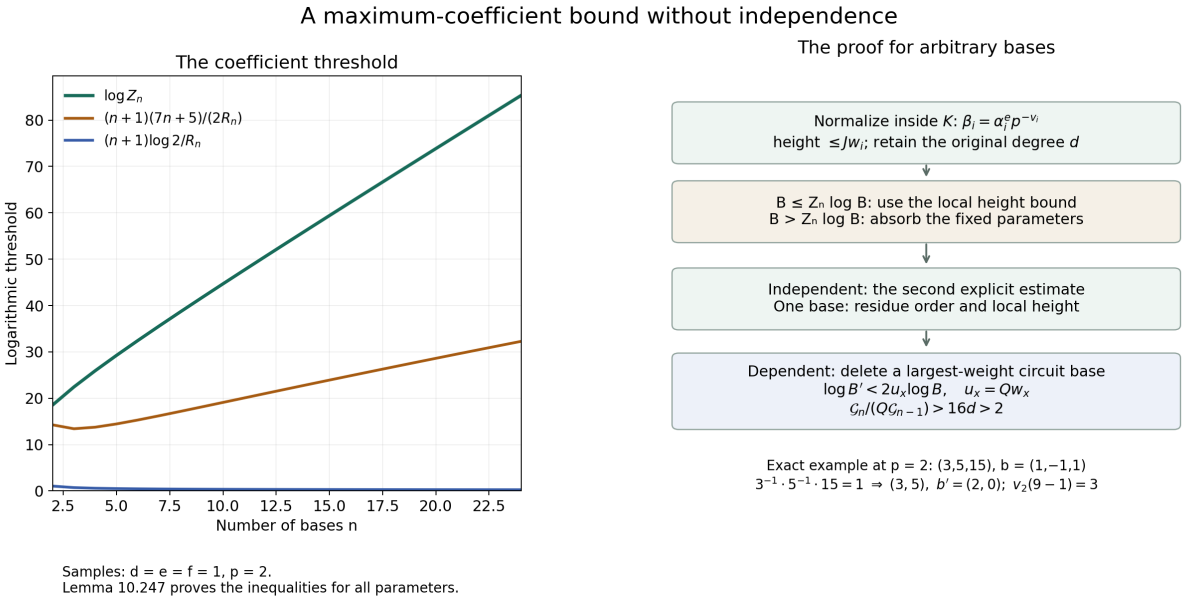


Figure 10.75. The left panel samples the three quantities in Lemma 10.247 for the rational field at two. Their inequalities for every rank, degree and prime are proved in the lemma. The right panel records the alternatives in Lemma 10.248 and Theorem 10.249. A dependent list loses one base while the increase in its logarithmic coefficient parameter is less than the decrease in the target constant. The displayed three-base example has exact valuation three. Corollary 10.250 returns to arbitrary original bases with the original field degree. Reproducible figure and exact comparisons. Human-source context: Theorem G of the free BIRS notes and Theorem I of Yu's free 2013 paper.

82. Exercises with solutions

1. **Easy.** Explain exactly how nonunit bases can be reduced to unit bases. Give an example showing that the conclusion of Theorem 10.8 cannot be extended unchanged to nonunits.

2. **Medium.** Derive (10.34) from (10.22), including the maximum in H.

3. **Medium.** Prove (10.36) using the residue order and the power laws. Explain where the assumption that p is odd enters.

4. **Medium.** For $K = 3, L = 4, p = 3, e = 1, t = 0, u = 0$, find the lower bound supplied by (10.10). Find the valuation precision required in its hypothesis, and the exact excess in the row expansion when n ordinary rows are used.

5. **Easy.** Show that (10.6) is attained for $K = 2, L = 4, R = 3$. Identify the maximizing list of first coordinates.

6. **Medium.** A proposed assertion for every prime p and pair of coprime positive integers x, y bounds $v_p(x^{p-1} - 1)$ by $283(p-1)\ln y \ln(xy)/(\ln p)^2 + 4$. Show why it is false as stated when $y = 1$, without using a logarithm estimate.

7. **Hard.** Adapt the determinant argument to the hypotheses of Theorem 10.11. Explain why its first-base precision, unit coefficient after removing the common p-part, and second-base condition at two are all recorded. Derive the power E^{-3} in its final bound.

8. **Medium.** Compute $M_2(4, 9)$ over $\mathbb{Q}$, prove (10.48) in this example, and compute the residue index (10.49) at five.

9. **Hard.** Derive the constant 58 in (10.55) directly from (10.53), including the roles of the chosen

height radius and the selected archimedean weight. For the rational bases 4,9, compute the full saturation index, its root-of-unity factor, and the diamond average in Figure 10.2.

10. **Medium.** Apply Theorem 10.21 to the rational unit 2 at three. Determine its residue order and exact valuation for every nonzero exponent with a nonzero power difference. Compare the bound with the normalization in a field of degree two having local degrees e=1, f=2.

11. **Medium.** At p=5 take $L_1 = 5z_1 + z_3$, $L_2 = 5z_2 + z_3$, $L_3 = -5z_1 - 5z_2 - z_3 + z_4$, and $L = L_1 + L_2 + L_3 = z_3 + z_4$. Compute the matrix in Lemma 10.23 and all its two-column upper minors. Choose a minimum-valuation derivative basis. Explain the denominator introduced by the other natural basis.

12. **Hard.** Set $r = 2, S = T = 90$, $D_0 = 32, D_1 = 2, D_2 = 3$, and $S_i = T_i = 30$. Show that Theorem 10.25 implies that a polynomial of these bounds with all the stated jets zero at $(s, 2^s, 3^s)$, $0 \le s \le 90$, is zero, for every nonzero rational vector $\boldsymbol{B}$. Check the integer degree inequalities and both possible one-dimensional relation bounds.

13. **Medium.** In Theorem 10.27 take $K = \mathbb{Q}, p = 5, n = 2$, bases $2, 3$, coefficients $1, 1$, and $L_1 = z_1 + z_2$. Use $\sigma = \ln 6, H = 10 \ln 5$. Compute the original saturated residue index, w_q, R_2, M_γ, and verify the one-base height-product hypothesis without a numerical approximation. What valuation is being bounded?

14. **Hard.** For $k = L = 2$, compute the four shifted-power polynomials in Figure 10.4 and the determinant of their coefficients of X, X^2, X^3, X^4. For $k = \ell = 2$, evaluate $v(k)^m \Theta(1/3; k, \ell, m)$, $0 \le m \le 4$, and check the exact q-power clearing bound with $q = 3, I = 1$.

15. **Medium.** Use Theorem 10.32 over $K = \mathbb{Q}$ with $r = 2, k = L = 1, S = T = 0$, $\vartheta = (2, 3)$, $A_1 = A_2 = 1$, $D_1 = Y_1\partial_{Y_1} - Y_2\partial_{Y_2}$, and exponents $(0, 0), (1, 0), (0, 1)$. Compute $N, M, \mathcal{B}$, exhibit a nonzero integral coefficient vector, and compare its exact projective height with (10.96). Form the ordinary polynomial P and identify which jets are imposed.

16. **Hard.** For the full node set with $p = 3, R = 3, \mu = 2, \theta = 1$, what precision in (10.100) suffices for conclusion $v_3(F(\rho x)) \ge 14$? For $p = 2, q = 3, R = 3$, compute $L_{-2}(0)$ in the q-deleted set and its valuation. Finally, why is the restriction $x \in \mathbb{Z}_p \subset \mathbb{Q}_p$ essential to the full-set cardinal bound? Use $p = 2, R = 1$ and an algebraic x with $x^2 = 2$.

17. **Medium.** In Proposition 10.37 take $p = 3, q = 2, I = 0, k = L = 2, \rho = 3$, and a single nonzero coefficient $c = 1$ at $\boldsymbol{m} = 0, a = 0, \ell = 2$. At prepared index zero, compute γ, Γ, and the polynomial $F(Z)$. Check (10.107) for $s = 0, j = 1$. If a torus factor with slope w is added, compare the sufficient slope hypothesis for $w = 9$ and $w = 3$.

18. **Hard.** At two, take the additive prepared polynomial $f(y) = \Delta(y; 2) = (y + 1)(y + 2)/2$, with $I = 0, k = 2, L = 1$ and no torus dependence. Show that its prepared values through order two at zero are integral, whereas its first ordinary divided jet has valuation -1. Explain the exact role of C_0. Next take full interpolation nodes with $p = 3, R = 3, \mu = 2, \theta = 1$, and weighted jet precision $\Lambda - jC$ with $\Lambda = 16, C = 2$. Compare Theorems 10.34 and 10.39.

19. **Medium.** At $p = 3$, take $q = 2, k = L = 1, I = 0, J = 1, c = 1$, no torus dependence, and $x = 1/2$. Compute the prepared value $\Delta(x; 1)$, the clearing exponent in (10.115), and the bound (10.116) with $X = 1/2, T = 0$. Distinguish the denominator cost at two from the valuation at three.

20. **Hard.** Let $K = \mathbb{Q}, p = 3, q = 2, J = 1, \vartheta = 10$. Use the exponential and logarithm to choose a square root $\eta \in \mathbb{Q}_3$ congruent to one. Determine the coefficient (10.117) for $F = \mathbb{Q}(\eta)$. At $s = 1, x = 1/2$, with $k = L = 1, I = 0$, take exponents $1, 3$ and coefficients $1, -1$. Factor the prepared value into its common local unit and a rational value, and compute its valuation. Compare the bounds from the root field and the common-factor rule, with $X = 1/2, T = 0$.

21. **Medium.** Use $I = 0, k = 2, L = 1, X = 2, r = 2, \Omega_1 = 1, T = 2$. Compute $\mathcal{P}_{0,2}(Z)$, $\mathcal{B}_0^*(2,2)$, and the earlier $\mathcal{B}_0(2,2)$. At $x = 2, a = 1, \ell = 1, \omega_1 = 1$, show that the three additive bounds are attained.

22. **Hard.** In Corollary 10.45, take $K = \mathbb{Q}, r = 1, k = 2, L = 1, S = 2, T = 0$, $\vartheta = 2$, and $\mathcal{M} = \{0, 1, 2\}$, so $A = 2$. Compute the separate-row bound (10.127) and the uniform averaged bound (10.128). Order the six columns by $(m, a) = (0,0), (0,1), (1,0), (1,1), (2,0), (2,1)$. Verify that $(2, -18, -57, 36, 1, 0)$ is an integral kernel vector at all five nodes, and compute its two projective heights. Explain why the zero row at $s = -2$ does not invalidate either bound.

23. **Medium.** In Theorem 10.47, take $q = 2, r = 1, F = \mathbb{Q}, \vartheta = 2, \eta = \sqrt{2}$. Show that at $s = 2$, the coefficients $A_0 = -2, A_1 = 1$ give a zero total sum whose separate coset coefficients are nonzero. Give a similar example with odd s in which the full-degree hypothesis fails. Finally, at $p = 7$, use the local square root $\eta \equiv 3 \pmod 7$ of two to show that finite valuation precision of a total sum need not transfer to its cosets.

24. **Hard.** Let $q = 2, p = 3, r = 2$, use one Euler argument $\omega(\boldsymbol{m}) = m_1$, and retain $m \equiv 1 \pmod 2$, with $m_* = 1$. Compute the affine binomial change $\Delta(2Z + 1; t)$ through order two and its inverse. Verify the minimum-valuation identity on the new jet vector $(1, 3, 9)$. Explain why it preserves total order when combined with an unchanged additive index. Then apply Corollary 10.49 over $K = \mathbb{Q}$ with $\alpha_0 = -1, \theta_1 = 2, \theta_2 = 3, P = 3, d_1 = 1, d_2 = 0$: identify F, the two phased roots and bases, their heights, and the root degree over F.

25. **Hard.** Use the divided support in Figure 10.8. Compute a_0, H, both subclasses and their phase factors after multiplication by ζ^s. If a coefficient of minimum valuation belongs to $\lambda_0 = (-1, 0)$, state the retained phase congruence and normalized root-of-unity factor. Explain why the subclass is chosen independently of s and the derivative index. Compare the coprime case $q = 2, G_0 = 5, a_* = 2$.

26. **Hard.** Over $\mathbb{Q}_5$, apply the Newton construction to $f(X) = X^2 + 1$, starting at $x_0 = 2$. Compute x_1, x_2, the three error valuations and the primitive fourth root ζ modulo 625. Take $K = \mathbb{Q}, \alpha_0 = -1, q = 2, \theta_1 = 2$, and $g = 2/\zeta$. Prove $v_5(g - 1) = 1$. For the target $\vartheta = 3$, verify the choices $t = 0, j = 3, P = 125, d_1 = -125$ in Lemma 10.53 and compute the exact valuation of $\gamma_1 - 1$.

27. **Medium.** In Corollary 10.54, assume the other hypotheses and $\mathcal{A}_{\mathrm{ph}}(4,4) = 11$. Take $p = 3, q = 2, R = 4, T = 5, \mu = 2, k = L = 2, \theta = 3, C_0 = 0$. Compute the exact input node set, n, B, M_0, T', and the least U meeting the exact first budget. State the resulting zeros. Compare both lines of the alternative larger-scale budget; explain why its first line alone does not give the same zero conclusion.

28. **Hard.** In Figure 10.9, verify that the individual γ_i lie outside $\mathbb{Q}$, whereas every normalized row entry is rational. Determine N, M_*, H_{ph}, and check the vector $(-b, C, -C, b)$ at all three nodes. Factor its torus polynomial, compute its two projective heights and its minimum 5-adic coefficient valuation, and compare its height with (10.152). Explain why the zero first row does not invalidate the bound.

29. **Hard.** Use the data of Figure 10.10. Compute the cancelled congruence and h_Λ, prove the degrees and chosen completions of E and F, and evaluate both 5-adic valuations of V. For $I = 0, k = L = 1, \lambda_0 = 0, c = (1, -1), s = 1, x = 1/2, \boldsymbol{t} = 0$, compute every term in (10.156). Compare the support-field budget with the budget from the full root field, and explain why equal local degrees do not justify using the original global degree.

30. **Medium.** Verify the auxiliary-prime choices in Corollary 10.9 for $(a, p) = (8, 3), (3, 2), (25, 2)$. Compute the exact original and transformed valuations, the rational heights and the cutoff H. Derive the all-power bound when $b = 1$, retaining the same $250000 \ln(5a) \ln 5$ coefficient. Explain why simply setting the second base to one would not prove this case.

31. **Hard.** Verify the two support heights in Figure 10.11 by clearing denominators in the monomial vectors and finding their primitive integer coordinates. Compute both signed heights of $\{(0,0),(1,0),(0,1)\}$. Extract the even-even coset from the diamond at $q=2$, use reference $(-2,0)$, and verify (10.162). At $P=1,S=1,k=L=1,T=0$, compare the separate-row coefficient budgets for the seven-point diamond support using its exact height and its coordinate box. Explain how a shifted original-coordinate box is handled when its centre is outside the image of B.

32. **Hard.** Verify the two initial kernels in Figure 10.12 and determine their exact row fields. Prove the discriminants 5 and 24 using the full integral bases of the two quadratic fields. Compute both projective coefficient heights, explain why their finite-place contributions vanish, and evaluate (10.171) with the actual row-field discriminant for each example. For $t=\sqrt{6}$, compare the one-generator discriminant bound with the bound using only the maximum support height. Explain what happens to the row field when $S=0$, and why $N>M_*$ and the later analytic inequalities still need checking.

33. **Medium.** Verify every lattice and phase coordinate in Figure 10.13. Compute $J, V/\Delta, g, H$, the guaranteed class size, and both sides of (10.180) at $k=2,L=1$. At $S=1,T=0$, compare the coefficient surplus certified by the exact class count with the coarser volume criterion. Then take $T=1$ and one nonzero Euler direction: compute M_*, and state precisely what the counting criterion does and does not prove. Explain why replacing the closed box by an open one would invalidate the integer-volume argument.

34. **Hard.** Use $K=\mathbb{Q}, p=3, r=2$, $(\alpha_1,\alpha_2)=(2,5)$, and $c_0=1.9, c_1=1.4494, c_3=1.3852, c_4=20.8$. Put $a_0=2+\ln 14$, $a_1=4.79$, $g_0=3(2a_0+a_1+\ln(2a_0+a_1))$, $h=g_0$, and $\tau=1/(4\cdot 10^{26})$. Use $\widehat{\vartheta}=3/2$ and $\sigma_1=\ln 2, \sigma_2=\ln 5$. Prove that $J=1,u=1,H=1$. Verify the floors in Figure 10.14 with rigorous bounds, construct a support of exactly $\lfloor D_1D_2\rfloor+1$ points without listing it, and compute $N, M_{\mathrm{full}}, M_*$. Use the integral identity $20N-57M_{\mathrm{full}}>0$ to verify the strict surplus. Identify the actual coefficient field and its discriminant cost.

35. **Hard.** First take $K=\mathbb{Q}(\zeta_3,\sqrt{2})$ and $p=2$. Prove that $d=4,e=2,f=2,u=1$, and verify equality in (10.192). Then return to all the rational parameters of Solution 34. Compute Ω_1, the actual clearing integer, both maximizing additive orders in (10.199) and (10.201), and the exact mean orders in (10.204). Reconstruct each signed component and net height bound in Figure 10.15, using (10.174) with the actual ratio $M_*/(N-M_*)$. Include the row-scaling bounds of Corollary 10.80. Explain why the coefficient field and all zeros are preserved by the alternative global clearing.

36. **Medium.** Reconstruct every exact lower-bound path and final depth in Figure 10.16. Distinguish these lower bounds from the actual valuation path for $\beta=1+\sqrt{2}$ in $\mathbb{Q}_2(\sqrt{2})$. Next let $\pi^3=3$, $L=\mathbb{Q}_3(\pi)$, and $\beta=1+\pi$. Show that the bound (10.210) is attained at $P=9$, and use the exponential coefficients to prove that the closed boundary radius 3^{ϑ} cannot replace 3^{θ} in Corollary 10.82. Describe the coherent principal square root of the phased coordinate for $\alpha=-\beta,\zeta=-1$.

37. **Hard.** Reconstruct Figure 10.17 over $K=\mathbb{Q}$ at $p=5$. Prove that the q-primary saturation lattice of $4,9$ has basis $B=\frac{1}{2}I$ and index $J=4$, and identify the even phase class of its generators $2,3$. Compute the exact prepared Euler range at reference 0, and again at reference $(8,0)$. Explain why the full box widths give a bound without a factor two, and why omitting J can even destroy integrality. Then return to Solution 34 and use (10.222)–(10.228) to certify the uniform initial height bound, retaining every field and scalar cost.

38. **Hard.** Reconstruct Figure 10.18 for $K=\mathbb{Q}(\sqrt{5})$, with original bases $5,7$ and coefficients $b_1=b_2=1$. Determine the full saturation lattice and its q-primary index at $q=2$, verify $B'=BU$, and compute both lists of generator heights. Identify the field and its discriminant. At $p=3$, determine e,f,u,ν,P and the phase classes. Use the constants of Solution 34, put

$g_0 = 3(2a_0 + a_1 + \ln(2a_0 + a_1) + \ln 2)$, $h = g_0$, $\sigma_1 = \ln 5, \sigma_2 = \ln 7$, and retain its τ. Certify the original g_{11}, g_{12}, a strict initial column surplus, and the full bound (10.235). Compare the generator field with the smallest field of the actual rows.

39. **Hard.** Reconstruct Figure 10.19 for the entire coefficient family (10.245). Obtain a rigorous common Euler-argument bound and a bound for $v_3(b_2)$, then determine every $R_j, T_j, \mu_j, n_j, B_j$. Certify both comparisons at all three integer steps, retaining the complete pointwise factorial and lcm costs. Explain how a ratio search bounds the enormous Euler binomial without constructing it, and why these integer comparisons do not establish the fractional descent.

40. **Hard.** Derive the five Hermite basis polynomials for $R_0 = 0, R_1 = 1, \mu_0 = 3, \mu_1 = 1$, checking every divided jet. Reconstruct the nested weights, total multiplicity and both strict comparisons in Theorem 10.95. Explain why equality of the chosen completions does not remove the quartic arithmetic factor, and why evaluating the denominator cost only at the scalar-maximizing order need not bound the joint maximum.

41. **Hard.** Expand $\binom{2n+a}{2}$ in the basis $1, n, \binom{n}{2}$, and give the inverse expansion of $\binom{n}{2}$ in the old argument. Explain the relevant integrality at three and two. Reconstruct all seventeen rows of the table in Theorem 10.98, retaining the odd-node loss, the true contracted Euler bound and both global field factors. Give the stage-one and stage-seventeen radii, orders and mixed multiplicities. Complete the polynomial contradiction.

42. **Hard.** Verify the smallest-rank rational constants in Theorem10.100 and prove that every displayed bound increases with its rank or subset-rank index. Explain why using $a_* = 7$ in every case would omit an original parameter case. Check the complete cancellation in the intermediate and final allocation ratios, retaining both factorials, all q^ν and $e\theta$ factors, and the real-to-integer floors. Explain how Corollary10.101 returns the new bases to the original field.

43. **Hard.** Derive (10.274) directly from (10.270) and the complete original parameter formulas. Identify where the old and new residue indices occur and why neither may be replaced by a fixed value. Prove the all-rank bound on $t^j M_j$, including the field condition at two, and reconstruct the bound $1/700$. Show that every independence and weighted-height condition in (10.277) survives composition with the integral vectors u_i. Verify the rank-two example and explain why its actual minimum rank is two.

44. **Hard.** Check the four minima in (10.281) by exact rational arithmetic. Prove the bound (10.285) by adding its three separate contributions. Enumerate the full rank-two derivative simplex through order six and show that its 28 rows have mean additive and Euler order two even when the additive degree is two. Derive (10.290), and explain why zero rows may be retained in the integral kernel argument. Finally take $k = 4, L = 1, p = 3, u = 2, s = 0, I = J = 0$ and the scalar column $F_{0,1}(X)$. Compute its prepared value, the value W of (10.298), and their exact valuation difference. Explain the cancellation in (10.297) and why it does not remove the global degree d_F.

45. **Hard.** Reproduce the certificate of Lemma 10.111 using only integer and rational arithmetic. Explain why checking $\psi(n) > 25n/26$ would be insufficient for the real-argument lower induction. Show that F has period 30030 and that its reflection sum is three. Derive $C_r^\sharp$, and quantify the improvement over C_r at rank two.

46. **Hard.** Derive (10.309) directly from the positive geometric series, retaining a prescribed additive order. Reconstruct every scalar and bound in Figure 10.26 with exact rational arithmetic before taking logarithms. Show that its peak is at $u = 1$, and explain why the remainder in (10.310) has a separate additive term. Verify the seven rational products in its proof and the two sharp-regime gaps in Lemma 10.114.

47 (additive divisibility and exact jet envelopes). At $p = 3, k = 4, L = 2$, compute $D_{\mathrm{a}}, \ell_p, \chi_p$. With $\beta = 0, \vartheta = 1, B = 0, M = 7$, evaluate the full integer minimum $K_u(M, B)$ for $0 \le u \le 8$, and

compare it with the linear-loss bound from C_0. Explain why no further lcm loss occurs after the breakpoint in the left panel of Figure 10.27. Next set $\beta = 3, B = 1, M = 5$, keeping the other data, and evaluate all nine envelopes again. Compare with the linear consequence of (10.318). Finally explain why a local q-unit in the jet proof does not allow deletion of $\Xi_{I,J,u} \ln q$ or of the global field degree in the zero criterion.

48. **Hard.** Reconstruct the strict original integer-extension comparison (10.332) in every field case. Explain why rank eight supplies a uniform estimate for all larger ranks, whereas checking finitely many ranks alone would not. Calculate the exact uniform I.2 margin and its difference from 1/200. Prove why the first q-denominator indicator does not spoil the increase from step zero to step one. Finally explain why the continuous depth curve in Figure 10.28 does not assert existence of an auxiliary stage at every plotted integer depth.

49. **Hard.** Use the exact original floor data of Solution 39 to calculate all three nested multiplicities in (10.336) at output order $O = 993090$. Calculate N_* both by annulus counts and by the telescoping formula. Compare it with the last zero block alone, and explain why this comparison concerns interpolation precision rather than actual nonzero function valuations. Prove the uniform four-term lower bound in (10.342), including its radius-floor correction.

50. **Hard.** Follow the complete first contraction through (10.344)–(10.348). In the original rational example, verify $R = 1374$, $M = 178095$, $B = 7$, and the q-deleted cardinality. At fixed additive order zero and $\beta = 0$, compute the two separate Hermite input losses. Explain why the fractional step retains $q^{h_\wedge}$, whereas the contracted integer step uses the field K, even though both selected local completions may already coincide.

51. **Hard.** In the small profile (10.355), list the nodes and their multiplicities, compute the four deleted-node residue counts modulo four, and derive the three cardinal-factor bounds from (10.354). Verify the total degree by both the annular and telescoping formulas. Explain why replacing the discrepancy two by one changes the proof.

52. **Continuous-depth comparisons and terminal closure (advanced).** Prove that a concave function on $[a, b]$ is bounded below by the lesser of its endpoint values. Explain why the positive coefficients of τ_j in (10.367) are needed when replacing them by conservative endpoint bounds. Derive both geometric sums used in (10.373). Finally, explain why the phase induction needs a closure estimate on $[1, L_D + 1]$, and compute the added clearing cost at its last endpoint.

53. **[Challenging: a complete original mixed profile.]** Take the illustrative rank-two III.2 parameters $q = 2, c = 0.5267, S_I = 100, T_I = 1000$. Compute H, the full integer radii, all original derivative cutoffs, the fractional output cutoff O, the three multiplicities μ_0, μ_1, μ_2, and the deleted increment E. Count the interpolation degree by disjoint node types and verify (10.356). Explain why the terminal closure at $J = L_D + 1$ uses a clearing cost $\ell(L_D + 2)$, and why a local completion of degree one does not remove q^r from the fractional comparison.

54. **The geometric endpoint and the preserved rank allowance (intermediate).** (a) For arbitrary $Q > 1, D > 0$, show that $I^* = \lfloor 3D/\ln Q \rfloor + 1$ is the least integer for which $Q^{I^*} > \exp(3D)$. Evaluate this integer when $Q = 9/8, D = 64\ln(9/8)$, and explain why equality $I^* = I_1$ is included in Theorem 10.141. These numbers test the stopping definition, rather than claim a complete choice of contradiction parameters. (b) Explain why the Laurent monomial shift in Proposition 10.31 preserves all ordinary hyperplane jets through the same total order. (c) If $L'_i = \sum_j u_{ij} L_j$ and $R_i = \sum_j |u_{ij}|\sigma_j$, prove the weighted-height condition in (10.277) for L'_i. Then verify that the factor $\Gamma_n^{r-m} M_r/M_m$ cancels the original rank allowance to give precisely Ψ_m, including $m = 1$.

55. **The multiplicity endpoint and final support (intermediate).** (a) For $a = 3, c = 269/500, \eta = 1 - c/a$, set $T_1 = a/(c\eta)$. Find the unique r_1 of Lemma 10.142 and compute $cT_\flat/a$. These scalar values illustrate the stopping definition, rather than a full choice of original contradiction parameters. (b) Show why replacing floor plus one by a ceiling in the definition

of I_2 fails at an integer quotient. (c) Verify the rational equality $5220/(7 \cdot 9 \cdot 2^{13}) = 145/14336$, and explain why the strict coordinate bound in Proposition 10.143 leaves only the origin in the integral torus support. Explain why the resulting nonzero additive polynomial cannot have the stated number of integer roots.

56. **The retained terminal jets (advanced).** (a) Compute R_0, N_* for $R_b = 25$ and $R_b = 26$, and prove the two mass inequalities for every positive integer R_b. (b) Explain why the outer cardinal factors lose at most B, rather than zero, and why this is already charged in the $M = 2, E = 0$ input bound. (c) Starting from $1 \le c\eta^r T_1/a$ and $T_1 < a/(cH)$, derive the upper input bound P_r and strict mass lower bound G_r in Theorem 10.144. Identify the term in C_r that retains the full torus and q-denominator cost. (d) Explain why the eight rank-eight endpoint rows cover every larger rank, and why a finite list of successful sampled ranks would not give that conclusion.

57. **Values without additional ordinary jets (advanced).** (a) Reconcile the zero-extra-jet convention with multiplicity $M = 1$ in Theorem 10.119. Explain why the cardinal and inverse-polynomial losses vanish, while the coefficient minimum β remains. (b) Derive $c_1 W_* \beta \le 1/(c_3 Y_- g_4)$ directly from the original formulas for S, T, W_* and $ef \le d$. (c) For $p = 5, q = 2, R = 2, x = 1/2$, compute all five $L_s(x)$ and their valuations, and verify the exact root-polynomial calculation in Figure 10.36. These values illustrate full-node normal interpolation; they are not a complete choice of logarithmic contradiction parameters. (d) Explain why the local analytic conclusion of Proposition 10.145 alone does not prove the fractional zero, even if its value belongs to the selected completion.

58. **The successor integer block (advanced).** (a) Prove that the multiplicity-stop index is at least eight for odd p and at least six in V, using the explicit band bounds on H in Theorem10.146. Show why this implies $Q^{I_1} > q$. (b) Take $p = 2, q = 3, R = 3$. Compute the four q-deleted cardinals at zero and their valuations; compare their actual loss with $B = \lfloor \log_2(2R + 1) \rfloor$. (c) Derive the bound on $I_3 \ln q$ in the theorem, retaining its floor-plus-one endpoint. Identify the denominator term that would be lost by using the old geometric range for J. (d) Explain why the theorem includes $J = I_3$, and what must still be proved to construct that family.

Exercise 59 (medium: the field of a prepared value). Use the exact phase data and principal roots of Theorem10.147. (a) Compute the surviving component spans and exact degrees of $U_1 U_2$, $1 + U_1$ and $V = 2 - U_1 - U_2$. (b) Prove the displayed minimal polynomial of V and its norm. (c) Compute its four valuations above73 without decimal root approximations. Explain why a selected completion equal to $\mathbb{Q}_{73}$ does not identify those valuations.

Exercise 60 (advanced: extra integer zeros at a fixed order). (a) For the illustrative floor data $q = 3, \theta = 1, U = 1001$, $\xi = 24/25$, compute $R_{\mathrm{s}}, R_{\mathrm{l}}$ and their node counts in Theorem10.149. Check $qR_{\mathrm{s}} \le R_{\mathrm{l}}$ and the strict normal-mass lower bounds. (b) Explain why all three stages use the same selected order, while the two integer comparisons use the original field and the fractional comparison retains q^r. Locate the term which is added outside that field factor. (c) Prove that the two-step depth increments in Lemma10.148 give an enclosure of every rank at least32. Why would a fixed value of m_{32}, or finite positive rank comparisons alone, not suffice?

61. **Hard: the explicit bound after a change of rank.** (a) Derive the exact expression for eU_r in Theorem 10.152 from (10.186), retaining γ, q^ν, q^u and both maxima. Substitute the profile (10.277) and identify every factor that cancels. Prove the bound on the remaining rank ratio for every $1 \le r \le n$. (b) Explain why the actual rank-one residue order gives $M_1 > g/t$, and verify the two strict fractions $1/400$ and $1/4000$ in that proof. (c) In the III.2 row, compute L_2, G_2, and show by rational arithmetic that its gap exceeds $4059/10^6$. Explain why this rank-two calculation controls every rank, and why retaining the lower bound $g_1 > 509/100$ matters.

62. **Hard: replacing the coefficient-height parameter.** (a) Prove $n! \le \mathrm{e}\sqrt{n}(n/\mathrm{e})^n$ from the concavity of the logarithm, without a Stirling formula. Use it to prove the infinite-rank part of

Lemma 10.153. Explain what the exact finite certificate covers and what it cannot prove alone. (b) Verify both alternatives $B/\ln B \le W_B$ and $B/\ln B > W_B$ in Theorem 10.154, including the small remainder $(d/t)\ln 2$, the coefficient renumbering and the singleton-height bound. (c) Over $\mathbb{Q}$, take $p = 3, a_1 = 4, b_1 = 3^k$ with $k \ge 1$. Compute $g, \delta, q^u, M_1, C_1^*, \Omega, H$, and the exact valuation. Derive the displayed bound with factor $1/2100$. Verify the two rational fractions used in the general rank-one proof.

63. **Hard: eliminating a dependent base with its weights.** (a) For $(2, 3, 4) \subset \mathbb{Q}^*$ at five, compute the greedy basis, L_3, L_2, the two modified height products and the residue index. Transfer coefficients $(1, 1, 2)$ through the exact circuit, and compute the valuation. Identify its coefficient-height branch. (b) In the general proof of Lemma 10.155 derive the normalized circuit envelope, first for terminal removal and then for both internal cases. Include the empty-list height bound and every degree power. (c) Derive the polynomial lower bound for $\ln(J_n/(n+5))$, and explain how the fourteen rational endpoints combine with it. Verify the final absorption inequality of Theorem 10.156. (d) Explain why removing either the first discarded base or an initial root of unity preserves both the greedy basis and its residue index; state the direct bound when the original list has rank zero.

64. **Hard.** For the indexed list $(4, 16)$ at five, with coefficients $(0, 5^k)$, compute the greedy rank-one basis, its original residue index, the length-two floor and weighted product. Give an exact one-base reduction and valuation. Then prove the cancellations in Lemma 10.157, explain why the maximum in the definition of s in the length-two scalar comparison is necessary, and verify the complete infinite-rank estimate in Theorem 10.158. State where a torsion complement changes the exponent budget.

65. **Medium.** For $S = \{3, 5, 7\}, p = 3$, compute the common factor, coprime exponents and exact valuation of $3^N 25 - 3^N 7$. Verify the residue index and constant displayed in Theorem 10.159, and show why a bound without the common-factor term would be false. For $x = 25, y = -7, p = 2$, verify the field data, sign coefficient, length-three floor and weighted product. Finally prove the exact $n = 0$ clause and derive Corollary 10.160.

66. In Theorem 10.161, compute the exact III.1 budget by using $d_{\min} = 2$, $\rho = 58$, $c_{\min}^{(2)} = 495$, $a^{(2)} = 7$, and $w_K/q^u \ge 1$. Prove its strict comparison with $1/4000$. Then determine the parameters for $K = \mathbb{Q}, p = 3, \beta = 4, b = 3^k$, $k \ge 1$, and verify both the exact valuation and the displayed second-constant estimate.

67. For the rank-one list $(4, 16)$ at five in $\mathbb{Q}$, find the second-constant height floor, the minimizing selected base, and its original residue index. For coefficients $(0, 5^k)$, $k \ge 0$, prove the exact valuation. Then prove the all-length inequality $\alpha_n > 3n/2$ from the two scalar facts $\alpha_2 > 3$ and $F_{n+1} > 3F_n$, and explain how the strict exponent and residue-height budgets add to $1/4000$.

68. **(Medium: ordering, residue order and the two cutoffs.)** For $p = 5$, $\alpha_1 = 1 + 5^m$, $\alpha_2 = 1 + 2 \cdot 5^m$, and $b_1 = b_2 = 1$, verify all hypotheses of Theorem 10.165, determine g, E, u, H_6, H_4, and calculate the exact difference valuation. Which displayed bound is smaller? Explain why the coefficient-ordering condition is automatic when both bases have the same deep precision, and why it cannot be dropped solely by interchanging a pair whose first base alone has that precision.

69. **(Medium: the two 2-adic unit classes.)** Apply the sign normalization of Corollary 10.166 to $(\alpha_1, \alpha_2) = (3, 5)$. Compute the valuations for $(b_1, b_2) = (1, 2)$ and $(2, 2)$. Prove that every odd b_1 gives valuation one, for every nonzero integer b_2. Explain how the limiting argument permits a prescribed logarithmic height equal to $\ln 2$.

70. **(Medium: fixed rectangles and the one-base case.)** Verify the exact margin $22/7 - 96/35 = 2/5$ and the coefficient $539/2 + 7/3 = 1631/6 < 283$ in Theorem 10.167. For $a = 8, b = 1, p = 3$, choose an allowed auxiliary prime c, check the valuation-preserving identity

and calculate the exact Fermat-difference valuation. Determine $v_3(8^n - 1)$ for all positive integers n.

71. **(Medium: congruence capacity and field data.)** In the 8-by-8 rectangle with $r + s \equiv 0 \pmod 2$, select all 32 admissible pairs and put $K = 4, L = 8$. Compute the exact maximum centered first-coordinate sum, the compressed occupancy bound and the relaxed bound G_1 of Lemma 10.168. Then take the independent rational bases $2, 3$ at $p = 5$: determine d, e, f, D, g in Theorem 10.170 and explain why no unramifiedness assumption or coefficient-ordering assumption is needed in its general statement.

72. **Hard: constants and coefficient imbalance.** (a) Derive the exact quotient C_2/C_1, and verify its seven rational lower bounds under $n \le Pt$. Calculate the smallest gap above $3/2$. Why does the comparison also require Lemma 10.172? Determine P over $\mathbb{Q}$ at three and at an unramified prime at least five. (b) For $(2, 5)$ at three with exponents $(3^k, 1)$, compute the coefficient ordering, residue index, height factor X and exact valuation. State which second estimate applies. (c) Prove the scalar logarithm bound used in Lemma 10.174, verify the exact positive exponential-series margin, and explain why the scalar inequality can be weak although Theorem 10.175 is strict. In the model of Figure 10.51, express both branches in terms of $z = \varepsilon r$, and show that $z \ge 0$ alone does not specify the allowable parameter interval.

73. **Hard: the congruence cost and the analytic gain.** (a) For $u_1 = 10, u_2 = 19$ in $\mathbb{Q}_3$, with j=2,m=1, compute the unit-product homomorphism on $\{0, \dots, 5\}^2$. Give its three fibres and the even-coordinate contraction of the zero fibre. Explain why the entire set of auxiliary polynomial indices E must be retained independently of that homomorphism. (b) Derive the rounding threshold of Lemma 10.177 and the bound on $p^{fm}(A + m)^{-r}$. In the $A_0 = 1/2$ row, verify both the rational gap between 1.3701 and its H endpoint and the gap between its F endpoint and 1/1.0568. Explain why endpoint calculations alone are insufficient. (c) In Corollary 10.179 take $p = 3, q = 2, I = 1, a = 1, k = \ell = 2, t_0 = 1$, with an empty product of the other prepared factors. Compute P exactly, including its coefficient valuations. Use N=2 and $c_\lambda = 1$ on the zero congruence fibre, with zero perturbations, to compare the uniform normalizing valuation with the sharper termwise one. Compute f(0), and explain the original field degree used in a later global height argument.

74. **Advanced: selection, projection and a complete input budget.** At three take $a_1 = 10$, $a_2 = 3281/500$, $(b_1, b_2) = (3, 1)$, $P = 1$, $q = 2$, $\nu = 0$, and $L_1(Z) = Z_1, L_2(Z) = Z_2$.

(a) Determine the exact valuations of $a_1 - 1, a_2 - 1, \Xi - 1$, and all quotient classes modulo 27 on $\{0, \dots, 5\}^2$. For the class containing zero, find the projected frequency and explain why the closed disc with $N = 2$ is allowed even though the first individual logarithm has valuation two.

(b) Take $k = L = 2$, the full nodes $-1, 0, 1$, and every prepared index of total degree at most two. Count the columns and the whole row list, prove that a nonzero integral kernel exists, and identify its row field. Explain how the actual count improves the height row factor to $3/5$. Verify one primitive integral kernel using the reproducible exact calculation supplied with Figure 10.53.

(c) Take $\mu = 3$ and base prepared index zero. Determine B, C_0, Γ and the output precision from Theorem 10.182. Explain which equalities in the numerical budget are harmless and which disc inequalities must be strict. Finally derive the growing Taylor precision in Lemma 10.181 directly from its exponential difference.

75. **Second-scale support and its exact integer floors (hard).** Over $K = \mathbb{Q}$, take $p = 3$, the four independent bases $2, 5, 7, 11$, weights equal to their logarithmic heights, and $h = 150$. Use the parameters of Theorem 10.184. Determine the optimizing depth, all four integer side lengths, the selected congruence class modulo 81, and the integers $k, L, D_{\mathrm{a}}, \lfloor S \rfloor, \lfloor T \rfloor$. Count the columns and the full prepared rows exactly and prove their required surplus. Check also the first full-node local input budget of Theorem 10.182 when the original distinguished coefficient is a 3-adic unit. Give a reproducible rational calculation and state precisely which assertion

about the full valuation estimate remains outside this finite check.

76. **Closing the small optimizer and retaining the first input (hard).** **(a)** At eleven take the independent rational bases $2, 3, 5$. Prove, using elementary exponential bounds, that Theorem 10.188 applies while the numerical range $n \le Pt$ in Theorem 10.173 does not. **(b)** At five take two independent rational units. Derive the strict $11/8$ comparison of their coefficient parameters and the exact remaining constant gap. Explain why the same calculation does not settle all higher-degree fields unramified at five. **(c)** For the four-base support of Solution 75, with original $b_n = B_\dagger = 1$, check the additional height condition of Theorem 10.190 and bound its first local input cost as a fraction of U_r. Keep the actual μ, B, C_0. Distinguish this analytic calculation from a proof of an algebraic zero.

Exercise 77 (degree allocations and the second rank allowance; intermediate). Use the parameters of Theorems 10.184 and 10.191. **(a)** Derive the exact unrounded lower expressions for the rank-zero and intermediate allocation ratios. Explain why the polynomial degree is kL, rather than $\mathcal{D}$, and check the factor $((m+1)!)^2$. **(b)** Verify the final universal bounds $2^{20}/29 > 8$, $1050/2^{36} < 10^{-6}$, and explain which conclusion still requires prepared zeros at the stopping depth. **(c)** For the rank-one argument, prove the inequality $\mathcal{R}_n A_n/t^2 \ge 1/n$ in both ranges of t. Recover the number 9261 and compare the two elementary contributions with $C_2 h \Omega_n/3000$.

Exercise 78 (complete averages and exact zeros; advanced). For the second-scale parameters of Section 64: (a) derive the coefficient C_r by averaging the additive, Euler and slack coordinates, and prove its stated deficit from one; (b) explain why using the Euler parameter a_E in the remainder allows the lcm coefficient $107/103$ to be kept separately; (c) in case I.2, at $y = 1/4, A_0 = 3/2$, evaluate the analytic and arithmetic bounds in Theorem 10.197 and prove their difference exceeds $41/1000$; (d) prove the floor inequality that supplies all ordinary input jets, and identify the two additional costs that prevent this integer argument from proving a fractional extension by itself.

Exercise 79 (uniform integer stages; advanced). (a) Derive the Bernstein coefficient identity in Theorem 10.198 and use it to prove $1 - 2y + 3y^2 \ge 11/16$ on $0 \le y \le 1/4$. (b) For r = 6 and q = 2, compute the final radius and derivative cutoff as exact multiples of S and T. Explain how the floors enter. (c) Verify both strict local-input margins below $15U_r/16$ and count the polynomials and coefficients in the finite comparison. (d) Prove that the stated positive tail increment at five controls every subsequent integer j. Explain why the result does not by itself supply the first fractional block.

80. **The full first fractional budget (advanced).** (a) In case I.2, show directly that the actual selected-column ratio is greater than $27/2$, so the row-to-surplus ratio is less than $2/25$.

(b) Verify the reserve in Lemma 10.200 for that case, including its strict remaining rational margin. (c) For the illustrative nested counts $S = 21/2, T = 1200, r = 3, q = 2, c_5 = 27/50$, compute all four radii, derivative cutoffs and multiplicities of Theorem 10.201, and their total interpolation mass. These are a counting example, not an assertion that this T meets the second-scale construction. Compare the mass with that of the final interval alone.

(c) In $\mathbb{Q}_7$, let $\alpha^2 = 2$ with $\alpha \equiv 3 \pmod 7$. Prove that $v_7(\alpha - 3) = 1$ and $v_7(-\alpha - 3) = 0$. Explain what this says about replacing the global root-field degree by its chosen local degree. (e) Explain which inequalities in the proof make the seven rank-eight comparisons cover all larger ranks.

81. **The first contracted block.** Use the exact counting data of Solution 80: $S = 21/2$, T = 1200, r = 3, q = 2, $c = 27/50$ and $\eta = 173/200$. Compute R, the input and output order floors, the interpolation multiplicity M, the number of q-deleted nodes and the cardinal degree. Check the available ordinary-jet order. Explain why the actual value field at an integer of the contracted family is K and why this does not remove its rational q-denominator.

82. **Saturation and the depth cost (advanced).** (a) For $h > 40, A > 5, x \ge 0$, prove $(h + x)(A + x) \le hA \exp x$ by comparing the coefficients of a quadratic. (b) At $q = 2, r = 3$, use $c_2 = 8750/5101$, $c_4 = 14$, $\lambda = 59/58$ and $\xi = 101/100$ to verify exactly that the linear

endpoint ratio in Lemma 10.205 is below 3/5. (c) Explain why the geometric assertions in Theorem 10.204 alone do not supply the zeros assumed in its rank conclusion.

83. **Hard.** In the counting illustration of Figure 10.62 take $q = 2, r = 3, I = 2, S = 21/2, \widehat{T} = 1200, c = 27/50$. Compute the three integer radii and derivative cutoffs in Theorem 10.206, both interpolation multiplicities, and both cardinal degrees. Verify the exact floor inequalities. Explain why the term $(\mu_j - 1)M_*$ has a bound independent of I after expressing it in terms of $\widehat{T}$, while the original coefficient loss contributes $1/\widehat{T}$. Do these counting data alone establish the existence of a depth-two auxiliary family?

84. **Mixed profiles (advanced).** In Figure 10.63 use $q = 2, r = 3, I = 1, S = 21/2, \widehat{T} = 1200, c = 27/50$. Compute the three radii, four multiplicities, deleted increment, and total interpolation degree, both by disjoint node types and by nested layers. Verify every required jet-order floor. Explain why the finite-rank arithmetic comparison in Theorem 10.207 uses $A_0 \le r$, and why Theorem 10.208 can supply the first integer block beyond $I \log Q \le 3B$ without discarding a denominator. Does Corollary 10.209 handle a family that reaches the derivative-order threshold first?

85. **The last integer block (advanced).** Use the counting data $q = 2, r = 3, c = 27/50, \eta = 173/200, S_1 = 40, T_1 = 12$. Find j_0, R, O from Lemma 10.211 and check the final jet-order floor in Theorem 10.210. Explain why $\mu = 1$ would also be allowed. Derive the constants 609/163840 and 1305/236 in Proposition 10.212. Which additional assertion is needed to turn that proposition into a contradiction for the starting family?

86. **A fixed order and two different integer intervals (advanced).**

(a) In case III.1 at r = 3, calculate L_r, m_r, j_r from Lemma 10.213 using exact rational comparisons. Identify the two vertices sufficient for the convex bound in Lemma 10.214 when B = 10.
(b) For the counting parameter $u = \zeta\chi\widehat{U}_r/N = 201/2$ and q = 2, calculate $R_{\mathrm{s}}, R_{\mathrm{l}}, n_{\mathrm{s}}, n_{\mathrm{l}}$. Check all three floor inequalities in Theorem 10.216.
(c) For a q-deleted interval of radius R = 40 with q = 2 and p = 5, count its nodes and find its separation bound. Write both analytic entries at multiplicity one. Explain why the derivative order is unchanged and why the original coefficient loss remains. Parts (b) and (c) use node-count examples, not complete auxiliary kernels.

87. **A smaller height floor (advanced).** (a) For the boundary parameters $a_1 = a_2 = 1/10, g = 1, B = 1200$ in Lemma 10.218, compute z,L,K,N. Check the lower bound on K and the strict determinant and coefficient margins. These are interpolation parameters, not an asserted example of algebraic numbers attaining all boundaries. (b) Separate the three possible padding cases in (10.992) and derive their bounds from the singleton and two-base height inequalities. Explain why squaring the singleton bound would be weaker. Verify the two exact fractions in (10.996) and the degree 2048 margin. (c) Why must a zero coefficient be treated separately? Derive both terms of (10.999), identifying which height bound absorbs each term.

88. **Original coefficients through a change of rank (advanced).**

(a) For a rank-r integral-form representation, prove (10.1007) after a second integral change of forms. Explain why clearing denominators of its rational coefficient vector does not, by itself, give the coefficient-height hypotheses needed for Theorem 10.152.
(b) Derive the rank quotient (10.1012), including the factorial and residue factors, and recover the uniform fraction 25/41 when r=2 and n is at least three.
(c) At five take the independent rational units $(2, 3, 11)$, original coefficients $(1, 1, 1)$, and forms $L_1 = z_1 + z_2, L_2 = z_3$. Compute the two saturation and residue indices, prove that weights $(\log 6, \log 11)$ satisfy (10.725), and find $v_5(\Xi - 1)$. Does this example prove that two is the least admissible rank?

89. **Hard: padded heights and a separate coefficient bound.**

(a) Prove the monotonicity in Lemma 10.224 also when one of the undistinguished coefficients is zero. Explain why increasing only the height product without updating the logarithmic coefficient parameter would not establish that lemma.
(b) Derive the three rational envelopes in Lemma 10.225 directly from (10.642), including the unramified quadratic field at two. Prove their decrease for every integer n at least two.
(c) Take $p = 5, (a_1, a_2) = (2, 3), (b_1, b_2) = (5^8, 1)$, $(w_1, w_2) = (1, \log 3)$, and $B_n = 1$. Write the two branches and the inflated last weight for $\varepsilon = 1/2$ and $\varepsilon = 5^{-8}$. Prove the exact valuation. Explain precisely why the same proof does not cover arbitrary dependent unit parts.

90. **Medium to hard: rational dependencies with a distinguished coefficient.**

(a) Find a primitive rational generator for $(4/9, 8/27)$. Compute the combined exponents for coefficients $(3, -2)$ and $(2, -1)$, and explain which product is excluded by the nonvanishing condition. At p=2 compute the valuation for bases $(-3, 9)$ and coefficients $(1, 1)$; explain why the even power in Theorem 10.228 handles its sign.
(b) Derive $\log |M| < 3U$ retaining B_n, and prove the factor 14/3 in (10.1045) for every prime and every $n \geq 2$. Identify where the proof uses rationality.
(c) Prove (10.1046) for every $k \geq 1$, and compute U from (10.1047) at $\varepsilon = 1/2$ and $\varepsilon = 5^{-k}$. Explain why the rank cases exhaust two rational bases but leave a case for $(2, 3, 6)$ at five.

91. **Hard: all rational ranks and the sign at two.** (a) For p=5, the bases (2,3,6), and coefficients (2,1,1), compute the determinant reduction in Lemma 10.230(a). Verify the exact product and its distance from one. (b) At p=2, use the bases (-3,27,5) and coefficients (1,1,1). Find a primitive relation among the first two vectors, choose an odd primitive coefficient for elimination, and double the whole relation as in Theorem 10.232. Check both valuations in the factored-power inequality. Repeat for coefficients (2,2,2), tracking the common additional factor two in the transferred coefficients. (c) Derive (10.1062), including the cancellation of the chosen relation coefficient. Explain why doubling the whole relation at p=2 preserves the minimum-valuation condition even though its minimum increases.

(d) Prove that the two distinguished-vector cases exhaust every intermediate rational rank, and explain why the determinant and circuit arguments do not by themselves prove the number-field refinement.

92. **Hard: the second estimate and an unchanged basis.** (a) For $K = \mathbb{Q}, p = 5$, use the sorted list $(2, 3, 4)$, coefficients $(1, 1, 2)$, and B=3. Find the greedy basis and its saturated residue index. Compute Ω_3, determine the two maxima M_3, A_3 in (10.1064), and evaluate the product's distance from one. Eliminate4 by an exact circuit and give the new coefficients. (b) Repeat the torsion-removal step for $(-1, 2, 3, 4)$ with all coefficients one. Track the two valuations and explain why the basis is unchanged. (c) Prove (10.1065) by maximizing its quotient in t. Use the exact sum 1957/720 to verify the positive margin in (10.1085) at two, explaining where the hypotheses on P and the residue degree enter. (d) In the circuit calculation, explain separately the cases in which the largest-height base is deleted, is a surviving basis member, or is a surviving nonbasis member. Why is the last case impossible when s=0?

93. **Hard: normalization without changing the field.** For the quadratic example in Section 79, verify the two original heights from their primitive minimal polynomials and verify the valuation of the product minus one by lifting the root of $X^2 - 2$ modulo 49. Derive the rational upper endpoint for the first row of Lemma 10.238 and explain why it controls every rank. Finally, at the same place, take the bases $(7, 1 + \sqrt{2})$ and coefficients $(0, 1)$. Check the coefficient condition and explain exactly why Theorem 10.239 does not apply to this last list.

94. **Dependence without a prime-to-p relation.** In the quadratic example of Section 80, use $\varepsilon = 1/2$, $B = B_n = 6$, and any allowed height weights. (a) Compute the norm of each of $a, c, 3$, prove their independence, and eliminate the third base from $(a, c, ac, 3)$ using $(-1, -1, 1, 0)$.

Give B', B_n', ε' and verify the equality of the second entries in (10.1117). (b) Repeat with the exact relation $(-7,-7,7,0)$, using the relation bound $\kappa = 7$. Check the minimum seven-adic valuation of the transformed coefficients. Explain why the proof does not need a relation coefficient prime to seven. (c) Lift the root 3 of $X^2 - 2$ modulo seven first to 49 and then to 343, and compute the valuation of $3^6(ac)^2 - 1$. With coefficients $(1,-1,1,6)$, show instead that elimination produces a zero coefficient while preserving the distinguished minimum valuation.

95. **Quadratic descent at two (hard).** Let K be a number field not containing ζ_3, and let $L = K(\zeta_3)$. If T is odd, $A \in K^*$, and $\beta^T = A$ with $\beta \in L$, construct a T-th root of A in K. For odd residue degree f, explain why the saturated residue factor satisfies $N^2/(m\delta_L) < N$. Verify the all-rank ratio in (10.1128) and the largest endpoint in (10.1129). Finally apply Theorem 10.246 to the rational list $(6, 11/2, 33)$, with all three coefficients one, $B = B_n = 1$, and $\varepsilon = 1/2$: determine the normalized units, their rank, and the exact valuation of the product minus one.

Solution 1. Put $q_i = v(\alpha_i)$. If $b_1q_1 \neq b_2q_2$, the ultrametric inequality gives the exact valuation $\min(b_1q_1, b_2q_2)$, so no cancellation estimate is needed. If they are equal, choose $\pi = p^{1/e}$ in an algebraic extension and set $\epsilon_i = \alpha_i\pi^{-eq_i}$. The exponents eq_i are integers, the ϵ_i are units, and

$$\alpha_1^{b_1} - \alpha_2^{b_2} = \pi^{eb_1q_1}(\epsilon_1^{b_1} - \epsilon_2^{b_2}).$$

The valuation is b_1q_1 plus the unit-difference valuation. The field degree, heights, and any dependence of the new bases must be checked anew. The linear offset cannot be discarded. For instance, 3 and 6 are multiplicatively independent but nonunits at three, and $v_3(3^m - 6^m) = m + v_3(1 - 2^m) \geq m$. A bound by a fixed multiple of $(1 + \ln m)^2$ cannot hold. Thus unit normalization reduces the cancellation problem; it does not preserve the unadjusted bound.

Solution 2. Use rational heights $h(a) = \ln a$, $h(b) = \ln b$, and the choices $\ln A_1 = \ln(5a)$, $\ln A_2 = \ln(5b)$. Coprimality makes both exponents vanish in a multiplicative relation. Also

$$\ln B' + \ln\ln p \leq \ln(p-1) + \ln(2/\ln 5) + \ln\ln p \leq 3\ln p.$$

Hence the maximum is $10\ln p$, not the smaller first term. Squaring it cancels two of the four denominator powers of $\ln p$; $2500 \cdot 100 = 250000$. This proves (10.34).

Solution 3. The residue field group has order $p - 1$, so the order q of a/b divides it. A nonmultiple of q gives valuation zero. For $n = qm$, prime-to-p powers preserve the valuation of $(a/b)^q - 1$, and each p-th power adds one because its initial valuation is an integer at least one. For odd p, one is strictly greater than $1/(p-1)$, so the boundary case does not occur. The same calculation for $p - 1 = q((p-1)/q)$ has no p-part. Combine the two identities and Corollary 10.9. At $p = 2$, valuation one lies on the boundary; for example $v_2(3^2 - 1) = 3$, whereas $v_2(3-1) + v_2(2) = 2$.

Solution 4. Here $N = 12$, $c_0 = 1$, and $\sum_i k_i = 12$. Formula (10.10) gives

$$v_3(\Delta) \geq 54 - \frac{12\ln 12}{\ln 3}.$$

Its hypothesis is $v_3(\Lambda) \geq 23/2$, equivalently at least twelve when the difference lies in $\mathbb{Q}_3$. Before the common logarithmic loss, the contribution from n ordinary rows is at least

$$(12-n)\frac{23}{2} + \frac{n(n-1)}{2} - 12 = 54 + \frac{(12-n)^2}{2}.$$

Thus its excess over the common minimum is exactly $(12-n)^2/2$. This is a lower bound for a summand's valuation, rather than an equality for that valuation.

Solution 5. Order the row labels as $0,0,1,1,2,2,3,3$, and take $r_i = 0,0,0,0,2,2,2,2$. Then $\sum_i r_i = 8$, $\sum_i l_i r_i = 20$, and the center is 12. The error is eight, equal to $NL(R-1)/8 = 8$. Reversing the chosen zero and two coordinates attains the negative error.

Solution 6. Take $p = 3$, $x = 3^{10} + 1$, $y = 1$. The pair is coprime and $p \nmid x$. Factor $x^2 - 1 = (x-1)(x+1)$. Its valuation is ten, because $x - 1 = 3^{10}$ and $x + 1 \equiv 2 \pmod 3$. The proposed right side is four, since $\ln y = 0$. This disproves the unrestricted statement. All the estimates used above have their hypotheses proved and stated separately.

Solution 7. Remove residue roots in the selected congruence class, and remove the common p-part of the coefficients by the deep-unit power law. The chosen base has $v(\log \epsilon_1) \ge E$. The assumption $p \nmid b_2'$ gives integral nodes $r + (b_1'/b_2')s$; it cannot be dropped from this construction when the first base alone supplies E. The second principal unit must already lie in the exponential domain because the exponent multiplier is now one. At odd primes its positive integer valuation does this automatically; at two the additional depth condition supplies it.

The pure minor of order n has the bound in the proof of (10.40), while complementary error rows cost at most $E \sum k_i$. A difference precision $E(N - 1/2)$ and the identity (10.14) give $ENK(L - 1)/2 - N \ln N / \ln p$. Compare this with the same centered polynomial height bound, now without any exponent multiplier. Use $a_i = \ln A_i/(E \ln p)$, $B = H_E/(E \ln p)$, and $G = p - 1$. The cardinalities and inequalities are exactly those checked in (10.40)–(10.42); the collision branch uses the residue order before applying $X - 1$ Liouville. The final budget is

$$EzB^2 = E(p-1)\frac{\ln A_1 \ln A_2}{E^2(\ln p)^2}\frac{H_E^2}{E^2(\ln p)^2} = \frac{p-1}{E^3(\ln p)^4}\ln A_1 \ln A_2 H_E^2.$$

This accounts for every power of E; replacing the established coefficient 1300 by 88000 gives the asserted weaker version.

Solution 8. If a rational β realizes $\boldsymbol{v}$ in (10.44), valuation at two gives $v_2(\beta) = 2v_1$, and valuation at three gives $v_3(\beta) = 2v_2$. Hence both coordinates lie in $\frac{1}{2}\mathbb{Z}$. Conversely every such vector is realized by $\beta = 2^{2v_1}3^{2v_2}$, with a common denominator T in (10.44). Thus the stated saturated lattice and the bases $2, 3$ are exact. If $(-1)^{a_0}2^{a_1}3^{a_2}$ is a rational square, prime valuations make a_1, a_2 even, and positivity makes a_0 even. The three square classes are independent, so the proved Kummer degree lemma gives degree eight. At five, the residue of two has order four, since $2^2 \equiv -1 \pmod 5$. The residue group generated by $-1, 2, 3$ is therefore the whole group of order four. Its index is one.

Solution 9. The choice $H = \exp(n/(2d))$ cancels the 2^n in the diamond volume and the $H^{2d} = \mathrm{e}^n$ in the count. The mixed complex term contributes the additional factor $\mathrm{e}^{-n/2} \le 1/\sqrt{\mathrm{e}} < 2/3$. Its two coefficients in (10.53) are less than 31 and 32, so their total is less than $31(2/3) + 32 = 158/3 < 58$. The real coefficient is less than 33. The selected archimedean weight is $f\ell/d$; only $\ell(d-f)/d$ remains for the other archimedean Hadamard factors. This exact division gives (10.50), before any numerical constant is chosen.

For rational 4,9, the valuations at two and three in (10.54) force $2v_1, 2v_2$ to be integers, for any realizing integer T. Conversely $\beta = 2^{2v_1}3^{2v_2}$ realizes every half-integral vector. Hence $M_{\mathbb{Q}} = \frac{1}{2}\mathbb{Z}^2$, its index is four, and $w_{\mathbb{Q}} = 2$, since the only rational roots of unity are $1, -1$. Thus the left side of (10.55) is eight. For $H = \mathrm{e}$, (10.56) gives $V = 2/(\ln 4 \ln 9)$ and average $4V = 8/(\ln 4 \ln 9)$. The three points $(-1/2, 0), (0, 0), (1/2, 0)$ lie in the open diamond, while the other points of its zero lattice coset lie outside. Their values are $1/2, 1, 2$, and the six signed values are distinct and have multiplicative height at most two, which is less than e.

Solution 10. The residue of 2 at three is -1, of order two. An odd exponent gives valuation zero. For even b, factor through 2^2=4; since v_3(4-1)=1, the proved power law gives $v_3(2^b - 1) = 1 + v_3(b)$, also for negative b by unit inversion. The height is $\ln 2$, so (10.59) with $d = e = f = 1$ gives $v_3(2^b - 1) \le v_3(b) + 4 \ln 2 / \ln 3$. In a field with $d = 2, e = 1, f = 2$, the ratio d/f is still one, the

residue of the rational unit is still -1, and $\operatorname{ord}_{\mathfrak{p}} = v_3$. The same expression therefore follows. A degree-only bound that discarded f would be larger, and a normalization omitting e in a ramified field would instead be incorrect.

Solution 11. Here $b = (0, 0, 1, 1)$, $B = (1, 1, 1)$, and

$$C = \begin{pmatrix} 5 & 0 & 1 \\ 0 & 5 & 1 \\ -5 & -5 & -2 \end{pmatrix}.$$

The upper minors for column pairs $(1, 2), (1, 3), (2, 3)$ are $25, 5, -5$, with valuations $2, 1, 1$. Choose columns $1, 3$. The derivations are $\delta_1 = 5(E_1 - E_3)$, $\delta_2 = 5(E_2 - E_3)$, $\delta_3 = E_1 + E_2 - 2E_3$, and $\delta_2 = -\delta_1 + 5\delta_3$. Their coefficients are 5-adically integral. Choosing columns $1, 2$ would instead give $\delta_3 = (\delta_1 + \delta_2)/5$. For example $\delta_2^2 = \delta_1^2 - 10\delta_1\delta_3 + 25\delta_3^2$, so the minimum-minor change preserves integrality for the second-order jets too. The rank and Cramer arguments prove the assertion at every order.

Solution 12. Both sums of allocations are 90, and $D_0 = T_2 + 2 = 32$ is largest. The $m = 0$ inequality is $31^2 = 961 > 32$. The $m = 1$ inequality is $31\binom{32}{2} = 15376 > 2 \cdot 32 \cdot 3 = 192$. The last inequality is $15376 > 6 \cdot 32 \cdot 2 \cdot 3 = 1152$. The bases $2, 3$ are independent by their prime valuations. If the polynomial were nonzero, the theorem would supply $\nu = 1$. Choose $A_1 = A_2 = 1$; then $C_{2,1} = 2$, and any nonzero integral relation vector has $R_1 \geq 1$. Its first possible bound is $31R_1 \leq 6$, and its second is $961R_1 \leq 384$. Both contradict $R_1 \geq 1$. Thus no nonzero polynomial can have the proposed jets.

Solution 13. Prime valuations show that the q-primary saturated lattice of $2, 3$ over $\mathbb{Q}$ is $\mathbb{Z}^2$: any realizing power forces both rational exponents to be integers. At five their residues, together with -1, generate all four nonzero classes. Thus $\delta(\boldsymbol{\alpha}) = 1$, while $w_q = 2$, $q = 2$, $e_{\mathfrak{p}} = f = d = 1$. We have $a = 28/3$ and $R_2 = 14\mathrm{e}(1 + \varepsilon_2)$.

Here $\gamma = 6$ has residue order one, so $\delta_\gamma = 4$ and $M_\gamma = \max\{5/(4(\ln 5)^2), \mathrm{e}\} = \mathrm{e}$. Indeed $\ln 5 > 1$ since $\mathrm{e} < 3$, and $\mathrm{e} > 2$. Also $M_2 \geq \mathrm{e}^2/4$. Consequently

$$R_2 \frac{M_2}{M_\gamma} \ln 2 \ln 3 \geq \frac{7}{2}\mathrm{e}^2(1 + \varepsilon_2) \ln 2 \ln 3 > 7.$$

The last inequality uses $\mathrm{e} > 2$, $\ln 2 > 1/2$ from its integral, and $\ln 3 > 1$. On the other hand $\sigma = \ln 6 < 2$, because the first four exponential terms give $\mathrm{e} > 8/3$, hence $\mathrm{e}^2 > 6$. This proves (10.77). The height of the rational number 6 is exactly $\ln 6$. The chosen H exceeds all three required lower bounds. The valuation is $v_5(6 - 1) = 1$. Taking $c = 1790$ matches the rational unramified case of the free preparation; Theorem 10.27 supplies its fully proved numerical bound.

Solution 14. Put $F_0 = (X^2 + 3X + 2)/2$, $F_1 = (X^2 + 5X + 6)/2$. The four polynomials are F_0, F_1, F_0^2, F_1^2, and their coefficient matrix is

$$\begin{pmatrix} 3/2 & 5/2 & 3 & 15 \\ 1/2 & 1/2 & 13/4 & 37/4 \\ 0 & 0 & 3/2 & 5/2 \\ 0 & 0 & 1/4 & 1/4 \end{pmatrix}.$$

Its lower left block is zero; the diagonal block determinants are $-1/2, -1/4$, giving $1/8$. Thus the polynomials are independent, despite their common root -2.

For the Taylor calculation, $v(2) = 2$. At $z = 1/3$, $\Delta(z; 2) = 14/9$, $\Delta'(z; 2) = 11/6$, and $\Delta''(z; 2) = 1$. Expanding its square at that point gives the prepared coefficients

$$\left(\frac{196}{81}, \frac{308}{27}, \frac{59}{3}, \frac{44}{3}, 4\right).$$

The exponent in (10.87) is $\ell(kI + v_3(k!)) = 2(2+0) = 4$. Multiplication by 3^4 gives the integers $196, 924, 1593, 1188, 324$, as required. There is no denominator at five, so this clearing factor has no cost in the 5-adic valuation.

Solution 15. Here $v(1) = 1$, $N = 3$, $M = 1$, $\Omega = 1$, and $\mathcal{B} = 2$. The only equation is $c_0 + c_1 + c_2 = 0$ at $(X, Y_1, Y_2) = (0, 1, 1)$. Take $c = (-1, 1, 0)$. Then $h_\infty(c) = 0$ and $h_2(c) = \frac{1}{2}\log 2$; the finite maximum norms are one. Formula (10.96) is $h_2(c) \le \frac{3}{4}\log 3 + \frac{1}{2}\log 2$, which holds strictly. Since $\boldsymbol{d} = (1, 1)$, the ordinary polynomial is $P = Y_1Y_2(X+1)(Y_1-1)$. It is nonzero and vanishes at $(0, 1, 1)$. Only the order-zero value is imposed: the D_1-derivative there equals one, so no first-order vanishing is asserted. Every nonzero coefficient is a local unit at every prime, so the optional normalization changes nothing.

Solution 16. In the first case $n = 7, B = 1, \kappa = 0$. Theorem 10.34 gives $v_3(F(\rho x)) \ge \min\{14, \Lambda - 1\}$, so $\Lambda \ge 15$ suffices. The larger logarithmic cost in Corollary 10.35 is sufficient but is not needed in this example.

The q-deleted set is $\{-2, -1, 1, 2\}$, and

$$L_{-2}(0) = \frac{(0+1)(0-1)(0-2)}{(-2+1)(-2-1)(-2-2)} = -\frac{1}{6}.$$

Its 2-adic valuation is -1. Thus the deleted-node cardinal factors need not be integral, whereas the full-node factors are integral at every $x \in \mathbb{Z}_p$. The bound $-B = -2$ is valid here.

For the last case the full nodes are $-1, 0, 1$, and $L_1(X) = X(X+1)/2$. The equation $x^2 = 2$ gives $v_2(x) = 1/2$; since $v_2(x) > 0$, the ultrametric inequality gives $v_2(x+1) = 0$. Therefore $v_2(L_1(x)) = -1/2$, although x is integral in its extension. It lies outside $\mathbb{Q}_2$, whose nonzero valuations are integers. The residue-count proof of Theorem 10.34 applies to $\mathbb{Z}_p$, and does not assert this cardinal integrality on the entire integral ball of $\mathbb{C}_p$.

Solution 17. Here $D = 4, \delta = 0$, $\gamma = 3^4(2!)^2 = 324$, and $\Gamma = 4$. The unscaled polynomial is $f(y) = ((y+1)(y+2)/2)^2$, so

$$F(Z) = (Z+3)^2(Z+6)^2 = Z^4 + 18Z^3 + 117Z^2 + 324Z + 324.$$

Its coefficients are 3-adically integral and its finite tail is zero, so it is normal. We have $f_1(0) = 3$ and $F_1(0) = 324$. Thus $v_3(F_1(0)) + 1 = 5 = \Gamma + v_3(f_1(0))$. The sufficient slope threshold is $\theta + 1/(p-1) = 3/2$. It holds for $w = 9$, whose valuation is two, and fails for $w = 3$, whose valuation is one. Under scaling, the corresponding exponential is $\exp(wZ/3)$; only the former meets the stated normal-series hypothesis.

Solution 18. Here $v(2) = 2$, and the prepared values at zero are

$$f(0) = 1, \qquad v(2)f_1(0) = 2 \cdot \frac{3}{2} = 3, \qquad v(2)^2 f_2(0) = 4 \cdot \frac{1}{2} = 2.$$

Their 2-adic valuations are $0, 0, 1$. Take $P_0 = 0$ and $\beta = 0$; the additive part of Lemma 10.38 has $C_0 = v_2(v(2)) = 1$. It gives $v_2(f_1(0)) \ge -1$, with equality. Prepared integrality does not imply ordinary divided-jet integrality. For order two the bound -2 is sufficient, while the actual valuation is -1. This also illustrates why the lcm cost must be retained.

For the interpolation example $n = 7, B = 1, \kappa = 0$. Theorem 10.39 gives $\min\{14, 16 - \max\{1, 2\}\} = 14$. If one first discards the difference between the two jet precisions, their common bound is only $\Lambda - C = 14$. Theorem 10.34 then gives $\min\{14, 14 - 1\} = 13$. Retaining each jet's precision saves the extra unit.

Solution 19. The value is $3/2$, with $v_3(3/2) = 1$. Here $D = N = 1$, $v_2(1!) = v_2(v(1)) = 0$, and $t_0 = 0$, so $\Xi_{0,1,0} = 1$. The clearing factor is two. The coefficient heights and torus term are zero, while $\mathcal{B}_0(1/2, 0) = 5/2$. Thus (10.116) gives

$$v_3(3/2) \le \frac{\ln(5/2) + \ln 2}{\ln 3} = \frac{\ln 5}{\ln 3}.$$

This exceeds one, since $5 > 3$. The denominator at two contributes to the global height budget, although it causes no loss in the local valuation at three.

Solution 20. Since $v_3(10-1) = 2 > 1/2$, the proved logarithm and exponential laws give $\eta = \exp(\frac{1}{2}\log 10) \in \mathbb{Q}_3$, $\eta^2 = 10$, and $v_3(\eta - 1) = 2$. The defining series have rational coefficients and converge in $\mathbb{Q}_3$. The number ten is not a rational square, by its odd prime valuations, so $[F : \mathbb{Q}] = 2$. The chosen completion is $\mathbb{Q}_3$: the embedded field contains the dense subfield $\mathbb{Q}$ and lies in $\mathbb{Q}_3$. Hence $e_F = f_F = 1$, and (10.117) is two, whereas $d/(ef) = 1$ for K.

Both exponents are congruent to $m_* = 1$ modulo two. Their differences give exponents $0, 1$ after division by two. The common-factor identity is

$$V = \frac{3}{2}(\eta - \eta^3) = \eta\frac{3}{2}(1 - 10) = -\eta\frac{27}{2}, \qquad V_0 = -\frac{27}{2} \in \mathbb{Q}.$$

The root is a local unit, so both values have 3-adic valuation three.

Here $N = 2$, $h_\infty(c) = 0$, $h_2(c) = \frac{1}{2}\ln 2$, so the coefficient norm term is $\ln 2$. Also $\mathcal{B}_0(1/2, 0) = 5/2$, $\Xi = 1$, $h(10) = \ln 10$, and the original exponent bound is $A = 3$. The root-field bound is therefore $2(\ln 2 + \ln 5 + 3\ln 10)/\ln 3 = 8\ln 10/\ln 3$. For the common-factor rule $\widehat{A} = 2$; it keeps the original field and gives $(\ln 2 + \ln 5 + 2\ln 10)/\ln 3 = 3\ln 10/\ln 3$. Both bounds exceed the actual valuation three. The different local-degree coefficients are accounted for explicitly; membership in the same local field alone does not remove the global field-degree factor.

Solution 21. Here $v(2) = 2$, and

$$\mathcal{P}_{0,2}(Z) = \frac{(4 + 2Z)(5 + 2Z)}{2} = 10 + 9Z + 2Z^2.$$

Here $E_h(1) = h + 1$. The maximum over total order at most two is $\mathcal{B}_0^*(2, 2) = 30$, attained at additive order zero and Euler order two. The other allocations give at most $9 \cdot 2 = 18$ and 2. The old bound is $\mathcal{B}_0(2, 2) = (7^2/2)3^2 = 441/2$. At the specified row, the polynomial whose coefficients are the prepared additive derivatives is exactly the displayed polynomial. Thus the values for $t_0 = 0, 1, 2$, with zero Euler order, are $10, 9, 2$. The order-dependent bounds are attained.

Solution 22. Here $D = 2, N = 6, J_0 = 1, M_* = 5, H = 2\ln 2$, and the discriminant is one. The scalar majorants for $|s| = 0, 1, 2$ are $3, 6, 10$, respectively. Since $\sum_{s=-2}^{2} |s| = 6$, (10.127) gives

$$h_2(c) \le 5\ln 6 + \ln 3 + 2\ln 10 + 24\ln 2.$$

The simpler (10.128), using $\mathcal{B}_0^*(2, 0) = 10$, is $h_2(c) \le 3\ln 6 + 5\ln 10 + 24\ln 2$. The first right side is smaller by $\ln(1000/108) > 0$. Using the maximum node size instead of its average would replace $24\ln 2$ by $40\ln 2$.

At the five nodes the row matrix is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & 0 & 1/4 \\ 1 & 3 & 1 & 3 & 1 & 3 \\ 3 & 6 & 6 & 12 & 12 & 24 \\ 6 & 10 & 24 & 40 & 96 & 160 \end{pmatrix}.$$

Multiplication by the given vector gives zero in every row. Its coordinates have greatest common divisor one, so all finite maximum norms are one. Consequently $h_\infty(c) = \ln 57$ and $h_2(c) = \frac{1}{2}\ln 4874$, which satisfy both bounds. The first row is zero because the two shifted additive polynomials share the root -2. The weighted Siegel lemma allows dependent and zero rows; $M_* = 5$ remains a valid upper bound for the number of equations.

Solution 23. At $s = 2$, the total sum is $-2 + \eta^2 = 0$. The coefficients in (10.131) are -2 in the even class and 1 in the odd class; both are nonzero. Multiplication by the even node does not permute

the residue classes. For failure of full degree take $\vartheta = 4, \eta = 2, s = 1$, with the same coefficients. Again the total sum is zero and the two coset coefficients are nonzero, while $[F(\eta) : F] = 1$.

At seven, $\eta = 3\exp(\frac{1}{2}\log(2/9)) \in \mathbb{Q}_7$: the argument $2/9$ lies in $1 + 7\mathbb{Z}_7$, so Proposition 9.4 proves convergence and $\eta^2 = 2, \eta \equiv 3 \pmod 7$. The sum $\eta - 3$ has valuation one. Indeed $(\eta-3)(\eta+3) = -7$, and $\eta + 3 \equiv 6 \pmod 7$ is a unit. Its even and odd coset terms $-3, \eta$ both have valuation zero. The degree over $\mathbb{Q}$ is two, but a finite precision statement for their sum does not imply that precision for each term.

Solution 24. The three changed binomials are

$$1, \qquad 2\Delta(Z;1), \qquad 4\Delta(Z;2) - \Delta(Z;1).$$

Thus, with rows and columns ordered by $0, 1, 2$, the change and inverse are

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 4 \end{pmatrix}, \qquad D^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 1/8 & 1/4 \end{pmatrix}.$$

Both are 3-adically integral. The vector $(1, 3, 9)$ becomes $(1, 6, 33)$; both have minimum valuation zero. In general p-integrality in both directions proves the equality for every vector, not just this example. Each row uses only indices no larger than its own, so keeping the additive index fixed preserves the total-order cutoff.

Here $F = \mathbb{Q}(i)$. With $\rho_1 = \sqrt{2}, \rho_2 = \sqrt{3}, \xi = i$, the phased roots are $\eta_1 = i\rho_1^3$ and $\eta_2 = \rho_2^3$, and their squares are $\vartheta_1 = -8, \vartheta_2 = 27$. Their heights are $3\ln 2, 3\ln 3$. The saturated Kummer theorem gives degree eight over $\mathbb{Q}$, and Corollary 10.49 gives degree four over F. Concretely $\rho_1 = (\eta_1/i)/2$ and $\rho_2 = \eta_2/3$, so the phased and unphased root fields over F coincide. The original two bases are local units at five or seven; at three the second one is not, so the local-unit version of Theorem 10.48 would require a different prime in this last example.

Solution 25. Here $a_0 = -1, H = 3$, and the possible coordinate sums are $-4, -1, 2$. The two classes are

$$b = 0 : (-2, 1), (-1, 0), (0, -1), (1, -2),$$
$$b = 1 : (-2, -2), (0, 2), (1, 1), (2, 0).$$

For a point of sum $u = -1 + 3b + 6v$, multiplication of its original phase ζ^{su} by ζ^s gives $i^{sb}(-1)^{sv}$. The red class has $v = 0$. In the blue class $v = -1$ at $(-2, -2)$ and $v = 0$ at the other points. Thus the two grouped contributions are Z_0 and $i^s Z_1$, with the sign at the first blue point included in Z_1.

The chosen minimum coefficient belongs to the red class, so $\varepsilon' = -1 \pmod 6$. Centering at $\lambda_0 = (-1, 0)$ makes every red exponent difference have coordinate sum zero; its normalized phase is exactly one. The minimum coefficient and its congruence class are fixed data of the coefficient vector. Extraction proves its zeros for every specified $s, \boldsymbol{t}$, so no new choice is made at each node or order. In the coprime example, $2^{-1} = 3 \pmod 5$, and $u \equiv -3 \cdot 2 \equiv 4 \pmod 5$. There is just one divided phase class and no second extraction.

Solution 26. Newton's formula gives

$$x_1 = 2 - \frac{5}{4} = \frac{3}{4}, \qquad x_2 = \frac{3}{4} - \frac{25/16}{3/2} = -\frac{7}{24}.$$

The errors $f(x_0), f(x_1), f(x_2)$ are $5, 25/16, 625/576$, with valuations $1, 2, 4$. The unique root $\zeta \equiv 2 \pmod 5$ therefore satisfies $\zeta \equiv -7/24 \equiv 182 \pmod{625}$. It has order four, since its square is -1 and its reduction is neither 1 nor -1. Here $G = 4, G_0 = 2, \zeta^2 = \alpha_0 = -1$.

The factorization $(2 - \zeta)(2 + \zeta) = 5$ and the unit $2 + \zeta \equiv 4 \pmod 5$ give $v_5(2 - \zeta) = 1$. As ζ is a unit, $v_5(g - 1) = 1$. The least t with $5^t > 1/4$ is zero; with $j = 3$, the lower depth $1 + 3 = 4$ is strictly

greater than $\vartheta + 1/4 = 13/4$. Therefore $P = 5^3 = 125$, $d_1 = -125$, and $\gamma_1 = 2^{125}\zeta^{-125} = g^{125}$. The exact p-power law, starting strictly above $1/4$, gives $v_5(\gamma_1 - 1) = 1 + 3 = 4$, not merely its lower bound. The individual phase is algebraic outside $\mathbb{Q}$; a support congruence modulo $G_0 = 2$ makes its normalized powers rational, as Theorem 10.51 proves.

Solution 27. The input nodes are $-3, -1, 1, 3$, so $n = 4 = 2(1 - 1/2)R$. Here $B = \lfloor \log_3 8 \rfloor = 1$, $M_0 = 1$, $T' = 4$, $D = 4$, and $Lv_3(k!) = 0$. The exact first budget is $U + 12 \geq 24 + 4 = 28$, hence $U \geq 16$. The exact output lower bound is $(8 - 4)3 = 12 > 11$. Therefore every integer $-4 \leq x \leq 4$ has all prepared zeros through total order four, including the previously omitted $-4, -2, 0, 2, 4$.

For the larger scale, the first line is $U + 14 \geq 28$, requiring only $U \geq 14$. Its corresponding lower bound is $12 - 4/2 = 10$, which does not exceed eleven. Thus the second line fails; this alternative pair does not certify the same zeros. These computations evaluate the stated conditional budgets, rather than construct their auxiliary coefficients.

Solution 28. Since $P = 125$ is odd, the two powers of ζ are $\pm\zeta$; neither nonzero rational multiple belongs to $\mathbb{Q}$, because $\zeta^2 = -1$ has no rational root. Their product phase is $\zeta^{-4P} = 1$, giving $\gamma_1\gamma_2 = b = 6^P$. On the support (j, j), $d \cdot (j, j) = -4Pj$ is divisible by $G_0 = 2$, so (10.151) gives b^{sj} at every positive or negative integer node.

Here $N = 4, D = 1, J_0 = 1, M_* = 3$, and $A_1 = A_2 = 3$, so $H_{\mathrm{ph}} = 3P(\ln 2 + \ln 3) = 3 \ln b$. The torus polynomial is

$$A(Y) = -b + CY - CY^2 + bY^3 = b(Y - b^{-1})(Y - 1)(Y - b).$$

It vanishes at $b^{-1}, 1, b$. Consequently $\Psi_0(s; 0) = (s + 1)A(b^s)$ is zero for $s = -1, 0, 1$. At -1 the additive factor alone already gives zero, and the vector supplies the additional torus zero. Directly the other two rows give $-b + C - C + b = 0$ and $-b + Cb - Cb^2 + b^4 = 0$.

Since $C = b^2 + b + 1 \equiv 1 \pmod{b}$, the integer coordinates have greatest common divisor one. Their heights are therefore

$$h_\infty(c) = \ln C, \qquad h_2(c) = \tfrac{1}{2} \ln(2b^2 + 2C^2).$$

Also $b \equiv 1 \pmod 5$, $C \equiv 3 \pmod 5$, so every coordinate is a 5-adic unit and the minimum valuation is zero. With $k = L = 1, T = 0$, the scalar majorant is $\mathcal{H}_0(|s|; 0) = |s| + 1$. The separate-row bound (10.152) is exactly

$$\ln 2 + \sum_{s=-1}^{1} (\ln 2 + \ln(|s| + 1) + 6|s| \ln b) = 6 \ln 2 + 12 \ln b.$$

It bounds the displayed height: $C \leq 3b^2$ and $b \geq 1$ give $h_2(c) \leq 2 \ln b + \frac{1}{2} \ln 20 \leq 12 \ln b + 6 \ln 2$, since $20 < 2^{12}$. The weighted Siegel proof permits zero and dependent rows. Counting the first row in $M_* = 3$ merely keeps a sufficient budget; it does not assert that the matrix has rank three.

Solution 29. Here $g = \gcd(2, 0) = 2$, $G_* = 1, d_{*1} = 0$, so the cancelled relation is $w = 0$. The two support vectors are $(0, 0), (0, 1)$ modulo two, and $h_\Lambda = 1$. Theorem 10.56 gives $E = \mathbb{Q}(\eta)$, with $\eta^2 = 6$, and degree two. A rational square has even prime valuations, whereas $v_2(6) = 1$, so $\sqrt{6}$ is not rational. The square classes of $-1, 6$ are independent: a relation $(-1)^a 6^b$ that is a rational square first forces b even by valuation at two, then a even by positivity. Thus the proved Kummer theorem gives $[F : \mathbb{Q}] = 4$.

The simple residue root of $X^2 - 6$ at 1 modulo five has unit derivative two, and the complete Newton proof in Lemma 10.53 supplies $\eta \in \mathbb{Q}_5$, $\eta \equiv 1 \pmod 5$. That same proof gives $i \in \mathbb{Q}_5$, $i \equiv 2 \pmod 5$. Hence the chosen closures of E, F are both $\mathbb{Q}_5$; their absolute indices are $e = f = 1$. Their global degrees are still two and four.

The normalized torus terms are $1, \eta$, and the prepared additive factor is $\Delta(1/2; 1) = 3/2$. Hence $V = \frac{3}{2}(1 - \eta)$. The identity $(1 - \eta)(1 + \eta) = 1 - 6 = -5$ and the unit $1 + \eta \equiv 2 \pmod 5$ give

$v_5(V) = 1$. Under the other embedding $\eta \mapsto -\eta$, the value is $\frac{3}{2}(1+\eta)$, a unit, with valuation zero. In particular a lower bound at one embedding cannot be counted twice.

Here $\delta = 0, N = 2, A_1 = 1$, $h_\infty(c) = 0, h_2(c) = \frac{1}{2}\ln 2$, $\mathcal{H}_0(1/2;0) = 3/2$, and $\Xi_{0,1,0} = 1$. The torus term is $2PXh(6) = \ln 6$. Therefore (10.156) is exactly

$$v_5(V) \le \frac{2}{\ln 5}\left(\ln 2 + \ln(3/2) + \ln 2 + \ln 6\right) = \frac{2\ln 36}{\ln 5}.$$

It bounds the exact valuation one, since $36^2 > 5$. Applying the same row estimate over F would instead give $4\ln 36/\ln 5$. The support field halves this sufficient budget. Its unchanged completion removes no further factor: $V \notin \mathbb{Q}$, and the product formula still sums over a global field of degree two.

Solution 30. Choose the smallest admissible prime. For $a = 8, p = 3$, two is dependent with eight and three is not a unit, so $c = 5$. The transformed difference is $63/25$, and $v_3(8^2-1) = v_3(63/25) = 2$. The heights are $\ln 8, \ln 5$, bounded by $\ln 40, \ln 5$. For $a = 3, p = 2$, two is excluded by p and three by dependence, so again $c = 5$. The difference is $3/5 - 1/5 = 2/5$, with valuation one, equal to $v_2(3-1)$. Here the heights are both $\ln 5$, bounded by $\ln 15, \ln 5$. For $a = 25, p = 2$, the smallest admissible prime is three. The difference is $25/3 - 1/3 = 8$, with valuation three, equal to $v_2(25-1)$. The heights are $\ln 25, \ln 3$, bounded by $\ln 125, \ln 5$.

For all these pairs, the proof gives $\ln(B' \ln p) \le 3\ln p$, so $H = 10\ln p$. For the all-power conclusion let $q \mid p-1$ be the residue order of a. If $q \nmid n$, the valuation of $a^n - 1$ is zero. Otherwise the proved prime-to-p and deep p-power identities at odd p give

$$v_p(a^n-1) = v_p(a^{p-1}-1) + v_p(n) \le 250000\ln(5a)\ln 5\,\frac{p}{(\ln p)^2} + v_p(n).$$

The prime p is odd in this last assertion, exactly as in Corollary 10.10. A base equal to one would be multiplicatively dependent with every base; Theorem 10.8 requires independence. The two rational bases constructed here satisfy that requirement while preserving the original valuation.

Solution 31. Order the diamond monomials as $1/4, 1/2, 1, 2, 4, 1/3, 3$. Multiplication by 12 gives $(3, 6, 12, 24, 48, 4, 36)$. Its coordinates have gcd one and maximum 48, so the projective height is $\ln 48$. For the box the smallest and largest monomials are $1/12$ and 12. Multiplication by 12 gives integer coordinates of gcd one, since one coordinate is one, and maximum 144. Its height is $\ln 144$. Direct local maxima give the same products $4 \cdot 4 \cdot 3 = 48$ and $12 \cdot 4 \cdot 3 = 144$.

For the asymmetric three-point support the vectors $(1, 2, 3)$ and $(6, 3, 2)$ are primitive; their heights are $\ln 3$ and $\ln 6$. They show why a negative node needs the negative support.

The even-even diamond points are $(-2, 0), (0, 0), (2, 0)$. Subtracting $(-2, 0)$ and dividing by two gives $M_1 = \{(0, 0), (1, 0), (2, 0)\}$. Its two heights are both $\ln 4$. It is contained in $C/2 + (1, 0)$, and $\ln 4 \le \frac{1}{2}\ln 48$ follows exactly from $16 \le 48$. Translation of the envelope is harmless; it does not change its height.

Here $N = 7, J_T = 1, M_* = 3, \Delta_{\mathbb{Q}} = 1$. The additive scalar majorants at $s = -1, 0, 1$ are $2, 1, 2$, while the exact signed torus costs are $\ln 48, 0, \ln 48$. Thus (10.164) gives

$$h_2(c) \le \frac{1}{2}\ln 7 + \frac{1}{4}\left(\frac{3}{2}\ln 7 + 2\ln 2 + 2\ln 48\right) = \frac{7}{8}\ln 7 + \frac{1}{2}\ln 96.$$

The coordinate-box calculation replaces 48 by 144; it gives $\frac{7}{8}\ln 7 + \frac{1}{2}\ln 288$. The exact-support bound improves this coefficient budget by $\frac{1}{2}\ln 3$. The mean-node bound (10.165) also uses $\ln 48$, but its uniform scalar majorant 2 loses a further $\frac{1}{4}\ln 2$ relative to the separate-row bound. The actual row at $s = -1$ is zero; allowing it the displayed factor is legitimate and does not assert that the row is nonzero.

Finally, for $|(Bu)_j - z_j| \le D_j$, the local bound includes the linear term $\sum_j z_j \ln|\alpha_j|_v$. It sums to zero by the product formula. No vector a with $Ba = z$ is required.

Solution 32. Put $C = t^2 + t + 1$. The vector $(-t, C, -C, t)$ has coordinate sum zero, and

$$-t + Ct - Ct^2 + t^4 = 0.$$

Therefore it annihilates both nonzero rows in the figure, and also the zero row. Its torus polynomial is

$$-t + CY - CY^2 + tY^3 = t(Y - t^{-1})(Y - 1)(Y - t).$$

For $t = 6$ all row entries are rational, so $E_0 = \mathbb{Q}$, even with ambient field $\mathbb{Q}(\sqrt{5})$. For $t = \sqrt{6}$, the entry $2t$ in the $s = 1$ row recovers t; hence $E_0 = \mathbb{Q}(\sqrt{6})$, of degree two because six is not a rational square.

Here are the integral-basis checks. For square-free m, write an algebraic integer in $\mathbb{Q}(\sqrt{m})$ as $\beta = x + y\sqrt{m}$. Its trace $a = 2x$ and norm are rational integers, by the earlier norm-and-trace proof. Thus $m(2y)^2 = a^2 - 4N(\beta)$ is an integer. If $2y = c/b$ in lowest terms, this implies $b^2 \mid m$, so $b = 1$. Hence $\beta = (a + c\sqrt{m})/2$ with integers a, c and $a^2 - mc^2 \equiv 0 \pmod 4$. For $m = 5$ this is exactly $a \equiv c \pmod 2$; for $m = 6$ it forces both a, c even. Conversely $u = (1 + \sqrt{5})/2$ and $t = \sqrt{6}$ are integral, satisfying $u^2 - u - 1 = 0$ and $t^2 - 6 = 0$. Their integer spans are integral and include every possible β. The full integral bases are consequently $(1, u)$ and $(1, t)$. Their trace matrices are

$$\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}, \qquad \begin{pmatrix} 2 & 0 \\ 0 & 12 \end{pmatrix},$$

of determinants 5 and 24.

In the rational kernel, $43 - 7 \cdot 6 = 1$, so its integer coordinates have gcd one. Its maximum is 43 and its Euclidean norm is $\sqrt{3770}$. Thus $h_\infty(c) = \ln 43$ and $h_2(c) = \frac{1}{2}\ln 3770$.

In the second kernel $C = 7 + t$, and the identity $C - (1 + t)t = 1$ shows that the integral coordinates generate the unit ideal. At every finite place their maximum norm is therefore one. At the two real embeddings the maximum coordinates are $7 + t$ and $7 - t$: the latter exceeds $t > 0$, since $49 > 24$. Their product is 43, giving $h_\infty(c) = \frac{1}{2}\ln 43$. The squared Euclidean norms are $122 + 28t$ and $122 - 28t$; their product is 10180. With both the square root and field-degree normalization, $h_2(c) = \frac{1}{4}\ln 10180$.

Both examples have $N = 4, J_T = 1, M_* = 3$. Their scalar majorants at $-1, 0, 1$ are $2, 1, 2$. For $t = 6$, the signed support heights are $3\ln 6$. Thus the bound retaining the actual discriminant in (10.171) is $6\ln 2 + 6\ln 6 = 6\ln 12$. An ambient-field application over $\mathbb{Q}(\sqrt{5})$ would add $\frac{1}{4}\ln 5$.

For $t = \sqrt{6}$, the monomial vector is $(1, t, 6, 6t)$; its signed support heights are both $\frac{3}{2}\ln 6$. Equality of the two signs also follows from translating the inverse support by three. The corresponding actual-discriminant bound is

$$h_2(c) \le 6\ln 2 + 3\ln 6 + \tfrac{1}{4}\ln 24.$$

These inequalities exceed the exact heights just calculated: for the first, $3770 < 12^{12}$; for the second, $10180 < 2^{24}6^{12} \cdot 24$.

The tower with the single generator $z_1 = t$ has $e_1 = 2$ and $h(t) = \frac{1}{2}\ln 6$, so (10.170) bounds its discriminant cost by $\frac{1}{2}\ln 2 + \frac{1}{2}\ln 6 = \frac{1}{2}\ln 12$. It bounds the exact cost because $24 < 12^2$. Using only the support maximum instead would give $\frac{1}{2}\ln 2 + \frac{3}{2}\ln 6$, a loss of $\ln 6$.

If $S = 0$, every torus factor is $W_\lambda^0 = 1$, so every row entry is rational even in the second example; the exact row field is $\mathbb{Q}$. In neither case does choosing a smaller row field change the number of required equations. The surplus $N > M_*$ still supplies the kernel, and extending its zeros still requires all the stated analytic and strict arithmetic comparisons.

Solution 33. The basis matrix and its inverse are

$$B = \begin{pmatrix} 1 & 1/2 \\ 0 & 1/2 \end{pmatrix}, \qquad B^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}.$$

Thus $B(\lambda_1, \lambda_2) = (\lambda_1 + \lambda_2/2, \lambda_2/2)$, its covolume is $1/2$, and the index over $\mathbb{Z}^2$ is $J = 2$. Equivalently its nontrivial coset is represented by $(1/2, 1/2)$, whose double is integral. The box has volume two and normalized volume four, so Theorem 10.66 supplies five points. The five displayed points are $(0,0), (1/2,1/2), (1,0), (3/2,1/2), (2,0)$, with exactly the stated integer basis coordinates.

Here $g = \gcd(4,2,2) = 2$ and $H = 2$. Their phases $2(\lambda_1 + \lambda_2) \pmod 4$ are $0, 2, 2, 0, 0$. The zero class retains $(0,0), (1,1), (2,0)$, of size three, equal to the guarantee $\lceil 5/2 \rceil$. At $kL = 2$, (10.180) gives $6 \le N \le 10$, and the actual chosen support has $N = 6$. Its strict volume bound is $N > 4$.

For $S = 1, T = 0$, $J_T = \overline{J}_T = 1$ and $M_* = 3$. The actual class count therefore certifies $N \ge 2M_*$, so $c_* = 2$ can be used in (10.176). The sufficient volume criterion certifies $c_* = 4/3$, since $4 = (4/3)\overline{M}$; it does not certify $c_* = 2$. This is a limitation of that coarser bound, not a failure of the exact class count.

At $T = 1$, with one nonzero Euler direction, the representative indices are $(0,0), (1,0), (0,1)$, so $J_T = 3$ and $M_* = 9$. The six columns no longer satisfy $N > M_*$. Thus this counting criterion does not supply the kernel for those requested jets. It makes no claim that an actual matrix of these rows has rank nine or that every kernel is impossible.

Finally an open interval of length one cannot contain two integers: their distance would be at least one, whereas any two points of that interval are at distance strictly less than one. Its volume nevertheless equals the covolume of $\mathbb{Z}$. The closed-body limiting argument is therefore essential at integer normalized volume.

Solution 34. Prime valuations at two and five prove both independence and saturation: if a rational β has $\beta^T = 2^{Ta}5^{Tb}$ with Ta, Tb integers, then $a = v_2(\beta)$ and $b = v_5(\beta)$ are integers. Thus the full and q-primary lattices are $\mathbb{Z}^2$, and $J = 1, \nu = 0$. The only rational roots of unity are $1, -1$, since a rational algebraic integer whose inverse is also integral is 1 or -1; hence $q^u = 2$. At three, $G = 2, G_0 = 1, \delta = 1, H = 1$, regardless of the phase coefficients. Here $e = f = d = 1, P = 3, c_2 = 7/4, \theta = (3/2)/(1+\tau)$, $\rho = 17$, $A = g_1 = 4 + \ln 3$, and $\gamma = 1$. The specified $g_0 > 39$ and $h \ge 3\ln 3$, as the bounds below also verify.

For a positive $z > 1$, put $v = (z-1)/(z+1)$. Integrating the geometric series for $2/(1-v^2)$ gives

$$\ln z = 2\sum_{j=0}^{n-1} \frac{v^{2j+1}}{2j+1} + R_n, \qquad 0 < R_n \le \frac{2v^{2n+1}}{(2n+1)(1-v^2)}.$$

The remainder estimate follows by replacing every later denominator by $2n+1$ and summing the remaining geometric powers. For an interval of positive z's, the monotonicity of $\ln z$ bounds it by applying these rational bounds at the two endpoints. Similarly the positive exponential series at two has

$$0 < \mathrm{e}^2 - \sum_{j=0}^{64} \frac{2^j}{j!} \le \frac{2^{65}}{65!}\frac{1}{1-2/66},$$

because the ratio of each successive omitted term is at most $2/66$. Using $n = 512$ in the logarithm bounds, exact rational interval operations in (10.185)–(10.186) give the following enclosures. Rounding each lower endpoint down and each upper endpoint up to a multiple of 10^{-60} preserves the inequalities at every operation. The reproducible source of Figure 10.14 implements these rational operations and tails.

Quantity	Lower bound	Upper bound
$g_0 = h$	50.136076	50.136077
S	379.288456	379.288457
$\mathcal{D}$	715255.319062	715255.319063
T	1796752.997002	1796752.997003
$\widetilde{D}_0$	35298.572063	35298.572064
D_1	67804.431507	67804.431508
D_2	29201.779183	29201.779184
$D_1 D_2$	1980010036.552128	1980010036.552129

Thus $s_0 = 379, t_0 = 1796752, k = 50, L = 35299, m = \lfloor D_1 D_2 \rfloor = 1980010036$. A concrete support of size $m + 1$ is

$$\Lambda = \left\{ \left(\left\lfloor \frac{j}{29202} \right\rfloor, j - 29202 \left\lfloor \frac{j}{29202} \right\rfloor \right) : 0 \le j \le m \right\}.$$

The enclosing integer grid has $67805 \cdot 29202$ points, at least $m + 1$, because each side count exceeds its corresponding real width. Integer division gives distinct points within $[0, D_1] \times [0, D_2]$. Every point has the same phase, since $H = 1$, and $(0, 0)$ is a permissible reference.

Take the nonzero Euler direction $Y_1 \partial_{Y_1} - Y_2 \partial_{Y_2}$; its exponent form $\lambda_1 - \lambda_2$ is nonzero on this support. The additive degree cap is $kL = 1764950 < t_0$. Therefore the representative count from (10.126) is

$$J_T = \binom{t_0 + 2}{2} - \binom{t_0 - kL + 1}{2} = 1613655870378.$$

This subtraction counts the excluded indices with additive order at least $kL + 1$, after shifting that order down by $kL + 1$. The exact column and row counts are

$$\begin{aligned} N &= 50 \cdot 35299 \cdot 1980010037 = 3494618714803150, \\ M_{\text{full}} &= 759 \binom{1796754}{2} = 1225148631539679, \\ M_* &= 759 J_T = 1224764805616902, \\ 20N - 57M_{\text{full}} &= 58902298301297 > 0. \end{aligned}$$

Since $c_{01} = 1.9 \cdot 3/2 = 57/20$, the last integer identity verifies the stronger full-row surplus in (10.190). The representative rows have a still larger surplus. This is a count of equations, not a claim that these rows are linearly independent.

The saturation basis is the identity, so the normalized integer torus values are rational powers of the rational bases, with a possible rational sign. Lemma 10.63 therefore gives $E_0 = \mathbb{Q}, d_0 = 1$; its discriminant is one. Both the discriminant and the generator contribution vanish. The scalar row-height contribution in (10.191) remains and must be retained.

Solution 35. At two, $F = \mathbb{Q}(\zeta_3)$ has unramified completion of degree two, since 2 has order two modulo three; use the prime-to-p proof cited in Lemma 10.73. Over this completion $X^2 - 2$ is Eisenstein: two is still a uniformizer, its constant coefficient has valuation one, and its middle coefficient is zero. Extensions of complete valued fields, Corollary 5.2, gives a totally ramified quadratic extension. Thus the resulting completion has degree four, ramification index two and residue degree two. The global field is generated by two quadratics, so its degree is at most four; the local/global dimension argument of Lemma 10.73 forces it to be at least four. Hence $d = 4$. The field contains ζ_3, but cannot contain ζ_9, whose degree six would have to divide four. Therefore $q^u = 3, u = 1$, and $d/e = 2 = 3^{u-1}(3 - 1)$.

For the rational example the support contains $(67804, 0)$ and $(0, 29201)$. Every coordinate is between the corresponding endpoints, so the Euler argument $\lambda_1 - \lambda_2$ has exact maximum absolute value

$\Omega_1 = 67804$. Thus $m = 67804$, $R = 379 + 2 \cdot 50 = 479$, $D_{\mathrm{a}} = 1764950$, and $t_0 = 1796752$. Prime powers give

$$\mathfrak{v} = \operatorname{lcm}(1, \ldots, 50) = 3099044504245996706400, \qquad 3^{50} = 717897987691852588770249.$$

Indeed the prime-power factors are $2^5, 3^3, 5^2, 7^2, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47$. Their product is the displayed integer. The prime list is checked by trial division by two, three, five and seven: every composite at most fifty has a prime divisor at most its square root, hence at most seven.

The separated and sharper maxima both occur at $u = D_{\mathrm{a}}$. For the sharper one it suffices to check the last step, since its ratios decrease:

$$\mathfrak{v} \cdot 31803 > 479 \cdot 1764950 \cdot 99607.$$

This holds for both $\mathfrak{v}$ and the larger 3^{50}. The remaining Euler order is 31802, and the sharper logarithm is

$$\ln \mathcal{H}_{\mathrm{fine}} = 1764950 \ln \mathfrak{v} - 35299 \ln(50!) + \ln \binom{99606}{31802}.$$

The factorial contribution is negative; omitting its sign would overstate this upper bound. The mean orders are obtained by summing $J_u = t_0 - u + 1$:

$$\overline{u} = \frac{820859713025}{1371417}, \qquad \overline{h} = \frac{1643236524559}{2742834}.$$

Use $\sum_{u=0}^{U} u = U(U+1)/2$ and $\sum_{u=0}^{U} u^2 = U(U+1)(2U+1)/6$ in $\sum u(t_0 - u + 1)$. The two polynomial sum identities follow by telescoping the consecutive squares and cubes, respectively.

Here $E_0 = \mathbb{Q}, d_0 = 1$, so the discriminant and field-generator terms vanish. The exact ratio is $M_*/(N - M_*)$, with both integers from Solution 34, and $Z = S\mathcal{D}$. Since $\sigma_i = h(\alpha_i)$, $PL_\alpha = \mathcal{D}/(c_1 c_2)$. The mean torus contribution divided by Z is therefore exactly

$$\frac{s_0(s_0+1)}{(2s_0+1)c_1 c_2 S}.$$

Substitute the sharper logarithm above, or (10.205), into the weighted coefficient bound. The rational series and outward interval operations from Solution 34 give these net upper bounds:

Scalar estimate	$h_2(\boldsymbol{c})/Z$ is at most
Sharper peak, actual lcm	0.269837353526
Sharper peak, 3^{50}	0.288952296008
Row mean, actual lcm	0.172796821752
Row mean, 3^{50}	0.179279284082
Alternative global rows, peak	0.128675661391
Alternative global rows, mean	0.124309602511

For the alternative peak, the exact integer rule (10.201) with clearing factor one gives $\widehat{u} = 3543$. Check its last increasing step and its next decreasing step with integer products. The alternative mean replaces A_{mean} by $A_{\mathrm{mean}} + 35299 \ln(50!)$, and removes $\overline{u} \ln \mathfrak{v}$, exactly as (10.207) prescribes.

Large binomial logarithms need no approximation of their integer coefficients: for a positive integer n, write $n = 2^b y$ with $1 \le y < 2$, and use $\ln n = b \ln 2 + \ln y$. The logarithm series in Solution 34 then has ratio at most $1/3$, even for these enormous integers. The figure program uses this reduction and exact rational outward intervals, retaining each component. The separate logarithmic contribution is less than 0.000000101556.

Finally, multiplication of a row by $(50!)^{35299}/\mathfrak{v}^u$ is multiplication by a nonzero rational number. It changes neither its zero equations nor its rational coefficient field. Its normalized projective height is invariant by the product formula. These global row estimates therefore preserve all required zeros; later local prepared derivatives retain their own original clearing convention.

Solution 36. Iterating $c \mapsto \min\{pc, c+1\}$ from $1/e$ gives the following exact lower bounds. The inequalities (10.209) determine the power; the separate unramified clause determines ϑ in the last row.

p, e	P	Guaranteed valuation path	ϑ	Final required depth
$2, 2$	8	1/2, 1, 2, 3	2	3
$2, 3$	8	1/3, 2/3, 4/3, 7/3	4/3	7/3
$3, 2$	3	1/2, 3/2	3/4	5/4
$3, 3$	9	1/3, 1, 2	3/2	2
$7, 2$	1	1/2	1/4	5/12
$5, 1$	1	1	3/4	1

For the first actual example, $X^2 - 2$ is Eisenstein, so the complete local-field proof linked in Solution 35 gives ramification index two. With $\beta = 1 + \sqrt{2}$, the first difference is $\beta^2 - 1 = 2(1 + \sqrt{2})$, of valuation one. The next factor is $\beta^2 + 1 = 4 + 2\sqrt{2} = 2\sqrt{2}(1 + \sqrt{2})$, of valuation $3/2$. Consequently $v_2(\beta^4 - 1) = 5/2$. The last step is above the threshold one, so Lemma 9.5 gives $v_2(\beta^8 - 1) = 7/2$. The actual path is therefore $1/2, 1, 5/2, 7/2$. Its extra gain at the boundary is permitted by the lower-bound path, but is not asserted for all units.

For the second example $X^3 - 3$ is Eisenstein, giving $e = 3$ and $v_3(\pi) = 1/3$. Direct expansion gives

$$\beta^3 - 1 = 3(1 + \pi + \pi^2), \qquad v_3(\beta^3 - 1) = 1.$$

The parenthesis is a unit since its residue is one. The next step is strictly above $1/(3-1) = 1/2$, hence $v_3(\beta^9 - 1) = 2$. Formula (10.209) gives $\kappa = 2, P = 9, \vartheta = 3/2$; thus the final depth $\vartheta + 1/2 = 2$ is attained.

Put $\ell = \log(\beta^9)$; its valuation is two. At the indices $n = 3^j$, the exponential coefficient $A_n = \ell^n/n!$ has

$$v_3(A_n) = 2n - \frac{n-1}{2} = \frac{3}{2}n + \frac{1}{2}.$$

Thus $v_3(A_n) - n\vartheta = 1/2$ along infinitely many indices. These weighted coefficients do not tend to zero, so the series is not in $\mathcal{A}_{3^{3/2}}$. At any $z \in \mathbb{C}_3$ of valuation $-3/2$, the terms of those indices still have valuation $1/2$, and the series does not converge. Such a point exists, for example a square root of 3^{-3}. Every smaller radius with $\theta < 3/2$ instead has the positive linear decay (10.213).

Finally the residue field here is $\mathbb{F}_3$. For $\alpha = -\beta$ and $\zeta = -1$, choose $a = 1$, so the phased coordinate is $V = \alpha^9\zeta^{-9} = \beta^9$. Its principal square root is $\eta = \exp(\ell/2) \in L$, of depth two. Fix $\xi^2 = -1$, and begin with any $\rho_0^2 = \alpha$. The ratio $\eta/(\rho_0^9\xi^{-9})$ is 1 or -1. Replacing ρ_0 by its negative when the ratio is -1 changes its ninth power by that sign and leaves its square unchanged. It therefore gives the exact choice $\rho^9\xi^{-9} = \eta$ required in (10.217).

Solution 37. If a rational power combination of $4, 9$ has a rational root, its valuations at two and three show that both coordinates of the corresponding exponent vector are half-integers. Conversely every half-integer pair produces a rational number $2^{2x}3^{2y}$. Thus the full saturation lattice, and hence its q-primary part at $q = 2$, is $(\frac{1}{2}\mathbb{Z})^2$. Its four cosets modulo $\mathbb{Z}^2$ prove $J = 4$, with basis $B = \frac{1}{2}I$; the corresponding generators are $2, 3$.

The rational roots of unity are $1, -1$, so $q^u = 2$. At five the residue group has order four. Choose the primitive residue lift ζ with residue two, whose existence and primitivity are proved in Lemma 10.53. Then $G_0 = 4/2 = 2$, and $2, 3$ have residues ζ, ζ^3, respectively. With $P = 1$, their phase coefficients are $-1, -3$; the zero phase class is therefore $\lambda_1 + \lambda_2 \equiv 0 \pmod 2$.

There are $5 \cdot 4 + 4 \cdot 3 = 32$ such points in $\{0, \dots, 8\} \times \{0, \dots, 6\}$. Their original coordinates are $(\lambda_1/2, \lambda_2/2)$, with full widths $4, 3$. For $(b_1, b_2) = (2, 3)$ and the identity matrix A_0, formula (10.219) gives

$$\omega = 4(3\mu_1 - 2\mu_2) = 6\lambda_1 - 4\lambda_2.$$

At reference zero its range is $[-24, 48]$, and both extremes occur at allowed points, $(0, 6)$ and $(8, 0)$. Hence $\Omega = 48$. At reference $(8, 0)$, subtract its value 48; the range becomes $[-72, 0]$ and the exact maximum absolute value becomes 72. The direct full-width estimate is $4(3 \cdot 4 + 2 \cdot 3) = 72$. It covers every reference in the same box because each difference of two original coordinates is bounded by its full width. The weighted estimate in Lemma 10.84 also holds, using $R_{\max} = 3/\ln 4 + 2/\ln 9$ and $\sum_i D_i\sigma_i = 4\ln 4 + 3\ln 9$.

Omitting J would give $3\mu_1 - 2\mu_2$. At the allowed point $(\lambda_1, \lambda_2) = (1, 1)$, this is $1/2$; its first rising binomial is $3/2$, which is not an integer. Thus the original denominator clearing is part of the integral prepared system.

For the mean-order panel, put $a = \overline{u}/t_0$. At $r = 2$, (10.204) gives $\overline{h}/t_0 = (1 - a)/2$ and $0 \le a \le 1/3$. The joint multiplier is $(1 - a)/2 + 2a = (1 + 3a)/2$, between $1/2$ and one. These are exactly the plotted lines; no numerical approximation of the means is used.

For the original rational parameters of Solution 34, $d = e = f = 1, \nu = 0$, $R_{\max} = 1/\ln 2 + 1/\ln 5$, and

$$W(1) = \frac{\ln 6}{\ln 2\,\ln 3}, \qquad g_{91} = 1 + \frac{1 + \ln W(1)}{g_0}.$$

The condition $h \ge \ln(R_{\max}/W(1))$ holds at the stated $h = g_0$. Its exact outward logarithm intervals are given by the series proved in Solution 34. The coefficient field is $\mathbb{Q}$, so the field-degree logarithm and generator/discriminant costs are zero. Keep $c_{01} = 57/20$, all g_2, g_5, g_8, and the strict θ from that solution. Substitution into (10.228) gives the following certified upper bounds.

Quantity	Upper bound
g_{91}	1.037012532704
$(E_{\mathrm{mean}} + \overline{u}\ln \mathfrak{v})/(t_0 h)$, actual lcm	0.336168252156
The same combined cost using 3^{50}	0.372348992541
The coefficient height $h_2(\boldsymbol{c})/Z$ in (10.228)	0.310817025826

For the two combined costs use the exact $\overline{u}, \overline{h}$ from Solution 35 and $E_{\mathrm{mean}} = \overline{h}\ln[\mathrm{e}(1 + 67804/\overline{h})]$. For the last line, the exact expression enclosed is

$$\frac{g_8}{2} + \frac{20}{37}\left[\frac{g_8}{2} + \frac{1 + 1/g_5}{c_1 c_4} + \frac{g_{91}}{c_1 c_3 \theta} + \frac{1 + 1/(2g_2)}{2c_1 c_2}\right].$$

The first logarithmic term is outside the row ratio; all other scalar and torus terms are inside it. The same rational outward interval rules apply to each positive logarithm and arithmetic operation, proving the displayed bounds.

Solution 38. Suppose a rational exponent vector belongs to the full saturation lattice. After taking a common positive integer power, its realizing element $\beta \in K^\times$ has a rational power $\beta^T = 5^a 7^b$. Conjugation gives $(\overline{\beta}/\beta)^T = 1$. This quotient is a root of unity in the real quadratic field, hence is 1 or -1. Writing $\beta = x + y\sqrt{5}$ shows that β is either rational or a rational multiple of $\sqrt{5}$.

In the rational case, the ordinary prime valuations at five and seven make both exponent coordinates integers. In the second case, dividing by $\sqrt{5}$ makes the first coordinate an integer plus $1/2$, and the second an integer. Conversely every such vector is realized by $(\sqrt{5})^{2v_1}7^{v_2}$. Thus

$$M = (\tfrac{1}{2}\mathbb{Z}) \times \mathbb{Z}, \qquad J = 2, \qquad B' = \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}.$$

This is already q-primary. Direct multiplication gives

$$\begin{pmatrix} 1/2 & 5 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -10 \\ 0 & 1 \end{pmatrix} = B', \qquad U^{-1} = \begin{pmatrix} 1 & 10 \\ 0 & 1 \end{pmatrix}.$$

The new generators are $\theta_1' = \sqrt{5}$ and $\theta_2' = (\sqrt{5})^{-10}(5^5 \cdot 7) = 7$. Their heights are $\frac{1}{2}\ln 5, \ln 7$; the old heights are $\frac{1}{2}\ln 5, 5\ln 5 + \ln 7$. For $\lambda' = (a, b)$, the old label is $(a - 10b, b)$, and both monomials are $(\sqrt{5})^a 7^b$. Both generator lists give $E = K$. The integral-basis proof in Solution 32 gives $\Delta_E = 5$.

At three, $X^2 - 5$ has irreducible separable reduction $X^2 + 1$. The complete quadratic proof in Cyclotomic, quadratic and Kummer extensions of local fields, Section 3, makes its completion unramified quadratic. Hence $e = 1, f = 2, d = 2$; both original bases are units. The only global roots of unity are $1, -1$, so $u = 1, \nu = 1$. The first admissible power is $P = 3$, since $p - 1 = 2 \le 2e$ and $3(p - 1) = 6 > 2e$. Thus $\vartheta = 3/2$, with the strict θ of Solution 34.

The residue field has order nine. Its element $\overline{\sqrt{5}}$ has square -1, so order four, whereas 7 has residue one. Choose the primitive residue lift ζ of order eight from Lemma 10.53 with $\overline{\sqrt{5}} = \overline{\zeta}^2$. Then $G_0 = 8/2 = 4$; the centered generator phases can be taken to be $d' = (-6, 0)$. Consequently $\delta = 2, H = 2$, and the two phase classes distinguish even and odd first coordinates. In the old basis the second phase is -60; $U^{\mathsf{T}}(-6, -60) = (-6, 0)$. Thus both bases have exactly the same phase at corresponding labels. The blue and brown points in the figure display the two classes.

Apply the outward rational interval method proved in Solution 34 to the parameters specified in the exercise, with $t = 2\ln 3, \rho = 58, x = \ln 2$, and $\delta = 2$. The basic linear-form matrix is the identity and $R_{\max} = 1/\ln 5 + 1/\ln 7 < 2$. Since $dW(d) = 2(\ln 6)^3 > 2$, the height condition of Theorem 10.86 holds at positive h. In the maximum defining $\mathcal{D}$, the second member exceeds the first. The resulting integer data are

Quantity	Exact value
$\lfloor S \rfloor, \lfloor T \rfloor$	400, 36977365
k, L	52, 579557
$\lfloor 2D_1D_2 \rfloor + 1$	597441303726
Selected points in either phase	298720651863
N	90025335312517639 32
M_{full}	547613916126766461
$80N - 171M_{\text{full}}$	626560702842464049729

Here is an explicit selection without listing the points. The integer-coordinate grid has 1201952 first-coordinate choices and 497060 second-coordinate choices. Take its first 597441303726 points, ordering the first coordinate fastest. For each full row, exactly half the first coordinates are even; the remaining initial segment also has even length. Each phase therefore contains the stated number of points. Their original coordinates lie in the box of widths D_1, D_2. Since $c_{01} = 1.9 \cdot 9/8 = 171/80$, the last line proves $N > c_{01}M_{\text{full}}$, with all floors retained.

The same outward interval arithmetic gives

Quantity	Certified upper bound
g_{11}	2975665.522410184771
$g_{12} = g_1/(2g_7) + 1/g_{11}$	0.000001658758
g_{91}	1.052658564765
Generator-field cost $\ln 5/(4Z)$	0.000000000069
Explicit tower bound $\ln 10/(2Z)$	0.000000000196
Centered bound $(\frac{1}{2}\ln 2 + \ln 7)/Z$	0.000000000390
Full $h_2(c)/Z$ bound (10.235)	0.510159606689

For a lower bound on g_{11}, the same calculation gives 2975665.522410184770. Also

$$5891891927.130371871238 \le Z \le 5891891927.130371871239.$$

The explicit tower first adjoins $\sqrt{5}$, of degree two and height $\frac{1}{2}\ln 5$; the generator 7 has degree increment one. Equation (8.33) therefore gives $\frac{1}{2}\ln 2 + \frac{1}{2}\ln 5 = \frac{1}{2}\ln 10$ as its discriminant bound. The centered maximum bound in (10.236) is $\frac{1}{2}\ln 2 + \ln 7$. Both are larger than the exact cost $\frac{1}{4}\ln 5$. The interval programme enclosed in the figure source uses only the rational positive series and geometric tail bounds of Solution 34, and verifies both instances of (10.232), the strict integer surplus, and every term in the last line.

Finally, within one phase, $\lambda_1 - \lambda_{01}$ is even. Since $P = 3$, each normalized torus monomial has a rational power of 5, a rational power of 7, and a rational sign. Every initial row is therefore rational, and its smallest field is $E_0 = \mathbb{Q}$, of discriminant one. The construction over the generator field E is valid and satisfies the stated uniform bound; applying Theorem 10.63 over E_0 improves its field cost to zero. The generator field and the actual row field are distinct objects.

Solution 39. Start with the real parameters and exact integer support of Solution 34, without replacing S, T by their floors inside the parameter formulas. Here $\eta = 1231/1500$, and the certified floor data are

j	$R_j = \lfloor 2^j S \rfloor$	$T_j = \lfloor \eta^j T \rfloor$
0	379	1796752
1	758	1474535
2	1517	1210101
3	3034	993090

For the three extensions, $\mu_j = T_j - T_{j+1} + 1$ is $322218, 264435, 217012$, and $n_j = 2R_j + 1$ is $759, 1517, 3035$. These choices use all available input derivative orders. Integer prime powers give $B_j = 6, 6, 7$. The actual $v_3(\mathfrak{v}) = 3$, and

$$v_3(50!) = \lfloor 50/3 \rfloor + \lfloor 50/9 \rfloor + \lfloor 50/27 \rfloor = 22.$$

For every allowed pair b_1, b_2, (10.245) bounds $|b_2|$ by $(\ln 2)W\exp(h)$. Taking logarithms, dividing by $\ln 3$, and applying the outward rational intervals of Solution 34 bounds the integer $v_3(b_2)$ by46. Thus $C_0 \le 46$ and $M_0 = 46$ is valid in every step.

The full-width Euler bound is the same one for the whole permitted family:

$$\Omega \le \left\lceil \frac{\mathcal{D}}{c_1 c_2 P} W \exp(h) \right\rceil \le 131384680780655552448464590 51.$$

The figure programme encloses $\exp(h)$ without a floating-point exponential. Beginning with term one, successively multiply by the outward interval for h and divide by j, through $j = 512$. If the

next term is a_{513}, the remaining positive terms are at most $a_{513}/(1-h/514)$, since every following ratio is at most $h/514<1$. Combine this with the rational logarithm intervals from Solution 34, and round the upper endpoint of the Euler bound up to an integer.

Use the proved $\mathfrak{v}<3^{50}$ for each individual scalar, with no averaging of its additive order. At step $j+1$, put

$$R=R_{j+1}+99,\quad m=\Omega,\quad D_{\mathrm{a}}=1764950,\quad T'=T_{j+1}.$$

The exact binary search (10.242), using the larger multiplier 3^{50}, gives maximizing additive orders $621676, 384993, 211681$. The respective remaining Euler orders are $852859, 825108, 781409$. All steps before these indices are increasing; the next step is decreasing. These tests involve only integer products, not factorials of the enormous Euler argument.

To bound the logarithm at that maximizing index, use

$$\ln\binom{D_{\mathrm{a}}}{u}\le D_{\mathrm{a}}\ln D_{\mathrm{a}}-u\ln u-(D_{\mathrm{a}}-u)\ln(D_{\mathrm{a}}-u),$$
$$\ln\binom{m+H}{H}\le H\,[1+\ln(1+m/H)]\qquad (H>0).$$

The first follows from the u-th term of the binomial expansion of $(t+(1-t))^{D_{\mathrm{a}}}=1$, taking $t=u/D_{\mathrm{a}}$; endpoint binomials equal one. For the second, the numerator product is at most $(m+H)^H$, and the integral proof in Lemma 10.76 gives $H!\ge(H/\mathrm{e})^H$. At $H=0$, its logarithm is zero. The exact maximum of the factorial expression has been located first; bounding its logarithm at that index therefore bounds the whole maximum.

Retain the negative term $-L\ln(50!)$. The upper bounds on $\ln\mathcal{H}_0^{\mathrm{fine}}/Z$, in step order, are

$$0.295755251977,\qquad 0.251149033062,\qquad 0.215672455561.$$

For all three steps, the coefficient/logarithmic contribution is

$$\frac{h_2(c)/Z+g_8/2}{\ln 3}\le 0.282940661808.$$

The normalized arithmetic bound at step $j+1$ is

$$\frac{1}{\ln 3}\left[\frac{h_2(c)}{Z}+\frac{g_8}{2}+\frac{\ln\mathcal{H}_0^{\mathrm{fine}}(R_{j+1},T_{j+1})}{Z}+\frac{R_{j+1}}{c_1c_2S}\right].$$

There is no q-denominator at these integer arguments. The full-width torus terms, after division by $\ln 3$, have upper bounds $0.717181076407, 1.435308301992, 2.870616603983$. Their sum with the coefficient and scalar terms gives exactly the arithmetic bounds in Theorem 10.93.

For the input precision, the common available lower bound is

$$\frac{U+G_0}{Z}\ge 7.294535093395.$$

The three required upper bounds are $1.406870557237, 2.262855922296, 3.678486939799$, respectively. Thus the sufficient inequality (10.113) holds at every step, including the full derivative loss $M_0=46$. The analytic gains are computed from

$$\frac{(n_j\mu_j-D_{\mathrm{a}})\theta-22L}{Z}.$$

The gaps over the arithmetic upper bounds exceed, in step order,

$$0.070283757014,\qquad 0.258541995694,\qquad 0.279197996096.$$

These certified positive gaps prove the three zero-forcing comparisons; no numerical estimate of an actual valuation was substituted.

Finally, at a first fractional argument, the scalar has the clearing cost (10.243), and its normalized torus value may belong to a larger global field. Both enter (10.244). Equality of the chosen completions does not remove that global degree, as shown in Theorems 10.42 and 10.56. The present comparisons use $I = J = 0, F = \mathbb{Q}$, so they prove the specified integer block. They do not verify these additional fractional costs or the subsequent support extraction and multiplicity contradiction.

Solution 40. The three nodes are $-1, 0, 1$ with respective multiplicities $1, 3, 1$, giving total degree five. The middle cardinal polynomial is $L_0(X) = 1 - X^2$; its inverse through degree two at zero is $1 + X^2$. Thus its three basis polynomials are

$$H_{0,0} = 1 - X^4, \qquad H_{0,1} = X - X^3, \qquad H_{0,2} = X^2 - X^4.$$

Each has the required one middle divided jet, with the other two equal to zero, and all vanish at the two end nodes. At the end nodes no inverse coefficient beyond degree zero is needed. Their cardinal polynomials are

$$H_{-1,0} = (X^4 - X^3)/2, \qquad H_{1,0} = (X^4 + X^3)/2.$$

They vanish to order three at zero, take value one at their own end node and zero at the other. All five degrees are at most four, so these jet checks and uniqueness prove the full interpolation formula.

For the original parameter family, subtract $T' = 993090$ from each available order and add one. This gives $803663, 481446, 217012, 1$. Cap the first multiplicity at 780000, leaving the other three unchanged. The integer node counts in the successive rings are $759, 758, 1518, 3034$. Therefore

$$N_* = 759 \cdot 780000 + 758 \cdot 481446 + 1518 \cdot 217012 + 3034 = 1286383318.$$

The same interval program of Solutions 34 and 39 gives the following outward bounds.

Quantity divided by Z	Relevant certified bound
Available input $U + G_0$	at least 7.294535093395
Required input $N_*\theta + 780000 \cdot 46$	at most 7.244899709935
Analytic excess $N_*\theta - G_0$	at least 7.100020502185
Arithmetic excess over the quartic field	at most 4.522335429896

The first comparison has margin greater than $0.049635383461Z$, and the second has margin greater than $2.577685072290Z$. No earlier zeros or their extra inner derivatives have been discarded.

The bounds on $\eta^{-3}S$ lie strictly between 686 and 687, so $\lfloor \eta^{-3}S \rfloor = 686$ and every target argument has absolute value at most 687. With $R = 786$ and the common Ω of Theorem 10.93, the exact consecutive-ratio search gives additive order 470396 and Euler order 522694. Use the entropy bounds of Solution 39 at this maximizing index and retain $-35299 \ln(50!)$. This gives the scalar-logarithm upper bound $0.208379911543Z$.

For the denominator, (10.243) here is

$$\Xi_{0,1,u} = \max\{0, 3424003 - 6u\},$$

because $v_2(50!) = 47$ and $v_2(\mathfrak{v}) = 5$. Its supremum is 3424003, contributing at most $0.008748404912Z$ to the logarithmic bracket. A scalar-maximizing index does not necessarily maximize the sum of its scalar logarithm and this decreasing, piecewise linear denominator term. The common supremum bounds every order and avoids that unsupported inference.

The remaining bracket terms have these upper bounds.

Term	Upper bound
Coefficient height and $\frac{1}{2}\ln N$	$0.310842088027Z$
Full-width torus height	$0.714102914710Z$

Multiply their sum with the scalar and denominator terms by $4/\ln 3$, using the unrounded rational enclosures. This gives the stated arithmetic upper bound. The global quartic field is proved in Theorem 10.95 even though its selected completion is $\mathbb{Q}_3$. The product formula sums over the global field; its degree cannot be replaced by the degree of that one completion. The result proves the exact first fractional range (10.250). Proposition 10.97 and Theorem 10.98 supply the subsequent contractions and comparisons.

Solution 41. Direct expansion gives

$$\binom{2n+a}{2} = 4\binom{n}{2} + (2a+1)n + \binom{a}{2}.$$

Writing $z = 2n + a$, the inverse is

$$n = \frac{z-a}{2}, \qquad \binom{n}{2} = \frac{1}{4}\binom{z}{2} - \frac{2a+1}{8}z + \frac{a(a+2)}{8}.$$

All denominators are powers of two, hence units at three. The inverse is not in general integral at two; at $a = 0$ its linear coefficient is $-1/8$. In higher degree the finite-difference proof of Lemma 10.30 supplies the same triangular filtration. Polynomial coefficients are retained as a subvector, so this inverse change does not enlarge their projective height.

For the numerical table, use the exact rational interval algebra of Solution 34 with $B_{\mathrm{arith}} = 10^{60}$. This precision parameter is distinct from the separation integer B in the interpolation lemmas. Multiplication and division round outwards. For logarithms reduce an argument by exact powers of two to the interval $[1, 2]$, put $y = (x-1)/(x+1)$, and use the positive series

$$\ln x = 2\sum_{j=0}^{511} \frac{y^{2j+1}}{2j+1} + \mathcal{R}, \qquad 0 \le \mathcal{R} \le \frac{2y^{1025}}{1025(1-y^2)}.$$

The same formula bounds $\ln 2$; integer powers restore the original logarithm. These remainder bounds and the earlier exact bounds for $\exp(h)$ make all parameter floors decisive. Use $k = 50, L = 35299$, $v_3(50!) = 22$, $v_2(50!) = 47$, and the proved bound $\mathfrak{v} < 3^{50}$.

For each of $I = 1, \dots, 17$, calculate (10.255) from the unfloored original S, T, then form the three input and analytic bounds (10.256)–(10.258). The mixed multiplicities satisfy all stated order inequalities at each stage. For each respective arithmetic output set $r = X/2^I + 99$, $D = 1764950$, $\Omega = \lfloor \Omega_0/2^I \rfloor$, and $V = 3^{50}$.

If $f(u)$ is the summand in the maximum (10.259), its exact ratio is

$$\frac{f(u+1)}{f(u)} = \frac{(D-u)V(O-u)}{r(u+1)(\Omega+O-u)}.$$

This decreases in u: both $(D-u)/(u+1)$ and $(O-u)/(\Omega+O-u)$ decrease, with the second constant when $\Omega = 0$. Comparing the numerator and denominator as rational integers locates the maximum by binary search, including endpoints. At its order u, put $H = O - u$, and use the proved binomial entropy bounds of Solution 39:

$$\begin{aligned}\ln f(u) \le{}& (D-u)\ln r - L\ln(50!) + D\ln D - u\ln u - (D-u)\ln(D-u)\\ &+ 50u\ln 3 + H\bigl(1 + \ln(1+\Omega/H)\bigr).\end{aligned}$$

Take $0 \ln 0 = 0$ and the last term zero for $H = 0$. Retain the negative factorial term. Insert this upper bound, the coefficient bracket bound $0.310842088027Z$, the complete common denominator $((I + J)D + 47L) \ln 2$, and the full torus term into (10.260). Use factor one for both integer outputs and four for the fractional output.

The following endpoint data make the calculation directly checkable.

Quantity	Stage 1	Stage 17
A_I	1374	18092274
R_I	2744	36184546
X_I	1242	16366762
O_0, O_1	993090, 814996	75, 61
O_2, O_3	668840, 548894	50, 41
μ_0, μ_1, μ_2	431116, 266102, 119946	32, 20, 9
Mixed N_*	1286895674	1266459164
Mixed separation B	7	16

At stage one the respective scalar maximizing orders are $517016, 343527, 372590$; at stage seventeen they are $61, 50, 41$. The stage-one common denominator exponents are $3424003, 3424003, 5188953$; the stage-seventeen ones are $31663203, 31663203, 33428153$.

Comparison at stage 1	Required input / Z, upper	Analytic gain / Z, lower	Arithmetic cost / Z, upper
Odd-to-full closure	1.387797139482	1.340382458724	1.100243509064
Full integer extension	2.246326479139	2.208922606590	1.727090951833
Mixed fractional step	7.192833169904	7.102853409486	3.989657490226

Use the unrounded interval endpoints when subtracting, then round the resulting gap down. At each stage, taking the maximum of the three required-input upper bounds and the minimum of the three gap lower bounds produces exactly the seventeen-row table in Theorem 10.98. Its largest required input is 7.192833169904, below the available lower bound 7.294535093395; the smallest zero-forcing gap is at least 0.240138949661. Thus every comparison passes with a strict margin. The reproducible figure program implements these rational interval operations and prints all fifty-one comparisons.

Finally the original widths are below $67805, 29202$. Dividing by 2^{17} puts both below one; since they are integer widths, both are zero. The retained coefficient vector is nonzero and gives the nonzero degree-at-most-1764950 polynomial (10.262). The stage-seventeen full closure gives 36184549 distinct integer roots. The elementary polynomial root bound contradicts this. No restriction to a single coefficient pair was made: all coefficient-height, Euler and precision estimates used the whole family (10.245).

Solution 42. The unramified case $p \geq 5, e = 1$ has $a_* = 7/2$ in (10.184). It must be retained in a uniform argument. With this weaker value, the common lower bound for g_5 at rank two is

$$\frac{893}{7250} \, 3 \left(\frac{7}{2} \right)^2 = \frac{131271}{29000} > 4.$$

Dividing that bound by r, its consecutive ratio is $(7/2)r(r+2)/(r+1)^2 \geq 28/9 > 1$. Thus $g_5 > 2r$ at every rank $r \geq 2$, and $1 + g_5^{-1} < 5/4$. For $g_4/[r(r+1)]$, the rank-two lower bound is

$$\frac{57}{116}\frac{(7/2)^2}{2} = \frac{2793}{928} > 1.$$

The consecutive ratio of $(7/2)^r/r$ is $(7/2)r/(r+1) \geq 7/3 > 1$. This gives $1 + \epsilon < \exp(1/2) < 2$.

The coarse degree-dominance comparison is

$$\frac{135143424}{6699000} > 20,$$

and the additive degree/output-order comparison follows from

$$\frac{47 \cdot 49 \cdot 39 \cdot 32768}{10000 \cdot 21 \cdot 44} > 4.$$

The complete singleton height bound from Theorem10.18 gives $d\sigma_i > 1/(87d^2)$. The stopping exponential contributes $\mathrm{e}^8(r+1)^2d^2$; its d^2 cancels that denominator before any constant is discarded. The remaining $\exp A/[A(A+39)]$ increases for $A \geq 5$.

For the intermediate allocation, let $\lambda = 1 + g_5^{-1}$, $E = (q\eta^{r+1})^I$, and use the upper additive degree

$$D_+ = \lambda\frac{S\mathcal{D}}{c_1c_4d(A+x)}.$$

The unrounded lower numerator is

$$\frac{2\overline{S}}{3r}\frac{(\overline{T}/(r+1))^{m+1}}{(m+1)!}.$$

Divide by $(m+1)!D_+\prod_{j\in J}\mathsf{D}_j$. Inserting $\overline{S}, \overline{T}, \mathsf{D}_j$ and cancelling produces

$$\frac{2B_0}{r}\frac{\prod_{j\in J}\sigma_j}{((m+1)!)^2}\left(\frac{\eta c_2 qPrd}{2e\theta t}\frac{E}{q^\nu}\right)^m.$$

One factorial comes from the binomial lower bound, and the other from the multiplicity theorem; neither may be removed. Every m-element subfamily satisfies its own height-product bound. After that bound is inserted, the stopping inequality and the proved elementary exponential bounds give

$$\frac{72}{725}\frac{(2048/3)^m}{(m+1)^3}.$$

At $m = 1$ this equals $147456/17400 > 8$. Its consecutive ratio is larger than $(2048/3)(2/3)^3 > 1$, so the comparison holds for every $m \geq 1$.

For the final allocation use the lower numerator

$$\frac{2\overline{S}}{3r}\frac{(\overline{T}/(r+1))^r}{r!}.$$

Divide by $(r+1)!D_+\prod_{j=1}^r\mathsf{D}_j$, and substitute the full expression for $\mathcal{D}$. The exact cancellation, before lower bounds on the stopping exponential, is

$$\frac{2q\eta^r q^{\nu+u}}{\gamma}\frac{q^{rI-r\nu}\eta^{(r-1)(r+1)I}}{3r(r+1)!\lambda(1+\epsilon)(2+g_2^{-1})c_0[2(r+1)]^r t^r M\ell_d}.$$

The factor $r!$ cancels against the one in $\mathcal{D}$. The factor $(r+1)!$ remains. Use $q^{rI}\eta^{(r-1)(r+1)I} > E^r$, $E > \exp(3(A+x))$, and $q^{\nu+u}/\gamma \geq 1$ before dropping any saturation factor. The resulting rank-dependent lower bound is

$$\frac{8(192/125)^r 16^{2r-1}}{135r(r+1)^2}.$$

At rank two this is

$$\frac{8 \cdot 192^2 \cdot 4096}{125^2 \cdot 135 \cdot 2 \cdot 9} > 31.$$

Its consecutive ratio is at least $(192/125)256(3/8) > 1$. These are analytical bounds for all ranks, rather than an inference from a finite rank check.

The floor allocations are formed from the real $\overline{S}, \overline{T}$. Their sums are integers at most those real quantities and thus at most their floors. Also $\lfloor y \rfloor + 1 > y$, including when y is an integer. This explains the strict inequalities used in every numerator.

Finally the multiplicity theorem supplies integral u_i, not an asserted integral basis of the whole relation lattice. They are rationally independent, span $\boldsymbol{B}$, and give $\beta_i = \prod_j \alpha_j^{u_{ij}} \in K$. Each is a unit at the chosen prime, and the height power and product laws give $h(\beta_i) \le R_i$. A relation among the β_i would be a relation among the original independent α_j; therefore the new bases are independent. Clearing denominators in the rational-span relation and killing the original finite root-of-unity phase proves (10.271). The arithmetic field of a fractional zero used earlier in the descent remains a separate quantity; it is not changed by this return of the new bases to K.

Solution 43. At two the reduction of ζ_3 has order three, so f is even and at least two. Hence $t > 1$ in every case. For any $j \ge 1$, the first member of $t^j M_j$ is $p^f/\delta \le p^f t$. The second is bounded by $p^f t$ because $\ln(t/j) \le t/j - 1$. This applies to the actual order at rank one as well as to the saturated residue index at larger ranks.

Write $D_+ = \lambda S \mathcal{D}/(c_1 c_4 d(A + x))$. The lower denominator in the bound (10.270) is

$$\frac{2\overline{S}}{3r} \frac{(\overline{T}/(r+1))^m}{m!}.$$

Before substituting $\mathcal{D}$, the resulting upper bound for the new weight product is

$$\frac{3r}{2q}(m+1)!m!\lambda \frac{\mathcal{D}}{c_1 c_4 d(A+x)} \left(\frac{2e\theta t}{c_2 q P d}\right)^m q^{m\nu} \frac{\eta^{(r+1)I-m}}{E^m}.$$

Substitute $\mathcal{D}/(c_1 c_4 d(A+x)) = (\gamma/q^{\nu+u})(1+\epsilon)(2+g_2^{-1})c_0[H_* r(r+1)d]^r M_r \ell_d \prod_j \sigma_j / r!$. Division by $\Gamma_n^{r-m}(M_r/M_m)\prod_j \sigma_j$ cancels all H_*, d, and old M_r factors, giving (10.274). The new M_m remains, so its actual residue index is still present before the uniform bound is applied. The two factorials arise separately from the multiplicity output and the binomial lower bound.

The factorial integral gives $r^r/r! < \mathrm{e}^r$, while $n \ge r$, $\gamma \le q^\nu$, and $q^u \ge 1$. The geometric stop contributes $\exp(-3(m+1)A - (2m+3)x)$. Combining these with $t^m M_m \le \exp(A)A$, $C < 45/2$, and $2\mathrm{e}/\eta < 25/3$ gives (10.275). At $A \ge 4 + \ln((r+1)d)$, its exponential factor is decreasing. Use $4 + \ln((r+1)d) \le 5(r+1)d$, $\ell_d \le d$, and

$$\frac{r(m+1)!m!}{(r+1)^{2m+1}} < m + 1 \le 2^m.$$

For the strict inequality, first maximize $r/(r+1)^{2m+1}$ over $r \ge m+1$, then use $(m+1)!m! \le (m+1)^{2m+1}$. Finally $\mathrm{e} > 2$ gives the exact envelope $(675/2048)(25/6144)^m$. Its largest value is $5625/4194304$, at $m = 1$, and $5625 \cdot 700 = 3937500 < 4194304$.

For the composed forms, independence modulo z_0 follows from independence of both the original projected rows and the integral vectors u_i. Rational span gives the original form representation. Coefficientwise triangle inequalities bound every new weighted row by R_i; integer products preserve the original field and chosen-prime units. The old product bound and the new one multiply with

exact cancellation of M_r, leaving Ψ_m with the new residue index. Since $m < r$, this contradicts the original minimum rank.

In the example $\Gamma_2 < 3 \cdot 7 \cdot (101/100) \cdot 3 < 64$. The two parts of M_2 are less than 3 and $9(11/10)/4 < 3$, respectively. Every M_1 is greater than 2, even when a torsion factor changes its actual residue order. Also $h(2)h(5) < (7/10)2 = 7/5$. Thus the rank-one profile allows a weight less than $64(3/2)(7/5) = 672/5$. The Euclidean calculation $103-101 = 2$, with 101 odd, gives $\gcd(101, 103) = 1$. Any proportional integral row is consequently a nonzero integer multiple of $(101, 103)$. Its required weight exceeds $511/3$, and $511 \cdot 5 > 672 \cdot 3$. No rank-one representation is admissible; the standard two rows have exactly the permitted product, so the minimum rank is two.

Solution 44. The minima, in the order in (10.281), are $20838501411/25000000$ in III.2, $1273657/6250$ in III.1, $42419/40$ in V, and $14271/10000$ in V. They exceed respectively $800, 202, 1000, 7/5$. These are obtained by multiplying the exact terminating decimals in each of the seven rows and comparing their rational numerators after clearing denominators.

The three contributions in Lemma 10.105 are, with common denominator $7(r+1)^2 b_r$, less than 10, $(3r+2)(r+1)$, and $5(3r+1)$. Their sum is $3r^2 + 20r + 17 = (r+1)^2[3 + 14/(r+1)]$. At rank two the result is $667/151200 < 1/225$; the residual factor $3 + 14/(r+1)$ decreases, giving the complete all-rank bound with 14^{2-r}.

For the simplex, at additive order $u = 0, \dots, 6$, there are $7-u$ choices of Euler order $h_{\mathrm{E}} = 0, \dots, 6-u$. Their total is $7+6+\cdots+1 = 28$. The sum of the additive orders is $0{\cdot}7+1{\cdot}6+2{\cdot}5+3{\cdot}4+4{\cdot}3+5{\cdot}2+6 = 56$, so their mean is two. Swapping the additive and Euler coordinates gives the same mean for the Euler order. The ten rows with $u > 2$ are zero when the additive degree is two, but retaining their equations imposes no new restriction. Assigning positive upper bounds to their zero entries still supplies legitimate row majorants, and Theorem 10.71 already supplies a strict column surplus for the full 28-row pattern.

Subtracting (10.289) from one over the common denominator $120(r+1)^2$ gives $108(r+1)-85(r-1) = 23r + 193$. Thus (10.290) holds for every rank, not just the plotted ranks.

For the scalar calculation,

$$F_{0,1}(X) = \frac{X^4 + 10X^3 + 35X^2 + 50X + 24}{24}, \qquad \mathfrak{v} = 12, \qquad \mathfrak{v}^2 \frac{F_{0,1}''(0)}{2!} = 210.$$

The factor $24/12^2$ in (10.298) gives $W = 35$. Hence $v_3(V) = 1$, $v_3(W) = 0$, and $Lv_3(4!) - uv_3(12) = 1 - 2 = -1$, exactly the stated valuation difference. For any permitted θ, $G_u = 2\theta - 1$, so $v_3(V) + G_u = 2\theta = v_3(W) + 2\theta$. The cancellation arises from multiplying the whole value by one rational scalar. This leaves its number field unchanged. The product formula still sums all embeddings and places of that global field, so the factor $d_F/(e_F f_F \ln p)$ remains.

Solution 45. The linked certificate starts from the fifteen displayed weights and the interval lists. Its exact rational check gives $\sum v_r/r = 0$. Each denominator divides 30030, so increasing an integer argument by 30030 changes F by $30030 \sum v_r/r = 0$. At the reflected argument $30030 - 1 - n$, each pair of floors sums to $30030/r - 1$. Summing gives $-\sum v_r = 3$, not one. The coefficient sum is -3; the mean value of F on a period is therefore $3/2$.

For logarithms, the sixteen down-rounded terms of (10.304) give a lower endpoint; add seventeen units of $1/10^{12}$ for an upper endpoint. Normalize each prime by its largest power of two and add the corresponding enclosure of $\ln 2$. The sieve and the assignments at every prime power produce cumulative L_n, U_n. Direct integer checks give the four strictly positive minima in Lemma 10.111. The lower check uses $26L_n > 25Q(n+1)$: for x just below $n+1$, $\psi(x) = \psi(n)$, and a comparison only with $25n/26$ would not imply $\psi(x) > 25x/26$. The two rational recurrence margins are respectively $6731/26000000$ and $11523/26000000$. They prove both induction inequalities for every later real interval.

The new mean coefficient is $(r-1+107/103)/(r+1)+17(r-1)/[24(r+1)^2]$, which is $C_r^\sharp$. At rank two it is $16871/22248$, while $C_2=841/1080$. Their difference is $21/1030$; it is an absolute improvement in the coefficient of TH_1, with the same full row set and coefficient field.

Solution 46. Multiplying $m+1$ positive geometric series gives coefficient $\binom{m+h}{h}$ at z^h. Its single contribution is no larger than the full sum. If $v(k)z\le 1$, multiplication by $v(k)^u z^{u+h}$ cannot increase this inequality. This proves (10.309) after setting $z=\exp(-a)$, without using an average derivative order.

In the figure, $v(4)=12$, $z=1/13$, and the exact scalar at order u is $12^u\binom{8}{u}15^{8-u}\binom{27-u}{10-u}/24^2$. The consecutive ratio is $12(8-u)(10-u)/[15(u+1)(27-u)]$. At zero its numerator is 960 and denominator 405, so the scalar increases. At one its numerator is 756 and denominator 780, so it decreases; both component ratios decrease thereafter. The maximum is therefore at one and equals 88976443359375.

The fixed-allocation upper bound before taking logarithms is $\binom{8}{u}15^{8-u}13^{10}(13/12)^{18}/24^2$. The uniform bound replaces its first two factors by 16^8, and is $1550293280266239621526953510 5521/3570467226624$. These formulas determine the entire left panel. The right-panel coefficient is $(1000/999)(3/8)(r-1)/[15(r+1)^2 2^I]$. Its maximum over real $r\ge 2$ occurs at three, because the difference from $1/8$ is $(r-3)^2/[8(r+1)^2]$. The maximum at depth zero is $25/7992<1/300$, and each further depth halves it in this illustration.

The additive term comes from the $r-1$ geometric factors in $m_I+1=\sum\Omega_j+r-1$. It remains even when their integer arguments are zero; an estimate only for $\sum\Omega_j$ would omit it. The seven products follow by multiplying the displayed c_1,c_3,q, $c_2=7/4$ or $13/9$, and the justified minimum P. Their smallest is $18772369/1250000>15$. Finally the exact differences are $4/103-125/3228=37/332484$ and $4/103-1/27=5/2781$, which prove the two caps used in Lemma 10.114.

Solution 47. We have $D_{\mathrm{a}}=8$, $\mathfrak{v}=12$, $\ell_p=1$, and $\chi_p=2v_3(24)=2$. In the first instance $\vartheta-\beta+B=1$, so the Euler-order minimum in (10.319) is at zero. The remaining expression is $\max\{-a,u-2\}$, for $0\le a\le\min\{6,8-u\}$. For $u=0,1$, the allowed range reaches the breakpoint $2-u$. For $u\ge 2$, the constant $u-2$ dominates already at zero. Thus the nine envelopes are

$$(K_0,\dots,K_8)=(-2,-1,0,1,2,3,4,5,6).$$

Here $C_0=1$; with $B=0,M=7$, the older contribution is $-(M-1)C_0=-6$ at every order. The new finite bound improves it by $u+4$. The left breakpoint occurs because γ_{u+a} acquires one unit of valuation for each further additive derivative. This exactly compensates that derivative's lcm factor. A higher Euler derivative still has its separate slope cost.

For the second instance, $\vartheta-\beta+B=-1$. Formula (10.319), with $H=\min\{4,8-u\}$, becomes

$$K_u(5,1)=\min_{0\le a\le H}\{\max\{-a,u-2\}+2a-4\}.$$

Both affine pieces strictly increase with a, so zero gives its minimum. Therefore

$$(K_0,\dots,K_8)=(-4,-4,-4,-3,-2,-1,0,1,2).$$

The interpolation contribution excluding $U-\beta+G_u$ and the separate deleted-node cost is $-(M-1)B+K_u=-4+K_u$. The original common loss gives $-4\max\{1,3\}=-12$. The sharpened linear loss gives $C_{\mathrm{sharp}}=2$, hence -8. The first three new contributions equal -8; the later ones are strictly larger. No minimum here is being asserted to equal an actual function's valuation.

The q-powers used in differentiation have p-adic valuation zero because $p\neq q$. Their rational denominators contribute at primes above q in the global product formula; their clearing factor therefore stays in (10.312). Likewise lying in an old local completion does not change the degree of

the actual number field containing the algebraic value. Both costs remain in $\mathcal{A}_u(x)$, and both strict numerical comparisons in Corollary 10.120 remain necessary.

Solution 48. All the input data are rational after Lemmas 10.121–10.123: the lower depths include the factor $1 + 10^{-6}$, not a rounded decimal. For each of the eight field rows and each $r = 2, \dots, 7$, substitute the original case constants, its actual c_5 band, and $C_r^\sharp$ into (10.331)–(10.332) at $j = 0$. These 48 differences give the first column of Lemma 10.125. Each larger rank uses the last c_5 band; the inequalities $C_r^\sharp < 1$, $\eta \le 1$, decreasing $\epsilon_r, \bar{g}_{12}$, increasing b_r, S_-, and $\bar{g}_9 = 107/103$ give the second column from eight rational substitutions at rank eight. These monotonic inequalities supply the missing universal quantifier.

For I.2 the uniform data are

$$\begin{aligned} &c_1 = 7247/5000, \quad c_3 = 3463/2500, \quad && c_4 = 104/5, \\ &C = 57/20, \quad Y_- = 1500000/1000001, \quad c_2 = 7/4, \quad && c_5 = 14/25. \end{aligned}$$

At rank eight, $b_8 = 54212659200/29$, $\epsilon_8 = 1189/3415397529600$, $S_- = 53856576/6875$, and $\bar{g}_{12} = 0$. Set $C_r^\sharp = \eta = 1$ only in the arithmetic upper bound, keeping $c_5 = 14/25$ in the gain. Direct substitution gives

$$\begin{aligned} \mathcal{L}_{\mathrm{I.2},\ge 8}(0) &= \frac{463042584276792145068493 07}{898513136237360739513600 0000}, \\ \mathcal{L}_{\mathrm{I.2},\ge 8}(0) - \frac{1}{200} &= \frac{137860161581117753116930 7}{898513136237360739513600 0000} > 0. \end{aligned}$$

Thus the smallest uniform margin exceeds $1/200$ strictly, although the lower-bound marker in the figure is exactly $5/1000$. The exact rational program lists all 56 enclosing differences without floating-point decisions.

Write $a = \bar{\lambda}\ell_q/(5c_4)$. The increasing continuous difference in Lemma 10.124 excludes the indicator cost. On $[0, 1]$, its increase exceeds $2(7/500) - 1/125 > 1/125$ at odd primes, and $3/20 - 3/50 > 3/100$ in V. These bounds exceed $a/(q-1)$, so subtraction of that full indicator still leaves a strict increase. On every later integer step the indicator is constant. This explains why it cannot simply be omitted at the first step.

Finally Figure 10.28 displays the geometric restriction $I \ln Q \le 3D$, under which the convex function $a_0 Q^{-I} + b_0 I$ is bounded by its endpoint chord. Actual stages must additionally satisfy $c_5 T_{I+1}/(r+1) \ge 1$, and a contracted stage must obtain its input zeros through fractional interpolation and coset extraction. A valid inequality at a candidate depth is different from existence of that stage. Corollary 10.127 provides the complete initial integer block without needing such later stages.

Solution 49. With the original floors of Solution 39, the output order is $O = 993090$. Thus

$$(\mu_0, \mu_1, \mu_2) = (803663, 481446, 217012).$$

The symmetric node counts in the three annuli are $759, 758, 1518$. Therefore

$$\begin{aligned} N_* &= 759(803663) + 758(481446) + 1518(217012) \\ &= 1304340501 \\ &= 3035(217012) + 759(322217) + 1517(264434). \end{aligned}$$

The last block alone contributes $3035(217012) = 658631420$. The added inner derivatives nearly double that degree. These are exact cardinal degrees in the normal quotient, which then supplies $N_*\theta$; no actual nonzero auxiliary valuation is inferred from the diagram.

For $r \geq 8$, reverse the geometric sum: $q^{-r}\sum_{j=0}^{r}(q\eta)^{j} = \eta^{r}\sum_{k=0}^{r}(q\eta)^{-k}$. Its first four reversed terms are at least $\eta^{r}\sum_{k=0}^{3} q^{-k}$. Integration gives $-\ln(1-c/(r+1)) \leq c/(r+1-c)$, hence $\eta^{r} > \exp(-c) > 1/E_{+}(c)$. For the radius correction, $1-H<1$ and every $\eta^{j}-H<1$; consequently $F_{r} \leq q(r+1)(2r+1)/(q^{r}S_{-}(r))$. Divide S_{-} by $(r+1)^{2}$. Its remaining factor is nondecreasing in each original case, while $(2r+1)/((r+1)q^{r})$ decreases. Its maximum at rank eight is exactly $153q/(q^{8}S_{-}(8))$. Combining the two estimates proves (10.342) for every larger rank.

Solution 50. The original certified first-stage radius has $\lfloor S_{1} \rfloor = 686$, so $R = 2(686+1) = 1374$. The floor $\lfloor T_{1} \rfloor = 993090$ alone gives $178094 < cT_{1}/3 < 178095$, since $c = 269/500$. Thus $M = 178095$. The earlier exact intervals also give $\lfloor \eta T_{1} \rfloor = 814996$, and $814996 + M - 1 = 993090$, verifying the required ordinary jets.

Because R is even, its q-deleted nodes are precisely its 1374 odd integers. Their total cardinal degree is $N_{*} = 1374(178095) = 244702530$. At three, $3^{7} = 2187 < 2R = 2748 < 6561 = 3^{8}$, so $B = 7$. At additive order zero with $\beta = 0$, the jet minimum $K_{0}(M,B)$ is zero: every allowed positive additive increment contributes at least $a(B-\ell_{p}) \geq 0$, where $\ell_{p} = 3$, and the remaining Euler contribution is nonnegative. The zero increment attains zero. The normal interpolation entry consequently has the two losses

$$MB = 1246665, \qquad (M-1)B = 1246658,$$

whose sum is 2493323. The first is the deleted-node cardinal cost; the second comes from the inverse coefficients paired with the jets. They have separate origins and both occur in (10.320).

Before extraction, Theorem 10.56 gives the actual normalized support field of degree $q^{h_{\Lambda}}$ over K. Its unchanged selected completion does not alter that global factor, which belongs in (10.340). After extraction, (10.141) and the phase congruence write each new integer torus term as a product of integral powers of the original θ_{i}^{P} and α_{0}; hence the value belongs to K. This algebraic identity justifies the return to degree d in (10.346). A statement about a selected local completion would not justify it. The separate q-denominator and normal-term bounds continue to apply in that original field.

Solution 51. The deleted inner nodes are $-8,-7,-5,-4,-2,-1,1,2,4,5,7,8$, each of multiplicity five. The other inner nodes are $-9,-6,-3,0,3,6,9$, each of multiplicity three. The outer nodes are the nine positive integers 10–18 and their negatives, each of multiplicity one. The annular degree is $12(5)+7(3)+18 = 99$. The telescoping degree is $(2\cdot 18+1)(1)+(2\cdot 9+1)(3-1)+12(5-3) = 99$.

Modulo four the deleted counts are $4,3,2,3$, in classes $0,1,2,3$. For example, the class zero has $-8,-4,4,8$, while the class two has only $-2,2$. Also $2^{5} = 32 \leq 36 < 64 = 2^{6}$, so $B = 5$, and $M = 5, E = 2$. Formula (10.354) gives $-5(5-5+2) = -10$, $-5(5-3+2) = -20$, and $-5(5-1+2) = -30$. Each Taylor-inverse coefficient of degree a has the separate bound $-5a$. The combined Hermite bound at $C = 0$ is consequently $\Lambda - EB - (M-1)B = \Lambda - 30$.

If a deleted increment contains the cardinal node, residue-count discrepancy two permits a loss of one after omitting that node; if it does not contain the node, the loss can be two. A claim of discrepancy one would replace these bounds by zero and one, respectively, and would discard the EB cost. The displayed counts disprove that claim. No change to the degree count or the Taylor-inverse loss justifies discarding the deleted cost.

Solution 52. Write $x = (1-t)a + tb$, $0 \leq t \leq 1$. Concavity gives $f(x) \geq (1-t)f(a) + tf(b) \geq \min\{f(a), f(b)\}$. For a twice differentiable function, $f'' \leq 0$ gives this concavity by subtracting the affine chord and observing that an interior negative minimum would contradict the decreasing derivative; alternatively integrate its decreasing derivative on the two sides. Each τ_{j} is a constant minus a positive sum of convex exponential-plus-affine functions. The mass is a positive affine combination of these cutoffs. Thus subtracting the convex arithmetic envelope gives the required

concave gap. At higher ranks the last-block mass has the same property, because its coefficient of τ_r is positive.

Replacing a cutoff by a lower bound decreases its mass contribution only when that coefficient is nonnegative. In (10.367), the coefficients for j=2 and j at least three are positive because $2(q^2-q)-2(q+1)/S_- > 0$ and $2(q^j-q^{j-1})-2/S_- > 0$. The first coefficient has no subtracted floor term. This justifies the direction of every finite endpoint substitution.

For $z = 1/q$, the finite geometric sum has remainder tending to zero, so $\sum_{j\ge1} q^{-j-1} = z^2/(1-z) = 1/[q(q-1)]$. Write $j = \sum_{k=1}^{j} 1$; all terms are nonnegative, and grouping them gives $\sum_{j\ge1} jz^{j+1} = z^2/(1-z)^2 = 1/(q-1)^2$. At q=3 these are $1/6, 1/4$. Summing the defining Δ_j with these upper bounds gives (10.373), including its separate $(r-1)Q^{-I}$ torus cost.

Fractional zeros are produced only at $I < I_0$, so $I \le I^* - 1 \le L_D$. Their phase passage constructs the successor J=I+1, which can equal $I^* = \lfloor L_D \rfloor + 1 > L_D$. Its deleted zeros still need full closure at order $\lfloor \eta T_J \rfloor$. The late endpoint arithmetic cost is bounded by the former late B_0 bound plus one extra ℓ_-, or ℓ_+ in the uniform rank range, because $\ell(L_D+1) = \ell L_D + \ell$. Its torus term remains less than 1/1000, since $Q^{-(L_D+1)} < \exp(-3D)$. Its input loss uses only $cT_J/a \ge 1$, giving $3c/(ac_3Y_-)$. Formula (10.369) therefore supplies the actual final full block, without extending the fractional comparison beyond its proved interval.

Solution 53. We have $\eta = 24733/30000$ and $H = 15129702640837/27000000000000$. The full radii are $R_1 = 2(\lfloor 100 \rfloor + 1) = 202$ and $R_2 = \lfloor 4 \cdot 100 \rfloor = 400$. The full derivative cutoffs are 824 and 679, the deleted cutoff is 1000, and $O = \lfloor 1000H \rfloor = 560$. Hence $\mu_0 = 441, \mu_1 = 265, \mu_2 = 120, E = 176$.

There are 202 deleted inner nodes, 203 inner multiples of 2, and 396 nodes in the outer annulus. Their respective multiplicities are 441, 265, 120. Thus

$$N = 202 \cdot 441 + 203 \cdot 265 + 396 \cdot 120 = 190397.$$

Equivalently, (10.356) gives $801 \cdot 120 + 405(265-120) + 202 \cdot 176 = 190397$. Every output of total order at most 560 has all these additional ordinary jets, including at each input cutoff: its largest resulting orders are 1000, 824, 679, respectively. The fractional numerator radius is $2(\lfloor 100/H \rfloor + 1) = 358$.

Integer clearing at depth J costs at most $\ell(J+1)$. Substitution of $J = L_D+1$ therefore gives $\ell L_D + 2\ell$; keeping only one residual ℓ would omit the extra terminal step. Finally Theorem 10.56 bounds the global degree by dq^{h_Λ}, while local degrees describe only the selected completion. Local degree one does not bound h_Λ by zero. The proved comparison uses $h_\Lambda \le r$ and retains the normalization $c_1 W_*/q^r$ throughout.

Solution 54. (a) Taking logarithms, the required strict inequality is $I > 3D/\ln Q$. The least integer with this property is its floor plus one. For the displayed values the quotient equals 192, so $I^* = 193$: integer 192 gives equality and does not suffice. If $I^* = I_1$, the minimum defining I_0 is still I^*. Corollary 10.140 includes its terminal stage, so the same zeros and the same strict geometric inequality apply.

(b) Write $P = Y^{\boldsymbol{d}}Q$. The torus Euler operator D_j acts on the factor $Y^{\boldsymbol{d}}$ by the scalar $c_j = \sum_i C_{ij}d_i$. Thus (10.93) replaces each D_j acting on Q by $D_j + c_j$. Expanding a product produces jets of no greater total order, with diagonal coefficient one. Replacing c_j by $-c_j$ gives the inverse change. The additive derivative commutes with this multiplication. At the evaluation point all torus coordinates are nonzero, so both multiplication by $Y^{\boldsymbol{d}}$ and its inverse are defined. Consequently both sets of jets vanish together through the identical cutoff.

(c) Let a_{jk} be the coefficient of z_k in L_j. By the triangle inequality and the old weighted-height condition,

$$\sum_{k=1}^{n}\left|\sum_{j=1}^{r}u_{ij}a_{jk}\right|h(a_k)\le\sum_{j=1}^{r}|u_{ij}|\sum_{k=1}^{n}|a_{jk}|h(a_k)\le\sum_{j=1}^{r}|u_{ij}|\sigma_j=R_i.$$

Theorem 10.102 and (10.277) now bound the new weight product strictly by

$$\frac{1}{700}\,\Gamma_n^{r-m}\frac{M_r}{M_m}\Gamma_n^{n-r}\frac{M_n}{M_r}\prod_k h(a_k)=\frac{1}{700}\Psi_m\prod_k h(a_k).$$

Every M_j is positive, so the cancellation is legitimate. At rank one M_1 uses the actual rank-one residue order specified in (10.272); keeping that factor in both expressions gives the same cancellation. Independence of the new forms and representation of the same nonzero linear form follow from Corollary 10.103. Thus this is a smaller admissible representation, which contradicts the least-rank choice.

Solution 55. (a) Here $\eta=1231/1500$. The value of $c\eta^jT_1/a$ is η^{j-1}: it is greater than one at $j=0$, equals one at $j=1$, and is less than one from $j=2$ onward. Thus $r_1=1$ and $cT_\flat/a=\eta=1231/1500$. This includes the lower endpoint in Lemma 10.142.

(b) If $(3D-I_1\ln Q)/\ln q=m$ is an integer, its ceiling is m. That choice gives $Q^{I_1}q^m=\exp(3D)$, whereas floor plus one gives $m+1$ and the required strict inequality. The latter also covers nonintegral quotients.

(c) Since $5220=36\cdot145$, $9\cdot2^{13}/36=2048$, and $7\cdot2048=14336$, the fractions agree. The sharper coordinate bound is $|m_j|<145/28672<1$ for integers m_j. Every coordinate is therefore zero. Proposition 10.31 preserves a nonzero polynomial, and the shifted-power independence of Lemma 10.29 preserves nonzeroness after this support collapse. Its degree is at most $D_{\mathrm{a}}=kL$, while it has $2R+1>(171/58)D_{\mathrm{a}}>D_{\mathrm{a}}$ distinct roots. Repeated division by $X-s$ at distinct roots makes their product divide the polynomial; its degree would be at least their number. This is impossible.

Solution 56. (a) At $R_b=25$, $R_0=24$ and $N_*=100$. At $R_b=26$, $R_0=24$ and $N_*=102$. Generally $24R_b/25-1<R_0\le24R_b/25$. Substitution into $N_*=2R_b+2R_0+2$ gives $(98/25)R_b<N_*\le(98/25)R_b+2$. The strict lower bound therefore includes both integral and nonintegral floor arguments.

(b) For an outer node, its multiplicity is one while the maximum multiplicity is two. The cardinal estimate proved in Lemma 10.94 is $-(M-\mu(s))B=-B$. For an inner node that part is zero, but the first-jet Taylor inverse can still cost B. Theorem 10.119 charges the combined loss with $(M-1)B=B$. Both intervals are full, so its separate q-deleted term is $EB=0$.

(c) Put $z=c\eta^rT_1/a\ge1$. Multiply the mass upper bound by z, use $R_b\le q^rS_1$, and substitute $c_1W_*S_1T_1\theta=qa$. Dividing by q^a gives the first two terms of P_r. The coefficient bound contributes $c_1/(7ab_rq^a)$. Since $M-1=1\le z$, the separation and retained-jet bound contributes at most $c\eta^r/(c_3Y_-aq^a)$, the last term. For the lower bound use $R_b>q^rS_1-1$ and then $a/T_1>cH$; this gives exactly G_r. The term $1/c_2$ in C_r is the full normalized q^a/c_2 cost. It must be retained along with the separate linear depth-indicator term.

(d) In each rank band at least eight, c is fixed. The proof gives increasing H,S_-,b_r, decreasing $\eta^r,\epsilon_r,\bar g_{12}$, the uniform $C_r^\sharp<1$ and $\bar g_9=107/103$, and decreasing r/q^{r+1}. These enclose every rank by the displayed endpoint formulas at eight. Exact rational substitution then checks those enclosing formulas. Successful evaluations at finitely many ranks alone would leave all other ranks unproved; it is the analytic enclosure that covers them.

Solution 57. (a) The only required divided jet at each node is order zero, which is the prepared value of the target selected order. Thus each node has multiplicity one. Since $M-\mu(s)=0$, the

full-interval cardinal is integral, and the inverse Taylor polynomial has degree $M-1=0$. Both of those losses are zero, and $E=0$ because no node is deleted. The normal baseline still contains $U-\beta$; removing the jet losses does not remove that coefficient cost.

(b) Use $\beta\ln p\le h\le H_1$ and $W_*=et/(S\mathcal{D})$. Substituting $S=c_3qadH_1/t$ and $\mathcal{D}=c_1\theta etT/(qa)$ gives $c_1W_*\beta\le f/(c_3d\theta T)$. The inequality $ef\le d$ bounds this by $1/(c_3e\theta T)=1/(c_3YT)$. Finally $Y\ge Y_-$ and $T\ge g_4$ give the claimed bound. All these are the original simultaneous field parameters.

(c) Direct substitution into $L_s(x)=\prod_{t\neq s}(x-t)/(s-t)$, in node order $-2,-1,0,1,2$, gives $3/128,-5/32,45/64,15/32,-5/128$. Their sum is one, and their 5-adic valuations are $0,1,1,1,1$. For $W(Z)=\prod_{s=-2}^{2}(Z-5s)$, the value is $W(5/2)=5^5(45/32)=140625/32$. Since $140625=9\cdot5^6$ and the denominator is prime to five, its valuation is six, at least the normal root-polynomial bound $N\theta=5$.

(d) The inequality concerns one selected analytic valuation. A zero criterion must compare it with an arithmetic upper bound for the actual algebraic value, including the full global degree, local degrees, scalar denominators and coefficient data. Membership in a selected completion neither supplies that bound nor implies equal valuations at other primes. Those are separate mathematical assertions which require their own proofs.

Solution 58. (a) At odd p, $cTH^8/a>(4500/29)(11/20)^8>1$, so the last index with $cTH^I/a\ge1$ is at least eight. Also $Q>11/10$, and $(11/10)^8>2$. In V, $cTH^6/a>(6750/29)(5/12)^6>1$, so $I_1\ge6$; here $Q>5/4$, and $(5/4)^6>3$. These are exact rational comparisons, including the original three c bands.

(b) The nodes in increasing order are $-2,-1,1,2$. Substitution in $L_s(0)=\prod_{t\neq s}(-t)/(s-t)$ gives $-1/6,2/3,2/3,-1/6$, whose sum is one. Their 2-adic valuations are $-1,1,1,-1$. Thus a deleted cardinal can have a negative valuation. The uniform B is two, whereas the actual largest loss here is one. With $W(Z)=\prod_s(Z-2s)$, $W(0)=64$, whose valuation six exceeds the normal bound $4\theta=4$ at $\theta=1$. This is the exact illustration in Figure10.37, rather than a full logarithmic-contradiction parameter choice. There is one value condition per node, and no further ordinary jet.

(c) The floor plus one gives $I_2\ln q\le3D-I_1\ln Q+\ln q$. Add $I_1\ln q$, and use $I_1\le3D/\ln Q$, to get $I_3\ln q\le3D\ln q/\ln Q+\ln q$. In particular the extra $\ln q$ endpoint remains. The complete denominator is $D_{\mathrm{a}}[J+1/(q-1)]$; J can exceed the earlier geometric range. Keeping only that old range would discard part of the depth cost. Its correct normal upper bound contributes $\bar{\lambda}[3\ell_q/\mathcal{L}(Q)+(1+1/(q-1))\ell_q/5]/c_4$.

(d) The scalar, denominator and torus estimates use the proved larger interval $I_1+1\le J\le I_3$, and both strict comparisons include its endpoint. Thus every missing integer zero follows for an admissible stage-I_3 family with the prescribed q-deleted zeros. Existence of such a family requires the original fractional zero at each preceding depth and its phase extraction. A local analytic valuation by itself does not prove that fractional zero or give its valuations at other primes. Theorem10.149 supplies the fractional zeros, and Corollary10.150 combines the actual phase construction with the integer closure.

Solution 59. (a) The supports are respectively $\{(1,1)\}$, $\{(0,0),(1,0)\}$ and $\{(0,0),(1,0),(0,1)\}$ in $\mathbb{F}_2^2$. Their span dimensions are one, one and two. Theorem10.147 gives degrees two, two and four. The constant term does not change a span.

(b) Put $W=U_1+U_2$. Then $W^2=221+2U_1U_2$ and $(U_1U_2)^2=10878$, so $(W^2-221)^2=43512$. Since $W=2-V$, expansion gives the polynomial in Theorem10.147. It has degree four, already proved to be the degree of V, so it is its minimal polynomial. Its constant term is the product of the four sign conjugates and is $3577=49\cdot73$.

(c) Both bases are congruent to one modulo73. Lesson9's exponential root law gives their roots congruent to one. Modulo 73^2, their values are $U_1 \equiv 2702$ and $U_2 \equiv 74$: squaring gives 74 and147 respectively, and each representative is congruent to one modulo73. Uniqueness at this precision follows by subtracting squares; the sum of two roots congruent to one is a unit at73. Thus $V \equiv 2 - 2702 - 74 = -2774 = -38 \cdot 73 \pmod{73^2}$, of exact valuation one. In the other three sign embeddings its residues are respectively2,2 and4, hence its valuations are zero. All four completions are $\mathbb{Q}_{73}$, since both signs of both roots lie in that field. Their embeddings of the global value are nevertheless different. The sum of the four valuations is one, as also follows from the norm; replacing it by four times the selected valuation would be incorrect. This example asserts no prescribed interval of auxiliary integer zeros.

Solution 60. (a) Here $\xi U = 24024/25$. The radii are $R_{\mathrm{s}} = \lfloor 7983/50 \rfloor = 159$, $R_{\mathrm{l}} = \lfloor 23999/50 \rfloor = 479$, with node counts319 and959. We have $3R_{\mathrm{s}} = 477 \le 479$. Their normal masses are respectively319 and959; they exceed $\xi U/3 - 2 = 7958/25$ and $\xi U - 2 = 23974/25$. They are also at most $\xi U/3$ and ξU. These data check the floor mechanism, not a complete original logarithmic-contradiction parameter choice.

(b) Full-node interpolation with multiplicity one requires only the prepared zero at the target order itself. Its inverse Taylor polynomial has degree zero, so no additional ordinary jet is consumed. Both integer values lie in K by the retained phase. At a fractional node, the value lies in the support field of Theorem10.56, whose selected completion has the same e,f but whose global degree can be q^r. Theorem10.147 can improve the factor for a particular value, but Theorem10.149 keeps its full valid upper factor. The normal addition G_u is bounded by Γ and is added after that product-formula arithmetic bracket. Multiplying Γ again by q^r would charge a different quantity.

(c) From the definition of m_r, multiply $cL_r H_r^{m_r}/(r+1) \ge 1$ by $a_*^2 (13/2)^2 H_{32}^{d_0} > 1$. Increasing H and the exact two-rank ratio give $cL_{r+2} H_{r+2}^{m_r+d_0}/(r+3) > 1$, hence $m_{r+2} \ge m_r + d_0$. Increasing Q then gives $\mu_{r+2} \le \mu_r q^2 / Q_r^{d_0} < \mu_r$. Thus even ranks are bounded by32 and odd ranks by33; the remaining cost terms decrease on those same sequences. Keeping m_{32} fixed would instead leave a growing numerator q^r, so would not be the proved enclosure. A finite list without the recurrence would leave all later ranks unchecked. The coefficient, denominator and phase hypotheses are still needed to construct each actual successor; Corollary10.150 supplies that induction from144,149,51 and146.

Solution 61. (a) Multiplication of $U_r = q^{r+1} S\mathcal{D}/(et)$ by e, followed by substitution for S, gives the prefactor $c_3 q^{r+2}(r+1) d(\mathsf{h}+x)/t^2$. The formula for $\mathcal{D}$ then gives $F_0 (qH_*)^r r^r (r+1)^{r+1} d^{r+2} \ell_d M_r A_r \mathsf{h}/(r! q^u t^2)$ times the weight product, because $\gamma(\mathsf{h}+x)(A_r+x)/q^\nu = \mathsf{h} A_r$. The profile multiplies this by $(\mathrm{e} H_* (n+1) d)^{n-r} M_n / M_r$ and by the original height product. Thus M_r disappears, the powers of H_* become H_*^n, and the powers of d become d^{n+2}. Dividing by the rank-n expression leaves exactly $(\mathrm{e}/q)^{n-r} (r^r/r!)/(n^n/n!)$ times $((r+1)/(n+1))^{r+1}$. Consecutive ratios of $k^k/k!$ are at least two, and $\mathrm{e} < 2q$; all three factors together are at most one. This algebraic rank-ratio estimate includes rank one, although its actual valuation estimate is established separately.

(b) The rank-one definition is $\delta = (p^f - 1)/g$, so the first entry of its maximum is $p^f g/((p^f - 1)t) > g/t$. The rank-one power identity kills precisely the q-primary torsion; its nonzero exponent numerator divides every original coefficient. The height lower bound makes the stated target greater than $800 d\mathsf{h}/t$, so its summand $2d\mathsf{h}/t$ is less than one four-hundredth of the target. After inserting the profile in its other summand and using $eq^u \le 2d$, the ratio is less than $8/(c\mathfrak{a}(n+1)^3 d\ell_d)$. At the universal minima this is at most $8/37800 < 1/4000$, since $32000 < 37800$. The two terms sum to less than $11/4000$ of the target, hence prove the strict bound with ample room.

(c) The exact III.2 values are $L_2 = 1599907960287/85000000$ and $G_2 = 1467/25$. Put $\Delta_2 = 3/(2L_2)$ and multiply $(1 + 10^{-26})(1 + \Delta_2)^2 (2 + 1/G_2)$ by $2.6 \cdot 1.432 \cdot 3.26 \cdot 18.2757 \cdot 4$. Subtracting from 1790 gives a positive rational greater than $4059/10^6$, as multiplication of its positive

denominators verifies. The exact calculation file preserves that fraction. The ratio of successive F_r is greater than four, so $\Delta_{r+1} < \Delta_r/14$; G_r increases and $\Delta_2 < 1$. Hence the enclosing coefficient decreases for every later rank, as proved in Lemma 10.151. The stronger lower bound for g_1 belongs to the actual III.2 parameter formula; discarding it needlessly enlarges the endpoint in the case with the smallest gap.

Solution 62. (a) On $[j, j+1]$, the chord of the concave logarithm lies below its graph. Sum the chord integrals from 1 to n: their value is $\ln(n!) - \frac{1}{2}\ln n$, whereas the integral of the logarithm is $n\ln n - n + 1$. Exponentiation proves the asserted factorial bound. Also $(n+2)\ln(1+1/n) > 1$ and $(n-1)\ln(1-1/n) > -1$, so the product of their two exponentials exceeds one. In the lower bound for W_B, these facts leave $W_B/(d\exp(a_0 n)) > (7899c/(7900\rho\mathrm{e}))A_n$. For $n \geq 512$, $\exp 6 < 513$ makes $A_n > 10$. The seven exact exponential upper intervals now establish its strict comparison with $\exp a_1$. The 3570 finite checks cover all seven rows and all integer ranks 2–511. The analytic argument, rather than a finite extrapolation, supplies every remaining rank.

(b) Put $\mathcal{T} = C_1^*\Omega H$. In the first branch, multiplication of the definition of W_B by $nd\ln B/t$ bounds $(d/t)nB\max h(a_j)$ by $(7899/7900)\mathcal{T}$, using $\ln B \leq H$. The strict lower target bound makes the remaining $(d/t)\ln 2$ less than $\mathcal{T}/7900$. Their sum is strictly below the target. In the other branch, $B > W_B \ln W_B$ gives $\ln B > \ln W_B + \ln\ln W_B$; Lemma 10.153 therefore bounds $G_1(n,d)$ by $(n+1)\ln B$. Choose the last coefficient to have minimum p-adic valuation, as required by Theorem 10.152. The singleton height bound yields $R_{\max}/(dW(d)) \leq \rho\mathrm{e}d^2 B$. Since $\ln\rho + 1 < 6 < n\ln B - 2\ln d$, this logarithmic entry is also less than $(n+1)\ln B$. Finally $\ln B_\dagger \leq \ln B$ and $(n+1)t \leq (n+1)H$. The complete maximum h, with the new coefficient ordering, is thus at most $(n+1)H$.

(c) The raw residue of four at three is one, so $g = 1$, $\delta = 2$. The rational q-primary roots are $\{1, -1\}$, giving $q^u = 2$. In M_1 the entries are $3/(2\ln 3)$ and $\mathrm{e}\ln 3$; the latter is larger. We have $\Omega = \ln 4$, $H = k\ln 3$, and

$$C_1^* = \frac{4 \cdot 636 \cdot 14\,\mathrm{e}(4 + \ln 2)}{\ln 3}.$$

Thus $C_1^*\Omega H/2100$ is exactly the right side of the worked example. The deep-unit power law gives $v_3(4^{3^k} - 1) = 1 + k$. Its bound exceeds $128k > 1 + k$. In the general rank-one proof the logarithmic term is less than $\mathcal{T}/2688$, and the residue term is less than $\mathcal{T}/10000$. Their exact sum is $793/1680000$, and $793 \cdot 2100 = 1665300 < 1680000$; hence the sum is less than $1/2100$. The raw residue index two in this example belongs to the rank-one theorem; replacing it by the saturated-basis residue index would change the parameter being proved.

Solution 63. (a) The prime-valuation vectors are $(1,0), (0,1), (2,0)$. Thus the selected basis is $(2,3)$, and at five its saturated residue group, including -1, has order four, so $\delta_\beta = 1$. The III.2 value is $\varkappa = 12$. Since $\ln 5 < 2$, $L_3 = 3/(12 \cdot 8) = 1/32$ and $L_2 = 2/(12 \cdot 7) = 1/42$. Every actual height exceeds both floors. Consequently $\Omega_3 = (\ln 2)(\ln 3)(\ln 4)$ and $\Omega' = (\ln 2)(\ln 3)$. The relation $4 \cdot 2^{-2} = 1$ has coefficient at four equal to one. Its transfer gives $b_1' = 1 - 2(-2) = 5$, $b_2' = 1$, and $\Xi = 2^5 \cdot 3 = 96$. Since $95 = 5 \cdot 19$, its valuation is one. Here B=3 and $B/\ln B < 3 < 8 < \mathcal{W}_3$, so the local-height branch applies.

(b) Write j=r-v+1, s=n-r-1. The circuit bound and height-product bound give the exact envelope for K/Ω_n displayed in the proof. At x=n, normalize by $BdH_n(a_x)/8$. The two appearances of $h(a_x)/H_n(a_x)$ are at most one. Insert L_n^{-s}, the second entry of the maximum in C_n, and $B > \mathcal{W}_n \ln B$. The degree identity $r + 3 + s = n + 2$ cancels all powers of d. The remaining rank expression is the terminal normalized envelope. If the largest-height base is selected and x<n, cancel its height and use the bound at length j-1; the envelope gains 1/e. At j=1 that height product is empty, and the root count of Theorem 10.17 supplies its bound. If the largest-height base is unselected, cancel its modified height, use s-1 floors, and gain $1/(\mathfrak{a}\mathrm{e})$. In

all cases the exponential base is less than $(1+5/n)/2$, its power is at most n-2, and the result is strictly below $107648/132237 < 1$.

(c) At n at least sixteen put y=1/n. The lower quadratic bounds for $\ln(1+y)$, $-\ln(1-y)$, $\ln(1+4y)$, and the upper cubic bound for $\ln(1+5y)$, after multiplication by their positive coefficients, sum to $1+(5/2)y-(119/3)y^2+(125/3)y^3$. Since $5/2-119/(3\cdot 16)=1/48$, this exceeds $1+y/48$. The fourteen exact checks at 2–15 give $J_n/(n+5) > 11/4 >$ e, completing every integer n. For the final absorption, $\mathfrak{a}/\varkappa \geq 13/17$, e $> 8/3$, $\varkappa \geq 9$ and n+5 at least seven give $104/51-1/672 > 2$. This is the inequality after subtracting the full additive coefficient-growth term $z/(4\mathrm{e}X_n)$; it is the required parameter comparison, not a test at one rank.

(d) The first discarded base is in the span of earlier retained bases, so removing it changes no later span or greedy rank choice. An initial root has zero rank and is discarded for the same reason. The ordered retained bases therefore remain identical, even as the complementary height floor changes; their actual residue index is unchanged. If a power of $\Xi \neq 1$ becomes one, the torsion part of Theorem 10.21 bounds its valuation by $(d/t)\ln 2$. The same direct bound applies when the full original list has rank zero.

Solution 64. Prime valuations show the list has rank one and first non-torsion base four. Its residue has order two at five, so the raw index is $(5-1)/2=2$. Here $\varkappa=12$, $X_2=2$, $L_2=1/42$, and both heights exceed the floor. Thus $\Omega_2=(\ln 4)(\ln 16)$. The relation $16\cdot 4^{-2}=1$ gives P=1 and $M=2\cdot 5^k$. Since $v_5(16-1)=1$, Lemma 9.5 gives $v_5(16^{5^k}-1)=1+k$.

For U, the powers of d cancel as $d^{n+2}d^{-2}d^{-(n-1)}d^{-1}=1$. The first factor is in C_n, the second is the singleton height bound, the third is the complementary floors, and the fourth comes from $U=t\mathcal{T}_n/d$. The factor ℓ_d cancels between the constant and height bound. The second maximum entry and the height bound leave e^{n-1}, while the root factor becomes w_K/q^u. This gives F_nA_nHR. For the residue-height contribution, the first maximum entry contributes a strict factor g, and $t^2 \leq A_nH$, $eq^u \leq 2d$ give exactly $E_n/(d\ell_dR)$.

At two, $E_2=14\varkappa/(81c\mathfrak{a}^2)$. The seven last-column bounds use the actual degree restrictions; for example I.2 gives $1/40068 < 1/40000$, and II gives $1/63630 < 1/40000$. All factors of the original residue order were retained. At every $n \geq 6$, the other power exceeds one, the logarithmic power exceeds e, and $(n+1)/(n+6) \geq 7/12$, giving $1456/459 > 3$. This proves the claim for infinitely many ranks, rather than extrapolating the four finite checks.

The maximum in s is needed because $D_0 \geq 1$. In I.2 the unmodified coefficient $2\rho\mathrm{e}/(7\varkappa)=17\mathrm{e}/63$ is less than one, so it cannot by itself give the asserted bound on D_0. Taking its maximum with one keeps the logarithmic comparison valid in every case.

For the logarithmic comparisons, the positive exponential series through degree five is $163/60 > 27/10$. To upper-bound a positive exponential at x, sum through degree 32 and bound the tail by its first term divided by $1-x/34$. Lower bounds need only the positive partial sum. The exact calculation checks $\exp(8/5) < 5$, $\exp(7/10) > 2$, $\exp 3 > 18$, $\exp 4 > 45$, $\exp(21/10) > 8$, $\exp(14/5) > 16$, $\exp(5/3) > 319/63$, and $\exp 5 > 58$, with strictly positive rational differences. This supplies all numerical logarithmic comparisons used in the proof.

In the infinite range $n \geq 3$, the coefficient of $\ln d$ in α_nA_nH is larger than $9n/8$. The remaining comparison at H=1 has difference at least $181n/90-5 \geq 31/30 > 0$; its derivative with respect to H is positive. Hence it holds for every H at least one. Finally, $\ln R \leq R-1$ and $\alpha_nA_nH > 2$ absorb the full product of modified heights. With no torsion bases, P is just the product of the exact relation coefficients. A torsion complement adds w_K, but removes at least one non-torsion coefficient factor. Its orders all divide w_K, so the resulting power eliminates every torsion base simultaneously. The torsion case $\Xi^P=1$ is handled by Theorem 10.21 and Lemma 10.155.

Solution 65. Coprimality of 25 and seven gives $g_0=3^N$; the coprime exponents at five and seven are $(2,-1)$. Their difference is 18, so the valuations are two for the coprime parts and N+2 for the

original integers. At three the residue of five is minus one and seven is one; the group including minus one has order two, so its index is one. For m=n=2 the rank factor in C_2 is $2^2 \cdot 3^4/(2! \cdot 2) = 81$, and the remaining field factors are exactly those in the displayed constant. The common factor cannot be omitted: N+2 grows linearly in N, whereas the remaining height parameter grows only like $\ln N$. For example $\ln N/N \to 0$ follows by integrating $1/x \le 1/\sqrt{x}$ for $x \ge 1$.

At two, the residue polynomial $X^2 + X + 1$ is irreducible over $\mathbb{F}_2$, so degree two forces $e = 1, f = 2$. The primitive cube root's residue generates all three units in the residue field. Thus the rank-two saturated index is one, even though both rational odd prime bases reduce to one. The sign base has coefficient one, the others have coefficients two and minus one, and the list length is three. Since $2 \ln 2 < 3$, $L_3 = 3/[34(3 + 5)2] = 3/544$. The sign has height zero, so its modified height is this floor. Multiplication by $\ln 5 \ln 7$ gives the weighted product. The actual difference is 32, of valuation five.

If n=0, the coprime magnitudes are powers of p. If one has positive p-valuation, the other has zero valuation, so their signed difference is a unit. If both have zero valuation, both magnitudes are one; distinctness makes their signs opposite and their difference is ± 2. This proves the stated exact alternatives, including the two-adic torsion case. Finally apply the decreasing function $z \mapsto p^{-z}$ to the strict excess-valuation bound. Its value at $K \max\{\ln B_M, t\}$ is $\min\{B_M^{-K \ln p}, p^{-Kt}\}$, proving the corollary.

Solution 66. In III.1 the strict lower bound for U_2/H is $8 \cdot 495 \cdot 7 \cdot (27/10) \cdot (16/3)/58 = 199584/29$. With $\mu = 23/16$, the logarithm share is $667/3193344$, and the original residue-order height share is less than $1/37422$. Their sum is $2257/9580032$. Exact subtraction gives $1/4000 - 2257/9580032 = 17251/1197504000 > 0$. The factor $(p - 1)/(p - 2) > 1$ increases the actual constant, so this covers all primes in III.1.

For the rational example, $d = e = f = 1$, $t = \ln 3$, $q^u = 2$, $g = 1$, $\delta_\beta = 2$, $c^{(2)} = 648$, $a^{(2)} = 7$, and $h(4) = \ln 4$. Since $\ln 3 < 11/10$, $A_1 = 4 + \ln 2$. Take $B = 3^k$, so $H = k \ln 3$. The maximum selects $\mathrm{e}(\ln 3)^2$, because $\ln 3 > 1$ and $\mathrm{e} > 8/3$. Substitution cancels all three factors of $\ln 3$, giving the displayed coefficient of $k \ln 4$. Lemma 9.5 gives $v_3(4^{3^k} - 1) = k + 1$. The bound exceeds $126k$, and $k + 1 \le 2k < 126k$ for every $k \ge 1$, verifying its strict inequality.

Solution 67. Here $d = e = f = 1$, $t = \ln 5$, $P = 1$, $\varkappa_2 = 25$, and $X_2 = 2$, since $1 < \ln 5 < 2$. Thus $L_2^{(2)} = 2/(25 \ln 5)$. Both $\ln 4$ and $\ln 16$ exceed this floor. The selected base is four, with residue order two and index $(5 - 1)/2 = 2$. The exact relation is $16 = 4^2$, giving $\Xi = 4^{2 \cdot 5^k}$; Lemma 9.5 proves its valuation $1 + k$. The minimizing weighted product is $(\ln 4)(\ln 16)$.

The constant factor defining α_n is independent of n, so $\alpha_n > 3 \cdot 3^{n-2}$. Induction proves $2 \cdot 3^{n-2} \ge n$: equality holds at two, and $3n \ge n + 1$ propagates the assertion. Hence $\alpha_n > 3n/2$ for every $n \ge 2$. Theorem 10.163 bounds $\ln(2|M|)/U$ strictly by $1/4000 - 1/100000 = 3/12500$, and Lemma 10.162 bounds $2egh(\beta)/U$ strictly by $1/100000$. Their sum is strictly below $1/4000$. Multiplication by $dU/t = \mathcal{T}_n^{(2)}$ gives the asserted valuation bound. The original residue order and the second height floor have both remained in the proof.

Solution 68. Both bases are positive 5-adic units and are congruent to one modulo five, so $g = 1$. Their difference from one has valuation m, so $E = m > 1/4$. They are coprime positive integers larger than one: subtracting twice the first from the second gives -1. Any multiplicative relation is trivial by comparing a prime dividing each base. Their coefficients give $u = 0$, with the second coefficient a unit. Rational heights are their logarithms, both larger than $m \ln 5$. Therefore

$$B' m \ln 5 = \frac{m \ln 5}{\ln(1 + 2 \cdot 5^m)} + \frac{m \ln 5}{\ln(1 + 5^m)} < 2.$$

Thus the logarithmic cutoff is less than $\ln 2 + 2/5 < 11/10$, while $4m \ln 5 > 5$. It follows that $H_6 = 6m \ln 5$ and $H_4 = 4m \ln 5$. The difference is -5^m, so its valuation is m. The two coefficient factors are $36.1 \cdot 6^2 = 1299.6$ and $53.8 \cdot 4^2 = 860.8$, respectively. The latter bound is smaller. Both estimates still exceed m, since the product of the two logarithms exceeds $m^2(\ln 5)^2$.

After division of both nonzero coefficients by their common p-part, at least one is a unit. If both bases have precision E, swapping the pair if necessary preserves the precision and makes the second coefficient a unit. If only the first base has that precision, the same swap may assign the analytic parameter E to a logarithm with smaller valuation. The determinant argument then loses its stated Taylor-order gain. The precision condition must be checked for the newly ordered first base; the swap itself does not establish it.

Solution 69. Choose $\eta_1 = -1, \eta_2 = 1$, so $\beta_1 = -3, \beta_2 = 5$ are both one modulo four and have precision two. They are independent by the prime exponents of three and five. The remaining sign is $(-1)^{b_1}$. If b_1 is odd, $3^{b_1} \equiv 3 \pmod 4$ and $5^{b_2} \equiv 1 \pmod 4$, including negative powers because inverses preserve these unit classes. The difference is two modulo four and its valuation is exactly one. In particular $3 - 25 = -22$ has valuation one. For even b_1, the sign is one and the deep-unit estimate applies. The example $3^2 - 5^2 = -16$ has valuation four.

If a logarithmic height is prescribed to be $\ln 2$, assigning $E = 1$ directly would violate the strict exponential-domain condition $E > 1$ in Theorem 10.165. Add $\varepsilon > 0$ to both prescribed heights and use $E_\varepsilon = \min\{2, (\ln A_1 + \varepsilon)/\ln 2, (\ln A_2 + \varepsilon)/\ln 2\} > 1$. Both sign-normalized bases have precision at least two. The uniform coefficient in the proof is independent of ε, and the height product and coefficient cutoff are continuous as ε decreases to zero. Passing to the limit therefore proves the stated closed-boundary estimate.

Solution 70. The difference of the two normalized determinant budgets is $110/35 - 96/35 = 14/35 = 2/5$. The determinant coefficient is $(1617 + 14)/6 = 1631/6$, whose gap below 283 is $67/6 > 0$. For $a = 8, p = 3$, the candidate two is dependent on a, and three is excluded by the unit condition, so take $c = 5$. The rational bases $8/5, 1/5$ are independent by comparing the prime exponents of two and five. The identity

$$(8/5)^2 - (1/5)^2 = 63/25$$

has valuation $v_3(63) = 2$; its denominator is a unit. The original difference $8^2 - 1 = 63$ also has valuation two. The order of eight modulo three is two. If n is odd, $8^n - 1 \equiv 1 \pmod 3$, so its valuation is zero. If n is even, the deep power law gives $v_3(8^n - 1) = v_3(8^2 - 1) + v_3(n/2) = 2 + v_3(n)$.

Solution 71. Every first coordinate $r = 0, \dots, 7$ appears four times. Pairing the increasing row labels $0, \dots, 7$, also each repeated four times, with the increasing coordinates maximizes the weighted sum. Indeed swapping an inverted pair changes it by the nonnegative product of the two coordinate differences. Thus the exact centered maximum is

$$4 \sum_{r=0}^{7} (r - 7/2) r = 4(140 - 98) = 168.$$

The congruence gives $g_0 = 1, g_1 = 2$, so $R' = 8, S' = 4$. Lemma 6.4 yields

$$\frac{32 \cdot 8 \cdot 8}{8} - \frac{32^2 \cdot 8}{24 \cdot 4} = \frac{512}{3}.$$

The relaxed residue-class bound is

$$G_1 = \frac{32 \cdot 8 \cdot 9}{8} - \frac{2 \cdot 32^2 \cdot 8}{24 \cdot 9} = \frac{5728}{27}.$$

It is less than the no-capacity bound $32 \cdot 8 \cdot 7/8 = 224$, though relaxing the compressed parameters loses some of the stronger bound $512/3$.

For the bases $2, 3$, the global field and the local field are $\mathbb{Q}$ and $\mathbb{Q}_5$, so $d = e = f = D = 1$. The residue of two has order four and already generates the whole residue group, so $g = 4$. Independence follows by comparing exponents at the primes two and three. The general proof keeps e throughout the analytic and one-place height estimates and uses only $ef \le d$. It handles whichever coefficient becomes a unit after removing their common p-part by interchanging the bases inside the analytic

argument; both principal factors have the same universally guaranteed precision p^t/e. Hence neither unramifiedness nor an asymmetric coefficient condition is imposed by Theorem 10.170.

Solution 72. (a) The first maximum becomes the second on multiplication by t^{n+1}. The factorial quotient is n, so the constant quotient is exactly $(c^{(2)}/c)(a^{(2)}\mathrm{e}/\mathfrak{a})^n(Pt/n)^{n-1}$. With $Pt/n \geq 1$, the least power outside III is the square of e/2; use $\mathrm{e} > 27/10$. In III write $x = (p-1)/(p-2)$, with $1 < x \leq 4/3$. The expression $(c_*/c)\mathrm{e}^n x^{1-n}$ increases with n for each x, and its n=2 value decreases with x. Using x=4/3 and $\mathrm{e} > 27/10$ gives the two III fractions. Subtracting 3/2 from the seven fractions gives, in order,

$$\frac{80817}{62600}, \quad \frac{3783}{10600}, \quad \frac{20001}{20200}, \quad \frac{411}{47840}, \quad \frac{144159}{716000}, \quad \frac{77361}{536000}, \quad \frac{86187}{41200}. \tag{10.657}$$

Cross-multiplication shows that the smallest is 411/47840. The coefficient parameters are different; comparing C_1, C_2 alone would not compare the full bounds. Lemma 10.172 supplies $\mathsf{h} \leq 3\mathsf{h}_2/2$, which is absorbed by the strict constant quotient. At three with e=1, the inequalities are $2 \leq 2 < 6$, giving $\kappa = 1, P = 3$. At an unramified prime $p \geq 5$, they are $(p-1)/p \leq 2 < p-1$, giving $\kappa = 0, P = 1$.

(b) The coefficients have valuations k and zero, so the last coefficient is correctly ordered. The prime valuations of two and five prove independence and that a rational element of their 2-primary saturation has integral prime exponents. Including minus one, the residues generate $\mathbb{F}_3^*$; thus $\delta(2,5) = 1, q^u = 2, d = e = f = 1$. The heights are $\ln 2, \ln 5$. Substitution of W(1) shows

$$\frac{1/h_{\min} + 1/h(5)}{W(1)} = \frac{(\ln 3)(\ln 10)}{(\ln 5)(\ln 6)} < 1, \tag{10.658}$$

because cancelling the common $(\ln 3)(\ln 5)$ term reduces the comparison to $\ln 3 < \ln 5$. Hence X=1. For k at least one, $3^k \equiv 3 \pmod 6$, and the powers of two modulo nine have period six. Therefore $5 \cdot 2^{3^k} - 1 \equiv 3 \pmod 9$, of exact valuation one. Since n=2 is less than $3 \ln 3$, Theorem 10.173 applies with the I.2 row, $c^{(2)} = 648, a^{(2)} = 7$, the original residue index one and the same field. Theorem 10.175 applies as well, with $B_n = 1, B = 3^k$.

(c) For s positive, the derivative of $u \exp(-su)$ is $\exp(-su)(1-su)$. Its maximum for u nonnegative is $1/(\mathrm{e}s)$, at u=1/s. Taking u=ln y gives the claimed bound. The partial exponential sum through degree six is 1957/720, whose difference from 125/46 is $11/16560 > 0$; therefore $8\eta\mathrm{e} > 1$. The scalar equality in the constant branch can occur when $A = \exp(H_0/(1+\eta))$, so a universal strict scalar statement is unnecessary. The strict inequality supplied by Theorem 10.152 persists after composing it with this weak scalar estimate.

In the plotted model $\Phi = 523/500$ and $Z = \max\{8\Phi, \exp(1000/523)\}$. The branches are $\Phi \ln(Z \exp 20/z)$ and z, and the scalar bound is 20. The parameter condition is exactly $0 < z \leq r = \exp 20$, since $0 < \varepsilon \leq 1$. Only a bounded subinterval of that positive interval is drawn; the proof, rather than the plot, supplies every allowable value.

Solution 73. (a) Modulo 27 the monomial is $1 + 9(\lambda_1 + 2\lambda_2)$. For λ_1 congruent to zero, one or two modulo three, the allowed λ_2 in the zero fibre are respectively congruent to zero, one or two. Thus that fibre consists of the four pairs in each of $\{0,3\}^2, \{1,4\}^2, \{2,5\}^2$. More generally, the fibre with image b is exactly $S_b = \{(x,y) : 0 \leq x, y \leq 5,\ y \equiv x - b \pmod 3\}$ for b=0,1,2. The other two fibres are obtained by assigning the indicated residue to $\lambda_1 + 2\lambda_2$; for each fixed first coordinate there are exactly two possible second coordinates in each fibre. Every fibre therefore has twelve points. Its even-even zero part is $\{(0,0),(2,2),(4,4)\}$, which contracts to $\{(0,0),(1,1),(2,2)\}$. The new monomials are 190^a, with $a = 0, 1, 2$, and are still one modulo 27.

The finite quotient has three possible monomial images, independent of E. Selecting one image therefore retains E in full. Fixing an auxiliary index as part of the equivalence relation would

multiply the possible number of classes by $|E|$ and would prevent a selected class from containing $E \times \{\lambda^0\}$ when $|E| > 1$. That is a different selection and has a different counting cost.

(b) For $x_0 = k + v > 1$, comparison of $\chi(k+1)/\chi(k) = \exp(-\rho)(A+k+1)/(A+k)$ with one gives $v \geq 1/\rho - 1/(\exp\rho - 1)$ for choosing k+1; at equality either choice has the same value. The continuous maximum is $\rho^{-1}\exp(-1+\rho A)$, so $\chi(m) > \chi(x_0)/\varsigma_2$ gives

$$\frac{p^{fm}}{(A+m)^r} < \varsigma_2^r \rho^r \exp(r - \rho A r) = \varsigma_2^r (\mathrm{e}t/r)^r p^{-fA}. \tag{10.678}$$

The two requested positive gaps are exactly

$$\frac{13701}{10000} - \frac{1370098893207}{10^{12}} = \frac{1106793}{10^{12}}, \qquad \frac{47314037827}{50000000000} - \frac{1250}{1321} = \frac{1843969467}{66050000000000}. \tag{10.679}$$

Finite endpoint calculations certify these numbers only. The derivative comparisons for H, z and F, the decreasing forced-ratio function, and the increase/decrease of the two rounding alternatives prove that these endpoints cover every allowed A and ρ.

(c) We have $\Delta(z/2+1;2) = (z^2+10z+24)/8$ and $V(2) = 2$. After squaring, differentiating and multiplying by two,

$$P(z) = \frac{(z^2+10z+24)(z+5)}{8} = \frac{z^3+15z^2+74z+120}{8}. \tag{10.680}$$

The coefficient valuations, in increasing powers of z, are $(1,0,1,0)$, so all coefficients are integral. The uniform choice uses D=4 and $C = D/(p-1) = 2$. At N=2 and $\beta = 0$ it gives a normalizing scalar of valuation $4(2+1/2) = 10$. Using the actual degree three and coefficient bound C=0 instead, Theorem 10.178 gives the sharper termwise sufficient valuation six. For every exponent in the zero fibre, the logarithmic frequency has valuation at least three, so both applications have $2 < 3 - 1/(3-1)$ as required. Since every exponential equals one at zero and the fibre has twelve terms, $f(0) = 12P(0) = 180$, with valuation two. The normalization concerns a single p-adic norm. A global height or product-formula estimate continues to use the actual original coefficient field and its degree; the choice of the local scalar does not replace either of them.

Solution 74. (a) The three differences are $9, 2781/500 = 27 \cdot 103/500$, and $\Xi - 1 = 3^8$, because $10^3(3281/500) = 6562 = 1 + 3^8$. Their valuations are therefore two, three and eight. Modulo 27 the monomial is $1 + 9\lambda_1$; the three classes are distinguished by $\lambda_1 \pmod 3$, with twelve points each. The zero class is $S_0 = \{0,3\} \times \{0,\dots,5\}$. Take the reference $(0,0)$. With $z_i = \log a_i$, the original and projected frequencies are

$$\gamma_\lambda = \lambda_1 z_1 + \lambda_2 z_2, \qquad \epsilon_\lambda = -\lambda_2 \log \Xi, \qquad \gamma_\lambda + \epsilon_\lambda = (\lambda_1 - 3\lambda_2) z_1. \tag{10.697}$$

Every quotient frequency has valuation at least three. This lower bound is attained at $(3,0)$, since $v_3(3\log 10) = 3$. The perturbations have valuation at least eight. Here $j = 2, m = 1, e = 1$, so $2 < 3 - 1/2$ and $8 > 2 + 1/2$ are both strict. These are the combined-frequency conditions. An assumption that each individual slope had valuation greater than $5/2$ would fail for z_1; Theorems 10.180 and 10.182 do not make that assumption. The circled pair in the figure has difference $(3,1)$: its original quotient is Ξ, and its projected frequency is zero.

(b) There are four polynomial indices at each of twelve exponents, hence 48 columns. The six indices $(t_0,t_1) = (0,0),(0,1),(0,2),(1,0),(1,1),(2,0)$ at each of three nodes give 18 rows. Every row entry is the rational number

$$\frac{2^{t_0}}{t_0!}\left[\frac{d^{t_0}}{dy^{t_0}}\left(\frac{(y+a+1)(y+a+2)}{2}\right)^{\ell}\right]_{y=s}\Delta(\lambda_1-3\lambda_2;t_1)\left(10^{\lambda_1}(3281/500)^{\lambda_2}\right)^{s}. \tag{10.698}$$

Here $a=0,1$, $\ell=1,2$, $s=-1,0,1$, and $\lambda\in S_0$. Thus the row field is $\mathbb{Q}$, and its discriminant cost is zero. Elimination of an 18-by-48 rational matrix gives a nonzero rational kernel because there can be at most 18 pivot columns. Clearing denominators and then dividing the common integer gcd gives a primitive integral vector. Its coefficient minimum at three is zero. The exact program builds this full matrix, produces one such vector, and substitutes it into all eighteen original rational equations; every sum is exactly zero. The row factor in (10.174) is $18/(48-18)=3/5$. This uses the actual finite counts. The volume-only bound in Proposition 10.183 is smaller: its column lower bound is $4\cdot 25/3=100/3$, which still exceeds $(11/6)18=33$.

(c) Distinct nodes have nonzero differences one or two, so B is zero. Also $V(2)=2$ and $b_n=1$, so $C_0=M_0=0$. The uniform degree bound is $D=4$, and the primitive coefficient vector has $\beta=0$. Therefore

$$\Gamma=4(2+1/2)=10,\qquad H+4(2+1/2)=8+10=18=3\cdot 3\cdot 2,\qquad v_3(f(x;0)),v_3(\Phi(x;0))\geq 8. \tag{10.699}$$

The equality of the two sides of the sufficient interpolation budget is permitted. The normality hypotheses remain the two strict inequalities verified in (a). No global height comparison or final logarithmic bound is inferred from this local example.

For the last assertion, let l be the exponential Taylor order. At order zero its error has valuation at least H. At positive order l, the binomial expansion in Lemma 10.181 has at least one perturbation in every term outside the first difference. It gives $H+(l-1)L-v_p(l!)\geq H+(l-1)N$, because $L>N+1/(p-1)$ and $v_p(l!)\leq (l-1)/(p-1)$. Multiplication by a polynomial of degree D leaves exponential order at least $t-D$ in an ordinary divided derivative of order t. Thus its lower bound is $\beta-C+H+\max\{0,(t-D-1)N\}$, as asserted.

Solution 75. Here $n=r=4, d=e=f=1, P=3, q=2, u=1$, $\vartheta=3/2$, and $J=\delta_r=1$. Unique factorization proves independence of the four bases and shows that their exponent lattice is already saturated: a rational power lying in the subgroup has integer valuations at the four indicated primes. Their heights are respectively $\log 2,\log 5,\log 7,\log 11$.

The real optimizer of Lemma 10.177 is $x_0=4/\log 3-3/2$, between two and three. To compare the two adjacent integers, the quotient of the objective at three by its value at two is $(9/7)^4/3<1$. Thus the maximizing depth is $m=2$, and

$$N=\frac{7/2}{1+10^{-26}/8},\qquad A_r=4+\log 5,\qquad \mathcal{R}_r=3. \tag{10.729}$$

The last identity follows by bounding $(\mathrm{e}/4)^4(\log 3)^5<3$. All these comparisons can be made with rational intervals, as described below. The deep units are $-8,-125,343,-1331$. They belong to $1+9\mathbb{Z}_3$ and, modulo 81, have the form $1+9c_i$ with

$$(c_1,c_2,c_3,c_4)\equiv(8,4,2,5)\pmod 9. \tag{10.730}$$

Consequently the actual unit-quotient image has order nine: the first coordinate already generates $\mathbb{Z}/9\mathbb{Z}$. The phase map is trivial. Substitution in Theorem 10.184 gives the following integer quantities, with every floor fixed by the rational intervals in the editable calculation:

Quantity	Exact value
$\lfloor D_1 \rfloor + 1$	9530854280
$\lfloor D_2 \rfloor + 1$	4104715517
$\lfloor D_3 \rfloor + 1$	3394958794
$\lfloor D_4 \rfloor + 1$	2755034737
k	150
L	8105949489
$D_{\mathrm{a}} = kL$	1215892423350
$\lfloor S \rfloor$	1638
$\lfloor T \rfloor$	353654158504

Here is an exact way to count the selected columns without traversing the box. A side containing the integers $0, \dots, a_i - 1$ contains $\lfloor a_i/9 \rfloor + 1$ integers in residue j when $j < a_i \bmod 9$, and $\lfloor a_i/9 \rfloor$ otherwise. Start with the residue count vector $(1, 0, \dots, 0)$ and convolve successively with these four nine-entry vectors, placing residue j of side i at $c_i j \bmod 9$. This is simply partitioning all sums according to their residue, so it counts each box point once. The classes $1, 3, 6, 8$ have the largest size,

$$|S_0| = 406568726782276943427706702458002323 65. \tag{10.731}$$

All other classes have one fewer point. Select class one and reference $(8, 0, 0, 0)$, since $8 \cdot 8 \equiv 1 \pmod 9$. Every one of the kL polynomial pairs is retained. The exact column count and the full row count are therefore

$$\begin{aligned} N_c &= 4943438344656267605751451579846978469027295172275 0, \\ M &= 2135896855592220094737330490491655575834866435515. \end{aligned} \tag{10.732}$$

Direct integer multiplication gives $10N_c > 27M$, the claimed $c_0 = 27/10$ surplus. Theorem 10.184 then supplies a nonzero global-integral kernel for every one of these original equations; this count does not discard any row to improve a height average.

For the first full-node input take $\mu = \lfloor T/10 \rfloor + 1 = 35365415851$, and all $2\lfloor S \rfloor + 1 = 3277$ integer nodes. Their separation bound is $B = \lfloor \log_3(3276) \rfloor = 7$, since $3^7 \le 3276 < 3^8$. The polynomial jet loss is $C_0 = v_3 V(k) = \lfloor \log_3 150 \rfloor = 4$, by the proved least-common-multiple formula for $V(k)$. For a 3-adic unit distinguished coefficient this gives $M_0 = \max\{B, C_0\} = 7$. With $U = 32S\mathcal{D}/t$, the sufficient inequality in Theorem 10.182 is

$$1 + U + D_{\mathrm{a}}(N + 1/2) \ge 3277\mu N + 7\mu. \tag{10.733}$$

Under the counterhypothesis $v_3(\Xi - 1) \ge U$, the strict normality hypothesis is also valid. The exact interval calculation bounds left minus right strictly above 10^{15}. It also gives $1 + U > N + 1/2$, as required for the strict normality inequality. Here the node set contains all its nodes, so $\eta_E = 0$. This verifies this first local input budget, including the degree, node and jet costs. It does not check every later stage of a global zero induction or the preservation of the weight allowance in Lemma 10.187. In particular it does not by itself prove the full second valuation estimate.

For completeness, the rational enclosures used in this solution have an elementary derivation. For $1 \le x \le 2$ put $z = (x - 1)/(x + 1)$, so $0 \le z \le 1/3$. Integrating the geometric series for $2/(1 - z^2)$ gives

$$0 \le \log x - 2\sum_{j=0}^{63} \frac{z^{2j+1}}{2j+1} \le \frac{2z^{129}}{129(1-z^2)}. \tag{10.734}$$

Indeed each remaining denominator is at least 129 and the remaining powers form a geometric series. Reduce an integer a to $a = 2^k x$, with $1 \le x < 2$, and add k copies of the enclosure for $\log 2$. Similarly the exponential series and its consecutive-term ratios give

$$0 < \mathrm{e} - \sum_{j=0}^{64} \frac{1}{j!} < \frac{1}{65!}\frac{1}{1-1/66}. \tag{10.735}$$

Every arithmetic operation is rounded outwards to multiples of 10^{-36}; multiplication takes the four endpoint products and division inverts a positive interval before multiplying. Thus interval inclusion is preserved by induction through every operation. The editable calculation checks that the two endpoints of each interval have the same floor, performs the nine-residue convolution with integers, and verifies the exact inequalities stated above. No sampled floating-point values are used for these certificates. □

Solution 76. (a) Over $\mathbb{Q}$ at eleven, $n = 3, P = e = f = 1$, $t = \log 11$, and $\vartheta = 9/10$. The three distinct prime valuations prove independence of the bases. The exponential bounds already proved give

$$\exp 2 < 9 < 11 < \left(\frac{27}{10}\right)^3 < \exp 3. \tag{10.762}$$

Thus $2 < t < 3$, so $n > Pt$. But $3/t < 3/2 = \vartheta + 3/5$, as required by Theorem 10.188. Equivalently the real optimizer $3/t - 9/10$ is less than $3/5$. The theorem proves the second valuation estimate for every original integer coefficient vector with its stipulated minimum-valuation ordering, not merely for a sample choice.

(b) In III.2 at five, $\vartheta = 3/4$, and the height parameters use $a_0 = 2 + \log 7$, $a_1 = a_2 = 146/25$, and $a_1^{(2)} = 83/25$. The elementary bounds of Theorem 10.189 give $G_1/3 < 827/50$, $G_2/3 > 303/25$, and therefore

$$\frac{11}{8}G_2 - G_1 > \frac{3}{8} > 0, \qquad \frac{C_2}{C_1} - \frac{11}{8} > \frac{899}{1754200} > 0. \tag{10.763}$$

The identical other entries of the two maxima preserve the $11/8$ comparison for h, h_2. This proves the full second estimate over $\mathbb{Q}$ at five. The III.1 constants are different: $c = 1794, c_* = 495, a_1 = 4.71$, and $a_1^{(2)} = 1.99$. Substituting the III.2 constants in that row would change the theorem being proved. No higher-degree conclusion follows from the rational-field comparison alone.

(c) For $n = r = 4, d = 1, p = 3$, the relevant row is I.2. Since $\log 21 < 4$ and $\log 5 < 2$,

$$G_2(4,1) = 5\big(4(2 + \log 21) + 2.76 + \log 5\big) < 145 < 150. \tag{10.764}$$

Also $5 \log 3 < 150$, since $\log 3 < 2$, and $\log B_\dagger = 0$. Thus the stated h meets every additional hypothesis. Retain the exact integers from Solution 75: $s_0 = 1638$, $\mu = 35365415851$, $B = 7, C_0 = 4, M_0 = 7$, and $N = (7/2)/(1 + 10^{-26}/8)$. The cost fraction is exactly

$$\frac{35365415851\,(3277N + 7)}{32S\mathcal{D}/\log 3}. \tag{10.765}$$

The rational logarithm intervals proved in Solution 75, with outward rounding through this expression, put it strictly between 8/100 and 9/100. The editable calculation records its rational endpoints. The all-rank theorem also bounds it by $437/2016 < 1/4$. The original kernel supplies the initial prepared zeros, so Theorem 10.190 yields the stated local valuation precision at every $x \in \mathbb{Z}_3$ under its counterhypothesis. To force an algebraic value to vanish one must still compare this precision strictly with its global height and the actual degree of its value field. A large local lower bound alone is not that comparison. □

Solution 77. (a) Write $D_* = S\mathcal{D}/(c_1c_4(N/\vartheta)dA)$. Corollary 10.185 gives $D_* < kL < \lambda D_*$. The powers of the prepared polynomial have degree at most kL; replacing this integer by $\mathcal{D}$ would not be justified by its definition. The strict floor inequalities give

$$\frac{(\bar{S}/3)(\bar{T}/(r+1))}{\lambda D_*} = \frac{q^2\eta c_4 dA}{3\lambda e\vartheta t} = B_0. \tag{10.802}$$

For an m-element subset J', divide $(2\bar{S}/(3r))(\bar{T}/(r+1))^{m+1}/(m+1)!$ by $(m+1)!\lambda D_* \prod_{j\in J'}(2Jq^{-I}D_j)$. The result is

$$\frac{2B_0}{r}\frac{\prod_{j\in J'}\sigma_j}{((m+1)!)^2}\left(\frac{\eta c_2 qrPd}{2eNt}\frac{E}{J}\right)^m. \tag{10.803}$$

One factorial comes from the binomial product; the other comes from the coordinate-degree side of the multiplicity inequality. Neither can be dropped during the cancellation.

(b) The first comparison is $1048576 > 232$. For the second, $1050\cdot 10^6 = 1050000000 < 68719476736 = 2^{36}$. The consecutive-ratio proof in Theorem 10.191 applies to all intermediate ranks, and the factor $35/2^{20} < 1$ in Corollary 10.192 applies to all output ranks. Thus these endpoint computations are supported by uniform monotonicity proofs. The degree inequalities hold from the parameters alone. The production of lower-rank bases with the asserted height product also requires the nonzero polynomial and prepared zeros stipulated in Corollary 10.192.

(c) If $t \le n$, use $\mathcal{R}_n \ge 1$ and $A_n \ge t$, giving $\mathcal{R}_nA_n/t^2 \ge 1/t \ge 1/n$. If $t > n$, use the second entry of the maximum:

$$\frac{\mathcal{R}_nA_n}{t^2} \ge \left(\frac{\mathrm{e}t}{n}\right)^n > 1 > 1/n. \tag{10.804}$$

The height-product lower bound (10.55) and the exact formula for C_2 then give $\mathcal{T} = C_2h\Omega_n \ge 9261dh/t$, since $406\cdot 49\cdot 27/58 = 9261$. The logarithmic contribution $2dh/t$ is less than $\mathcal{T}/4000$, as $8000 < 9261$. For the height contribution use the actual residue order $g < \mathcal{R}_1$, the second rank allowance and $eq^u \le 2d$. Its ratio to $\mathcal{T}$ is less than

$$\frac{4}{406\cdot 7\cdot 2\cdot 27\cdot 3} = \frac{4}{460404} < \frac{1}{100000}. \tag{10.805}$$

The sum is therefore less than $1/4000 + 1/100000 = 13/50000 < 1/3000$; the final comparison is $39000 < 50000$. All coefficient-height inputs concern the original integer coefficients, as required in Proposition 10.193. □

Solution 78. (a) Put $t_0 = \lfloor T \rfloor$. The additive coordinate, the $r-1$ Euler coordinates, and the slack are $r+1$ symmetric nonnegative coordinates summing to t_0. Their means are each $t_0/(r+1)$. This remains true for the full equation list because its zero rows have been retained. The mean lcm contribution is at most $(107/103)Ht_0/(r+1)$; the mean Euler contribution is at most $(1+3/[8(r+1)])H(r-1)t_0/(r+1) + E_0$. Adding, and using $t_0 \le T$, gives

$$C_r=\frac{r+4/103}{r+1}+\frac{3(r-1)}{8(r+1)^2},\qquad 1-C_r=\frac{483r+1101}{824(r+1)^2}. \tag{10.833}$$

Subtracting $483/[824(r+1)]$ from the deficit leaves $618/[824(r+1)^2]>0$. Deleting zero rows before this average would destroy the permutation argument.

(b) With $z=\exp(-a_E)$, positivity of the binomial series bounds the Euler logarithm by a_Ev+E_0, independently of the additive order. The additive lcm logarithm is bounded by $(107H/103)u$. A single larger coefficient a suffices for individual values; the separate coefficients give the smaller mean in part (a). Replacing the Euler remainder by the smaller remainder at $\exp(-a)$ while keeping coefficient a_E would not be justified.

(c) Here $q=2, c_1=7/10, c_3=6/5, c_4=14$, $c_2=8750/5101$, $c_5=27/50$, and $\gamma=12/5$. Use the exact e_0, $\varepsilon=13/18000$, and $\mathcal{N}=\lambda(3/2+101/200)/(c_1c_4)$ from the theorem. Rational substitution gives

$$\mathcal{L}=\frac{3411}{1120},\qquad \mathcal{A}=\frac{47193470266316287}{15709515479280000},\qquad \mathcal{L}-\mathcal{A}=\frac{650420179955213}{15709515479280000}. \tag{10.834}$$

To compare the last fraction with $41/1000$, multiply its positive denominator by 41 and its numerator by 1000. Their difference is $6330045304733000>0$. The other three endpoint differences, and the concavity reduction, are necessary for the uniform case-I.2 conclusion; a single rank-three computation alone would not prove it.

(d) For real x, z, writing each as its integer part plus a number in $[0,1)$ gives $\lfloor x\rfloor+\lfloor z\rfloor\le\lfloor x+z\rfloor$. Apply this with $x=\eta T$, $z=(1-\eta)T$. Since $\mu-1=\lfloor(1-\eta)T\rfloor$, every target of order at most $\lfloor\eta T\rfloor$ has all ordinary jets through $\mu-1$ among the original prepared equations. At a fractional node, the scalar q-denominator in (10.243) may be nonzero, and the consistent torus roots may generate a larger global value field. Both costs belong in (10.244). Neither has been bounded by this depth-zero integer calculation.

Solution 79. (a) For $u=y/b$, expand the proposed Bernstein sum and interchange its two finite sums. The contribution of a_kb^k is

$$\sum_{i=k}^{n}\frac{\binom{i}{k}}{\binom{n}{k}}\binom{n}{i}u^i(1-u)^{n-i}=u^k\sum_{i=k}^{n}\binom{n-k}{i-k}u^{i-k}(1-u)^{n-i}=u^k. \tag{10.850}$$

This proves the identity. The weights are nonnegative on the unit interval and their sum is one. For the proposed quadratic, b = 1/4 and n = 2 give coefficients $1, 3/4, 11/16$. The polynomial is therefore at least $11/16$, attained at y = 1/4.

(b) Here $\eta=1-(27/50)/7=323/350$. The final real radius is $64S$ and the derivative cutoff is $(323/350)^6T$. The actual nodes are the integers of absolute value at most $\lfloor 64S\rfloor$, and the prepared indices have total order at most $\lfloor(323/350)^6T\rfloor$. At each prior step the difference of the real cutoffs is $c_5\eta^jT/(r+1)$. Taking its floor and adding one gives μ_j, and the floor-sum inequality supplies every required jet through μ_j-1.

(c) Direct subtraction gives

$$\frac{15}{16}-\frac{20907}{22400}=\frac{93}{22400}>0,\qquad \frac{15}{16}-\frac{2323}{2520}=\frac{79}{5040}>0. \tag{10.851}$$

There are seven field cases, four step numbers, and two endpoints of the fractional-linear function: $7\cdot4\cdot2=56$ polynomials. Their degrees are at most 3,4,5,6, hence their Bernstein lists have 4,5,6,7 entries. The total is $7\cdot2(4+5+6+7)=308$. These are coefficient checks; the convex-combination identity, rather than a list of sampled ranks, proves positivity throughout each required interval.

(d) The tail has the form $F_j = \mathfrak{a} q^{j+1} - \mathfrak{b} - \mathfrak{c}(j+1)$ with $\mathfrak{a} > 0$ and q greater than one. Thus

$$F_{j+1} - F_j = \mathfrak{a}(q-1)q^{j+1} - \mathfrak{c} \geq \mathfrak{a}(q-1)q^6 - \mathfrak{c} > 0 \quad (j \geq 5). \tag{10.852}$$

The lower bound for F_5 controls the entire tail. In the local-input proof the step number satisfies $j \leq r-1$, which supplied $q^{j-r} \leq 1/q$. A further step has not been included in that estimate. Moreover fractional prepared scalars may have a q-denominator and their torus roots may enlarge the global field of the value. The arithmetic comparison here used neither cost because every target was an integer at depth zero. They must be included before a fractional extension can be concluded.

Solution 80. (a) Here $p = 3, A_0 \geq 3/2$. Since $27 > 25$, one has $3\sqrt{3} > 5$. Lemma 10.199 therefore gives

$$\frac{N_c}{M} > \frac{27}{10}3\sqrt{3} > \frac{27}{2}, \qquad \frac{M}{N_c - M} < \frac{1}{27/2 - 1} = \frac{2}{25}. \tag{10.869}$$

(b) The upper prefactor multiplied by the proposed reserve is

$$631\frac{51}{50} = \frac{32181}{50}, \qquad 648 - \frac{32181}{50} = \frac{219}{50} > 0. \tag{10.870}$$

Thus the multiplier still fits strictly below the stated second constant, in every rank covered by the original prefactor bound.

(c) Here $\eta = 173/200$. Exact floors give

$$\begin{aligned} (R_0, R_1, R_2, R_3) &= (10, 21, 42, 84), \\ (\lfloor T \rfloor, \lfloor \eta T \rfloor, \lfloor \eta^2 T \rfloor, \lfloor \eta^3 T \rfloor) &= (1200, 1038, 897, 776), \\ \lfloor \eta^4 T \rfloor &= 671, \\ (\mu_0, \mu_1, \mu_2, \mu_3) &= (530, 368, 227, 106), \\ N_* &= 21(530) + 22(368) + 42(227) + 84(106) = 37664. \end{aligned} \tag{10.871}$$

Using only the final interval would give $169(106) = 17914$, omitting 19750 valid jet conditions. The inequalities $\lfloor \eta^4 T \rfloor + \mu_j - 1 = \lfloor \eta^j T \rfloor$ check every required order exactly.

(d) The root exists by successive lifting: if $a_m^2 \equiv 2 \pmod{7^m}$ and $a_m \equiv 3 \pmod 7$, there is a unique digit $t \in \{0, \dots, 6\}$ such that $a_{m+1} = a_m + t7^m$ solves the congruence modulo 7^{m+1}, because $2a_m \equiv 6 \pmod 7$ is invertible. The compatible sequence converges in $\mathbb{Q}_7$ to the indicated root. Now $(\alpha - 3)(\alpha + 3) = -7$ and $\alpha + 3 \equiv 6 \pmod 7$. The second factor is a unit, so the first has valuation one. Also $-\alpha - 3 \equiv 1 \pmod 7$ is a unit. The global field $\mathbb{Q}(\sqrt{2})$ has degree two, while the chosen completion has degree one; its two embeddings give different valuations for the same algebraic expression. Chosen-completion membership alone therefore supplies no equality of conjugate valuations that would cancel the global factor.

(e) The function $(1 - c/v)^v$ increases, and its limit is bounded using the explicit positive exponential series. These facts bound both $\mathfrak{h}$ and its inverse in the required directions. The radius-floor loss decreases with rank after division by $q^r S_-$, while S_-, T_-, b_r, q^r increase and ε_r, e_r decrease. The proof keeps the last four terms of a positive reversed geometric sum. Finally the fractional-linear dependence on A_0 requires only its two endpoints. These are uniform inequalities, so the tail rows prove all larger ranks rather than sampling them.

Solution 81. The fraction is $\mathfrak{h} = (173/200)^4 = 895745041/1600000000$. Consequently

$$\begin{aligned} S_1 &= \frac{16800000000}{895745041}, & T_1 &= \frac{2687235123}{4000000}, \\ R &= 2(18+1) = 38, & \lfloor T_1 \rfloor &= 671, \\ M &= \left\lfloor \frac{27T_1}{200} \right\rfloor + 1 = 91, & \lfloor \eta T_1 \rfloor &= 581. \end{aligned} \tag{10.883}$$

There are 19 positive odd nodes and 19 negative odd nodes in the interval, so $n_E = 38$ and $n_E M = 3458$. The largest required total jet order is $581 + 91 - 1 = 671$, exactly the available input order. Among the 77 integers in the interval, the other 39 are even, including zero. Theorem 10.202 fills these at order 581 while retaining the odd-node zeros through order 671. These counts illustrate the interpolation formula; the small S and T are not asserted to satisfy the full parameter construction.

For the field assertion, Theorem 10.51 first separates the actual root-monomial classes, including its second phase partition when needed. After restriction and division of the exponents, the remaining phases are powers of the root of unity in K, and the torus bases belong to K. Theorem 10.55 therefore puts the integer prepared value in K. The additive polynomial is nevertheless evaluated at s/q, so its rational denominator must still be cleared. The bound $\Xi_{1,0,u} \le D_{\mathrm{a}}(1 + 1/(q-1))$ is included in the arithmetic comparison. Membership of a rational number in K imposes no integrality condition on that number.

Solution 82. Dividing the first inequality by hA gives $1 + (1/h + 1/A)x + x^2/(hA)$. Its linear coefficient is smaller than $9/40 < 1$, and its quadratic coefficient is smaller than $1/200 < 1/2$. The positive exponential series gives the required inequality, strictly for $x > 0$ and with equality at zero.

For (b), $Q \ge 2(173/200)^4 = 895745041/800000000$. Consequently $\log Q > 191490082/1695745041$. An exact upper bound for the linear endpoint ratio is $(3 \cdot 59 \cdot 101 \cdot 8750 \cdot 7 \cdot 1695745041)/(58 \cdot 100 \cdot 5101 \cdot 10 \cdot 14 \cdot 4 \cdot 191490082)$. Multiplication by the positive denominator verifies that this is less than $3/5$. Adding the bound $1/1000$ for the exponential endpoint term still gives less than one. Convexity then covers every intermediate depth; the separate $1/(q-1)$ denominator term is retained in Lemma 10.205.

For (c), degree and allocation inequalities control the possible multiplicity output. They do not assert that the auxiliary polynomial vanishes at any particular point. The rank conclusion explicitly uses the full prepared integer zeros through $\bar{S}$ and order $\bar{T}$, and those zeros require their analytic and arithmetic comparisons. The parameter construction supplies only the specific blocks stated in Theorem 10.203.

Solution 83. Here $\eta = 173/200$, $\mathfrak{h} = (173/200)^4$, and the exact contracted parameters are

$$S_2 = \frac{26880000000000000000}{802359178476091681}, \qquad T_2 = \frac{2407077535428275043}{6400000000000000}. \tag{10.915}$$

Integer division of the positive numerators by their denominators gives the following values. All radii count both endpoints.

j	$\lfloor 2^j S_2 \rfloor$	$\lfloor \eta^j T_2 \rfloor$	μ_j for the next step	Cardinal degree
1	67	325	44	5940
2	134	281	38	10222
3	268	243	—	—

The two degrees are $135 \cdot 44$ and $269 \cdot 38$. The jet requirements are $281 + 43 = 324 \le 325$ and $243 + 37 = 280 \le 281$. Thus no unprovided derivative is used.

At general depth, the two opposite powers cancel:

$$(\mu_j-1)M_*\le\frac{c\eta^j\mathfrak{h}^I\widehat{T}}{r+1}\frac{\mathfrak{h}^{-I}H}{\log p}=\frac{c\eta^j\widehat{T}H}{(r+1)\log p}. \tag{10.916}$$

The separate original bound is $v_p(b_n)\le H/\log p$. Factoring $\widehat{T}H/\log p$ out of the sum therefore leaves $c\eta^j/(r+1)+1/\widehat{T}$, exactly the bracket in the input estimate of Theorem 10.206. Replacing μ_j-1 by an unnecessarily rounded upper bound would lose this exact cancellation.

These data demonstrate integer counting and jet availability. They do not construct the required number field, support or kernel at depth two. The theorem's application there also needs its starting integer block and the stipulated phase-family hypotheses. The depth-one starting block is supplied by Theorems 10.202–10.203; the stronger q-deleted data survive the integer extensions.

Solution 84. The exact parameters are

$$\eta=\frac{173}{200},\qquad S_1=\frac{16800000000}{895745041},\qquad T_1=\frac{2687235123}{4000000},\qquad O=\lfloor T_2\rfloor=376. \tag{10.934}$$

The radii are 38, 75 and 150. The input floors are 671, 581, 502 and 434, so the four multiplicities are 296, 206, 127 and 59. Their jet-order checks are the exact equalities $376+295=671$, $376+205=581$, $376+126=502$, and $376+58=434$. The deleted increment is $296-206=90$.

There are 38 odd inner nodes, 39 even inner nodes including zero, 74 nodes in the first annulus and 150 in the second. Thus

$$\begin{aligned}N_I&=38\cdot296+39\cdot206+74\cdot127+150\cdot59\\&=301\cdot59+151(127-59)+77(206-127)+38\cdot90\\&=37530.\end{aligned} \tag{10.935}$$

The first line counts disjoint node types. The second starts with the full outer layer, then adds two full inner increments and the deleted increment. In particular the inner full layer contains $2\cdot38+1=77$ nodes, not just the 39 multiples of q. The multiplicity hypothesis holds because $T_2>376$ and $(27/50)376/4>1$. These are counting data; they do not specify a complete number field and auxiliary kernel.

The optimizer gives $r/t>A_0+3/5$, with $t>1$, hence $A_0<r$. A fractional-linear arithmetic bound can therefore be checked at the two endpoints a_* and r for each fixed small rank. An unrestricted endpoint at infinity would ignore this hypothesis and may give an insufficient bound.

The depth-I fractional value and the following depth-$I+1$ integer value have the same scaled radius and the same total denominator depth $I+1$ in (10.243). The latter has a smaller derivative order and lies in K. Thus the already proved bound at depth I supplies its arithmetic cost, even if the new depth is the first one beyond the logarithmic threshold. No denominator is removed in this passage.

Corollary 10.209 constructs the family only through $\min(I_G,I_M)$. If $I_M<I_G$, it has not justified a further application of the mixed step: the required derivative order can be too small. The distinct fixed-order argument of Theorems 10.216–10.217 supplies the continuation to its endpoint.

Solution 85. The five successive threshold quantities are

$$\frac{c\eta^jT_1}{4}=\frac{81}{50},\quad\frac{14013}{10000},\quad\frac{2424249}{2000000},\quad\frac{419395077}{400000000},\quad\frac{72555348321}{80000000000}\quad(j=0,1,2,3,4). \tag{10.950}$$

The last one at least 1 occurs at $j_0 = 3$. Hence $R = 2\lfloor 8 \cdot 40 \rfloor = 640$ and $O = \lfloor (173/200)^4 \cdot 12 \rfloor = 6$. The input interval has radius $\lfloor 8 \cdot 40 \rfloor = 320$ and order $\lfloor (173/200)^3 \cdot 12 \rfloor = 7$. Its interpolation multiplicity is $\mu = \lfloor 419395077/400000000 \rfloor + 1 = 2$, so the jet-order test is exactly $6 + 2 - 1 = 7$. These are counting data, not a complete number-field auxiliary-kernel instance.

If $\mu = 1$, only the ordinary jet of order zero is used at each node. The floor inequality still holds, and the loss $(\mu - 1)M_*$ is zero. The separate original $v_p(b_n)$ loss remains; it must not be discarded with the jet loss.

The torus-degree estimate is

$$\frac{2436}{5} d^2 \exp(-3A) < \frac{2436}{5 \cdot 16 \cdot 2^{13}} = \frac{609}{163840} < 1. \tag{10.951}$$

Here the factors $16, 2^8, 2^5$ come respectively from $a^2 \geq 16$, $\exp(8) > 2^8$, and $\exp(A) > 2^5$. For the root count, $\lambda = 59/58$ gives $45/(8\lambda) = 45 \cdot 58/(8 \cdot 59) = 1305/236 > 1$. Both estimates are strict. The first leaves a nonzero polynomial in the additive variable; the second gives more distinct roots than its degree.

One must construct every intervening selected family through I_3, retaining a nonzero coefficient subvector, complete fibres and the full integer interval through order O. Lemma 10.211 gives the starting family at I_1. Theorems 10.216–10.217 supply the intervening fractional zeros and full integer successors; the endpoint contradiction alone does not prove those assertions.

Solution 86. (a) Here $a = 4, c = 27/50, c_0 = 27/10, c_4 = 14$, $c_2 = 3500/2083$, $P_0 = 1$, and $\max(1, 5(a_* + 3/5)/r) = 9/4$. Substitution gives

$$L_3 = \frac{11202105600000000}{262099267823}, \qquad \mathfrak{h} = (173/200)^4 = \frac{895745041}{1600000000}. \tag{10.975}$$

Multiplying rational numbers and comparing integer numerators gives

$$\frac{17}{10} < \frac{cL_3\mathfrak{h}^{14}}{4} < \frac{43}{25}, \qquad \frac{95}{100} < \frac{cL_3\mathfrak{h}^{15}}{4} < \frac{97}{100}. \tag{10.976}$$

The powers decrease strictly, so $m_3 = 14$. Also $164/100 < 8/(2\mathfrak{h})^{14} < 165/100$, which places μ_3 strictly between 1 and 2. Thus $j_3 = 1$. For B = 10 the two required vertices are $(14, 14)$ and $(30/\log Q, 30/\log Q)$. The third vertex is $(14, (30 + 14\alpha)/\log 2)$, with $\alpha = -\log \mathfrak{h}$. Its exponential term is the same as that at the second required vertex, and its linear term is smaller. Convexity on their triangle gives the entire enclosure.

(b) The two radii are $R_{\mathrm{s}} = \lfloor 197/8 \rfloor = 24$ and $R_{\mathrm{l}} = \lfloor 199/4 \rfloor = 49$. Their node counts are 49 and 99. Precisely,

$$\frac{193}{4} < 49 \leq \frac{201}{4}, \qquad \frac{197}{2} < 99 \leq \frac{201}{2}, \qquad 2 \cdot 24 \leq 49. \tag{10.977}$$

Multiplication of the first two comparisons by N gives the exact mass inequalities. These formulas do not order either new radius relative to the original R. Each known interval can be used as its own interpolation input, so such an ordering is unnecessary.

(c) There are forty odd integers in $[-40, 40]$, twenty of each sign. Since $5^2 \leq 80 < 5^3$, the separation bound is two. The raw analytic entries of Theorem 10.182, after subtracting β, are

$$40N - Q_D, \qquad \kappa + \chi\widehat{U}_r - v_p(b_n) - 2. \tag{10.978}$$

Only ordinary jets of order zero are used: $\mu - 1 = 0$. Thus every output prepared order up to O was already present in the input. The q-deleted separation loss occurs once. The projection error still contains the original $v_p(b_n)$; setting the ordinary-jet loss to zero does not remove that distinct term.

Solution 87. (a) We have $z = 1/100, L = 3122, K = 94, N = 293468$. Thus $K > 85$, and the factorial comparison used to bound $\log \beta$ applies. The determinant margin is $3848/1125 - (1/200 + 136/75 + 7/225{+}13/40) > 0$. The bound including the common p-part is $588/25{+}7/30{+}1/12 = 7151/300 < 24$; the collision coefficient is $1/12 + 49/6000 + 7/1200 = 73/750 < 1$.

(b) With no padding the ratio is one. With one padded height, say the first, it is $\epsilon t/(dh(a_1))$, bounded by $58\mathrm{e}\epsilon t d\ell_d/w$. If both heights are padded, the ratio is $\epsilon^2 t^2/(d^2\Omega)$, bounded by $29\mathrm{e}^2\epsilon^2 t^2 d\ell_d/w$. Squaring the singleton bound would introduce $d^2\ell_d^2/w^2$ instead of $d\ell_d/w$, and would miss the independent two-base height-product information.

At d=64 the sufficient upper bound is

$$\frac{3480}{218295(27/10)^2(8/5)^2}\frac{64}{46/5}\frac{(291/5+6)^2}{291/5} = \frac{3320210000}{7889661549} < \frac{1}{2}. \tag{10.1000}$$

At the upper endpoint of the middle range it is

$$\frac{696}{218295(27/10)^3(8/5)^3}\frac{2048}{12}\frac{(687/10+15/2)^2}{687/10} = \frac{37419280000}{65596294071} < \frac{3}{5}. \tag{10.1001}$$

Multiplying positive denominators checks both inequalities. For large degrees, substitute $\log d \geq 11(69/100)$ in $(123/100)(549/50 + \log d) - (76/5 + \log d)$; the result is $511/10000$. Its slope as a function of $\log d$ is $23/100 > 0$, so it remains positive for every larger degree.

(c) The two-logarithm determinant was stated for two nonzero coefficients, and its collision argument divides by their gcd to obtain two positive coprime integers. With one nonzero coefficient, the one-base power law instead gives $(34/33)d\mathsf{h}_2/t + 8dh(a)/t$. The first summand contains no height, so divide by the two-base product lower bound. In the second summand the active height cancels; use the singleton lower bound for the other base. Substitution of $C_2 = 218295\mathrm{e}^4 d^4\ell_d A/w$ gives the two fractions in (10.999), each with its stated field and torsion factors. Their sum is less than $1/1000$ of the target.

Solution 88. (a) If the second row coefficients are v_{lj}, the composed original coefficient of z_k is $\sum_{j,i} v_{lj}u_{ji}a_{ik}$. Its absolute value is at most $\sum_{j,i} |v_{lj}||u_{ji}||a_{ik}|$. Multiply by $h(a_k)$ and sum to obtain the required weight $\sum_j |v_{lj}|s_j$. The actual residue factor at the middle rank cancels in the product allowance. Denominator clearing can introduce large new integer coefficients. It does not identify their maximum or minimum nonzero absolute value with the original $R_{\max}$ or $B_\dagger$. Lemma 10.221 instead keeps the original Euler arguments and the original minimum-valuation coefficient at every step.

(b) Write the constant as

$$C_2(j) = \frac{c^{(2)}}{P}(a^{(2)}\mathrm{e}P)^j F(j)\frac{d^{j+2}\ell_d}{q^u t^3}\mathcal{R}_j A_j. \tag{10.1018}$$

After multiplying by $\Gamma_2^{n-r}\mathcal{R}_n/\mathcal{R}_r$, the factors $\mathcal{R}_r$ cancel and the powers of d and P become exactly those at rank n. The remaining exponential quotient is $(\mathrm{e}/q)^{n-r}$; the factorial quotient is $F(r)/F(n)$. Finally $A_r/A_n \leq 1$. Since $F(j+1)/F(j) > \mathrm{e}$, the ratio is strictly below q^{r-n} when r<n. At r=2 multiply this by 3/2, for the coefficient parameter, and 100/123, for the constant comparison. The largest resulting bound is $(3/2)(100/123)/2 = 25/41$, at n=3 and q=2.

(c) Valuations at the rational primes 2,3,11 prove independence. The original full exponent lattice is saturated, so its q-primary index is one. For the pair $(6, 11)$, a rational power product has equal integral valuations at 2 and 3 and an integral valuation at 11; its two rational exponents

are therefore integers. This saturation index is also one. The original residues generate all of $\mathbb{F}_5^\times$, whereas both 6 and 11 reduce to one. Including the rational torsion root -1, the generated residue groups have orders four and two. Thus $\delta(2,3,11)=1$, $\delta(6,11)=2$.

Here $d=e=f=P=1, q=2, a^{(2)}=7$, and $\Gamma_2=7\mathrm{e}^2/2>25$, since $\mathrm{e}>27/10$. Also $\mathcal{R}_3\ge 5$, while

$$\mathcal{R}_2=\max\{5/2,\mathrm{e}^2(\log 5)^3/4\}<\frac{9}{4}\left(\frac{13}{8}\right)^3<10. \tag{10.1019}$$

The latter upper bound exceeds $5/2$ as well. The elementary bounds $\log 2>2/3, \log 3>1, \log 6<2$ imply

$$\Gamma_2\frac{\mathcal{R}_3}{\mathcal{R}_2}\log 2\log 3>\frac{25}{3}>\log 6. \tag{10.1020}$$

Multiplying by $\log 11$ proves (10.725). The individual weighted conditions are equalities. Finally $\Xi=66$ and $\Xi-1=65=5\cdot 13$, so the valuation is one. This constructs an admissible rank-two representation. It does not exclude a rank-one representation; no least-rank assertion follows from that construction alone. □

Solution 89. (a) If $b_j=0$, its logarithmic entry is $\log(\omega(d)|b_n|/w_j)$, with a positive argument because $b_n\neq 0$. In a varying coordinate it is still of the form $\log(A+D/u)$ used in (10.1023); D can vanish. The derivative on the logarithmic branch is at least H_i-1, and the branches meet continuously. This proves the same monotonicity. Merely increasing the product would give a bound with the old coefficient parameter; it would not justify the smaller parameter computed from the new weights.

(b) For odd primes, insert $\max\{p/\delta(\boldsymbol{a}),\mathrm{e}^n t^{n+1}/n^n\}<(n+1)p$ into (10.642). The factors $q^u=2$ and the leading 2 cancel. Divide by (10.1024) and use (10.1026). The remaining coefficient is $(c^{(2)}/P)/(256\mathrm{e})$, the exponential factor is $(7P/256)^n$, and the remaining rank factor is $(n+1)^3(A/t)/(n^{3/2}\log^2(2n))$. At primes at least five use $c^{(2)}/P\le 2228/3, A/t\le n+2$; at three use $c^{(2)}/P=216, A/t\le 2(n+2)$. These give respectively $10n^3(7/256)^n$ and $6n^3(21/256)^n$. Their n=2 values are 245/4096 and 1323/4096, and their successive ratios are at most 189/2048 and 567/2048. At two, retain $d=2, f=2, P=4, q^u=3$ and the maximum bound 6. The exact constant before the factorial estimate is $203(104\mathrm{e})^n/(16\mathrm{e})^{2(n+1)}$. The second inequality of (10.1026) leaves $203(13/32)^n/(256\mathrm{e})$ times $(n+1)^2A/(tn^{3/2}\log^2(2n))$. The estimates in the proof give the scalar factor 345303/262144, which is less than 4/3. Thus the final envelope is $(4/3)n^2(13/32)^n$, with initial value 169/192 and ratio at most 117/128. Every starting value and ratio is below one; induction, rather than a finite numerical sample, proves all ranks.

(c) Put $r=5^8=390625$. At $\varepsilon=1/2$, $L=51+\log 500$; the two branches are $(\log 3)(51+\log 500)$ and $390625/2$, and $w_2^*=390625/(2(51+\log 500))$. This is the larger weight: $\log 500<7$, $\log 3<2$, and $390625/2>116$ suffice. At $\varepsilon=1/r$, $L=51+\log(250r)$, the second branch is one, and $w_2^*=\log 3$, since $1/L<1<\log 3$. To justify the rough estimates just used, $\mathrm{e}^7>(8/3)^7>500$ and $\mathrm{e}^2>(8/3)^2>3$. The congruences in the worked example give valuation exactly one. For a dependent list the application of Theorem 10.223 is unavailable; padding heights changes no multiplicative relations. In particular it cannot turn (1,3,3) into an independent list or bound new coefficients arising from eliminating a relation. Those are additional proof obligations.

Solution 90. (a) The valuation vectors at two and three are (2,-2) and (3,-3). Their primitive integral direction is (1,-1), so take $\beta=2/3$, with exponents (2,3). For (3,-2) the combined exponent is zero and the product is one; it is excluded. For (2,-1) the combined exponent is one and the product is 2/3. At two, $(-3)9=-27$, and $-27-1=-28$ has valuation two. The primitive positive generator is 3, its exponent is three, and the sign is negative. Squaring gives 3^6. Here $v_2(3^2-1)=3$, so the power identity gives $v_2(3^6-1)=3$, consistent with the inequality in (10.1043).

(b) The exact coefficient transfer gives $|M|<2nB_n\Omega r$. Since $r\le U/\varepsilon$, its logarithm is less than $\log(2n)+\log B_n+\log(\varepsilon^{-1})+\log\Omega+\log U$. The first three terms together are less than L,

which is at most U; each remaining logarithm is less than U. This proves the strict factor three. The generator height is at most Ω, and $\Omega < U/((n+1)\log p)$. After adding the height term and $\log 2 < U$, (10.1043) is bounded by the first line of (10.1045). Since $2/(n+1) \le 2/3$ and $\log p < p-1$, its bracket is less than $(14/3)(p-1) < 14p/3$. Rationality supplies an integral primitive valuation vector representing an actual positive rational number, the lower height bound $h(\beta) \ge \log 2$, and the elementary residue and local-height bounds in the rational field. These assertions cannot be transferred unchanged to arbitrary number fields.

(c) Modulo four, $3 \cdot 5^k + 5$ is zero. At k=1 it is 20; for k greater than one its quotient by five is $3 \cdot 5^{k-1} + 1$, prime to five. Therefore $v_5(M/4) = 1$, and the power identity for $2^4 = 16$ gives $v_5(2^M - 1) = 1 + 1 = 2$. At $\varepsilon = 1/2$, the two branches defining U are $\Omega(92 + \log(2500\Omega))$ and $5^k/2$; U is their maximum. At $\varepsilon = 5^{-k}$, they are $\Omega(92 + \log(1250\Omega 5^k))$ and one. The first is larger, since $\Omega = (\log 8)(\log 4) > 1$ and its logarithmic factor exceeds 92. Two rational unit parts have rank at most two. Rank two is independence, and lower ranks are covered by Theorem 10.228. The three bases (2,3,6) at five have rank two: their first two valuation vectors are independent and the third is their sum. This is an intermediate dependent rank, which neither Theorem 10.226 nor Theorem 10.228 covers. Theorem 10.232 treats it.

Solution 91. (a) At primes two and three the first two valuation vectors are the standard basis, while the last is their sum. Take $d_0 = 1, D = 2$. Cramer's numerators give exponents $M_1 = 2(2+1) = 6$ and $M_2 = 2(1+1) = 4$. The original product is $2^2 \cdot 3 \cdot 6 = 72$, and its square is $2^6 3^4 = 5184$. Both $72 - 1 = 71$ and $5184 - 1 = 5183$ are prime to five, so both valuations are zero. The third base is redundant, so Proposition 10.231 applies directly.

(b) The first two valuation vectors at three are 1 and 3. Take $(k_1, k_2) = (-3, 1)$, whose product is $(-3)^{-3}27 = -1$. Eliminate the second index. Doubling this primitive relation gives $\kappa = (-6, 2, 0)$ and $\kappa_x = 2$. For coefficients (1,1,1), the surviving coefficients are (8,2), and the exact identity is $(-405)^2 = (-3)^8 5^2 = 164025$. We have $v_2(-405 - 1) = 1$, whereas $v_2(164025 - 1) = v_2(8 \cdot 20503) = 3$. The minimum coefficient valuation increases from zero to one, with the distinguished coefficient still attaining it. For original coefficients (2,2,2), the product is 164025 and the surviving coefficients are (16,4). Its square is $(-3)^{16}5^4$. The original difference has valuation three; the square has valuation four, since 164025+1 is twice an odd number. The minimum coefficient valuation increases from one to two. Thus both examples satisfy the factored-power inequality and the required minimum condition, including the common factor at two.

(c) Substitute $B'_n = \kappa_x B_n$, $\varepsilon' = \kappa_x \varepsilon/(2K)$, and $V' = V/w_x$ into the definition of T for length n-1. The factors κ_x cancel, leaving a ratio $(2K/w_x)\exp(D_{n-1} - D_n)/p$ to the original T. Taking logarithms gives (10.1062). Since $\log(2K) < L$, its value is less than 2L. The other branch of U is preserved exactly. At two every doubled circuit coefficient is divisible by two. If the chosen primitive coefficient is odd, the pivot has valuation exactly one. Every transferred coefficient is divisible by $2^{v_2(b_n)+1}$, and the distinguished one has exactly that valuation. Hence its minimum condition persists. One must track this common factor rather than incorrectly claiming that the pivot itself remains odd.

(d) Either the last vector is in the span of the preceding ones, or adjoining it raises rank by one. In the latter case, dependence of the whole list forces dependence among those preceding vectors. This is exactly the circuit needed for induction, and its support avoids the distinguished coefficient. For rational numbers the prime-valuation map has only signs as its kernel, and bounds each coordinate by the logarithmic height divided by $\log 2$. In a number field this kernel can contain units of infinite order. For example, in $\mathbb{Q}(\sqrt{2})$, the number $1 + \sqrt{2}$ and its inverse $\sqrt{2} - 1$ are algebraic integers, as the monic equations $X^2 - 2X - 1 = 0$ and $X^2 + 2X - 1 = 0$ show. Their product is one, so their finite-prime valuations vanish; the powers of $1 + \sqrt{2} > 1$ are distinct. Additional archimedean and unit information is therefore needed in general.

Solution 92. (a) The valuation vectors at two and three are $(1,0), (0,1), (2,0)$, so the greedy basis

is $(2,3)$. It is already saturated over the rationals: a rational root of an integral combination has integral valuations at both primes. The residue of two generates all four nonzero classes at five, so $\delta_\beta = 1$. Here P=1, t=log5, and $L_3 = 3/(25\log 5) < 2/25 < \log 2$. Therefore $\Omega_3 = (\log 2)(\log 3)(\log 4)$, $H = \log 5$, and $A_3 = 4 + \log 4$. The second entry in M_3 is $\mathrm{e}^3(\log 5)^4/27 < 5$, so $M_3 = 5$. The strict inequality follows from the positive-series enclosures in (10.1069); the exact calculation gives a positive rational lower bound for $135 - \mathrm{e}^3(\log 5)^4$. The product is96 and $v_5(96-1) = v_5(95) = 1$. The circuit $4 \cdot 2^{-2} = 1$ deletes4 with coefficient one, leaving exponents $(1 + 2\cdot 2, 1) = (5,1)$ on $(2,3)$; their product is again96. This example lies in the small-coefficient branch: $B/\log B < 3$, whereas (10.1071) gives $W_3 > 200\mathrm{e}^3$.

(b) The first base is torsion of order two, and the greedy basis is still $(2,3)$. The original product is -24, with $v_5(-24-1) = 2$. Squaring deletes the torsion base and yields $2^2 3^2 4^2 = 576$; its distance from one is575=$25 \cdot 23$, also of valuation two. Thus the factored-power inequality is equality here. The old modified product is $L_4(\log 2)(\log 3)(\log 4)$, because all three positive-base heights exceed $L_4 = 4/(25\log 5)$. The new product omits that factor, and the selected basis and its residue index remain exactly the same. A bound B=3 becomes a valid coefficient bound $B^2 = 9$, although the actual new coefficients are only two.

(c) Before $t_0 = (n/\mathrm{e})^{n/(n+1)}$, the comparison quotient is t^j; after that point it is $n^n \mathrm{e}^{-n} t^{j-n-1}$. The first increases and the second is nonincreasing for $j \le n+1$, proving the bound at their common endpoint. The ratio comparison following (10.1065) proves the stated strict upper bound. At two, P is at least4. A primitive cube root has a nontrivial reduction of order three, so $3 \mid 2^f - 1$ and f is at least two. Hence $Pt \ge 4\log 4 > 16/3$. Together with $\mathrm{e} > 8/3$, this gives $(4\mathrm{e}Pt)^{-1} < 9/512$. Finally the exponential series through degree six is1957/720 and the remaining terms are positive. Direct rational arithmetic gives $13(1957/720)^2 - 96 - 9/512 = 25049/1036800 > 0$.

(d) If the maximal-height base is the deleted base, normalize the coefficient bound by $BdH_n(a_x)/8$; the two ratios $h(a_x)/H_n(a_x)$ are at most one. If it is a surviving selected base, cancel its height before applying the independent product bound: the required independent list has length j-1, and its contribution loses a factor e. If it is a surviving nonbasis member, cancel its actual height against its modified height and use one fewer floor. This changes the powers to $\varkappa^{s-1}/P^{r+1}$ and t^{s+1}/X_n^{s-1}. There are n-r nonbasis indices in total. One is the deleted index and another is this surviving maximal-height index, so n-r is at least two and s=n-r-1 is at least one. These are exactly the three normalizations bounded by (10.1081).

Solution 93. The first polynomial is $X^2 - 14X - 49$, with roots $7(1 \pm \sqrt{2})$. Both absolute values exceed one, since $\sqrt{2} > 8/7$, so the logarithmic height is $\frac{1}{2}\log 49 = \log 7$. The second primitive polynomial is $49X^2 - 84X + 34$. Its roots are $(6 \pm \sqrt{2})/7$; the positive-sign root is greater than one and the other is between zero and one. Its height is therefore $\frac{1}{2}\log(7(6+\sqrt{2}))$. The polynomials are irreducible since their roots generate the quadratic field, and their coefficients have greatest common divisor one.

Write a lift of 3 as $3 + 7k$. Modulo 49, its square minus two is $7 + 42k$. Thus $1 + 6k \equiv 0 \pmod 7$, giving $k \equiv 1$ and the root 10 modulo 49. For $\Xi = 8 + 7\sqrt{2}$, the residue of $\Xi - 1$ is 28 modulo 49. It is divisible by seven but not 49, so its prime-ideal valuation is exactly one.

At n=2 the first envelope is less than

$$\frac{3627}{1024}\left(\frac{21}{128}\right)^2 \frac{135}{14} = \frac{30847635}{33554432} < 1.$$

Its positive margin is 2706797/33554432. The successive ratio is less than 7/18, as proved in (10.1092); induction therefore bounds every later rank too. A finite plot alone would not establish that quantifier.

For the final list, $b_2 = 1$ has minimum seven-adic valuation zero, while $v_7(b_1) = +\infty$. The total

valuation is zero and the product is $1+\sqrt{2}\neq 1$. However, its normalized bases are $(1,1+\sqrt{2})$, a dependent list. The independence hypothesis in Theorem 10.239 fails. This example neither violates that theorem nor supplies a proof for the dependent case.

Solution 94. (a) Conjugation sends $\sqrt{2}$ to $-\sqrt{2}$, giving norms $-1, 34, 9$. If $a^r c^s 3^t = 1$, taking the product with its conjugate and then the valuations at seventeen and three gives $s = t = 0$. Since $a > 1$ in the chosen real embedding, $r = 0$. A torsion relation reduces to this case after raising to its order, so the same list is independent modulo torsion. The first relation has $k_x = 1$ at x=3 and $\kappa = 1$; (10.1116) gives $(2, 2, 6)$. As $c_3(4,2)/c_3(3,2) = 16$, the new bounds are $B' = 12$, $B'_n = 6$, and $\varepsilon' = 1/64$. Thus

$$\frac{\varepsilon' B'}{B'_n c_3(3,2)} = \frac{1}{32 c_3(3,2)} = \frac{1}{2c_3(4,2)} = \frac{\varepsilon B}{B_n c_3(4,2)}.$$

The universal proof uses $\kappa = AR$; any smaller verified relation bound also satisfies its comparisons. Here the explicit relation supplies that smaller bound.

(b) Now $k_x = 7$, and the reduced coefficients are $(14, 14, 42)$. Take $B' = 84$, $B'_n = 42$, and again $\varepsilon' = 1/64$. The ratio B'/B'_n is still two, so the displayed equality is unchanged. All three new coefficients have seven-adic valuation one, including the distinguished coefficient. The product of the reduced bases is the seventh power of the original product; its difference from one has valuation at least that of the original difference. The proof requires only that k_x have minimum valuation among the relation coefficients. Its valuation can be positive.

(c) Write the first lift as $3 + 7z$. Modulo 49 its square minus two is $7(1 + 6z)$, so z=1 gives the root 10. Next write $10 + 49z$. Modulo 343 its square minus two is $49(2 + 20z)$, so z=2 gives 108. Now $ac = 8 + 7\sqrt{2} \equiv 78 \pmod{343}$, and

$$3^6 \cdot 78^2 - 1 \equiv 245 = 5 \cdot 49 \pmod{343}.$$

The valuation is therefore exactly two. For the alternate coefficient vector, the relation with $k_x = 1$ gives $(2, 0, 6)$. Its last coefficient has valuation zero, equal to the minimum of zero, infinity and zero. This nonidentity product is allowed by Theorem 10.242; the zero coefficient does not create a new dependence assumption.

Solution 95. Let $z = N_{L/K}(\beta)$, so $z^T = A^2$. Then $\theta = z^{(T+1)/2}/A$ belongs to K and satisfies $\theta^T = A$. Applying this to every vector in the 3-primary saturation proves equality of the two lattices and allows the roots representing a saturated basis to be chosen in K.

For odd f, these roots reduce to a group of order $g \mid N-1$. The 3-primary root-of-unity group has order $m \mid N+1$. The orders are coprime, so the combined group has order mg, and $\delta_L = (N^2-1)/(mg)$. Thus $N^2/(m\delta_L) = N^2 g/(N^2-1) \leq N^2/(N+1) < N$. The factor m is essential to this calculation.

Direct cancellation of factorials gives

$$\frac{T_{n+1}}{T_n} = \left(1 + \frac{1}{n+1}\right)^{n+1} \frac{(n+2)\sqrt{n}}{(n+1)^{3/2}}.$$

The first factor is less than e. The logarithmic derivative of the second is $(2-n)/(2n(n+1)(n+2))$, so its maximum for $n \geq 2$ occurs at two. Its square there is $32/27 < 121/100$. This proves (10.1128). In the rational row of (10.1129), the endpoint is bounded by

$$\frac{406}{1024}\left(\frac{13}{16}\right)^2\left(\frac{3}{8}\right)^4 \frac{135}{14}\frac{81}{8} = \frac{4340987235}{8589934592} < 1.$$

The positive margin is $4248947357/8589934592$. The successive ratio is less than $143/160 < 1$, so this endpoint controls every larger rank in that row. The other rows use the smaller endpoints and ratios displayed in (10.1129).

For the rational example, $d = e = f = 1$. The valuations of $(6, 11/2, 33)$ at two are $(1, -1, 0)$, so (10.1087) gives $\boldsymbol{\beta} = (3, 11, 33)$. This list has rank two: prime valuations prove that three and eleven are independent, and their product is thirty-three. The original product is 33^2, hence

$$\Xi - 1 = 33^2 - 1 = 1088 = 2^6 \cdot 17, \qquad v_2(\Xi - 1) = 6.$$

The minimum coefficient valuation is zero. The weights $(\log 6, \log 11, \log 33)$ are the original rational heights and exceed Q^{-1}, where $Q = 16\mathrm{e}^2$. Theorem 10.246 therefore applies with the stated bounds. Neither the nonunit original inputs nor the dependence of their normalized units requires a change to the field degree or height floor.

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