

Effective lower bounds I: Baker's theorem and its arithmetic tools

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An effective logarithm estimate combines a rank argument with a comparison of arithmetic and analytic size. We begin with the rank test: how many exact zeros would force an exponential polynomial to be zero? We then choose polynomial bases that preserve its coefficients, prove the denominator bounds needed to clear its derivatives, and account for the height cost of normalizing the logarithmic form.

Theorem 4.1 states the target, including the algebraic constant and arbitrary fixed logarithm branches. The next lesson, *Effective lower bounds II: proof of Baker's theorem*, gives its induction and parameter choices. The tools proved here include a degree-adapted polynomial family with a derivative denominator independent of the evaluation point; this supplies an alternative to the interval denominator in the classical construction.

We use the qualitative theorem from *Baker's theorem II: extrapolation and the end of the proof*, elementary polynomial algebra and integer factorization. The algebraic-integer criterion is proved in *Algebraic integers and rings of integers*, Proposition 1.1. The freely accessible [Waldschmidt 2000] supplies the effective logarithm estimates and Matveev polynomial method discussed here. The logarithm methods of Baker, Feldman and Matveev are credited in the references. All the arithmetic arguments used below are supplied in full.

1. The theorem, with its quantifiers

For an algebraic number γ , let $H(\gamma)$ be the maximum absolute coefficient of its primitive irreducible polynomial in $\mathbb{Z}[X]$, with positive leading coefficient. This is the **naive height**. Thus

$$H(p/q) = \max(|p|, q) \quad (\gcd(p, q) = 1, q > 0), \quad H(0) = H(1) = H(i) = 1.$$

It differs from the logarithmic absolute Weil height used in the first lesson.

Theorem 4.1 (target: Baker's effective lower bound). Fix an integer $n \geq 1$, an integer $d \geq 1$, nonzero algebraic numbers $\alpha_1, \dots, \alpha_n$ of degree at most d and naive height at most A , where $A \geq 2$, and a determination ℓ_j of each logarithm:

$$e^{\ell_j} = \alpha_j.$$

There is an effectively computable $C > 0$, depending on n, d, A and these determinations, with the following property. For every $B \geq 2$ and every tuple of algebraic numbers $\beta_0, \dots, \beta_n$, each of degree at most d and naive height at most B , the form

$$\Lambda = \beta_0 + \beta_1 \ell_1 + \dots + \beta_n \ell_n$$

satisfies

$$\Lambda = 0 \quad \text{or} \quad |\Lambda| > B^{-C}. \tag{4.1}$$

The constant is uniform in B and in all the allowed coefficients. No rational independence assumption on the logarithms is required.

“Effective” means that the proof gives a terminating procedure for choosing C from the fixed algebraic data and the logarithm branches. It does not promise that the resulting number is small. Increasing C preserves (4.1), which is useful for absorbing fixed numerical factors.

The separation into zero and nonzero cases matters. For example, $2 \log 2 - \log 4 = 0$ on the real branches. A lower-bound theorem cannot exclude that relation. The proof will either find a bounded integer relation and reduce the number of logarithms, or force the proposed very small form to be zero.

For principal logarithms, or a specified bounded family of branches, the proof gives a power of $\log A$:

$$C = C'(n, d)(\log A)^{\kappa(n)} \quad (4.2)$$

in the principal case, after increasing the effective constant if necessary. The next lesson establishes this dependence. For arbitrary branches the leading constant also depends on their branch integers. This qualification is necessary; it is not removable by an adjustment depending only on n, d .

To see why, set both bases equal to 1, take $A = B = 2$, and use the coefficients $\sqrt{2}, -1$. Write

$$(1 + \sqrt{2})^r = p_r + q_r \sqrt{2}, \quad p_r, q_r \in \mathbb{Z}.$$

Choose logarithms $2\pi i q_r$ and $2\pi i p_r$ of the two bases. The nonzero form becomes

$$2\pi i(q_r \sqrt{2} - p_r) = -2\pi i(1 - \sqrt{2})^r,$$

whose absolute value tends to zero while the degrees and naive heights stay fixed. Hence a common C over all branch choices is impossible. The fixed-branch quantifier in Theorem 4.1 avoids this problem.

2. What the estimate gives

The effective theorem is the target of the construction below; its proof is completed in the next lesson. The resulting transcendence measures for logarithms and the fixed-base multiplicative bound are proved after that argument, in *Effective lower bounds II: proof of Baker’s theorem*, Section 10.

Later applications need sharper height dependence than a single power of $\log A$. The individual-height product $\prod_j \log A_j$, and the different coefficient and branch hypotheses of the modern estimates, are treated in [Waldschmidt 2000] and [Waldschmidt 2003]. The later lesson *Linear forms in many logarithms: the modern estimates and how to use them* gives the precise modern estimates and their proofs. The present theorem first establishes the decisive polynomial dependence on the coefficient height.

3. Plan the final rank test

Before estimating an auxiliary function, decide what enough zeros would imply. Suppose its values have the form

$$F(i) = \sum_{s=0}^{l-1} P_s(i) w_s^i, \quad \deg P_s < k.$$

There are kl polynomial coefficients. Evaluation at $i = 0, \dots, kl - 1$ is a square linear map from those coefficients to the values of F . If its matrix is invertible, vanishing of all those values forces every coefficient to vanish. In a construction with a nonzero coefficient vector, that is the contradiction we want.

The two possible failures have different meanings. Coincident nodes combine frequency blocks, so the values can no longer distinguish their coefficients. When $k \geq 2$, a zero node destroys the polynomially weighted columns. The next formula identifies both failures exactly. It also gives the complete rank test when each frequency carries several polynomial coefficients.

Lemma 4.6. Let $k, l \geq 1$, and let $w_0, \dots, w_{l-1}$ be distinct nonzero complex numbers. Order the columns by pairs (s, r) , first $s = 0, \dots, l-1$, then $r = 0, \dots, k-1$. The matrix with rows $i = 0, \dots, kl-1$ and entries

$$A_{i,(s,r)} = i^r w_s^i, \quad 0^0 = 1,$$

has determinant

$$\det A = \left(\prod_{s=0}^{l-1} w_s^{k(k-1)/2} \right) \left(\prod_{r=0}^{k-1} r! \right)^l \prod_{0 \leq s < t < l} (w_t - w_s)^{k^2}. \quad (4.15)$$

In particular, it is nonzero.

Proof. Put $N = kl$ and $K_0 = k(k-1)/2$. The ordinary Vandermonde determinant at nodes $z_0, \dots, z_{N-1}$ is $\prod_{a < b} (z_b - z_a)$. Indeed, its polynomial determinant changes sign under exchanging nodes, is divisible by each difference, and has the same total degree as that product; the coefficient obtained by choosing the successive highest powers is 1.

Apply this identity to the nodes

$$z_{s,r} = w_s + r\varepsilon \quad (0 \leq s < l, \ 0 \leq r < k)$$

in our column order, and compare the coefficient of the lowest nonzero power ε^{lK_0} . Taylor expansion of the columns shows that in each block the derivative orders must be distinct, since equal orders give repeated columns. The smallest total order is K_0 , attained exactly by $0, \dots, k-1$. Its coefficient within that block is

$$\det \left(\frac{r^t}{t!} \right)_{0 \leq t, r < k} = \frac{\prod_{a < b} (b - a)}{\prod_{t=0}^{k-1} t!} = 1.$$

Thus the coefficient on the determinant side is the determinant of the confluent matrix

$$F_{i,(s,r)} = (i)_r w_s^{i-r}, \quad (i)_r = i(i-1) \cdots (i-r+1),$$

with $(i)_0 = 1$. On the product side, the within-block differences contribute $\left(\prod_{r=0}^{k-1} r! \right)^l \varepsilon^{lK_0}$; each pair of different blocks contributes $(w_t - w_s)^{k^2}$. Consequently

$$\det F = \left(\prod_{r=0}^{k-1} r! \right)^l \prod_{s < t} (w_t - w_s)^{k^2}. \quad (4.16)$$

The falling factorials $(i)_0, \dots, (i)_r$ form a triangular monic basis for polynomials in i of degree at most r . Write

$$i^r = \sum_{t=0}^r S(r, t) (i)_t, \quad S(r, r) = 1.$$

Then the r -th column of the s -block of A is $\sum_{t \leq r} S(r, t) w_s^t F_{(s,t)}$. This triangular column change has determinant $\prod_{r=0}^{k-1} w_s^r = w_s^{K_0}$ for each block. Multiplying (4.16) by these factors proves (4.15). $\square$

For $k = l = 2$, the exact formula is

$$\det \begin{pmatrix} 1 & 0 & 1 & 0 \\ w_0 & w_0 & w_1 & w_1 \\ w_0^2 & 2w_0^2 & w_1^2 & 2w_1^2 \\ w_0^3 & 3w_0^3 & w_1^3 & 3w_1^3 \end{pmatrix} = w_0 w_1 (w_1 - w_0)^4. \quad (4.17)$$

At $w_0 = 2, w_1 = 3$, the determinant is 6. Nonzero nodes are needed: when $k \geq 2$, a node $w_s = 0$ makes the column with $r = 1$ identically zero.

Recover four coefficients from four values

Take

$$F(i) = (a + bi)2^i + (c + di)3^i, \quad y_j = F(j).$$

The determinant 6 calculated above proves uniqueness. Solving the same system gives the more informative identities

$$\begin{aligned} a &= -27y_0 + 36y_1 - 15y_2 + 2y_3, \\ b &= -9y_0 + \frac{21}{2}y_1 - 4y_2 + \frac{1}{2}y_3, \\ c &= 28y_0 - 36y_1 + 15y_2 - 2y_3, \\ d &= -4y_0 + \frac{16}{3}y_1 - \frac{7}{3}y_2 + \frac{1}{3}y_3. \end{aligned}$$

For example, $(y_0, y_1, y_2, y_3) = (3, 9, 32, 119)$ recovers $(a, b, c, d) = (1, -1, 2, 1)$. These identities can be verified by inserting the four rows in (4.17); they give an explicit inverse, rather than a numerical rank calculation.

The count of values is essential. With only $y_0 = y_1 = y_2 = 0$, the nonzero choice $(a, b, c, d) = (6, \frac{3}{2}, -6, 1)$ is possible and gives $y_3 = 3$. Thus three zeros do not yet force the four coefficients to vanish. In the effective proof, extrapolation must reach the number of rows required by the rank test. A small analytic estimate before that point would not finish the argument.

The polynomial coefficient blocks need not initially be written in monomials. We next construct a basis that retains the same coefficient count while accommodating the integer-valued polynomials used for denominator control.

4. Preserve the polynomial coefficients

Lemma 4.5. Let K be a field of characteristic zero, $P \in K[X]$ a polynomial of degree $n > 0$, and $0 \leq m \leq n$. The $n + 1$ polynomials

$$P(X), P(X + 1), \dots, P(X + m), \quad 1, X, \dots, X^{n-m-1} \quad (4.14)$$

are linearly independent over K . If $m = n$, the second list is empty. They consequently form a basis of $K[X]_{\leq n}$.

Proof. Suppose $\sum_{j=0}^m A_j P(X + j)$ has degree at most $n - m - 1$. For $0 \leq t \leq m$, the coefficient of X^{n-t} in $P(X + j)$, as a polynomial $q_t(j)$, has degree t and leading coefficient $a_n \binom{n}{t} \neq 0$. Therefore $q_0, \dots, q_m$ form a basis of the polynomials of degree at most m in j .

Vanishing of those highest $m + 1$ coefficients gives $\sum_j A_j q_t(j) = 0$ for all t . By the basis assertion it gives $\sum_j A_j j^t = 0$ for $t = 0, \dots, m$. The Vandermonde matrix at the distinct elements $0, \dots, m$ is invertible, so every $A_j = 0$. The remaining monomials are independent. $\square$

For $P(X) = X^2$, $m = 1$, the basis is $X^2, (X + 1)^2, 1$. Subtracting the first and third elements from the second produces $2X$; characteristic zero permits division by 2.

The characteristic assumption is essential. In characteristic p , $P(X) = X^p - X$ satisfies $P(X + 1) = P(X)$, so even two of the proposed translates coincide. All coefficient fields in the effective proof are subfields of $\mathbb{C}$, where Lemma 4.5 applies. Exercise 3 gives the complete induction proof by finite differences.

Read the basis as a finite-difference calculation

For $P(X) = X^3$ and $m = 2$, the proposed basis is

$$X^3, \quad (X + 1)^3, \quad (X + 2)^3, \quad 1.$$

Its first difference is $(X + 1)^3 - X^3 = 3X^2 + 3X + 1$. Its second difference is

$$(X + 2)^3 - 2(X + 1)^3 + X^3 = 6X + 6.$$

Together with 1, these recover X , then X^2 , and finally X^3 . Each difference removes one highest degree. Lemma 4.5 formalizes that mechanism for any polynomial with nonzero leading coefficient; Solution 3 gives the induction in full.

This explains a practical division of labour. The translate basis preserves all polynomial coefficients needed by the rank test. The choice of its underlying polynomial controls the arithmetic size of those coefficients and their derivatives. We now choose that polynomial.

5. Form normalized derivatives

For the auxiliary construction, derivatives must satisfy two requirements at once: their values must be small enough analytically, and their rational denominators must be cheap enough to clear. The normalization by $m!$ makes the first requirement a coefficient estimate. A product of consecutive factors makes the second requirement accessible to divisibility arguments.

After Lemma 4.3, Section 6 will give a stronger common denominator independent of the integer evaluation point. We first prove the direct interval bound because its positive coefficient expansion shows exactly how the normalized derivative is formed, and because the effective argument can use it without changing its original parameter choices.

For the classical construction, use

$$\Delta(x; k) = \frac{(x+1) \cdots (x+k)}{k!}, \quad \Delta(x; k, l, m) = \frac{1}{m!} \frac{d^m}{dx^m} \Delta(x; k)^l, \quad (4.7)$$

where $k \geq 1$ and $l, m \geq 0$ are integers. For a nonnegative integer x , set

$$\nu(x; k) = \text{lcm}(x+1, \dots, x+k).$$

The normalized derivative is exactly the coefficient of t^m in $\Delta(x+t; k)^l$.

Lemma 4.3. For every integer $x \geq 0$,

$$\nu(x; k)^m \Delta(x; k, l, m) \in \mathbb{Z}_{\geq 0}, \quad 0 \leq \Delta(x; k, l, m) \leq 4^{l(x+k)}. \quad (4.8)$$

The integer is positive when $0 \leq m \leq kl$, and is zero when $m > kl$.

Proof. First $\Delta(x; k) = \binom{x+k}{k}$ is a positive integer. Regard the list $x+1, \dots, x+k$ as repeated l times. Expanding

$$\Delta(x+t; k)^l = \Delta(x; k)^l \prod_{j=1}^k \left(1 + \frac{t}{x+j}\right)^l$$

gives

$$\Delta(x; k, l, m) = \Delta(x; k)^l \sum_{\substack{\text{choices of } m \\ \text{positions in the repeated list}}} \frac{1}{(x+j_1) \cdots (x+j_m)}. \quad (4.9)$$

There are $\binom{kl}{m}$ choices, with the sum read as zero when $m > kl$. Multiplication by $\nu(x; k)^m$ clears every denominator, proving integrality and the positivity assertion. For $l = 0$, the only nonzero derivative is $m = 0$, with value 1.

Since $x+j \geq 1$, each reciprocal product is at most 1. The inequalities $\binom{x+k}{k} \leq 2^{x+k}$ and $\binom{kl}{m} \leq 2^{kl}$ therefore give

$$\Delta(x; k, l, m) \leq 2^{l(x+k)+kl} \leq 4^{l(x+k)}.$$

□

For example, at $k = l = 2$,

$$\Delta(x; 2)^2 = \frac{x^4 + 6x^3 + 13x^2 + 12x + 4}{4}.$$

At $x = 1$, its normalized derivatives for $m = 0, \dots, 5$ are

$$9, \quad 15, \quad \frac{37}{4}, \quad \frac{5}{2}, \quad \frac{1}{4}, \quad 0.$$

Here $\nu(1; 2) = 6$, so multiplying the m -th entry by 6^m gives an integer. The final zero illustrates why an assertion of positivity for *all* orders would be false.

6. Compare denominator costs

Lemma 4.4. For all integers $x \geq 0$, $k \geq 1$,

$$\nu(x; k) \leq \left(\frac{16(x+k)}{k} \right)^{2k}. \quad (4.10)$$

Proof. Write $V(N) = \text{lcm}(1, \dots, N)$. We first prove

$$V(N) \leq 8^N \quad (N \geq 1). \quad (4.11)$$

For a prime power p^a , the factorial valuation formula is

$$v_p(N!) = \sum_{j \geq 1} \left\lfloor \frac{N}{p^j} \right\rfloor,$$

because multiples of p^j contribute one extra factor of p . Put $u = \lfloor N/2 \rfloor$, $v = \lceil N/2 \rceil$. If the highest power $p^a \leq N$ is at most v , its full contribution to $V(N)$ already divides $V(v)$. Otherwise $p^a > v$, but $p^{a-1} \leq N/2 \leq v$. The term $j = a$ in $v_p(N)$ is $1 - 0 - 0 = 1$, and every other floor difference is nonnegative. Hence

$$V(N) \mid \binom{N}{u} V(v).$$

Using $\binom{N}{u} \leq 2^N$, induction gives

$$V(N) \leq 2^N 8^{\lceil N/2 \rceil} \leq 8^N;$$

the exponent inequality $N + 3\lceil N/2 \rceil \leq 3N$ holds for $N \geq 2$, including $N = 3$. The base $N = 1$ is immediate.

Now set $N = x + k$ and split $\nu(x; k) = \nu' \nu''$, according as the prime divisor is at most k or greater than k . For $p \leq k$, let $a_0 = \lfloor \log_p k \rfloor$. Every interval of k consecutive integers contains a multiple of p^{a_0} , so $V(k) \mid \nu'$. If p^a is the highest power of p dividing an integer in our interval, then $p^a \leq N$ and $p^{a_0} > k/p$. Thus

$$p^{a-a_0} < \frac{pN}{k}$$

whenever an extra power occurs; the weak inequality with this upper bound also covers $a = a_0$. Therefore

$$\nu' \leq V(k) \prod_{p \leq k} p (N/k)^{\pi(k)} \leq 64^k (N/k)^k, \quad (4.12)$$

where $\prod_{p \leq k} p \mid V(k)$, $\pi(k) \leq k$, $N/k \geq 1$, and (4.11) were used.

A prime $p > k$ divides at most one of the consecutive integers $x + 1, \dots, x + k$. Its valuation in their product is consequently its valuation in ν'' , while $v_p(k!) = 0$. Hence $\nu'' \mid \binom{N}{k}$. Finally

$$\binom{N}{k} \leq \frac{N^k}{k!} \leq (eN/k)^k. \quad (4.13)$$

For the last inequality, $\sum_{j=1}^k \log j \geq \int_1^k \log t \, dt = k \log k - k + 1$ suffices, also at $k = 1$. Combining (4.12)–(4.13) gives

$$\nu(x; k) \leq (64e)^k (N/k)^{2k} \leq (16N/k)^{2k},$$

since $e < 4$. $\square$

The point of (4.10) is its dependence on the ratio x/k . If x is comparable with k , then $\log \nu(x; k) = O(k)$. A crude factorial denominator would have logarithm $O(k \log k)$; losing that extra factor would spoil the effective auxiliary-function count.

Here are three different parameter regimes in the original bound, keeping their dependence visible:

Evaluation point	Consequence of (4.10)	Dependence on k
$x = 0$	$\log \nu \leq 2k \log 16$, sharpened by (4.11) to $k \log 8$	linear
$x = k$	$\log \nu \leq 2k \log 32$	linear
$x = k^2$	$\log \nu \leq 2k \log(16(k+1))$	$O(k \log k)$

The last row prevents a mistaken uniform $O(k)$ assertion when the evaluation interval moves far from zero. Lemma 4.7 below supplies a uniform denominator for normalized derivatives even in that regime; it does not assert that the interval LCM itself becomes small.

A denominator independent of the evaluation point

The preceding interval estimate grows with x/k . A different divisibility argument removes this dependence from the derivative denominator. It also accommodates arbitrary polynomial degrees. This is the polynomial family introduced by Matveev and treated in [Waldschmidt 2000].

For integers $a \geq 0$ and $b \geq 1$, write $a = bq + r$, with $0 \leq r < b$, and define

$$D_b(Z; a) = \left(\frac{Z(Z+1) \cdots (Z+b-1)}{b!} \right)^q \frac{Z(Z+1) \cdots (Z+r-1)}{r!}.$$

An empty product is 1. This polynomial has degree exactly a . In particular, for fixed b , the polynomials $D_b(Z; 0), \dots, D_b(Z; A)$ form a basis of $\mathbb{Q}[Z]_{\leq A}$: their distinct degrees and nonzero leading coefficients make the coefficient matrix triangular and invertible.

Lemma 4.7 (uniform derivative denominators and analytic size). Let $a, C \geq 0$, $b \geq 1$ be integers, and put $V(b) = \text{lcm}(1, \dots, b)$. For every integer u , including negative integers, and every $0 \leq c \leq C$,

$$\frac{V(b)^C}{c!} D_b^{(c)}(u; a) \in \mathbb{Z}.$$

For every $z \in \mathbb{C}$,

$$\sum_{c=0}^C \binom{C}{c} |D_b^{(c)}(z; a)| \leq C! e^{a+b} \left(1 + \frac{|z|}{b}\right)^a.$$

Derivatives of order greater than a are zero.

Proof. Each consecutive-factor quotient is integer-valued on $\mathbb{Z}$. For $u \geq 1$ it is $\binom{u+t-1}{t}$, and for $u \leq 0$ it is $(-1)^t \binom{-u}{t}$, for a block of length t . These formulas include its zeros.

To prove the sharper derivative assertion, expand the coefficient of T^c in $D_b(u+T; a)$. Every term is obtained by deleting c factors from the numerator's a linear factors, then dividing the surviving product by $(b!)^q r!$. No extra combinatorial denominator appears, since that coefficient is $D_b^{(c)}(u; a)/c!$.

Fix a prime p , and let $L = \lfloor \log_p b \rfloor$. For $1 \leq j \leq L$, each block of b consecutive integers contains at least $\lfloor b/p^j \rfloor$ multiples of p^j , and the shorter block contains at least $\lfloor r/p^j \rfloor$. Set

$$t_j = q \lfloor b/p^j \rfloor + \lfloor r/p^j \rfloor.$$

Deleting c factors leaves at least $\max(t_j - c, 0)$ such multiples. A surviving product that is zero contributes the integer zero. Every nonzero surviving product has p -valuation at least $\sum_{j=1}^L \max(t_j - c, 0)$. The denominator has valuation $\sum_{j=1}^L t_j$, while $V(b)^C$ has valuation CL . Therefore its cleared term has valuation at least

$$\sum_{j=1}^L (C + \max(t_j - c, 0) - t_j) \geq 0,$$

because $c \leq C$. For $p > b$, the denominator has valuation zero. Every term is consequently an integer after multiplication by $V(b)^C$, and so is their sum. This proof uses intervals of integers and remains valid when u is negative.

For the analytic bound, set $W = |z| + b - 1$. Expanding the differentiated product gives

$$|D_b^{(c)}(z; a)| \leq \frac{c! \binom{a}{c} W^{a-c}}{(b!)^q r!} \quad (0 \leq c \leq a).$$

If $W = 0$, the expression is interpreted by the underlying polynomial: a term with exponent zero has value 1, including at zero. Since $\binom{C}{c} c! = C! / (C - c)! \leq C!$, summing and then completing the binomial sum gives

$$\sum_{c=0}^C \binom{C}{c} |D_b^{(c)}(z; a)| \leq \frac{C! (W + 1)^a}{(b!)^q r!}.$$

The elementary factorial inequality $t! \geq (t/e)^t$, proved in (4.13), yields

$$\frac{1}{(b!)^q r!} \leq \frac{e^a}{b^a} \left(\frac{b}{r}\right)^r \leq \frac{e^{a+b}}{b^a}.$$

At $r = 0$ the middle factor is read as 1. Otherwise $r \log(b/r) \leq b/e < b$, by maximizing $t \log(1/t)$ on $0 < t \leq 1$. Substitution of $W + 1 = |z| + b$ proves the stated bound. The degree gives the final zero assertion. $\square$

For the original polynomial, $D_k(x + 1; k; l) = \Delta(x; k; l)$. Taking $C = m$ gives the useful specialization

$$V(k)^m \Delta(x; k, l, m) \in \mathbb{Z} \quad (x \in \mathbb{Z}).$$

At $k = l = 2, x = 1$, multiplication of the five nonzero normalized derivatives by 2^m gives

$$9, \quad 30, \quad 37, \quad 20, \quad 4.$$

The interval denominator in Lemma 4.3 used 6^m at this point. At $x = -1$, the normalized derivatives instead are

$$0, \quad 0, \quad \frac{1}{4}, \quad \frac{1}{2}, \quad \frac{1}{4};$$

the same 2^m still clears them. Positivity from Lemma 4.3 is confined to its nonnegative evaluation points; the integer-valued assertion in Lemma 4.7 has no such restriction.

Together with (4.11), this gives a uniform logarithmic denominator cost at most $km \log 8$. The general family lets us keep the polynomial degree a and the block size b as separate parameters. A large degree need not force a factorial-size derivative denominator. The analytic bound records the corresponding size cost explicitly; it is this pair of bounds that makes the alternative basis useful.

7. Budget the coefficient heights

The rank and denominator arguments leave one bookkeeping question: how much does division by an algebraic coefficient change the degree and height? We first prove the elementary sum and product estimate, then compute its effect on a lower-bound exponent.

Lemma 4.2. If α, β have degree at most d and naive height at most H , where $H \geq 2$, then each of $\alpha + \beta, \alpha\beta$ has degree at most d^2 , and

$$H(\alpha + \beta), H(\alpha\beta) \leq (1 + d^2)^{d^2} H^{4d^2}. \quad (4.6)$$

In particular, $\log H(\alpha + \beta)$ and $\log H(\alpha\beta)$ are at most $c(d) \log H$, with

$$c(d) = d^2 \left(4 + \frac{\log(1 + d^2)}{\log 2} \right).$$

For $H \geq 1$, the same logarithmic conclusion holds with $\log(2H)$.

Proof. Suppose $aX^s + \cdots + a_s$ is the primitive minimal polynomial of α . Here $1 \leq a \leq H$ and $s \leq d$. If $|\alpha| \geq 1$, its defining equation gives

$$a|\alpha|^s \leq H \sum_{j=0}^{s-1} |\alpha|^j \leq sH |\alpha|^{s-1}.$$

Thus every conjugate of α has size at most dH ; the same holds for β . Also $a\alpha$ is an algebraic integer: substituting X/a into the defining polynomial and multiplying by a^{s-1} gives a monic integral polynomial in X .

Let $b \leq H$ be the leading coefficient for β , and set $q = ab$. Both

$$q\alpha\beta = (a\alpha)(b\beta), \quad q(\alpha + \beta) = b(a\alpha) + a(b\beta)$$

are algebraic integers. The compositum $\mathbb{Q}(\alpha, \beta)$ is spanned by the products of powers $1, \alpha, \dots, \alpha^{s-1}$ and $1, \beta, \dots, \beta^{t-1}$; consequently either new number γ has degree $D \leq st \leq d^2$.

The monic minimal polynomial of $q\gamma$ has integral coefficients. Evaluating it at qX gives an integral polynomial with root γ and leading coefficient q^D . By Gauss's lemma, the positive leading coefficient c of the primitive minimal polynomial of γ divides q^D . Hence $c \leq H^{2D}$.

Every conjugate of $\gamma = \alpha\beta$ has size at most $d^2 H^2$. A conjugate of $\alpha + \beta$ has size at most $2dH \leq d^2 H^2$, since $H \geq 2$. Expanding

$$c \prod_{j=1}^D (X - \gamma^{(j)})$$

shows that each coefficient is at most

$$c \prod_j (1 + |\gamma^{(j)}|) \leq H^{2D} (1 + d^2 H^2)^D \leq (1 + d^2)^{d^2} H^{4d^2}.$$

This proves (4.6). Taking logarithms and using $\log H \geq \log 2$ gives the displayed $c(d)$. For $1 \leq H < 2$, apply the result with $2H$. $\square$

The restriction or padding at height one is real: $H(1) = 1$, but $H(1 + 1) = 2$. A statement $\log H' \leq c(d) \log H$ without padding cannot hold at $H = 1$. Inversion is particularly simple: reversing the minimal polynomial gives $H(\alpha^{-1}) = H(\alpha)$ for $\alpha \neq 0$, with unchanged degree. Combining this fact with Lemma 4.2 handles every finite sum, product and quotient appearing in the later induction.

Account for the constant when dividing by a coefficient

The effective proof normalizes a nonzero logarithm coefficient to -1 . A degree or height estimate alone does not finish that step: after obtaining a lower bound for the normalized form, we must multiply back by the coefficient and absorb every fixed factor into an exponent of B .

Lemma 4.8 (normalization budget). Suppose $\beta_0, \dots, \beta_n$ have degree at most d , naive height at most $B \geq 2$, and $\beta_n \neq 0$. Put

$$r = 4d^2, \quad K_d = (1 + d^2)^{d^2}, \quad \gamma_j = -\beta_j/\beta_n \quad (0 \leq j < n).$$

Each γ_j has degree at most d^2 and height at most $K_d B^r$. For fixed logarithms define

$$\Lambda_* = \gamma_0 + \sum_{j=1}^{n-1} \gamma_j \ell_j - \ell_n = -\Lambda/\beta_n.$$

If $|\Lambda_*| > (K_d B^r)^{-C_*}$, with $C_* > 0$, then

$$|\Lambda| > B^{-C}, \quad C = 1 + rC_* + \frac{\log d + C_* \log K_d}{\log 2}.$$

Proof. Inversion preserves naive height and degree, and changing sign preserves them too. Apply Lemma 4.2 to $-\beta_j$ and β_n^{-1} . This proves the asserted degree and height bounds, including zero coefficients.

The root bound proved in Lemma 4.2, now applied to β_n^{-1} , gives $|\beta_n^{-1}| \leq dB$, hence $|\beta_n| \geq (dB)^{-1}$. Consequently

$$|\Lambda| = |\beta_n| |\Lambda_*| > d^{-1} K_d^{-C_*} B^{-(1+rC_*)}.$$

For a fixed $M = dK_d^{C_*} \geq 1$ and $B \geq 2$, we have $M \leq B^{\log M / \log 2}$. Absorbing precisely this factor gives the displayed exponent. $\square$

If every logarithm coefficient is zero, the form is simply β_0 . It is either zero, or the same inverse root bound gives

$$|\beta_0| \geq (dB)^{-1} \geq B^{-1 - \log d / \log 2}.$$

Increasing the exponent slightly makes the final inequality strict, as in Theorem 4.1. This also covers the constant-only case without dividing by a nonexistent logarithm coefficient.

The normalization budget keeps the coefficient-height variable separate from the fixed bases and branches. The theorem's exponent may deteriorate when we normalize or eliminate a coefficient, but it remains independent of B . In the next lesson the analytic gain must exceed all the denominator and height costs proved here.

8. Exercises

1. **Easy.** Prove directly that $\Delta(x; k) = \binom{x+k}{k}$ is an integer for $x \geq 0$. Explain the cases $x = 0$ and $k = 1$.
2. **Medium.** Prove (4.10) with an explicit absolute constant, without the prime number theorem. State where you use that the interval contains exactly k consecutive integers.
3. **Medium.** Prove Lemma 4.5 by induction on $\deg P$, using $Q(X) = P(X) - P(X+1)$. Treat $m = 0$ and $m = n$ explicitly.
4. **Hard.** Compute (4.17) by column operations. Explain separately the factors $w_0 w_1$ and $(w_1 - w_0)^4$, and verify the numerical case $w_0 = 2, w_1 = 3$.
5. **Medium.** At $k = l = 2$, compare the interval and uniform denominators for the normalized derivatives at $x = 1$. Compute the derivatives at $x = -1$, and explain why the uniform integer-valued assertion survives although positivity no longer holds at every order.
6. **Hard.** Suppose all coefficients in Lemma 4.8 are rational of naive height at most B . Prove that each normalized coefficient has height at most B^2 , and show that a bound $|\Lambda_*| > (B^2)^{-C_*}$ gives $|\Lambda| > B^{-(1+2C_*)}$. Compare this exponent with the general bound at $d = 1$.

9. Solutions

Solution 1. The product in (4.7) is $(x+k)!/x!$; dividing by $k!$ gives the binomial coefficient counting the k -element subsets of a set of size $x+k$. Hence it is integral. At $x=0$ it equals 1, and at $k=1$ it equals $x+1$. This also proves its positivity.

Solution 2. The factorial valuation argument in Section 6 proves $V(N) \mid \binom{N}{\lfloor N/2 \rfloor} V(\lfloor N/2 \rfloor)$, hence $V(N) \leq 8^N$. Split the interval LCM into primes $p \leq k$ and $p > k$. An interval of length k contains a multiple of every prime power at most k , so $V(k)$ divides its small-prime part. Any extra power is at most pN/k , where $N = x+k$; thus that part is at most $64^k(N/k)^k$. A prime $p > k$ has at most one multiple in the interval, so the large-prime part divides the product of the interval divided by $k!$, namely $\binom{N}{k} \leq (eN/k)^k$. Multiplication gives $(64e)^k(N/k)^{2k} \leq (16N/k)^{2k}$. Both uses of the interval length are essential. This proves the assigned bound with $c=16$, including $k=1$.

Solution 3. For $m=0$, P and the lower-degree monomials are independent by their leading coefficients. For degree 1 and $m=1$, the difference $P(X) - P(X+1)$ is a nonzero constant, so the two translates are independent.

Assume the result for degree $n-1$, and take $1 \leq m \leq n$. Suppose

$$R(X) = \sum_{j=0}^m A_j P(X+j)$$

has degree at most $n-m-1$, where a bound of -1 means the zero polynomial. Its leading coefficient gives $\sum_j A_j = 0$. Put $C_j = A_0 + \dots + A_j$. Telescoping gives the exact identity

$$R(X) = \left(\sum_{j=0}^m A_j \right) P(X+m) + \sum_{j=0}^{m-1} C_j (P(X+j) - P(X+j+1)).$$

Since $Q(X) = P(X) - P(X+1)$ has degree $n-1$ in characteristic zero, the induction hypothesis with parameter $m-1$ says that its m translates, together with the monomials up to degree $n-m-1$, are independent. Therefore every $C_j = 0$. Successive differences give $A_0 = \dots = A_{m-1} = 0$, and the vanishing total gives $A_m = 0$. When $m=n$, the induction uses no lower-degree monomials and the same argument works.

Solution 4. Write $a = w_0$, $b = w_1$, and factor a from column 2 and b from column 4. The remaining matrix has columns

$$C(a), C'(a), C(b), C'(b), \quad C(z) = (1, z, z^2, z^3)^\top.$$

Set $\delta = b - a$, and use

$$C(b) = C(a) + \delta C'(a) + \frac{\delta^2}{2} C''(a) + \frac{\delta^3}{6} C'''(a),$$

$$C'(b) = C'(a) + \delta C''(a) + \frac{\delta^2}{2} C'''(a).$$

Subtract $C(a) + \delta C'(a)$ from the third column and $C'(a)$ from the fourth. In the last two columns, the coefficient determinant relative to $C'''(a), C''''(a)$ is

$$\det \begin{pmatrix} \delta^2/2 & \delta \\ \delta^3/6 & \delta^2/2 \end{pmatrix} = \delta^4/12.$$

Meanwhile $\det[C(a), C'(a), C''(a), C'''(a)] = 1 \cdot 1 \cdot 2 \cdot 6 = 12$, since its matrix is triangular. The remaining determinant is δ^4 , and restoring the factors gives $ab(b-a)^4$. For $a=2, b=3$, the original rows are

$$(1, 0, 1, 0), (2, 2, 3, 3), (4, 8, 9, 18), (8, 24, 27, 81),$$

and the formula gives 6.

Solution 5. At $x = 1$ the normalized derivatives are $9, 15, 37/4, 5/2, 1/4$, followed by zeros. The interval LCM is 6; its cleared values are integers by Lemma 4.3. The uniform LCM is $V(2) = 2$, and the cleared values are $9, 30, 37, 20, 4$. Thus the same derivatives admit the smaller denominator 2^m . Write $x = -1 + t$. Then

$$\Delta(-1 + t; 2)^2 = \frac{1}{4}t^2(1 + t)^2 = \frac{1}{4}t^2 + \frac{1}{2}t^3 + \frac{1}{4}t^4.$$

Its normalized derivatives at -1 are the coefficients $0, 0, 1/4, 1/2, 1/4$, so multiplication by 2^m gives $0, 0, 1, 4, 4$. The first two entries vanish even though their orders are at most the polynomial degree. Lemma 4.3 did not assert positivity at negative points; Lemma 4.7 proves integer-valued cleared derivatives there directly by prime valuations.

Solution 6. Write $\beta_j = p_j/q_j$ and $\beta_n = p_n/q_n$ in lowest terms, with positive denominators and $p_n \neq 0$. All numerator and denominator absolute values are at most B . The unreduced fraction for $-\beta_j/\beta_n$ has numerator $-p_jq_n$ and denominator q_jp_n , each of absolute value at most B^2 . Reduction cannot increase either, proving the height bound. Also $|\beta_n| = |p_n|/q_n \geq 1/B$. Therefore

$$|\Lambda| = |\beta_n||\Lambda_*| > B^{-1}(B^2)^{-C_*} = B^{-(1+2C_*)}.$$

At $d = 1$, Lemma 4.8 has $r = 4$, $K_d = 2$, and exponent $1 + 5C_*$. The sharper rational calculation avoids the deliberately coarse general compositum and conjugate bounds. Both estimates are uniform in the same variable B .

Prerequisites and continuation

Theorem 4.1 is stated here and fully proved in *Effective lower bounds II: proof of Baker's theorem*. Every arithmetic tool used to prepare that proof has been proved above. The existing *Number fields* lesson supplies the elementary algebraic-integer criterion at exactly the required generality. The later many-logarithm lesson owns the modern quantitative refinements; the brief historical discussion above supplies neither a substitute proof nor an additional prerequisite for the next lesson.

References

- [Waldschmidt 2000] Michel Waldschmidt, *Diophantine Approximation on Linear Algebraic Groups*, Springer, 2000. Author's full text. Effective homogeneous and nonhomogeneous estimates, integer-valued polynomial bases and Matveev's derivative-denominator method.
- [Waldschmidt 2003] Michel Waldschmidt, "Linear Independence Measures for Logarithms of Algebraic Numbers," in *Diophantine Approximation*, Lecture Notes in Mathematics 1819, Springer, 2003, 249–344. Author's full text. Baker's auxiliary-function method, normalized derivations and quantitative extrapolation.

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