

8. On the polynomial representation of the invariants of a system of arbitrarily many ground forms

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At the end of his contribution to the Schwarz Festschrift, “On the Invariants of a System of Arbitrarily Many Ground Forms,” D. Hilbert conjectures that these invariants can be expressed polynomially in the finitely many invariants of the system (J, PJ) , where J denotes a complete system of invariants of the linearly independent ground forms, and the invariants PJ arise from J by polar processes.

In I below, I show that this conjecture is indeed correct, as a simple consequence of the reduction theorem asserting that every form in arbitrarily many rows of n variables each can be represented as a sum of polars of forms that contain only n fixed rows and are themselves derived from the original form by polar processes. This reduction theorem, first formulated by F. Mertens, is a special case of the generalization by A. Capelli, F. Mertens, and J. Deruyts of the Clebsch–Gordan series expansion to forms in arbitrarily many rows of n variables each; that generalization was proved by formally transforming the differentiation processes.¹¹⁹

In II, I give a more conceptual proof of the reduction theorem by treating the question as a problem concerning the equivalence of linear families of forms. This viewpoint, and an essential part of the arguments used, are taken from a paper “On Algebraic Module Systems”¹²⁰ and from related unpublished theorems of E. Fischer, whose use he kindly permitted.

Finally, in III, I show how the corresponding considerations lead also to the most general series expansions.

I.

Let θ be a form homogeneous in each of the $N > n$ rows

$$A_1, A_2, \dots, A_N,$$

where in general the row A_k consists of the n elements

$$A_k^{(1)}, A_k^{(2)}, \dots, A_k^{(n)}.$$

The coefficients of θ may be indeterminates or elements of a given field of rationality. Let P further denote a polynomial — with rational-number coefficients — in the polar processes

$$P_{hk} = \left(A_h \frac{\partial}{\partial A_k} \right) = A_h^{(1)} \frac{\partial}{\partial A_k^{(1)}} + A_h^{(2)} \frac{\partial}{\partial A_k^{(2)}} + \dots + A_h^{(n)} \frac{\partial}{\partial A_k^{(n)}},$$

where $P_{hk}P_{ij}\theta$ means that P_{hk} is applied to $P_{ij}\theta$.

The reduction theorem just mentioned is then expressed by the identity

$$\theta = \sum PZ, \tag{1}$$

where the forms Z contain only the rows

$$A_1, A_2, \dots, A_n$$

and are derived from θ by polar processes P .

Now consider the system of ground forms (F') , consisting of the N forms of equal order and equal number of variables

$$F_1, F_2, \dots, F_N,$$

whose coefficients are taken respectively as the N rows

$$A_1, A_2, \dots, A_N.$$

Let (J) denote a complete system — consisting of finitely many invariants — for $F_1, \dots, F_n$, such that every invariant of $F_1, \dots, F_n$ is a polynomial expression in the invariants (J) . Hilbert’s conjecture then says that

¹¹⁹F. Mertens: “Über eine Formel der Determinantentheorie” (formula 5), Sitzb. d. Ak. d. Wiss. Wien, vol. 91, II. Abt. (1885), and, more fully and with applications, “Über ganze Funktionen von m Systemen von je n Unbestimmten,” Monatsh. f. Math. u. Phys., vol. 4 (1893). For some of Capelli’s proofs (beginning in 1882), see, for example, Math. Ann. 37 or the autographed “Lezioni sulla teoria delle forme algebriche” (1902). J. Deruyts: *Essai d’une théorie générale des formes algébriques*, Mém. d. la Soc. Roy. des Sciences de Liège (2) 17 (1892).

¹²⁰E. Fischer: Über algebraische Modulsysteme und lineare homogene partielle Differentialgleichungen mit konstanten Koeffizienten, §§ 1–4, J. f. M. 140 (1911).

a complete system for the simultaneous invariants S of $F_1, \dots, F_N$ is given by the finitely many invariants (J, PJ) , where PJ denotes all invariants derivable from J by polar processes P .

Indeed, every simultaneous invariant S of (F') with rational-number coefficients is a form homogeneous in each of the N rows $A_1, \dots, A_N$, so identity (1) applies to it. Since applying polar processes P to S is known to produce invariants again, the forms Z are likewise invariants; because they contain only the rows $A_1, \dots, A_n$, they are simultaneous invariants of $F_1, \dots, F_n$ and hence, by hypothesis, polynomial functions of the invariants (J) . Thus identity (1) becomes

$$S = \sum PY,$$

where the Y are monomials in the invariants (J) . By definition, carrying out the processes P on Y consists of successively applying the simple polar operations P_{hk} a finite number of times. The product rule for differentiation turns these into sums of products of invariants in (J) and (PJ) . The same is true when P_{hk} is applied to a product of elements of (J) and (PJ) ; hence PY is likewise a polynomial expression in (J, PJ) . *This proves the polynomial representability of all invariants S in terms of (J, PJ) .*

II.

Before proving the reduction theorem, we recall some facts about linear families of forms. A *linear family of forms over K* means a system of forms of equal degree, complete in the sense that, together with θ_1 and θ_2 , it always contains $c_1\theta_1 + c_2\theta_2$, where c_1, c_2 are elements of a prescribed field of rationality K , and K contains the coefficient field of the θ 's. Among these forms there are always finitely many — say ρ — linearly independent over K , while every $\rho + 1$ forms satisfy a linear relation with coefficients in K ; the family has *rank ρ over K* .¹²¹ We shall use the immediate fact that if $\mathcal{L}$ and $\mathcal{T}$ are two linear families over K of the same rank ρ , and $\mathcal{T}$ is a subfamily of $\mathcal{L}$, then $\mathcal{L}$ and $\mathcal{T}$ are identical (equivalent).

With these preliminaries, we turn to the reduction theorem. The identity from I,

$$\theta = \sum PZ,$$

becomes an analogous identity for $P\theta$ when the same polar process P is applied to both sides. Thus the reduction theorem simultaneously represents every polar of θ as a sum of the special polars PZ . Conversely, these special polars certainly belong to the totality of all polars, so the reduction theorem asserts that the two linear families of forms are equivalent; in particular, the same holds for the subfamilies whose forms have the same degree as θ in each individual row. To prove this equivalence, we insert an intermediate family determined by the forms Z . For simplicity, we first give the proof for three binary rows ($N = 3, n = 2$) and then briefly indicate the parallel general proof.

Thus let $\theta(x, y, z)$ have degrees α, β, p , respectively, in the rows

$$x_1x_2; \quad y_1y_2; \quad z_1z_2;$$

and initially let the coefficients of θ be indeterminates a (or rational numbers). We consider the following three linear families of forms.

1. The family $\mathcal{L}$, consisting of all forms $f(x, y, z)$ that arise from θ by polar processes P and have the same degree as θ in each of the rows x, y, z ; in particular, $\mathcal{L}$ contains θ . Regarding $f(x, y, z)$ as a form in x, y, z and the indeterminates a , its coefficient field is the field R of rational numbers. One must also take $K = R$, since $c_1P_1 + c_2P_2$ is again a process P only when c_1, c_2 are rational numbers. Let $\mathcal{L}$ have rank ρ over R .

2. The family $\mathcal{S}$, consisting of all forms $g(\lambda; x, y)$ defined by

$$g(\lambda; x, y) = f(x, y, \lambda_1x + \lambda_2y),$$

or, in terms of polar processes,

$$\begin{aligned} g(\lambda; x, y) &= \frac{1}{p!} \left[(\lambda_1x + \lambda_2y) \frac{\partial}{\partial z} \right]^p f(x, y, z) \\ &= g_0(xy)\lambda_1^p + g_1(x, y)\lambda_1^{p-1}\lambda_2 + \dots + g_p(xy)\lambda_2^p, \end{aligned}$$

¹²¹More general linear families of forms — whose rank need not be finite — are obtained by dropping either the restriction of equal degree or the requirement that K contain the coefficient field of the θ 's.

where

$$i!(p-i)!g_i(xy) = \left(y \frac{\partial}{\partial z}\right)^i \left(x \frac{\partial}{\partial z}\right)^{p-i} f(xyz).$$

¹²² Thus the $g_i(xy)$ provide the forms Z in identity (1). The g are forms with rational-number coefficients in x, y, λ and the indeterminates a ; let $\mathcal{S}$ have rank σ over R .

3. The family $\mathcal{T}$, obtained from $\mathcal{S}$ by the inverse polar process, just as $\mathcal{S}$ is obtained from $\mathcal{L}$; the forms h of $\mathcal{T}$ are defined by

$$\begin{aligned} h(x, y, z) &= \frac{1}{p!} \left[z \left(\frac{\partial}{\partial \lambda_1} \frac{\partial}{\partial x} + \frac{\partial}{\partial \lambda_2} \frac{\partial}{\partial y} \right) \right]^p g(\lambda; xy) \\ &= h_0(xyz) + h_1(xyz) + \cdots + h_p(xyz), \end{aligned}$$

where, by 2,

$$h_i(xyz) = \left(z \frac{\partial}{\partial x}\right)^{p-i} \left(z \frac{\partial}{\partial y}\right)^i g_i(xy).$$

The individual h_i , and consequently also h , are thus forms PZ or sums of such forms; let $\mathcal{T}$ have rank τ over R .

The construction of the forms $h(x, y, z)$ of $\mathcal{T}$ shows that they arise from θ by polar processes P and have the same degree as θ in each of the rows x, y, z . Thus $\mathcal{T}$ is a subfamily of $\mathcal{L}$, and by the fact noted above it remains only to prove equality of their ranks. This proof makes no further use of the special structure of the $f(xyz)$ as polars of one and the same form θ .

a) *To compare the ranks of $\mathcal{L}$ and $\mathcal{S}$, we use Lemma a): $f(x, y, \lambda_1 x + \lambda_2 y) = 0$ necessarily implies $f(x, y, z) = 0$.* This follows immediately from the identity among three binary rows

$$(xy)z + (yz)x + (zx)y = 0, \quad (xy) = (x_1 y_2 - x_2 y_1), \dots,$$

which gives

$$f[x, y, (zy)x + (xz)y] = (xy)^p f(x, y, z).$$

Thus, substituting $\lambda_1 = (zy)$ and $\lambda_2 = (xz)$ into $f(x, y, \lambda_1 x + \lambda_2 y) = 0$ (identically in λ), and using $(xy) \neq 0$, gives necessarily $f(x, y, z) = 0$.

By this lemma, the forms of $\mathcal{S}$ and $\mathcal{L}$ correspond one-to-one. Indeed,

$$f_1(x, y, \lambda_1 x + \lambda_2 y) = g(\lambda; xy), \quad f_2(x, y, \lambda_1 x + \lambda_2 y) = g(\lambda; xy)$$

necessarily implies

$$f_1(xyz) = f_2(xyz).$$

Moreover, every relation in $\mathcal{S}$ is preserved among the uniquely corresponding forms in $\mathcal{L}$, and conversely. For

$$c_1 g^{(1)}(\lambda; xy) + c_2 g^{(2)}(\lambda; xy) + \cdots + c_r g^{(r)}(\lambda; xy) = 0$$

necessarily implies, by the lemma,

$$c_1 f^{(1)}(x, y, z) + c_2 f^{(2)}(x, y, z) + \cdots + c_r f^{(r)}(x, y, z) = 0.$$

This proves that $\mathcal{L}$ and $\mathcal{S}$ have equal rank: $\rho = \sigma$.

b) *To compare the ranks of $\mathcal{S}$ and $\mathcal{T}$, we use the corresponding Lemma b): $h(x, y, z) = 0$ necessarily implies $g(\lambda; xy) = 0$.* By 3, the equation $h(x, y, z) = 0$ is equivalent to the vanishing of the individual differential monomials

$$\left(\frac{\partial}{\partial \lambda_1} \frac{\partial}{\partial x_1} + \frac{\partial}{\partial \lambda_2} \frac{\partial}{\partial y_1} \right)^{p_1} \left(\frac{\partial}{\partial \lambda_1} \frac{\partial}{\partial x_2} + \frac{\partial}{\partial \lambda_2} \frac{\partial}{\partial y_2} \right)^{p_2} g(\lambda; x, y), \quad p_1 + p_2 = p.$$

¹²²As in I,

$$\left(t \frac{\partial}{\partial x} \right) = t_1 \frac{\partial}{\partial x_1} + t_2 \frac{\partial}{\partial x_2},$$

and hence in particular

$$\begin{aligned} \left[(\lambda_1 x + \lambda_2 y) \frac{\partial}{\partial z} \right] &= (\lambda_1 x_1 + \lambda_2 y_1) \frac{\partial}{\partial z_1} + (\lambda_1 x_2 + \lambda_2 y_2) \frac{\partial}{\partial z_2}, \\ \left[z \left(\frac{\partial}{\partial \lambda_1} \frac{\partial}{\partial x} + \frac{\partial}{\partial \lambda_2} \frac{\partial}{\partial y} \right) \right] &= z_1 \left(\frac{\partial}{\partial \lambda_1} \frac{\partial}{\partial x_1} + \frac{\partial}{\partial \lambda_2} \frac{\partial}{\partial y_1} \right) + z_2 \left(\frac{\partial}{\partial \lambda_1} \frac{\partial}{\partial x_2} + \frac{\partial}{\partial \lambda_2} \frac{\partial}{\partial y_2} \right). \end{aligned}$$

Further differentiation also yields the vanishing of the differential monomials

$$\begin{aligned} & \left(\frac{\partial}{\partial x_1} \right)^{\alpha_1} \left(\frac{\partial}{\partial x_2} \right)^{\alpha_2} \left(\frac{\partial}{\partial y_1} \right)^{\beta_1} \left(\frac{\partial}{\partial y_2} \right)^{\beta_2} \\ & \cdot \left(\frac{\partial}{\partial \lambda_1} \frac{\partial}{\partial x_1} + \frac{\partial}{\partial \lambda_2} \frac{\partial}{\partial y_1} \right)^{p_1} \left(\frac{\partial}{\partial \lambda_1} \frac{\partial}{\partial x_2} + \frac{\partial}{\partial \lambda_2} \frac{\partial}{\partial y_2} \right)^{p_2} g(\lambda; xy). \end{aligned}$$

Consequently every linear combination of these differential monomials vanishes, and hence so does every form

$$f \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial \lambda_1} \frac{\partial}{\partial x} + \frac{\partial}{\partial \lambda_2} \frac{\partial}{\partial y} \right) g(\lambda, xy).$$

Choose f in particular so that

$$f(x, y, \lambda_1 x + \lambda_2 y) = g(\lambda; xy),$$

Then one also obtains

$$g \left(\frac{\partial}{\partial \lambda}; \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) g(\lambda, x, y) = 0;$$

or, on writing

$$g(\lambda; xy) = \sum G_i \lambda_1^{\nu_1} \lambda_2^{\nu_2} x_1^{\alpha_1} x_2^{\alpha_2} y_1^{\beta_1} y_2^{\beta_2},$$

where the G_i are linear forms with rational-number coefficients in the indeterminates a , one has

$$\sum \nu_1! \nu_2! \alpha_1! \alpha_2! \beta_1! \beta_2! G_i^2 = 0,$$

and hence $g(\lambda; xy) = 0$.

Exactly as in a), Lemma b) implies that $\mathcal{S}$ and $\mathcal{T}$ have equal rank: $\sigma = \tau$.

Thus a) and b) give $\rho = \tau$; since $\mathcal{T}$ is a subfamily of $\mathcal{L}$, this proves that the two linear families are equivalent. Hence every form of $\mathcal{L}$, and in particular θ , can be represented as a linear combination with rational-number coefficients of ρ linearly independent forms $h(x, y, z)$:

$$\theta = \sum c_i h^{(i)}(x, y, z) = \sum PZ.$$

This identity, derived for indeterminate coefficients a of θ , remains valid when the indeterminates a are replaced by elements of any field of rationality. Thus *the reduction theorem is proved in full generality for three binary rows*.

The general proof for N rows of n variables each is entirely parallel. Let $\theta(A)$ have degree α_i in the row A_i . We again consider three linear families of forms:

1. the family $\mathcal{L}$, consisting of all forms $f(A)$ that arise from $\theta(A)$ by polar processes P and have the same degree as $\theta(A)$ in each row A_i ;
2. the family $\mathcal{S}$, consisting of all forms $g(\lambda; A_1 \cdots A_n)$ obtained from $f(A)$ by the substitutions

$$A_{n+1} = \lambda_{n+1}^{(1)} A_1 + \lambda_{n+1}^{(2)} A_2 + \cdots + \lambda_{n+1}^{(n)} A_n,$$

$$\vdots$$

$$A_N = \lambda_N^{(1)} A_1 + \lambda_N^{(2)} A_2 + \cdots + \lambda_N^{(n)} A_n;$$

the coefficients of the individual monomials in the λ 's in $g(\lambda; A_1 \cdots A_n)$ give the forms Z in identity (1);

3. the family $\mathcal{T}$, consisting of all forms $h(A)$ defined by

$$\begin{aligned} h(A) = & \left[A_{n+1} \left(\frac{\partial}{\partial \lambda_{n+1}^{(1)}} \frac{\partial}{\partial A_1} + \cdots + \frac{\partial}{\partial \lambda_{n+1}^{(n)}} \frac{\partial}{\partial A_n} \right) \right]^{\alpha_{n+1}} \\ & \cdots \left[A_N \left(\frac{\partial}{\partial \lambda_N^{(1)}} \frac{\partial}{\partial A_1} + \cdots + \frac{\partial}{\partial \lambda_N^{(n)}} \frac{\partial}{\partial A_n} \right) \right]^{\alpha_N} g(\lambda; A_1 \cdots A_n). \end{aligned}$$

The form $h(A)$ is a sum of forms PZ .

The identity among every $n+1$ rows ensures that Lemma a) remains valid; Lemma b) remains valid as well, since the number of variables played no role in its proof. Thus $\mathcal{L}$ and $\mathcal{T}$ have the same rank and, since $\mathcal{T}$ is a subfamily of $\mathcal{L}$, are identical. Consequently,

$$\theta = \sum PZ$$

holds for indeterminate coefficients and therefore for coefficients in any field of rationality; *this proves the reduction theorem in general*.

III.

We briefly show how analogous arguments also yield the *general series expansion* (for $N \leq n$). Let Z be a form separately homogeneous in the n rows $A_1, \dots, A_n$ of n quantities each, and let Δ be the determinant of these rows. We first prove the identity corresponding to (1):

$$Z = \sum PH \pmod{\Delta}, \quad (2)$$

where the forms H contain only the rows $A_1, \dots, A_{n-1}$ and are derived from Z by processes P .

The proof converts the relations derived in II into congruences modulo Δ . For simplicity, take three ternary rows x, y, z and assume the coefficients to be indeterminates.

Define the three linear families $\mathcal{L}, \mathcal{S}, \mathcal{T}$ as in II for binary rows, and let their ranks be ρ, σ, τ . Let ρ^* and τ^* denote the ranks of $\mathcal{L}$ and $\mathcal{T}$ modulo Δ ; that is, $\mathcal{L}$ contains ρ^* , but not $\rho^* + 1$, forms linearly independent modulo Δ . Then, as in II, we have

Lemma a): $f(x, y, \lambda_1 x + \lambda_2 y) = 0$ necessarily implies $f(xyz) \equiv 0 \pmod{\Delta}$, and conversely. Indeed, the identity among four ternary rows,

$$\Delta_1 x - \Delta_2 y + \Delta_3 z = \Delta t, \quad \Delta_1 = (yzt), \dots,$$

always gives a linear relation modulo Δ among three ternary rows:

$$\Delta_1 x - \Delta_2 y + \Delta_3 z \equiv 0 \pmod{\Delta},$$

and hence, as in II,

$$f(x, y, -\Delta_1 x + \Delta_2 y) \equiv \Delta_3^p f(x, y, z) \pmod{\Delta}.$$

Since Δ is irreducible and does not divide Δ_3 , the identity $f(x, y, \lambda_1 x + \lambda_2 y) = 0$ (identically in λ) therefore necessarily implies $f(x, y, z) \equiv 0 \pmod{\Delta}$. The converse is clear from $\Delta(x, y, \lambda_1 x + \lambda_2 y) = 0$. Thus, as in II, $\rho^* = \sigma$.

Lemma b), with its proof, remains valid; hence $\sigma = \tau$.

To prove $\tau = \tau^*$, we use

Lemma c): $h(x, y, z) \equiv 0 \pmod{\Delta}$ necessarily implies $h(x, y, z) = 0$. As a polar, $h(x, y, z)$ is annihilated by the familiar Ω -process; this is expressed by the differential equation

$$\Delta \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) h(x, y, z) - \left(\frac{\partial}{\partial x} \frac{\partial}{\partial y} \frac{\partial}{\partial z} \right) h(x, y, z) = 0.$$

¹²³ If $h(x, y, z) = \Delta(x, y, z) \cdot k(x, y, z)$, then further differentiation shows that

$$k \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \Delta \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) = h \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) h(x, y, z),$$

also vanishes, and hence so does $h(x, y, z)$. Thus $\rho^* = \sigma = \tau = \tau^*$; because $\mathcal{T}$ is a subfamily of $\mathcal{L}$, identity (2) is proved. The same argument proves the result for n variables.

Applying identity (2) to the multiplier of Δ gives an expansion

$$Z = \varphi_0 + \Delta \varphi_1 + \Delta^2 \varphi_2 + \dots + \Delta^\mu \varphi_\mu,$$

where each φ_i is annihilated by the Ω -process. Repeating the process i times gives — since, in general, $\Omega(\Delta \cdot \psi) = c_1 \psi + c_2 \Delta \cdot \Omega(\psi)$ —

$$c \cdot \Omega^i(Z) \equiv \varphi_i \pmod{\Delta};$$

while (2) gives

$$c \cdot \Omega^i(Z) \equiv \sum PH_i \pmod{\Delta},$$

where the H_i are derived from $\Omega^i(Z)$ in the same way as H is derived from Z . Lemma c), extended to n variables, therefore gives $\varphi_i = \sum PH_i$, and consequently

$$Z = \sum PH + \Delta \cdot \sum PH_1 + \Delta^2 \cdot \sum PH_2 + \dots + \Delta^\mu \cdot \sum PH_\mu \quad (3)$$

¹²³By 3 one has

$$- \left(\frac{\partial}{\partial \xi} \frac{\partial}{\partial \eta} \left(\frac{\partial}{\partial \lambda_1} \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \lambda_2} \frac{\partial}{\partial \eta} \right) \right) \left(x \frac{\partial}{\partial \xi} \right)^{\alpha-1} \left(y \frac{\partial}{\partial \eta} \right)^{\beta-1} \left[z \left(\frac{\partial}{\partial \lambda_1} \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \lambda_2} \frac{\partial}{\partial \eta} \right) \right]^{p-1} g(\lambda; \xi, \eta),$$

and $\Omega(h)$ therefore vanishes because the determinant factor vanishes.

as the most general expansion of a form in n rows of n variables each in terms of polars of forms in only $n - 1$ rows.

The expansion for forms in N rows of n variables ($N < n$) is then obtained, as is well known, by a simple substitution. Put, for $N = 3$ and $n > 3$,

$$u = \xi_1 x + \xi_2 y + \xi_3 z, \quad v = \eta_1 x + \eta_2 y + \eta_3 z, \quad w = \zeta_1 x + \zeta_2 y + \zeta_3 z,$$

and expand $H(u, v, w) = Z(\xi, \eta, \zeta)$ by (3). Since ξ, η, ζ occur only in the combinations u, v, w , one then has

$$\left(\eta \frac{\partial}{\partial \xi}\right) = \left(v \frac{\partial}{\partial u}\right) \cdots, \quad \Omega = \sum_{ikl} \left(\frac{\partial}{\partial u} \frac{\partial}{\partial v} \frac{\partial}{\partial w}\right)_{ikl} (xyz)_{ikl} = \nabla_{uvw}(xyz).$$

Under the specialization $\xi_1 = \eta_2 = \zeta_3 = 1$ and $\xi_2 = \xi_3 = \cdots = \zeta_2 = 0$, the form $H(u, v, w)$ becomes $H(x, y, z)$, while $\nabla_{uvw}(xyz)$ becomes, up to a numerical factor, $\nabla_{xyz}(xyz) = \nabla_{xyz}$, since applying $(y\partial/\partial x)$ and the analogous operators to the determinants $(xyz)_{ikl}$ makes them vanish. The corresponding statement holds for N rows, so (3) yields the expansion

$$H = \sum P\Phi + \sum P\Phi_1 + \cdots + \sum P\Phi_\mu, \quad (4)$$

where the Φ_i are obtained from $\nabla_{A_1 \dots A_N}^i H$ by processes P and contain only the rows $A_1, \dots, A_{N-1}$, apart from the determinant combinations occurring in ∇^i , which, as noted, play no role in polarization.

Replacing these determinants in ∇^i by indeterminates p , one can again expand the resulting forms by (4), finally obtaining

$$H = \left[\sum P\psi(p) \right]_{p=\text{determinants of } A_1 \dots A_N}, \quad (5)$$

where the ψ are obtained from the original form by the processes P , $\nabla_{A_1 \dots A_N}(p)$, $\nabla_{A_1 \dots A_{N-1}}(p)$, and so forth. These expansions and their combinations — for example, applying (3), followed by (4) and (5), to the forms Z occurring in (1) — exhaust all known expansions for forms in arbitrarily many rows of n cogredient variables each.

Erlangen, 5 January 1915.