

3. On the Invariant Theory of Forms in n Variables²⁹

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In the projective invariant theory of forms in n variables the main problem has been solved: the finiteness of the system of forms has been proved. A series of further questions, however, have not been treated, namely those concerned with methods for investigating the connection among forms. I would like briefly to communicate the results that I have obtained with respect to such questions, but for the sake of clarity I shall first sketch the questions in the ternary domain, where they are known.

I have first in mind the two fundamental theorems of the symbolic method: the theorem that all invariant formations can be represented symbolically, and the second theorem that all relations existing among invariants are obtained by successive application of a small number of symbolic identities; thus the symbolic method gives all relations without leaving the domain of invariants.³⁰ Built upon this are the processes for generating the forms, the symbolic folding process and the nonsymbolic differentiation processes, the theory of series expansions, and the like. In this connection I should also mention a theorem proved in my dissertation,³¹ namely that all foldings can be generated by successive application of the two possible “cogredient” foldings; by this, for a symbolic product $s_x t_x u_\sigma u_\tau$, I mean the two foldings (stu) and $(\sigma\tau x)$. On this theorem one can build the theory of reducers and of the reduction of systems of forms, in analogy with the binary domain, as I showed in my dissertation.

²⁹Lecture delivered at the Salzburg meeting of natural scientists, 1909.

³⁰Study: *Methoden zur Theorie der ternären Formen*. II. § 6. (Leipzig, Teubner, 1889.)

³¹*Über die Bildung des Formensystems der ternären biquadratischen Form*. § 1. (Crelle’s Journal Bd. 134. 1908.)

When we now pass to the n -ary domain, even the first theorem of the symbolic method holds only in a conditional measure. It is well known that already for quaternary forms there occur, besides the point and plane coordinates x and u , line coordinates p_{ik} , the subdeterminants from two rows of point coordinates. Analogously, in the n -ary domain we have rows of variables p_ρ , whose individual elements are the subdeterminants from ρ rows x , and rows of symbols S^ρ , whose elements behave like subdeterminants from ρ rows s cogredient to the u . The symbolic representation by determinants is known to hold only when the symbol rows S^ρ are resolved into the rows s and these are distributed among different determinants; only such aggregates of determinants as depend on the S^ρ then have real meaning. But which aggregates of determinants accomplish this cannot be surveyed, and thereby all the other theorems mentioned in the ternary domain are lost.

I would now like to state a simple principle by which one obtains a symbolic representation explicit in the rows S^ρ , and thereby also recovers the remaining theorems. This is an application of the familiar theorem on corresponding matrices: “To each row of variables p_ρ there corresponds one-to-one another row $q_{n-\rho}$ whose individual elements are subdeterminants from $n - \rho$ rows u connected with the ρ rows x by equations $(u | x) = 0$.” Correspondingly, to each symbol row S^ρ there is assigned another $S_{n-\rho}$, behaving as if it were composed from $n - \rho$ rows cogredient to the x .

The theorem also permits a second formulation, suited for applications: “To every matrix product $(v^{(1)} \dots v^{(\rho)} | y^{(1)} \dots y^{(\rho)})$,³² where the rows of the second matrix are contragredient to those of the first, there is assigned a determinant; and conversely each determinant can be transformed into a matrix product with a prescribed number ρ of rows.” Thus determinant and matrix product appear as fully equivalent, and the first theorem of the symbolic method assumes the form: “All invariant formations can be represented symbolically by matrix products.” From two symbol rows S^σ , T^τ one can now, with the aid of rows of variables, very easily form a matrix product explicitly containing these symbol rows and hence representing an invariant formation of the form $(S^\sigma | p_\sigma)(T^\tau | p_\tau)$, namely the following:

$$(q_{n-\sigma-\lambda} S^\sigma | T_{n-\tau} p_{\tau-\lambda}), \quad (\lambda = 1, 2, \dots, \tau \text{ or } n - \sigma). \quad (1)$$

More important is the converse: that all invariant formations can be represented explicitly by matrix products, so that the above process is the extension of the binary and ternary folding process. To prove this converse it is necessary to transfer the second theorem of the symbolic method to the n -ary domain, namely to set up the totality of the independent indecomposable identities in which all rows S^ρ , p_ρ are regarded as indecomposable. These identities arise from the known ones containing only rows x and u ,³³ by replacing the product theorem for determinants by a system of product theorems for matrices, and by contracting different matrix products into a single one by means of precisely these product theorems. One obtains in this way two dualistic formulas, of which only one is given:

$$(S_1^{\rho_1} S_2^{\rho_2} \dots S_k^{\rho_k} | x p_{\rho-1}) = \sum_{i=1}^k \varepsilon (S_1^{\rho_1} \dots S_{i-1}^{\rho_{i-1}} S_{i+1}^{\rho_{i+1}} \dots S_k^{\rho_k} q_{n-\rho+1} | S_{n-\rho_i} x), \quad (2)$$

$$\rho = \sum_{i=1}^k \rho_i < n, \quad \varepsilon = \pm 1.$$

The only specialization contained in these formulas, namely that a row x enters, cannot be avoided; for completely general symbol and variable rows we arrive at a “decomposition identity,” an identity in which a row p_ρ is decomposed into the individual rows x . The simplest special case of the decomposition identity was already used by Clebsch in his “fundamental problem of invariant theory” to derive the elementary covariants in the theory of series expansions. With the aid of these identities, which mediate the transition from the non-explicit determinant expressions to the matrix product, one can in fact show that the desired explicit symbolic representation is obtained with the matrix representation.

For the folding process, however, a theorem entirely analogous to that in the ternary domain holds, and the reduction theorems can likewise be built upon it analogously. If in (1) one denotes the number λ as the defect of the folding, and calls foldings for which $\sigma = \tau$ cogredient, then all foldings can be generated by successive application of the cogredient foldings of defect 1. The proof of this theorem rests essentially on the fact that the most general forms can be replaced by “normal forms,” that is, by forms that vanish identically under the application of all invariant differential operations. In the quaternary domain this latter fact was proved by Mertens;³⁴ our knowledge of the most general differential operations—which, as is well

³²That is, the sum over the products of corresponding determinants, formed from the matrix of the ρ rows v and that of the ρ rows y .

³³E. Pascal in *Memorie d. R. Acc. d. Lincei*. (4) 5 (1888), p. 375.

³⁴Über invariante Gebilde quaternärer Formen. Wiener Berichte. Bd. 98. 1889.

known, arise from the folding processes by replacing the symbol rows by differential symbols—allows us, using the decomposition identity, to transfer Mertens’s proof almost verbatim to the n -ary domain.