

2. On the Formation of the form System of the Ternary Biquadratic form

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Introduction.

Work by Gordan, Maisano, and Pascal⁷⁸ deals with the form system of the ternary biquadratic form. Gordan sets up the complete form system, consisting of 54 formations, of the special automorphic form

$$f = x_1^3 x_2 + x_2^3 x_3 + x_3^3 x_1$$

on the basis of principles similar to those that he had given for form systems in the binary domain.

In Maisano's work, for the general biquadratic form, the forms up to and including the fifth order⁹ are set up, together with some invariants, covariants, and contravariants of higher order, according to the method used by Gordan in volume I of the *Mathematische Annalen* for the ternary cubic form. Pascal, using Maisano's results, is chiefly concerned with the question of the decomposition of the biquadratic form into factors¹⁰.

The aim of the following investigations is to set up the form system for the general ternary biquadratic form; in this work, namely, only the principal foundations are given, and a so-called "relatively complete system"¹¹ is established.

The work is closely connected with Gordan's work; however, the principles that were stated there only quite generally first had to be worked out in detail. With the aid of a theorem on the connection among the foldings and by introducing the "form sequence" (§ 1), the reduction theorems are sharply formulated and completed (§ 3), while the recursive establishment of special series expansions (§ 2) gives the computational means for actually carrying out the reductions.

The basic idea for forming systems of ternary forms is the same as in the binary domain. Starting from a first relatively complete system – the system of forms taken over from the binary case – one passes, according to a definite law, to systems with ever higher modulus, until either the system of a modulus – the modulus taken as ground form – becomes finite and known, or until a modulus can be reduced to forms having invariants as factors. By transvection of the relatively complete system with the system of the modulus, in the first case the absolutely complete system arises, while in the second case the relatively complete and absolutely complete systems coincide. By Hilbert's general proof the finiteness of form systems, this procedure must necessarily come to an end.

In our case the modulus (abc) of the first relatively complete system (§ 4) is reduced to the moduli

$$\Delta = (abc)^2 a_x^2 b_x^2 c_x^2 \quad \text{and} \quad \nu = (abu)^4$$

(§ 5), and then the relatively complete system modulo ν is found (§ 6). These results are already given in Gordan's work for the general biquadratic form; our manner of deriving them, however, is more easily transferred to forms of higher degree.

⁷A short extract from the Introduction and Chapter I appeared in the *Sitzungsberichte of the physikalisch-medizinischen Sozietät Erlangen 1907*, pp. 176–179.

⁸P. Gordan, on the complete form system of the ternary biquadratic form

$$f = x_1^3 x_2 + x_2^3 x_3 + x_3^3 x_1.$$

(*Math. Annalen*, vol. XVII (1880), pp. 217–233.) G. Maisano, 1) *Sistemi completi dei primi cinque gradi della forma ternaria biquadratica e degli invarianti, covarianti e contravarianti di sesto grado.* (Giorn. di Battaglini XIX.) 2) *Sui covarianti indipendenti di 6° grado nei coefficienti della forma biquadratica ternaria.* (Rend. Circ. Mat. di Palermo I, 1887.) E. Pascal, *Contributo alla teoria della forma ternaria biquadratica e delle sue varie decomposizioni in fattori.* (Memoria premiata dalla R. Accademia delle scienze fisiche e matematiche di Napoli. 1905.)

⁹By "order" the dimension in the coefficients is to be understood, and by "degree" the dimension in the variables.

¹⁰In his second short note, Maisano attempts to prove a linear dependence of the three covariants of order 6 and degree 6. In contrast to this not quite complete proof, Pascal believes he has proved the linear independence of the three covariants of order 6, and likewise that of the three invariants of order 9. That, however, a linear relation does in fact exist between the three covariants on the one hand and the three invariants on the other is shown by formulas (13), § 11, and (22), § 17 note, of the present work, where these relations are given explicitly. According to a communication from Pascal, the contradiction is explained by a numerical error in the expression for his contravariant p (called ϱ here), p. 46 of his paper.

¹¹For the terminology see § 3a.

As the sequence of moduli we now choose the forms (§ 7)

$$\nu, \quad \nu(\nu) = (\nu\nu_1x)^4 = s_x^4, \quad \nu(s) = (ss'u)^4, \dots$$

In forming the relatively complete system modulo s , two quadratic forms occurring in the system, u_ϱ^2 and t_x^2 , are adjoined as moduli (§§ 9 and 10), and accordingly the relatively complete system modulo (s, ϱ, t) is formed. It is then shown (§ 17) that the modulus $(ss'u)^4$ is reducible to the moduli ϱ and t . The next higher system thereby passes over into a relatively complete system modulo (ϱ, t) . Since, however, the simultaneous system of two quadratic forms is finite and known, and in the general case consists of 20 formations¹², the termination described above is reached with the establishment of the relatively complete system modulo (ϱ, t) . The transvection of this system with the system of (ϱ, t) in order to form the absolutely complete system is reserved.

As the relatively complete system modulo (ϱ, t) one finds 331 formations, which in the appended table are ordered according to their degree in the variables x and u .

Finally we give an overview of the symbols introduced:

$$\begin{aligned} f &= a_x^4, \\ \theta &= \theta_x^4 u_\vartheta^2 = (abu)^2 a_x^2 b_x^2, \\ K &= K_x^6 u_k^3 = (a\theta u) u_\vartheta^2 a_x^3 \theta_x^3, \\ j &= u_j^6 = (a\theta u)^4 u_\vartheta^2, \\ \Delta &= \Delta_x^6 = a_\vartheta^2 a_x^2 \theta_x^4 = (abc)^2 a_x^2 b_x^2 c_x^2, \\ N &= N_x^8 u_\eta = (a\Delta u) a_x^3 \Delta_x^5, \\ \nu &= u_\nu^4 = (abu)^4, \\ H &= H_x^2 u_\eta^4 = (\nu\nu_1x)^2 u_\nu^2 u_{\nu_1}^2, \\ L &= L_x^3 u_l^6 = (\nu\eta x) H_x^2 u_\nu^3 u_\eta^3, \\ g &= g_x^6 = (\nu\eta x)^4 H_x^2, \\ \sigma &= u_\sigma^6 = H_\nu^2 u_\nu^2 u_\eta^4, \\ s &= s_x^4 = (\nu\nu_1x)^4, \\ Z &= Z_x^4 u_\zeta^2 = (ss'u)^2 s_x^2 s_x'^2, \\ \varrho &= u_\varrho^2 = \theta_\nu^4 u_\vartheta^2, \\ t &= t_x^2 = a_\eta^4 H_x^2, \\ i &= a_\nu^4, \\ J &= s_\nu^4. \end{aligned}$$

Chapter I. General Theorems on Ternary Forms.

§ 1. The folding process. form sequences.

The fundamental process for producing forms is the folding process, which in the ternary domain can be defined as follows. Let there be given a symbolic product

$$s_x^m t_x^n u_\sigma^\mu u_\tau^\nu.$$

If one replaces the pairs of factors

$$s_x t_x \quad u_\sigma u_\tau \quad s_x u_\sigma \text{ or } s_x u_\tau \quad t_x u_\tau \text{ or } t_x u_\sigma$$

respectively by

$$(stu) \quad (\sigma\tau x) \quad s_\sigma \text{ or } s_\tau \quad t_\tau \text{ or } t_\sigma,$$

folding

$$\text{I} \quad \text{II} \quad \text{III} \quad \text{IV},$$

¹²Cf. Clebsch–Lindemann, *Vorlesungen über Geometrie*. (Vol. I, p. 288 ff.) System of two cogredient forms. Gordan, “Über Büschel of Kegelschnitten.” (*Math. Ann.* vol. XIX, p. 530.) System of two contragredient forms.

then the resulting forms have arisen from the original one by folding¹³.

Here the expression $s_x^m t_x^n u_\sigma^\mu u_\tau^\nu$ can be regarded either as an actual product of two forms $S \cdot T$, or as a single form with several rows of symbols:

$$s_x^m t_x^n u_\sigma^\mu u_\tau^\nu = A_x^{m+n} u_x^{\mu+\nu}.$$

In the first case we speak of the folding of the form S with T , in the second case of the “folding of the form A in itself.” For brevity we always assume in what follows, as indeed is the case for all forms later to be considered, that

$$s_\sigma^\lambda = 0, \quad t_\tau^\lambda = 0 \quad (a)$$

for any λ and x distinct from 0, and in conjunction with any other foldings. This says that the form $s_x^m t_y^n u_\sigma^\mu u_\tau^\nu$ is a so-called “normal form,” that is, it vanishes under application of the Ω -process, both with respect to the variables x and u and with respect to the variables y and v .

Thus condition (a) represents no restriction, but can always be achieved¹⁴.

For the connection among the individual foldings the following theorem holds:

Theorem I. Foldings I and II are fundamental foldings from which, independently of the order of composition, foldings III and IV can be composed. In other words: in order to form all forms arising by folding from a given expression, one has to apply only foldings I and II.

Proof: By the identity theorem, respectively the product theorem for matrices, taking (a) into account, one obtains

$$\begin{aligned} (\widehat{st}\sigma x) &= -t_\sigma, & (\widehat{\sigma\tau}su) &= -s_\tau, \\ (\widehat{st}\tau x) &= s_\tau, & (\widehat{\sigma\tau}tu) &= t_\sigma, \\ (stu)(\sigma\tau x) &= s_\tau + t_\sigma - u_x s_\tau t_\sigma. \end{aligned}$$

Thus by composing foldings I and II one obtains foldings III and IV, and this holds both when applying folding II to folding I, i.e. to the factors $(stu)u_\sigma u_\tau$, and when applying folding I to folding II, to the factors $(\sigma\tau x)s_x t_x$.

By a *form sequence*¹⁵ we mean an initial form together with all forms that have arisen from it by folding in itself, and we denote the form sequence by the initial form. A “higher form” is to mean here any form with more folding in itself, while among the equally entitled foldings s_τ and t_σ one has to be normalized as higher. According to the law of formation of the form sequence it follows that:

- 1) Each form of the form sequence is linear in the symbols of the initial form.
- 2) The form sequence is uniquely determined for initial forms with two rows each of contragredient symbols; for initial forms with more rows of symbols it is to be uniquely normalized by distinguishing special foldings among those connected by the identity theorem.

By Theorem I, a form sequence $s_x^m t_x^n u_\sigma^\mu u_\tau^\nu$ ($m > n$; $\mu > \nu$) can be arranged according to the following rectangular scheme:

$$\begin{array}{c|c|c|c|c} s_x^m t_x^n u_\sigma^\mu u_\tau^\nu & (stu) & (stu)^2 & \dots & \star \\ (\sigma\tau x) & t_\sigma, s_\tau & t_\sigma(stu), s_\tau(stu) & \dots & \vdots \\ (\sigma\tau x)^2 & t_\sigma(\sigma\tau x), s_\tau(\sigma\tau x) & t_\sigma^2, t_\sigma s_\tau, s_\tau^2 & \dots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (\sigma\tau x)^\nu & t_\sigma(\sigma\tau x)^{\nu-1}, s_\tau(\sigma\tau x)^{\nu-1} & t_\sigma^2(\sigma\tau x)^{\nu-2}, t_\sigma s_\tau(\sigma\tau x)^{\nu-2}, s_\tau^2(\sigma\tau x)^{\nu-2} & \dots & \vdots \\ \vdots & t_\sigma(\sigma\tau x)^\nu & t_\sigma^2(\sigma\tau x)^{\nu-1}, t_\sigma s_\tau(\sigma\tau x)^{\nu-1} & \dots & \vdots \\ \vdots & \vdots & t_\sigma^2(\sigma\tau x)^\nu & \dots & \vdots \end{array}$$

and the correspondingly continued lower part

$$\begin{array}{c|c|c} \star^{(*)} & (stu)^n & \dots \\ t_\sigma(stu)^{n-1}, s_\tau(stu)^{n-1} & s_\tau(stu)^n & \dots \\ t_\sigma^2(stu)^{n-2}, t_\sigma s_\tau(stu)^{n-2}, s_\tau^2(stu)^{n-2} & t_\sigma s_\tau(stu)^{n-1}, s_\tau^2(stu)^{n-1} & s_\tau^2(stu)^n. \end{array}$$

¹³Gordan, *Math. Annalen* vol. XVII, p. 219.

¹⁴Cf. Gordan, *Math. Annalen* vol. V, p. 104.

¹⁵Cf. Clebsch, “Über eine Fundamentalaufgabe der Invariantentheorie” (Abh. of the Gött. Ges. d. Wiss. vol. XVII): § 17 and end of § 18. The “form sequence” differs from the “properly reduced equivalent system” introduced there by the principle of ordering according to higher forms.

Here one obtains, when one moves

- 1) one column to the right, all forms arising by folding I from the adjacent forms,
 - 2) one row downward, all forms arising by folding II from the forms above,
- and hence all forms arising by folding.

An arbitrary form of the total form sequence, $t_\sigma^\kappa s_\tau^\lambda (stu)^\mu$ or $t_\sigma^\kappa s_\tau^\lambda (\sigma\tau x)^\nu$, forms the beginning of a new form sequence, consisting of all those forms of the rectangular scheme with $t_\sigma^\kappa s_\tau^\lambda (stu)^\mu$ or $t_\sigma^\kappa s_\tau^\lambda (\sigma\tau x)^\nu$ as left upper corner, which have the factor $t_\sigma^\kappa s_\tau^\lambda$.

Supplementary remark. Theorem I has an exception in the case where the numbers n and μ (respectively m and ν) become equal to zero, so that foldings I, II, and IV no longer exist, while folding III (respectively IV) still does. In this case we designate as the form sequence the diagonal member of the scheme:

$$\begin{array}{ccc} s_x^m u_\tau^\nu & & t_x^n u_\sigma^\mu \\ & s_\tau & t_\sigma \\ & s_\tau^2 & t_\sigma^2 \\ & \ddots & \\ & s_\tau^\nu & t_\sigma^\mu. \end{array}$$

In accordance with the absence of foldings I and II, the series expansion according to polars of the form sequence in this case always reduces to the initial member; however, the theorems on reducers remain valid.

§ 2. Series expansions according to polars of the form sequence.

For forms with n variables two kinds of series expansions have been set up explicitly¹⁶:

- 1) for forms with one row each of contragredient variables, $s_x^m u_\tau^\nu$, a series progressing by powers of u_x , whose coefficients are “normal forms”;
- 2) for forms with two rows of cogredient variables, $s_x^m t_x^n$, an expansion according to polars of the elementary covariants (polars of the form sequence $s_x^m t_x^n$), which in the ternary case reads as follows¹⁷:

$$s_x^m t_x^{n-\lambda} t_y^\lambda = \sum_{\varrho} \frac{\binom{m}{\varrho} \binom{\lambda}{\varrho}}{\binom{m+n-\varrho+1}{\varrho}} [(st\widehat{xy})^\varrho s_x^{m-\varrho} t_x^{n-\varrho}]_{y^{\lambda-\varrho}}. \quad (\text{I})$$

We combine the two expansions by setting up, for forms with two rows each of contragredient variables,

$$s_x^m t_x^{n-\lambda} t_y^\lambda u_\sigma^\mu u_\tau^{\nu-\kappa} v_\tau^\kappa,$$

expansions according to powers of u_x , whose coefficients are composite polars of the form sequence $s_x^m t_x^n u_\sigma^\mu u_\tau^\nu$, taken with respect to the variables x and u .

It is enough to set up the series for two contragredient rows of symbols (respectively variables), since, by combining each two rows of symbols (variables) into a single new row, forms with more rows of symbols (variables) can be reduced to this case.

In order to ensure that all forms of the form sequence contain only two rows of symbols, respectively variables, we specialize:

$$\begin{array}{ll} \text{a) } \sigma = \widehat{st}, & \text{a') } y = \widehat{uv}, \\ \text{b) } s = \widehat{\sigma\tau}, & \text{b') } v = \widehat{xy}, \end{array}$$

and carry out the expansion for the specialization a), a').

By direct transfer of expansion I one obtains composite polars for forms $(stu)^\mu s_x^m t_x^{n-\lambda} t_y^\lambda$, whereas for forms $(stu)^\mu u_\tau^{\nu-\kappa} v_\tau^\kappa s_x^m t_x^{n-\lambda} t_y^\lambda$, instead of composite polars, terms of such a kind arise whose evaluation makes

^{*)} Here, and similarly in what follows, the lower part marked with an asterisk is to be set at the correspondingly marked place at the upper right of the scheme.

¹⁶ Cf. Gordan, “Über Kombinanten,” §§ 2 and 5 (*Math. Ann.* vol. V).

¹⁷ Cf. Study, *Methoden to the Theorie of the ternären Formen* (Teubner 1889) II. § 7. In distinction to the derivation there, ours gives the method for the rapid computational determination of the numerical coefficients $C_{pr\varrho}$. The Clebsch–Study expansion, upon specialization a), a'), would read

$$(stu)^\mu u_\tau^{\nu-\kappa} v_\tau^\kappa s_x^m t_x^{n-\lambda} (tuv)^\lambda = \sum_{p,r,\varrho} C_{pr\varrho} \left[s_\tau^\varrho (stu)^{\mu+r-\varrho} \right]_{\widehat{uv}^{\lambda-r+\varrho} v^{\nu-\varrho}, u^p, (vw=\widehat{xw})},$$

and therefore does not permit the simplification that occurs already in the course of our calculation, once one knows that all terms with the factor u_x are to be omitted.

necessary the introduction of ever new auxiliary variables that are to be set equal only at the end, thereby complicating the calculation unnecessarily.

We therefore take an indirect route, by representing the composite polars of all forms of the form sequence, that is, expressions

$$[s_\tau^\varrho(stu)^{\mu-\varrho+r}]_{y^\lambda}^{v^\kappa},$$

by the initial members of these polars, and by inversion of the resulting recursive system of equations obtaining the desired expansion according to polars.

By the transfer principle, for the λ -th polar of a form $s_x^m t_x^n : [s_x^m t_x^n]_{y^\lambda}$ ($\lambda \leq n$) the following expansion holds (the case $\lambda > n$ is reduced to the first by the expansion of $[s_y^m t_y^n]_{x^{m+n-\lambda}}$)¹⁸:

$$[s_x^m t_x^n]_{y^\lambda} = \sum_r (-1)^r \frac{\binom{m}{r} \binom{\lambda}{r}}{\binom{m+n}{r}} (st\widehat{xy})^r s_x^{m-r} t_x^{n-\lambda} t_y^{\lambda-r},$$

and correspondingly for the κ -th polar of $u_\sigma^\mu u_\tau^\nu : [u_\sigma^\mu u_\tau^\nu]_{v^\kappa}$

$$[u_\sigma^\mu u_\tau^\nu]_{v^\kappa} = \sum_\varrho (-1)^\varrho \frac{\binom{\mu}{\varrho} \binom{\kappa}{\varrho}}{\binom{\mu+\nu}{\varrho}} (\sigma\tau\widehat{uv})^\varrho u_\sigma^{\mu-\varrho} u_\tau^{\nu-\kappa} v_\tau^{\kappa-\varrho}.$$

By multiplication it follows, with introduction of the specialization a) and a'),

$$[s_x^m t_x^n (stu)^\mu u_\tau^\nu]_{uv}^{v^\kappa} = \sum_{r,\varrho} c_{r\varrho} (st\widehat{uv})^r (st\widehat{uv})^\varrho s_x^{m-r} t_x^{n-\lambda} (tuv)^{\lambda-r} (stu)^{\mu-\varrho} u_\tau^{\nu-\kappa} v_\tau^{\kappa-\varrho},$$

and from this, by expansion according to powers of u_x with the first member distinguished ($\sum'_{r,\varrho}$ means that the member $r=0, \varrho=0$ is to be omitted) and taking account of (a) on p. 26,

$$\begin{aligned} & s_x^m t_x^{n-\lambda} (tuv)^\lambda (stu)^\mu u_\tau^{\nu-\kappa} v_\tau^\kappa \\ &= [s_x^m t_x^n (stu)^\mu u_\tau^\nu]_{uv}^{v^\kappa} \widehat{\lambda} \\ &= \sum_{r,\varrho} c_{r\varrho} s_\tau^\varrho (stu)^{\mu-\varrho+r} u_\tau^{\nu-\kappa} v_\tau^{\kappa-\varrho} s_x^{m-r} t_x^{n-\lambda} (tuv)^{\lambda-r+\varrho} v_x^r \\ &+ u_x \sum_{\varrho} \sum_{r=1}^{\lambda} c_{r\varrho} s_\tau^\varrho (stu)^{\mu-\varrho+r-1} (stv) u_\tau^{\nu-\kappa} v_\tau^{\kappa-\varrho} s_x^{m-r} t_x^{n-\lambda} (tuv)^{\lambda-r+\varrho} v_x^{r-1} \\ &\vdots \\ &- (-1)^\lambda u_x^\lambda \sum_{\varrho} c_{\lambda\varrho} s_\tau^\varrho (stu)^{\mu-\varrho} (stv)^\lambda u_\tau^{\nu-\kappa} v_\tau^{\kappa-\varrho} s_x^{m-\lambda} t_x^{n-\lambda} (tuv)^\varrho, \end{aligned} \tag{II}$$

where

$$c_{r\varrho} = (-1)^{r+\varrho} \cdot \frac{\binom{m}{r} \binom{\lambda}{r}}{\binom{m+n}{r}} \cdot \frac{\binom{\mu}{\varrho} \binom{\kappa}{\varrho}}{\binom{\mu+\nu}{\varrho}} \quad \begin{array}{l} \text{for } \lambda \leq n, \\ \kappa \leq \nu. \end{array}$$

and correspondingly for the remaining combinations of the numbers $\lambda, n; \kappa, \nu$. Thus the expression $s_x^m t_x^{n-\lambda} (tuv)^\lambda (stu)^\mu u_\tau^{\nu-\kappa} v_\tau^\kappa$ is reduced to a composite polar, and, by applying the identity

$$(stv)u_\tau = (stu)v_\tau - s_\tau(tuv),$$

to a sum of analogously formed expressions corresponding to higher forms of the form sequence.

By iteration one arrives at the expansion

$$s_x^m t_x^{n-\lambda} (tuv)^\lambda (stu)^\mu u_\tau^{\nu-\kappa} v_\tau^\kappa = \sum_{p,r,\varrho} u_x^p C_{pr\varrho} \cdot [s_\tau^\varrho (stu)^{\mu-\varrho+r}]_{uv}^{\lambda-r+\varrho} v_{\nu-\varrho+p}^{\kappa-\varrho+p} v_x^{r-p}. \tag{III}$$

The numerical coefficients $C_{pr\varrho}$ are computed uniquely and recursively from the coefficients $c_{r\varrho}, c'_{r\varrho}$ of the system of equations arising by iteration of (II) (cf. the example on p. 42). Analogous expansions arise for the other specializations.

¹⁸Gordan, *The Resultante binärer Formen* (Chap. I, § 5). Rend. Circ. Mat. di Palermo XXII. 1906.

§ 3. Reduction theorems.

The known theorems on the reduction of forms and form systems shall now be brought into connection with §§ 1 and 2, for which some definitions taken over from the theory of binary forms are necessary.

a) We call a system of forms a *relatively complete system* modulo a prescribed sequence of forms if it has the property that all formations arising by folding an arbitrary product of these forms can be expressed as integral rational functions of the forms of the system, up to an additive term consisting of expressions that have arisen by folding the system forms with the system of the modulus¹⁹.

b) We call a form *reducible* when it can be expressed by forms having invariants as factors or by “higher forms,” that is, by higher forms of the total form sequence to which the form belongs, or by forms containing the symbols of the modulus in higher order.

The theorems on reducers²⁰ can now be stated in their most general form as follows:

Definition: A *reducer* is a reducible form sequence.

*Theorem II. If the initial form of a form sequence is reducible because one of its members has arisen by folding with a reducer (has a reducer as factor), and if the final form of the form sequence has arisen from exactly this member by folding, then the total form sequence is reducible*²¹.

Proof: The form sequence arises from the initial member by folding in itself, hence by higher folding with the reducer on the one hand, and by folding the reducer in itself on the other hand. In both cases, by § 2, all forms of the form sequence can be represented as polars (transvections) over the form sequence of the reducer.

The inference from the initial form to the total form sequence no longer holds for the remaining reduction methods. These are:

1) so-called “double reduction.”

By this we mean the reduction, in two different ways, of an expression that has two mutually independent reducers as factors, whereby relations arise between the higher forms to which one has reduced.

By systematic application of “double reduction” one must, on the one hand, obtain all relations²²; on the other hand, if “double reduction” applied to different expressions supplies no new relations, one has both a criterion for the independence of the higher forms and a check on the calculation.

2) *Folding with decomposable forms.*

By “decomposable forms” are to be understood – with a slight generalization of the usual concept – forms that can be expressed by products of forms of lower degree and by “higher” forms. Products of forms pass into products under one folding; likewise products of nonlinear forms under twofold folding in the specializations a') and b') (§ 2), according to the product theorem:

$$a_y b_y a_x b_x = \frac{1}{2} a_y^2 b_x^2 + \frac{1}{2} b_y^2 a_x^2 - \frac{1}{2} (a b \widehat{y x})^2.$$

First polars (transvections) and certain second polars of decomposable forms are therefore again decomposable forms. It follows that:

A member of a onefold (respectively twofold) transvection over a decomposable form, arising by onefold (respectively twofold) folding with a decomposable form, itself becomes a decomposable form:

a) if the next-higher forms in the form sequence of the decomposable form decompose or become reducible, or also if, under onefold folding, the specializations $y = \widehat{uv}$ or $v = \widehat{xy}$ occur;

b) if the remaining onefold foldings lead to higher forms (cf. § 12, B. H_k and $H_k(k\eta x)$).

Such a member of a transvection can also decompose:

c) if the remaining onefold foldings lead to lower forms that decompose independently of the folding with the decomposable form, that is, that can be expressed by products and by the form to be reduced, although

¹⁹Gordan–Kerscheneister, *Vorlesungen über Invariantentheorie*. Vol. II. p. 227.

²⁰Gordan, *Math. Annalen* vol. XVII, p. 222.

²¹The simplification that results from Theorem II can be seen in the following example. For the special form $f = x_1^3 x_2 + x_2^3 x_3 + x_3^3 x_1$, $a_y^2 a_x^2 u_y^2$ is a reducer. It follows by Theorem II that the form sequence

$$\theta_\nu(\vartheta \nu x) \theta_x^3 u_\vartheta u_\nu^2 = a_\nu^2(abu) - a_\nu b_\nu(aba),$$

is reduced, since $\theta_\nu^4 = a_\nu^2 b_\nu^2 (abu)^2$ has arisen from the member $a_\nu^2(abu)$ by folding. In Gordan, loc. cit., p. 231, by contrast, the reduction of the forms

$$\theta_\nu(\vartheta \nu x), \quad \theta_\nu(\vartheta \nu x)^2, \quad \theta_\nu^2, \quad \theta_\nu^2(\vartheta \nu x), \quad \theta_\nu^2(\vartheta \nu x)^2$$

is carried out individually.

²²Thus, for example, the following were found by “double reduction”: the reduction of the form a_η^4 (§ 10), the only form included superfluously in Maisano's system; and also the relations mentioned in the note to the Introduction between the three covariants and the three invariants.

in each case one must first verify by calculation that the numerical coefficient of the member concerned does not vanish identically. This is nothing other than “double reduction” of the lower form (cf. § 23, C. $H_q(\theta Hu)(\theta su)$).

Decomposable forms arise by all onefold foldings with functional determinants²³, and moreover by certain higher foldings. One has

$$\begin{aligned} (sty\widehat{x})s_y &= \frac{1}{2} t s_y^2 - \frac{1}{2} s t_y^2 + \frac{1}{2} (sty\widehat{x})^2, \\ (sty\widehat{x})^2 s_y^2 &= \frac{1}{3} t s_y^3 - \frac{1}{3} s t_y^3 + s_y (sty\widehat{x})^2 - \frac{1}{3} (sty\widehat{x})^3. \end{aligned}$$

Analogous formulas hold for the dualistic forms

$$(\sigma\tau\widehat{\nu}u)v_\sigma \quad \text{and} \quad (\sigma\tau\widehat{\nu}u)v_\sigma^2.$$

From the theorems of this section the following rule follows for setting up all irreducible forms that arise from the folding of S with T :

Beginning with the lowest foldings, proceed along any previously fixed path (for example row by row, or symmetrically with respect to the diagonal member) in the formation of the form sequence ST until one reaches a form that has a reducer as factor. The form sequence defined by this form is to be omitted as soon as the final form satisfies the condition stated in Theorem II. After all reducible form sequences have been eliminated, all remaining reducible forms, and likewise all decomposable forms, are to be removed by applying double reduction. Places in the total form sequence that are covered twice by independently reducible form sequences give rise to the reduction of higher forms (cf. § 11, D).

²³cf. Gordan, loc. cit. § 3.

Theorem III. *The forms taken over from the binary domain, that is, those forms which arise from the system forms of the corresponding binary form by bordering according to Clebsch's transfer principle, form a relatively complete system modulo (abc) .*

Proof. To a binary relation asserting that the forms $Q'_1, \dots, Q'_n$ form a complete system of a binary ground form,

$$Q' - F(Q') = 0 = \sum_{\lambda} \{ (ab)c_x + (bc)a_x + (ca)b_x \} \varphi_{\lambda}^*,$$

there corresponds, by bordering, the ternary relation

$$Q - F(Q_i) = \sum_{\lambda} u_x \cdot (abc) \varphi_{\lambda}.$$

This is precisely the defining equation for a relatively complete system modulo (abc) . For the biquadratic form, the system of the forms taken over consists of the following forms:

$$\begin{aligned} f &= a_x^4, & \theta &= \theta_x^4 u_{\theta}^2 = (abu)^2 a_x^2 b_x^2, & K &= K_x^6 u_k^3 = (a\theta u) u_{\theta}^2 a_x^3 \theta_x^3, \\ j &= u_j^6 = (a\theta u)^4 u_{\theta}^2, & \nu &= u_{\nu}^4 = (abu)^4, \end{aligned}$$

among which the first four form a relatively complete system modulo $((abc), \nu)$.

§ 5. Reduction of the modulus (abc) to the moduli Δ and ν .

The modulus (abc) , given only by a bracket factor, is to be reduced, in agreement with the definition (§ 3, a), to forms of the system. In other words, the form sequence (abc) is to be normalized uniquely (cf. § 1).

$$a_s^2 a_x u_x = \frac{1}{3} a_s^3 u_x = \frac{1}{3} S \cdot u_x. \quad (2)$$

$$\left. \begin{aligned} (a\Delta u)^2 &= a_s - \frac{S}{6} u_x^2, \\ (a\Delta u)^3 &= 0. \end{aligned} \right\} \quad (3)$$

$$\left. \begin{aligned} \theta(s) &= (ss, x)^2 = 2a_t + \frac{1}{3} S \cdot \theta - \frac{1}{3} T u_x^2, \\ \theta(t) &= (tt, x)^2 = -\frac{1}{3} S \cdot a_t + \frac{2}{3} T \cdot a_s + \frac{1}{18} S^2 \cdot \theta - \frac{1}{18} u_x^2 \cdot S \cdot T. \end{aligned} \right\} \quad (4)$$

Sequence of moduli: $(abc); \Delta; s; t$ (the system of the modulus t consists of the single form t).²⁴

It will be shown that the forms

$$\Delta = (abc)^2 a_x^2 b_x^2 c_x^2, \quad \nu = (abu)^4$$

suffice as moduli, that is, that the forms

$$\begin{aligned} \Delta &= (abc)^2 a_x^2 b_x^2 c_x^2, & a_{\nu} &= (abc)(bcu)^3 a_x^3, \\ a_{\nu}^2 &= (abc)^2 (bcu)^2 u_x^2, & i &= a_{\nu}^4 = (abc)^4 \end{aligned}$$

define the form sequence (abc) . In the form sequence a_{ν} , the form a_{ν}^3 is missing according to the identity

$$a_{\nu}^3 = (abc)^3 a_x (bcu) = \frac{1}{3} u_x \cdot (abc)^4 = \frac{1}{3} i \cdot u_x.$$

For reducing a modulus to a higher one, according to the definition, we have to reduce the most general foldings, or all special foldings coming into consideration, with the modulus to foldings with the higher modulus.

In the present case the most general foldings arise from the expressions

$$(abc) a_x^3 b_y^3 c_z^3, \quad (abc) a_x^2 b_y^2 c_z^2 a_z b_x c_x$$

(taken over from the cubic forms),

$$(abc) a_x b_y c_z a_x^2 b_x^2 c_x^2$$

²⁴Gordan-Kerschensteiner, loc. cit., p. 134.

(taken over from the quadratic forms), and from the polarisation of these expressions.

For the calculation of these expressions by the polars of Δ and of the form sequence a_ν , respectively of their initial terms, we take the indirect route, as in § 2. We expand the polar terms

$$(abc)^2 a_x^2 b_y^2 c_z^2, \quad a_\nu(\nu xy)(\nu yz)(\nu zx) a_x a_y a_z = (abc)(bc\hat{x}u)(bc\hat{y}z)(bc\hat{z}x) a_x a_y a_z \quad \text{etc.}$$

as linear aggregates of expressions with the symbolic factor (abc) and obtain, by inverting the system of equations, the desired relations, as well as a relation between the polars taken according to different combinations of the variables. The identity theorem and the product theorem for determinants are used in the evaluation.

The system of equations is, for $(abc)a_x^3 b_y^3 c_z^3$:

$$\begin{aligned} 1) \quad & \sum_3 a_\nu(\nu yz)^3 a_x^3 = 6(abc)a_x^3 b_y^3 c_z^3 - 6 \sum_3 (abc)a_x^3 b_y^2 b_z c_z^2 c_y^*, \\ 2) \quad & 3(abc)^2 a_x^2 b_y^2 c_z^2 \cdot (xyz) - \frac{1}{2} \sum_3 a_\nu^2(\nu yz)^2 \nu_x^2(xyz) = 2 \sum_3 (abc)a_x^3 b_y^2 b_z c_z^2 c_y \\ & - 4 \sum_3 (abc)a_x a_y a_z b_y^2 b_z c_z^2 c_x, \\ 3) \quad & a_\nu(\nu xy)(\nu yz)(\nu zx) a_x a_y a_z = 2 \sum_3 (abc)a_x a_y a_z b_y^2 b_z c_z^2 c_x, \\ 4) \quad & \sum_3 a_\nu(\nu yz)^3 u_x^3 - \frac{3}{2} \sum_3 a_\nu^2(\nu yz)^2 \nu_x^2(xyz) + \frac{1}{6} i \cdot (xyz)^3 = 6 \sum_3 (abc)a_x a_y a_z b_y^2 b_z c_z^2 c_x. \end{aligned}$$

It follows for

$$(abc)a_x^3 b_y^3 c_z^3,$$

and by analogous calculation for

$$\begin{aligned} & (abc)a_x^2 b_y^2 c_z^2 a_z b_x c_x, \quad (abc)a_x b_y c_z a_z^2 b_x^2 c_x^2 : \\ (abc)a_x^3 b_y^3 c_z^3 &= \frac{3}{2}(abc)^2 a_x^2 b_y^2 c_z^2(xyz) + \frac{3}{2} a_\nu(\nu xy)(\nu yz)(\nu zx) a_x a_y a_z \\ & - \frac{1}{12} i \cdot (xyz)^3, \\ \text{(IV.) } (abc)a_x^2 b_y^2 c_z^2 a_z b_x c_x &= \frac{2}{3}(abc)^2 a_x^2 b_y^2 c_z^2 \cdot (xyz) \\ & + \frac{1}{3} a_\nu(\nu xy)(\nu yz)(\nu zx) a_x^3 - \frac{1}{6} a_\nu^2(\nu xy)(\nu zx) a_x^2 \cdot (xyz), \\ (abc)a_x b_y c_z a_z^2 b_x^2 c_x^2 &= \frac{1}{6}(abc)^2 a_x^2 b_z^2 c_z^2 \cdot (xyz). \end{aligned}$$

We have therefore obtained a relatively complete system modulo (Δ, ν) , consisting of the four forms f, θ, K, j . It follows from this that all transvections of f over θ not taken over from the binary domain, as well as the transvection $(a\theta u)^2$ reducible in the binary case to $(ab)^4$, can be expressed by the symbols Δ and ν .²⁵

We obtain the reduction formulas fundamental for what follows by specializing expansion IV, or more shortly by direct calculation (interchanging the symbols), for $a_\vartheta(a\theta u)^2$, $a_\vartheta^2((a\theta u)^2)$, $(a\theta u)^2$, according to the following ansatz:

$$\begin{aligned} a_\vartheta(a\theta u)^2 &= (abc)(bcu)(abu)^2 - \frac{1}{3}(abc)(bcu)\{(bcu)a_x - u_x(abc)\}^2, \\ a_\vartheta^2(a\theta u)^2 &= (abc)^2(abu)^2 - \frac{1}{3}(abc)^2\{(bcu)a_x - u_x(abc)\}^2, \end{aligned}$$

for $((a\theta u)^2)^*$:

$$\begin{aligned} (abu)a_y^3 b_x^3 &= \frac{3}{2} u_\vartheta(\vartheta yx) \theta_y^2 + \frac{1}{4} (\nu yx)^3, \\ ((abu)(acu))b_x^3 &= \frac{3}{2} u_\vartheta(\vartheta \hat{a}ux)(\theta au)^2 + \frac{1}{4} (\nu \hat{a}ux)^3. \end{aligned}$$

²⁵ $\sum_3$ refers to the sum of three terms obtained from the initial term by cyclic interchange of the variables. The equations also hold individually for each term of the sum.

$$\left. \begin{aligned} (a\theta u)^2 &= \frac{1}{6}\nu \cdot f - \frac{2}{3}u_x \cdot a_\nu + \frac{2}{3}u_x^2 \cdot a_\nu^2 - \frac{i}{18}u_x^4, \\ a_\vartheta &= \frac{1}{3}u_x \cdot \Delta, \\ a_\vartheta(a\theta u) &= 0, \\ a_\vartheta^2 &= \Delta, \\ (a\theta u)^3 &= 0, \\ a_\vartheta(a\theta u)^2 &= -\frac{5}{6}a_\nu + \frac{7}{6}u_x \cdot a_\nu^2 - \frac{i}{9}u_x^3, \\ a_\vartheta^2(a\theta u) &= 0, \\ (a\theta u)^4 &= j, \\ a_\vartheta(a\theta u)^3 &= 0, \\ a_\vartheta^2(a\theta u)^2 &= \frac{2}{3}a_\nu^2 - \frac{i}{9}u_x^2. \end{aligned} \right\} \quad (1)$$

We add the formula, likewise obtained from the reduction of the modulus (abc) :

$$a_\nu^3 a_x u_\nu = \frac{1}{3}i \cdot u_x. \quad (2)$$

From formulas (1.) and (2.) and from the formulas formed analogously for the ground form ν , all later reduction formulas are derived by polarisation, except for the relations for the decomposed forms. From formulas (1.) we see:

The form sequence leads:

$$\begin{aligned} a_\vartheta(a\theta u)^2 &\text{ to symbols } \nu \text{ alone,} \\ a_\vartheta &\text{ to symbols } \Delta \text{ and } \nu, \\ (a\theta u)^2 &\text{ to symbols } j, \Delta \text{ and } \nu, \\ (a\theta u)^2 \text{ under folding I} &\text{ to symbols } j \text{ and } \nu, \\ (a\theta u)^2 \text{ under folding II} &\text{ to symbols } \Delta \text{ and } \nu. \end{aligned}$$

We can therefore replace:

1. modulo (Δ, ν) : foldings with j are replaced by corresponding foldings with $(a\theta u)^2$ or $(a\theta u)^3$;
2. modulo (ν) : foldings with Δ are replaced by corresponding foldings with a_ϑ or $a_\vartheta(a\theta u)$, or also by special foldings with $(a\theta u)^2$ (which lead only to a single folding I).

In particular:

$$\begin{aligned} (a\theta u)^2 \theta_y^2 \theta_x^2 &= \frac{1}{3}(jyx)^2 \pmod{\nu}, & (a\theta u)^2 u_y \theta_y &= -\frac{1}{6}(jyx)^2 \pmod{\nu}, \\ (a\theta u)^2 \nu_y^2 &= \frac{1}{3}(\Delta u \nu)^2 \pmod{\nu}, & (a\theta u)(a\theta \nu) u_\vartheta \nu_\vartheta &= -\frac{1}{6}(\Delta u \nu)^2 \pmod{\nu}, \\ (a\theta u)^2 u_\nu^2 \vartheta_\nu^2 &= \frac{1}{3}(\Delta u \nu)^2 \Delta_\nu \pmod{\nu}. \end{aligned}$$

§ 6. Reduction of the modulus (Δ, ν) to the modulus (ν) .

The reduction formulas of the preceding paragraph give us the means to set up the relatively complete system modulo ν directly; we shall see that to the system of the transferred forms one has only to add the forms Δ and $(a\Delta u) = N$ (analogously as for the cubic form, and indeed valid in general).

For the reduction of the modulus Δ we consider all special foldings coming into consideration, that is, the foldings of the relatively complete system modulo (Δ, ν) with the system of Δ , and begin with the forms lowest in the coefficients, the foldings of f with Δ .

For the reduction of the forms $(a\Delta u)^2$, $(a\Delta u)^3$, $(a\Delta u)^4$, we replace the symbols Δ by lower forms of the form sequence according to § 5; that is, we develop the expressions

$$a_\vartheta b_\vartheta(b\theta u)^2, \quad a_\vartheta(a\theta u)b_\vartheta(b\theta u)^2, \quad a_\vartheta(a\theta u)b_\vartheta(b\theta u)^3$$

- 1) by polars of the form sequence a_{ϑ} (symbols Δ and ν),
 - 2) by polars of the form sequence $b_{\vartheta}(b\theta u)^2$ (symbols ν),
- and obtain the reduction formulas (calculation below):

$$\begin{aligned}
\text{a) } (a\Delta u)^2 &= \frac{1}{2}(\vartheta\nu x)^2 - \frac{2}{5}f \cdot a_{\nu}^2 - \frac{4}{5}u_x \cdot \theta_{\nu}(\vartheta\nu x)^2 \\
&\quad + u_x^2 \left\{ \frac{1}{6}i \cdot f - \frac{1}{40}S \right\}, \\
\text{b) } (a\Delta u)^3 &= -\frac{3}{10}\theta_{\nu}(\vartheta\nu x), \\
\text{c) } (a\Delta u)^4 &= \frac{21}{10}\theta_{\nu}^2 - \frac{3}{10}\Pi - \frac{6}{5}u_x \cdot \theta_{\nu}^3 + \frac{1}{10}u_x^2 \cdot \theta.
\end{aligned} \tag{3}$$

(The same results are also reached by the double reduction of the expressions $a_{\vartheta}^2(b\theta u)^2$, $a_{\vartheta}^2(b\theta u)^3$, $a_{\vartheta}^2((a\theta u)^2(b\theta u)^2)$ and $a_{\vartheta}^2((a\theta u)(b\theta u))^3$ after elimination of a_{ϑ}^2 .)

From formulas (3.) it follows that $(a\Delta u)^2$ is a reducer with respect to the modulus ν . Hence:

1) The reduction of the system of Δ : the form sequence $(\Delta\Delta'u)^2 = a_{\vartheta}^2((a\Delta u)^2)$ has the reducer $(a\Delta u)^2$ as a factor.

2) The reduction of the system arising by folding with the modulus Δ :

The form sequence $(\theta\Delta u)^2$ has the reducer $(a\Delta u)^2$ as a factor.

For the forms $(\theta\Delta u)$ and Δ_{ϑ} one obtains:

$$\begin{aligned}
(a\Delta u)^2(abu) &= (\theta\Delta u) - u_x \cdot \Delta_{\vartheta}(\theta\Delta u), \\
(a\Delta u)(a\Delta b) &= -\frac{1}{2}\Delta_{\vartheta} + \frac{1}{2}u_x \cdot \Delta_{\vartheta}^2.
\end{aligned}$$

From the reducer $(\theta\Delta u)$, however, follows the reduction of the system arising by folding with the modulus Δ .

The relatively complete system modulo ν therefore consists of the six forms:

$$f, \theta, K, j, \Delta, N.$$

As an example of the sequence expansion in § 2 we give the derivation of formula (3.)a in full; in later calculations the final formula III will be set down directly. (Polars of vanishing forms have already been omitted during the calculation; the first line gives the values inserted according to Expansion II, and the second line gives the final values found recursively, beginning with the last equation of the system.) It follows, according to Expansion II:

$$\begin{aligned}
1) \quad m &= 3, \quad n = 4, \quad \lambda = 2, \quad \mu = 0, \quad \nu = \chi = 1, \\
a_{\vartheta}b_{\vartheta}(b\theta u)^2 &= [a_{\vartheta}]_{b^2}b + \frac{6}{7}\{a_{\vartheta}b_{\vartheta}(a\theta u)(b\theta u) - u_x a_{\vartheta}b_{\vartheta}(a\theta b)(b\theta u)\} \\
&\quad - \frac{1}{7}\{a_{\vartheta}(a\theta u)^2b_{\vartheta} - 2u_x a_{\vartheta}(a\theta u)(a\theta b)b_{\vartheta} + u_x^2 a_{\vartheta}(a\theta b)^2b_{\vartheta}\} \\
&= \frac{2}{3}(a\Delta u)^2 + \frac{1}{5}[a_{\vartheta}(a\theta u)^2]b + \frac{1}{30}f[a_{\vartheta}^2(a\theta u)^2] - \frac{2}{5}u_x[a_{\vartheta}(a\theta u)^2]b^2 \\
&\quad - \frac{1}{15}u_x[a_{\vartheta}^2(a\theta u)^2]b + \frac{1}{5}u_x^2[a_{\vartheta}(a\theta u)^2]b^3 + \frac{1}{30}u_x^2[a_{\vartheta}^2(a\theta u)^2]b^2; \\
2) \quad m &= 2, \quad n = 3, \quad \lambda = 1, \quad \mu = 1, \quad \nu = \chi = 1, \\
a_{\vartheta}(a\theta u)b_{\vartheta}(b\theta u) &= \frac{2}{5}\{a_{\vartheta}(a\theta u)^2b_{\vartheta} - u_x a_{\vartheta}(a\theta u)(a\theta b)b_{\vartheta}\} + \frac{1}{2}a_{\vartheta}^2(b\theta u)^2 \\
&\quad - \frac{1}{5}\{a_{\vartheta}^2(b\theta u)(a\theta u) - u_x a_{\vartheta}^2(a\theta b)(b\theta u)\} \\
&= \frac{1}{2}(a\Delta u)^2 + \frac{2}{5}[a_{\vartheta}(a\theta u)^2]b + \frac{1}{15}[a_{\vartheta}^2(a\theta u)^2]f \\
&\quad - \frac{2}{5}u_x[a_{\vartheta}(a\theta u)^2]b^2 - \frac{1}{10}u_x[a_{\vartheta}^2(a\theta u)^2]b + \frac{1}{30}u_x^2[a_{\vartheta}^2(a\theta u)^2]b^2;
\end{aligned}$$

$$3) \quad m = 2, \quad n = 3, \quad \lambda = 1, \quad \mu = 1, \quad \nu = 1, \quad \chi = 2,$$

$$\begin{aligned} a_{\vartheta}(a\theta b)b_{\vartheta}(b\theta u) &= \frac{2}{5}\{a_{\vartheta}(a\theta b)(a\theta u)b_{\vartheta} - u_x a_{\vartheta}(a\theta b)^2 b_{\vartheta}\} \\ &= \frac{2}{5}[a_{\vartheta}(a\theta u)^2]b^2 + \frac{1}{30}[a_{\vartheta}^2(a\theta u)^2]b - \frac{2}{5}u_x[a_{\vartheta}(a\theta u)^2]b^3 - \frac{1}{30}u_x[a_{\vartheta}^2(a\theta u)^2]b^2; \end{aligned}$$

$$4) \quad m = 1, \quad n = 2, \quad \lambda = 0, \quad \mu = 2, \quad \nu = \chi = 1,$$

$$\begin{aligned} a_{\vartheta}(a\theta u)^2 b_{\vartheta} &= [a_{\vartheta}(a\theta u)^2]b + \frac{2}{3}a_{\vartheta}^2(a\theta u)(b\theta u) \\ &= [a_{\vartheta}(a\theta u)^2]b + \frac{1}{6}[a_{\vartheta}^2(a\theta u)^2]f - \frac{1}{6}u_x[a_{\vartheta}^2(a\theta u)^2]b; \end{aligned}$$

$$5) \quad m = 1, \quad n = 2, \quad \lambda = 0, \quad \mu = 2, \quad \nu = 1, \quad \chi = 2,$$

$$\begin{aligned} a_{\vartheta}(a\theta u)(a\theta b)b_{\vartheta} &= [a_{\vartheta}(a\theta u)^2]b^2 + \frac{1}{3}a_{\vartheta}^2(a\theta b)(b\theta u) \\ &= [a_{\vartheta}(a\theta u)^2]b^2 + \frac{1}{12}[a_{\vartheta}^2(a\theta u)^2]b - \frac{1}{12}u_x[a_{\vartheta}^2(a\theta u)^2]b^2; \end{aligned}$$

$$6) \quad m = 1, \quad n = 2, \quad \lambda = 0, \quad \mu = 2, \quad \nu = 1, \quad \chi = 3,$$

$$a_{\vartheta}(a\theta b)^2 b_{\vartheta} = [a_{\vartheta}(a\theta u)^2]b^3;$$

$$7) \quad m = 2, \quad n = 4, \quad \lambda = 2, \quad \mu = \nu = \chi = 0, \quad (\text{according to Expansion I}),$$

$$a_{\vartheta}^2(b\theta u)^2 = [a_{\vartheta}^2]_{ub^2} + \frac{1}{10}\{[a_{\vartheta}^2(a\theta u)^2]f - 2u_x[a_{\vartheta}^2(a\theta u)^2]b + u_x^2[a_{\vartheta}^2(a\theta u)^2]b^2\};$$

$$8) \quad m = 1, \quad n = 3, \quad \lambda = 1, \quad \mu = 1, \quad \nu = \chi = 0, \quad (\text{Expansion I}),$$

$$a_{\vartheta}^2(a\theta u)(b\theta u) = \frac{1}{4}[a_{\vartheta}^2(a\theta u)^2]f - \frac{1}{4}u_x[a_{\vartheta}^2(a\theta u)^2]b;$$

$$9) \quad m = 1, \quad n = 3, \quad \lambda = 1, \quad \mu = \chi = 1, \quad \nu = 0, \quad (\text{Expansion I}),$$

$$a_{\vartheta}^2(a\theta b)(b\theta u) = \frac{1}{4}[a_{\vartheta}^2(a\theta u)^2]b - \frac{1}{4}u_x[a_{\vartheta}^2(a\theta u)^2]b^2.$$

From 1) and 4) it follows:

$$\begin{aligned} (a\Delta u)^2 &= \frac{6}{5}[a_{\vartheta}(a\theta u)^2]b + \frac{1}{5}f[a_{\vartheta}^2(a\theta u)^2] + \frac{3}{5}u_x[a_{\vartheta}(a\theta u)^2]b^2 - \frac{3}{20}u_x[a_{\vartheta}^2(a\theta u)^2]b \\ &\quad - \frac{3}{10}u_x^2[a_{\vartheta}(a\theta u)^2]b^3 - \frac{1}{20}u_x^2[a_{\vartheta}^2(a\theta u)^2]b^2, \\ (a\Delta u)^2 &= -a_{\nu}b_{\nu} + \frac{3}{5}fa_{\nu}^2 + \frac{4}{5}u_x a_{\nu}^2 b_{\nu} - \frac{3}{20}u_x^2 a_{\nu}^2 b_{\nu}^2 - \frac{1}{12}u_x^2 i f. \end{aligned}$$

(For conversion into the symbols θ cf. § 8 and § 9.) Formula (3.)a can be computed still more briefly by direct transfer of Expansion I, introducing the auxiliary variable $w = x\hat{u}b$ whereas already for formulas (3.)b and (3.)c the path taken here becomes the simpler one; for (3.)b one knows in advance, by § 9, that all terms with factor u_x are to be omitted. For the calculation of (3.)b and (3.)c one obtains:

$$\begin{aligned} (3.)b) \quad 1) \quad a_{\vartheta}(a\theta u)b_{\vartheta}(b\theta u)^2 &= \frac{1}{2}(a\Delta u)^3 + \frac{4}{5}[a_{\vartheta}(a\theta u)^2]_{ub}b + \frac{7}{360}[a_{\vartheta}^2(a\theta u)^2]_{ub^2}, \\ 2) \quad a_{\vartheta}(a\theta u)b_{\vartheta}(b\theta u)^2 &= [a_{\vartheta}(a\theta u)^2]_{ub}b + \frac{13}{36}[a_{\vartheta}^2(a\theta u)^2]_{ub^2}; \\ (3.)c) \quad 1) \quad a_{\vartheta}(a\theta u)b_{\vartheta}(b\theta u)^3 &= \frac{1}{2}(a\Delta u)^4 + \frac{6}{5}[a_{\vartheta}(a\theta u)^2]_{ub^2}b + \frac{53}{120}[a_{\vartheta}^2(a\theta u)^2]_{ub^3} \\ &\quad - \frac{6}{5}u_x[a_{\vartheta}(a\theta u)^2]_{ub^3}b^2 - \frac{3}{5}u_x[a_{\vartheta}^2(a\theta u)^2]_{ub^2}b + \frac{19}{120}u_x^2[a_{\vartheta}^2(a\theta u)^2]_{ub^3}b^2, \\ 2) \quad a_{\vartheta}(a\theta u)b_{\vartheta}(b\theta u)^3 &= \frac{3}{4}[a_{\vartheta}^2(a\theta u)^2]_{ub^2}. \end{aligned}$$

Supplementary remark: the transvectants of f over j that are reducible according to § 5 are computed

analogously. One obtains

$$\begin{aligned}
g_\nu &= -\theta_\nu + u_x \left(\frac{9}{2}\theta_\nu^2 - \frac{1}{2}H \right) - 3u_x^2\theta_\nu^3 + \frac{1}{2}u_x^3\theta, \\
g_\nu^2 &= \frac{21}{10}\theta_\nu^2 - \frac{3}{10}H - 2u_x\theta_\nu^3 + \frac{1}{2}u_x^2\theta, \\
g_\nu^3 &= -\frac{7}{10}\theta_\nu^3 + \frac{1}{2}u_x^2\theta, \quad g_\nu^4 = \frac{3}{5}\theta. \\
g_\nu^3 &= -\frac{7}{10}\theta_\nu^3 + \frac{1}{2}u_x^2\theta, \quad g_\nu^4 = \frac{3}{5}\theta.
\end{aligned} \tag{4}$$

From (3.) and (4.) it follows, by transfer from the binary domain, that

$$(\theta\theta'u)^2 = \frac{1}{3}j \cdot f \pmod{\nu}.$$
²⁶

The remaining forms of the form sequence $(\theta\theta'u)^2$ and the form θ'_ν are reducible to symbols ν . We can therefore replace:

$$\text{mod } \nu : \quad \text{foldings with } f \cdot j \text{ by the corresponding foldings with } (\theta\theta'u)^2.$$

Furthermore,

$$(\vartheta\vartheta'x)^2 = \frac{4}{3}f \cdot \Delta, \quad \theta_{g\nu}(\vartheta\vartheta'x) = -N.$$

Chapter III. The relatively complete system modulo (s, ϱ, t) .

§ 7. Overview of the formation of the system.

In order to pass from the relatively complete system modulo ν to the relatively complete system with the next higher modulus, one has, by Definition § 3, to form the system of ν with respect to this higher modulus and to fold it with the forms of the relatively complete system modulo ν . But $\nu = u_\nu^4$, as a contravariant of the fourth degree, is dual to the form f . If therefore we regard ν as the ground form, we obtain a relatively complete system of six forms modulo $\nu(\nu) = (\nu\nu_1x)^4 = s_x^4$.

Thus, for the formation of the relatively complete system modulo s (System III), we have to transvect

$$\begin{aligned}
\text{System I:} \quad & f, \theta, K, j, \Delta, N \quad \text{over} \\
\text{System II:} \quad & \nu, H = (\nu\nu_1x)^2u_\nu^2u_{\nu_1}^2, L = (\nu\eta x)H_x^2u_\nu^3u_\eta^3, \\
& g = (\nu\eta x)^4H_x^2, \sigma = H_\eta^2u_\nu^2u_\eta^4, (\nu\sigma x).
\end{aligned}$$

It will be shown (formula (13.)) that the form g is reducible, and hence that System II consists of only five forms.

In the course of the calculation the quadratic forms

$$\varrho = u_\varrho^2 = \theta_\nu^4u_\vartheta^2, \quad t = t_x^2 = a_\eta^4H_x^2$$

are adjoined as moduli, so that one obtains a relatively complete system modulo (s, ϱ, t) ; and the modulus (ϱ, t) is to be regarded as a higher modulus than the modulus s .

The arrangement of the individual forms is to follow the order in the coefficients. We normalize the folding

$$\theta_\nu(K_\nu, \theta_\nu, \dots)$$

as higher than the folding

$$H_y(H_y, L_y, \dots).$$

Consequently, by § 1 and § 3b, the order of the individual forms of the same order is uniquely determined.

The formation of the forms of the same order is carried out in tabular form:

I. Indication of which forms of Systems I and II are to be folded with each other in order to obtain the prescribed order in the coefficients.

²⁶Gordan-Kerschensteiner, loc. cit., p. 181.

II. Listing of the irreducible forms according to the scheme of the form sequence.

III. Reduction of the reducible or decomposable form sequences or forms, that is, of those places where the total form sequence breaks off, thereby proving that all irreducible constructions have been exhausted by those indicated.

IV. Consequences: 1) indication of newly arising reducers; 2) indication of those forms which do not enter, in products with forms of the same system, into folding with forms of the other system.

It will be shown that only the following occur:

1) foldings of one form of System I with one form of System II;

2) foldings of powers of θ , respectively H , or of two forms of one system with one form of the second system.

We shall obtain the following scheme for folding:

System I	folded with	System II
1. order f		2. order ν
2. ,, θ		4. ,, H
3. ,, K, j, Δ		6. ,, L, σ
4. ,, $N, \theta^2, f \cdot j$		8. ,, $(\nu\sigma x), H^2$
5. ,, $\theta \cdot K$		10. ,, $H \cdot L$
6. ,, $\theta^3, \Delta \cdot j$		12. ,, H^3 .

For checking the calculation, besides the “double reduction”, the two special forms serve:

$$\begin{aligned}
 & f = \sum x_1^3 x_2, \quad u_\nu^2 = \frac{i}{6} u_x^2, \quad s = \frac{i}{3} f, \\
 \text{I. } & a_\eta^2 = -\frac{1}{9} i \theta, \quad H_\nu^2 = \frac{i}{6} \theta, \quad \sigma = -\frac{i}{6} j, \\
 & g = -\frac{2}{3} i \Delta, \quad \varrho = 0, \quad t = 0, \\
 & f = \sum x_1^4, \quad s = \frac{4}{3} i f, \quad a_\eta^2 = \frac{2}{9} i \theta, \\
 \text{II. } & \Delta_\nu^2 = \frac{1}{15} i \theta, \quad \sigma = \frac{4}{3} i j, \quad g = \frac{4}{3} i \Delta, \\
 & \varrho = 0, \quad t = 0.
 \end{aligned}$$

Supplementary remark: for the ground form ν , formulas (1.), (2.), (3.), (4.) correspond to the following:

$$\begin{aligned}
 (\nu\eta x)^2 = A & \left| \begin{array}{l} H_\nu(\nu\eta x) = 0 \\ H_\nu(\nu\eta x)^2 = B \end{array} \right| \begin{array}{l} H_\nu^2 = \sigma \\ H_\nu^2(\nu\eta x) = 0 \end{array} \\
 (\nu\eta x)^3 = 0 & \left| \begin{array}{l} H_\nu(\nu\eta x)^2 = B \\ H_\nu(\nu\eta x)^3 = 0 \end{array} \right| \begin{array}{l} H_\nu^2(\nu\eta x) = 0 \\ H_\nu^2(\nu\eta x)^2 = C \end{array} \\
 (\nu\eta x)^4 = g & \left| \begin{array}{l} H_\nu(\nu\eta x)^3 = 0 \end{array} \right|
 \end{aligned} \tag{5}$$

$$A = \frac{1}{6} s\nu - \frac{2}{3} u_x s_\nu + \frac{2}{3} u_x^2 s_\nu^2 - \frac{J}{18} u_x^4, \quad B = -\frac{5}{6} s_\nu + \frac{7}{6} u_x s_\nu^2 - \frac{J}{9} u_x^3, \quad C = \frac{2}{3} s_\nu^2 - \frac{J}{9} u_x^2.$$

$$s_\nu^3 u_x s_\nu = \frac{1}{3} u_x s_\nu^4 = \frac{1}{3} J u_x. \tag{6}$$

$$\begin{aligned}
 \text{a) } (\nu\sigma x)^2 &= \frac{1}{2} (Hsu)^2 - \frac{2}{5} \nu \cdot s_\nu^2 - \frac{4}{5} u_x s_\eta (Hsu)^2 + u_x^2 \left\{ \frac{1}{6} J \cdot \nu - \frac{1}{40} (ss'u)^4 \right\}, \\
 \text{b) } (\nu\sigma x)^3 &= -\frac{3}{10} s_\eta (Hsu),
 \end{aligned} \tag{7}$$

$$\begin{aligned}
 \text{c) } (\nu\sigma x)^4 &= \frac{21}{10} s_\eta^2 - \frac{3}{10} Z - \frac{6}{5} u_x s_\eta^3 + \frac{1}{10} u_x^2 s_\eta^4. \\
 \text{a) } g_\nu &= -s_\eta + u_x \left(\frac{9}{2} s_\eta^2 - \frac{1}{2} Z \right) - 3u_x^2 s_\eta^3 + \frac{1}{2} u_x^3 s_\eta^4, \\
 \text{b) } g_\nu^2 &= \frac{21}{10} s_\eta^2 - \frac{3}{10} Z - 2u_x s_\eta^3 + \frac{1}{2} u_x^2 s_\eta^4, \\
 \text{c) } g_\nu^3 &= -\frac{7}{10} s_\eta^3 + \frac{1}{2} u_x s_\eta^4, \\
 \text{d) } g_\nu^4 &= \frac{3}{5} s_\eta^4.
 \end{aligned} \tag{8}$$

§ 8. Forms of the third order (System III).

Folding of f with ν .

Irreducible forms:

$$\begin{array}{cccc} a_\nu & \cdot & \cdot & \cdot \\ \cdot & a_\nu^2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & a_\nu^4 = i. \end{array}$$

Reduction:

$$a_\nu^3 = \frac{1}{3} i u_x \quad (\text{formula (2.)}).$$

Consequences: 1) a_ν^3 is a reducer. 2) f enters into folding with ν only in products with contragredient forms (j). The same holds for ν with respect to f .

§ 9. Forms of the fourth order (System III).

Folding of θ with ν .

Irreducible forms:

$$\begin{array}{cccc} (\theta\nu x) & \theta_\nu & \cdot & \cdot \\ (\theta\nu x)^2 & \theta_\nu(\theta\nu x) & \theta_\nu^2 & \cdot \\ \cdot & \theta_\nu(\theta\nu x)^2 & \cdot & \theta_\nu^3. \end{array}$$

reductions: From the reducer a_ν^3 , by double reduction of the expressions $a_\nu^3(abu)$, $a_\nu^3 b_\nu$, $a_\nu^3 b_\nu(abu)$ according to the ansatz:

$$\begin{aligned} \theta_\nu^2(\vartheta\nu x) &= a_\nu^2(\widehat{ab\nu x})(abu) - \frac{1}{3}(\nu\nu_1 x)^3 = a_\nu b_\nu(\widehat{ab\nu x})(abu) + \frac{1}{6}(\nu\nu_1 x)^3 \\ &= a_\nu^3(abu) - a_\nu^2 b_\nu(abu) = 2a_\nu^2 b_\nu(abu) = \frac{2}{3}a_\nu^3(abu), \\ \theta_\nu^2(\vartheta\nu x)^2 &= a_\nu^2(\widehat{ab\nu x})^2 - \frac{1}{3}(\nu\nu_1 x)^4 = a_\nu b_\nu(\widehat{ab\nu x})^2 + \frac{1}{6}(\nu\nu_1 x)^4 \\ &= if - 2a_\nu^3 b_\nu + a_\nu^2 b_\nu^2 - \frac{1}{3}s = 2a_\nu^3 b_\nu - 2a_\nu^2 b_\nu^2 + \frac{1}{6}s = \frac{2}{3}if - \frac{2}{3}a_\nu^3 b_\nu - \frac{1}{6}s, \\ \theta_\nu^3(\vartheta\nu x) &= a_\nu^2 b_\nu(\widehat{ab\nu x})(abu) = a_\nu^3 b_\nu(abu). \end{aligned}$$

The formulas obtained are:

$$\left. \begin{array}{l} \theta_\nu^2(\theta\nu x) = 0 \\ \theta_\nu^2(\theta\nu x)^2 = \frac{4}{9}if - \frac{1}{6}s \end{array} \right| \left. \begin{array}{l} \theta_\nu^3(\theta\nu x) = 0 \\ \theta_\nu^4 u_x^2 = u_\varrho^2 = \varrho \end{array} \right| \quad (\text{adjoined modulus, § 7}), \quad (9)$$

Consequences: 1) $\theta_\nu^2(\theta\nu x)^2$ is a reducer. 2) ν enters into folding with θ only in products with contragredient forms (g).

§ 10. Forms of the fifth order (System III).

$$\text{Folding of } \left\{ \begin{array}{l} K, j, \Delta \text{ with } \nu, \\ f \text{ with } H. \end{array} \right.$$

Irreducible forms:

$$\begin{array}{ccccccc} & \cdot & K_\nu & \cdot & \cdot & (j\nu x) & \Delta_\nu \\ \text{A)} & (k\nu x)^2 & \cdot & K_\nu^2 & \cdot & (j\nu x)^2 & \cdot \\ & (k\nu x)^3 & K_\nu(k\nu x)^2 & \cdot & \cdot & (j\nu x)^3 & \cdot \\ & \cdot & K_\nu(k\nu x)^3 & \cdot & \cdot & \cdot & \cdot \\ & & & & & & \\ & & (aHu) & (aHu)^2 & \cdot & \cdot & \\ \text{D)} & a_\eta & a_\eta(aHu) & a_\eta(aHu)^2 & \cdot & \cdot & \\ & \cdot & a_\eta^2 & \cdot & \cdot & \cdot & \end{array}$$

reductions:

- A) form sequence K_ν^3 : reducer a_ν^3 .
 form $K_\nu^2(k\nu x)^2$: reducer $\theta_\nu^2(\vartheta\nu x)^2$.

$$(k\nu x), \quad K_\nu(k\nu x), \quad K_\nu^2(k\nu x)$$

are decomposable forms according to § 3.

B) form

$$(j\nu x)^4 = (\widehat{a}\theta\nu x)^4 = a_\nu^4\theta + \theta_\nu^4 f - 4a_\nu^3\theta_\nu - 4a_\nu\{a_\nu\theta_x - (\widehat{a}\theta\nu x)\}^3 \\ + 6a_\nu^2\{a_\nu\theta_x - (\widehat{a}\theta\nu x)\}^2 = \varrho f + \frac{5}{3}i\theta - 6a_\nu^2(\widehat{a}\theta\nu x)^2 + 4a_\nu(\widehat{a}\theta\nu x)^3,$$

and therefore

$$(j\nu x)^4 = \varrho f + \frac{5}{3}i\theta \pmod{(\Delta, \nu)}, \quad (\text{cf. § 5 end}).$$

C) form Δ_ν^4 : reducer θ_ν^4 .

forms Δ_ν^2 and Δ_ν^3 : reducer a_ν^3 or θ_ν^4 by replacing the form Δ by lower forms of the form sequence (§ 5 end), i.e. by double reduction of the expressions

$$a_\vartheta a_\nu^3, \quad a_\vartheta a_\nu^3 \theta_\nu, \quad a_\vartheta \theta_\nu^4.$$

The double calculation of Δ_ν^3 gives rise to the reduction of the higher form a_η^3 .

Reduction formulas:

$$\begin{aligned} \text{a)} \quad \Delta_\nu^2 &= \frac{3}{5}a_\eta^2 - \frac{3}{10}(asu)^2 + \frac{1}{3}i\theta + \frac{1}{5}u_x^2(2a_\varrho^2 - t), \\ \text{b)} \quad \Delta_\nu^3 &= -\frac{3}{10}a_\varrho + \frac{3}{5}u_x \left(\frac{13}{10}a_\varrho^2 - \frac{2}{5}t \right), \\ \text{c)} \quad \Delta_\nu^4 &= a_\varrho^2 - \frac{2}{5}t. \end{aligned} \tag{10}$$

Derivation of the formulas:

$$\begin{aligned} \text{a)} \quad a_\vartheta a_\nu^3 &= \frac{1}{3}i\theta = [a_\vartheta]_{\nu^3} + \frac{6}{7}[a_\vartheta^2]_{\nu^2} + \frac{6}{5}[a_\vartheta(a\theta u)^2]_{\nu^2} \widehat{\nu}x^2 + \frac{11}{30}[a_\vartheta^2(a\theta u)^2]_{\nu} \widehat{\nu}x^2 \\ &\quad - \frac{6}{7}u_x[a_\vartheta^2]_{\nu^3} - \frac{1}{6}u_x[a_\vartheta^2(a\theta u)^2]_{\nu^2} \widehat{\nu}x^2, \\ \frac{1}{3}i\theta &= \Delta_\nu^2 - \frac{2}{3}u_x\Delta_\nu^3 - a_\nu a_{\nu_1}(\nu\nu_1 x)^2 + \frac{2}{5}a_\nu^2(\nu\nu_1 x)^2 + \frac{1}{5}u_x^2 a_\nu^2 a_{\nu_1}(\nu\nu_1 x)^2; \\ \text{b)} \quad a_\vartheta a_\nu^3 \theta_\nu &= 0 = [a_\vartheta]_{\nu^4} + \frac{9}{14}[a_\vartheta^2]_{\nu^3} + \frac{3}{5}[a_\vartheta(a\theta u)^2]_{\nu^2} \widehat{\nu}x^2 + \frac{17}{120}[a_\vartheta^2(a\theta u)^2]_{\nu} \widehat{\nu}x^2 \\ &\quad - \frac{9}{14}u_x[a_\vartheta^2]_{\nu^4} - \frac{1}{24}u_x[a_\vartheta^2(a\theta u)^2]_{\nu^2} \widehat{\nu}x^2, \\ 0 &= \frac{5}{6}\Delta_\nu^3 - \frac{1}{2}u_x\Delta_\nu^4 - \frac{1}{4}a_\eta^3 + \frac{1}{20}u_x a_\eta^4, \\ \text{b')} \quad a_\vartheta \theta_\nu^4 &= a_\varrho = a_\vartheta \theta_\nu \{a_\nu - (a\theta\nu x)\}^3 \\ &= -\frac{3}{2}[a_\vartheta]_{\nu^3} + \frac{3}{5}[a_\vartheta(a\theta u)^2]_{\nu^2} \widehat{\nu}x^2 - \frac{37}{120}[a_\vartheta^2(a\theta u)^2]_{\nu} \widehat{\nu}x^2 \\ &\quad + \frac{3}{2}u_x[a_\vartheta^2]_{\nu^4} + \frac{49}{120}u_x[a_\vartheta^2(a\theta u)^2]_{\nu^2} \widehat{\nu}x^2, \\ a_\varrho &= -\frac{3}{2}\Delta_\nu^3 + \frac{3}{2}u_x\Delta_\nu^4 - \frac{11}{20}a_\eta^3 + \frac{7}{20}u_x a_\eta^4; \\ \text{c)} \quad a_\vartheta^2 \theta_\nu^4 &= a_\varrho^2 = [a_\vartheta^2]_{\nu^4} + \frac{3}{5}[a_\vartheta^2(a\theta u)^2]_{\nu^2} \widehat{\nu}x^2, \\ a_\varrho^2 &= \Delta_\nu^4 + \frac{2}{5}a_\eta^4. \end{aligned}$$

D) reduction of a_η^3 by double reduction of Δ_ν^3 (C).

The remaining reductions are obtained by a computation dualistically opposite to that of § 9.

Reduction formulas:

$$\left. \begin{aligned} a_\eta^2(aHu) &= -\frac{1}{6}(asu)^3, \\ a_\eta^3 &= -a_\vartheta + \frac{3}{5}u_x a_\vartheta^2 + \frac{1}{5}u_x t, \end{aligned} \right| \begin{aligned} a_\eta^2(aHu)^2 &= \frac{4}{9}i\nu - \frac{1}{6}(asu)^4, \\ a_\eta^3(aHu) &= 0, \\ a_\eta^4 &= t \quad (\text{adjoined as a modulus}). \end{aligned} \quad (11)$$

Consequences:

- 1) $(j\nu x)^4$, Δ_ν^2 , $a_\eta^2(aHu)$ are reducers.
- 2) ν enters only in products with contragredient forms in foldings with K, j, Δ ; the same holds for f in relation to H , and for j in relation to ν , since $\theta_\nu(\theta \cdot j, \nu, x^3)$ becomes reducible according to § 11 B.

§ 11. Forms of the sixth order (System III).

$$\text{Folding of } \begin{cases} \theta^2, f \cdot j, N \text{ with } \nu, \\ \theta \text{ with } H. \end{cases}$$

Irreducible forms:

$$\begin{aligned} &\text{A) } (\vartheta^2 \nu x)^3, \quad \theta_\nu(\vartheta^2 \nu x)^3 \quad \text{B) } a_\nu(j\nu x), \quad a_\nu^2(j\nu x) \quad \text{C) } N_\nu \\ &\text{D) } \begin{array}{c} \left(\begin{array}{c} (\vartheta \eta x) \\ (\vartheta \eta x)^2 \\ \cdot \end{array} \right) \left| \begin{array}{c} (\theta Hu) \\ \theta_\eta \\ H_\vartheta(\vartheta \eta x), \theta_\eta(\vartheta \eta x) \\ \theta_\eta(\vartheta \eta x)^2 \end{array} \right| \left(\begin{array}{c} (\theta Hu)^2 \\ H_\vartheta(\theta Hu), \theta_\eta(\theta Hu) \\ \theta_\eta^2 \\ \cdot \end{array} \right) \left| \begin{array}{c} \cdot \\ \theta_\eta(\theta Hu)^2 \\ \theta_\eta^2(\theta Hu) \\ \cdot \end{array} \right. \end{array} \end{aligned}$$

reductions:

A) $(\vartheta^2 \nu x)^4$ decomposes according to formula (3.)a:

$$(a\Delta \vartheta x)^4 = \frac{1}{2}(\vartheta^2 \nu x)^4 - \frac{3}{5}f \cdot a_\nu^2(\nu \vartheta x)^2 = \frac{1}{3}\Delta^2 + f\Delta_\vartheta^2.$$

B) $a_\nu(j\nu x)^2$ decomposes according to formula (9.)a, indem wir according to § 6 end $f \cdot j$ by $(\theta \theta' u)^2$ ersetzen, and the reducibility of θ'_ϑ take account of:

$$\frac{1}{3}a_\nu(j\nu x)^2 = (\theta \theta' \nu x)^2 \theta_\nu = \theta_\nu^3 \theta - \theta_\nu^2 \theta'_\nu = \theta_\nu^3 \theta + \theta_\nu^2(\widehat{j\nu \theta' u}) = \theta_\nu^3 \theta - \frac{2}{3}\theta_\nu^3 \theta.$$

forms $a_\nu(j\nu x)^3$ and $a_\nu^2(j\nu x)^2$: reducer a_j and $(j\nu x)^4$:

$$a_\nu(j\nu x)^3 = a_j(j\nu x)^3 - (j\nu x)^3(j\nu \widehat{au}),$$

$$a_\nu^2(j\nu x)^2 = a_j^2(j\nu x)^2 - 2a_j(j\nu x)^2(j\nu \widehat{au}) + (j\nu x)^2(j\nu \widehat{au})^2.$$

C) form sequence N_ν^2 ; reducer Δ_ν^2 . $(n\nu x)$ and $N_\nu(n\nu x)$ decompose as a transvection over functional determinants according to § 3.

D) Overview of the reductions:

a) form sequence $\theta_\eta^2 H_\vartheta$: reducer $a_\eta^2(aHu)$. (Leads to forms with higher symbols.)

b) form sequence $\theta_\eta H_\vartheta$: reducer $a_\eta^2(aHu)$ by double reduction. (Leads to forms with higher symbols and to higher forms of the total form sequence, i.e. to the form sequence θ_η^2 .) Instead of $\theta_\eta H_\vartheta$ we consider the form sequence

$$\theta_\eta(\vartheta \eta x)(\theta Hu) = \theta_\eta H_\vartheta + \theta_\eta^2,$$

ersetzen in it the symbols θ by the lower form (abu) , gelangen so to the double reduction of the form sequence

$$a_\eta^2(aHu)(abu) = \theta_\eta H_\vartheta + \frac{3}{2}\theta_\eta^2 \mod s$$

up to the terminal form

$$a_\eta^3 b_\eta(bHu)(abu) = \theta_\eta^3 H_\vartheta + \frac{1}{2}\theta_\eta^4.$$

c) form sequence θ_η^3 : By double reduction of those forms which simultaneously belong to the form sequences $\theta_\eta H_\vartheta$ and $\theta_\eta^2 H_\vartheta$ - i.e. the forms of the form sequence $\theta_\eta^2 H_\vartheta$ - to forms with higher symbols on the one hand, and to forms with higher symbols and to the higher form sequence θ_η^3 on the other hand.

d) forms $\theta_\eta^2(\vartheta\eta x)$, $\theta_\eta^2(\vartheta\eta x)^2$, $\theta_\eta^3(\vartheta\eta x)$: By double reduction by means of the reducers a analogous to the computation of § 9.

e) forms $\theta_\eta^2(\theta Hu)^2$ and $\theta_\eta^2(\vartheta\eta x)^2$: By double reduction of the expressions $\Delta_\nu^2(a\Delta u)^4$ and $(a\Delta\nu x)^4$ by means of the reducers Δ_ν^2 and $(a\Delta u)^4$.

f) forms H_ϑ and H_ϑ^2 : Decomposition of forms. Is obtained for H_ϑ^2 by double reduction of the expression $\Delta_\nu^2(a\Delta u)^2$ or also by direct calculation of the products $(a_\nu^2)^2$, $a_\nu^2 b_{\nu_1}$ by forms of the system. (The analogous calculation of $(a_\nu)^2$ gives a relation between decomposable forms.)

g) Reduction of higher forms by the fact that forms of the form sequence $\theta \cdot H$ admit two reduction possibilities (belong to two reducible form sequences), namely:

g : by double reduction of $\theta_\eta^2(\vartheta\eta x)^2$; allows reduction d) and e) zu.

$s_\vartheta^2(\theta su)$: by double reduction of $\theta_\eta^3(\vartheta\eta x)$; allows reduction c) and d) zu.

$s_\vartheta^2(\theta su)^2$: by double reduction of $\theta_\eta^2 H_\vartheta^2$; allows reduction a) and has the reducers $\theta_\nu^2(\vartheta\nu x)^2$ as a factor.

s_ν^2 : by double reduction of $\theta_\eta^3 H_\vartheta$; allows reduction a) and c) zu.

The proof of the independence of the remaining forms is obtained by double reduction of the form sequence $\Delta_\nu^2(a\Delta u)^2 = \theta_\eta^2$.

Execution of the reductions:

The existence of reductions a), b), c), d) is clear a priori, since according to the stated law of formation the relations arising from double reduction are independent of one another. For reductions e), f), g), it must be shown by computation that no identities arise.

Proceeding symmetrically with the diagonal term in the form sequence $\theta \cdot H$ up to the form θ_η , and partly indicating the calculation, we also give the formulas obtained by reductions a), b), c), d), since they are used partly in reductions e), f), g), and partly later.

1) H_ϑ and H_ϑ^2 according to f). According to the ansatz²⁷

$$\begin{aligned} a_\nu b_{\nu_1}^2 &= \nu a_\nu b_\nu^2 - (aHu)(bHu)b_\eta \sim \nu a_\nu b_\nu^2 - f a_\eta (aHu)^2 - (abu)(aHu)b_\eta + (abu)^2 b_\eta, \\ a_\nu^2 b_{\nu_1}^2 &= b_\nu^2 \{a_\nu u_\nu - (a\nu\nu_1 u)\}^2 \sim \nu \{a_\nu^2 b_\nu^2 - 2a_\nu b_\nu (aHu)(bHu) - \frac{1}{6}(asu)^2 (bsu)^2\} \\ &\sim \nu a_\nu^2 b_\nu^2 - 2a^2 (aHu)(abu) - \frac{1}{6}(asu)^2 (bsu)^2 + (\vartheta\eta x)^2 (\theta Hu)^2 \end{aligned}$$

after substitution of the value of $\theta_\eta H_\vartheta$, the formulas arise:

$$\begin{aligned} \text{a)} \quad H_\vartheta &\sim 2a_\nu a_\eta^2 + 2\nu b_\nu (\vartheta\eta x)^2 + 2f a_\eta (aHu)^2 - 2\theta_\eta, \\ \text{b)} \quad H_\vartheta^2 &\sim (a^2)^2 + 2\theta_\eta^2 + \frac{2}{3}(\theta su)^2 + \frac{1}{12}s\nu - \frac{1}{9}if\nu - \frac{1}{6}f(asu)^4. \end{aligned} \tag{12}$$

2) $\theta_\eta H_\vartheta$ according to b):

$$\begin{aligned} a_\eta^2 (aHu)(abu) &\sim [a_\eta^2 (aHu)]_{bu} + \frac{1}{2}f[a_\eta^2 (aHu)^2] = a_\eta b_\eta (aHu)(abu) \\ &+ a_\eta (\widehat{ab}\eta x)(abu)(aHu) \sim \theta_\eta (\vartheta\eta x)(\theta Hu) + \frac{1}{2}\theta_\eta^2 + \frac{1}{4}(\nu\eta x)^2. \end{aligned}$$

According to the formulas (5.) and (11.) follows:

$$\theta_\eta H_\vartheta \sim -\frac{3}{2}\theta_\eta^2 - \frac{1}{4}(\theta su)^2 - \frac{1}{12}s\nu + \frac{2}{9}if\nu.$$

3) $\theta_\eta H_\vartheta(\theta Hu)$ is computed according to b), analogously to $\theta_\eta H_\vartheta$:

$$\theta_\eta H_\vartheta(\theta Hu) \sim -\theta_\eta^2(\theta Hu) - \frac{1}{6}(\theta su)^3.$$

4) $\theta_\eta H_\vartheta(\vartheta\eta x)$ according to b); $\theta_\eta^2(\vartheta\eta x)$ according to d) (cf. § 9):

$$\theta_\eta H_\vartheta(\vartheta\eta x) \sim -\theta_\eta^2(\vartheta\eta x) - \frac{1}{6}s_\vartheta(\theta su) \sim -\frac{1}{3}(\vartheta\varrho x) - \frac{1}{6}s_\vartheta(\theta su),$$

²⁷The sign $\sim$ in place of the equality sign means that products arising from u_x with higher forms, other than those arising by foldings III and IV, are omitted.

$$\theta_\eta^2(\vartheta\eta x) \sim \frac{2}{3}a_\eta^3(abu) \sim -\frac{1}{3}(\vartheta\varrho x).$$

5)

$$\begin{array}{lll} \theta_\eta H_\vartheta^2 \text{ according to b)} & \theta_\eta^2 H_\vartheta \text{ according to a)} & \theta_\eta^2 H_\vartheta \text{ according to b)} \\ \text{from } a_\eta^2(aHb)(bHu) & \text{from } a_\eta^3(Hab)(abu) & \text{and thereby } \theta_\eta^3 \text{ according to c)} \\ & & \text{from } a_\eta^3(bHu)(abu) \end{array}$$

$$\theta_\eta H_\vartheta^2 \sim -\frac{3}{2}\theta_\eta^2 H_\vartheta - \frac{1}{4}s_\vartheta(\theta su)^2 + \frac{1}{6}s_\nu - \frac{4}{9}ia_\nu \sim -\theta_\varrho - \frac{1}{6}s_\nu - \frac{1}{9}ia_\nu,$$

$$\theta_\eta^2 H_\vartheta \sim \frac{2}{3}\theta_\varrho - \frac{1}{6}s_\vartheta(\theta su)^2 + \frac{2}{9}s_\nu - \frac{2}{9}ia_\nu,$$

$$\theta_\eta^3 H_\vartheta \sim \frac{2}{3}\theta_\varrho - \frac{2}{3}\theta_\eta^3 + \frac{5}{36}s_\nu, \quad \theta_\eta^3 \sim \frac{1}{4}s_\vartheta(\theta su)^2 - \frac{1}{8}s_\nu + \frac{1}{3}ia_\nu.$$

6) $\theta_\eta^2(\theta Hu)^2$ according to e):

$$\Delta_\nu^2(a\Delta u)^2 = [(a\Delta u)^4]_{\nu^2} = -\frac{12}{5}\theta_\eta^2(\theta Hu)^2 - \frac{3}{10}\sigma - \frac{1}{20}(\theta su)^4 + \nu\varrho \quad (\text{according to formula (3.)c}),$$

$$\Delta_\nu^2(a\Delta u)^4 = [\Delta_\nu^2]_{aa}^4 = -\frac{3}{5}\theta_\eta^2(\theta Hu)^2 + \frac{1}{5}\sigma - \frac{3}{10}(\theta su)^4 + \frac{1}{3}ij \quad (\text{according to formula (10.)a}),$$

$$\theta_\eta^2(\theta Hu)^2 = -\frac{1}{6}\sigma + \frac{1}{12}(\theta su)^4 - \frac{1}{9}ij + \frac{1}{3}\nu\varrho.$$

7) $\theta_\eta^2(\vartheta\eta x)^2$ according to d) and e). From this reduction of the higher form g .

By a computation analogous to that in § 9, it follows:

$$\begin{aligned} \theta_\eta^2(\vartheta\eta x)^2 &= \frac{1}{2}if - \frac{1}{2}a_\eta^2b_\eta - \frac{1}{12}g = \frac{2}{3}tf - \frac{2}{3}a_\eta^3b_\eta - \frac{1}{6}g \\ &= -\frac{1}{6}g - \frac{1}{3}(\vartheta\varrho x)^2 + \frac{4}{15}fa_\varrho^2 + \frac{8}{15}ft \quad (\text{according to formula (11.)}). \end{aligned}$$

By the corresponding computation as above follows:

$$((a\Delta\nu x)^4) = [(a\Delta u)^4]_{\nu x^4} = \frac{21}{10}\theta_\eta^2(\vartheta\eta x)^2 + \frac{7}{10}s_\vartheta^2 - \frac{3}{10}g \quad (\text{according to formula (3.)c}),$$

$$\begin{aligned} (a\Delta\nu x)^4 &= -\frac{1}{3}i\Delta + 6[\Delta_\nu^2]a^2 - 4[\Delta_\nu^3]a + \Delta_\nu^4f \\ &= -\frac{36}{5}\theta_\eta^2(\vartheta\eta x)^2 - \frac{9}{5}s_\vartheta^2 - \frac{3}{5}g - \frac{3}{5}(\vartheta\varrho x)^2 \\ &\quad + \frac{5}{3}i\Delta + \frac{37}{25}fa_\varrho^2 + \frac{74}{25}ft \quad (\text{according to formula (10.)}). \end{aligned}$$

Thus

$$\begin{aligned} \frac{93}{10}\theta_\eta^2(\vartheta\eta x)^2 &= -\frac{3}{10}g - \frac{5}{2}s_\vartheta^2 - \frac{3}{5}(\vartheta\varrho x)^2 \\ &\quad + \frac{5}{3}i\Delta + \frac{37}{25}fa_\varrho^2 + \frac{74}{25}ft, \end{aligned}$$

and by comparing the two values of $\theta_\eta^2(\vartheta\eta x)^2$ according to g):

$$0 = g - 2s_\vartheta^2 + 2(\vartheta\varrho x)^2 + \frac{4}{3}i\Delta - \frac{4}{5}fa_\varrho^2 - \frac{8}{5}ft. \quad (13)$$

8) $\theta_\eta^2 H_\vartheta(\theta Hu)$ according to a) and b), from this $\theta_\eta^3(\theta Hu)$ according to c) (forms with the Faktor u_x vanish):

$$\theta_\eta^2 H_\vartheta(\theta Hu) = -\frac{1}{6}s_\vartheta(\theta su)^3 - \frac{1}{3}(\nu\varrho x),$$

$$\theta_\eta^2 H_\vartheta(\theta Hu) = -\frac{1}{3}\theta_\eta^3(\theta Hu) - \frac{1}{3}(\nu\varrho x),$$

$$\theta_\eta^3(\theta Hu) = \frac{1}{2}s_\vartheta(\theta su)^3.$$

9) $\theta_\eta^2 H_\vartheta(\vartheta\eta x)$ according to a) and b), from this $\theta_\eta^3(\vartheta\eta x)$ according to c). $\theta_\eta^3(\vartheta\eta x)$ according to d), from this reduction of the higher form $s_\vartheta^2(\theta su)$ according to g) (forms with the Faktor u_x vanish):

$$\begin{aligned}\theta_\eta^2 H_\vartheta(\vartheta\eta x) &= \frac{1}{3}(atu), \\ \theta_\eta^2 H_\vartheta(\vartheta\eta x) &= \frac{1}{3}(atu) + \frac{1}{3}\theta_\eta^3(\vartheta\eta x) - \frac{1}{6}s_\vartheta^2(\theta su), \\ \theta_\eta^3(\vartheta\eta x) &= \frac{1}{2}s_\vartheta^2(\theta su), \\ \theta_\eta^3(\vartheta\eta x) &= \frac{2}{5}\theta_\varrho(\vartheta\varrho x) + \frac{1}{5}(atu), \\ s_\vartheta^2(\theta su) &= \frac{4}{5}\theta_\varrho(\vartheta\varrho x) + \frac{2}{5}(atu).\end{aligned}\tag{14}$$

10) $\theta_\eta^2 H_\vartheta^2$ according to a) and by reducer $\theta_\nu^2(\vartheta\nu x)^2$, $\theta_\eta^3 H_\vartheta$ according to a) and c), θ_η^4 according to c), from this reduction of the higher forms $s_\vartheta^2(\theta su)^2$ and s_ν^2 according to g):

$$\begin{aligned}\text{According to a)} \quad \theta_\eta^2 H_\vartheta^2 &= [a_\eta^2(aHu)^2]b^2 + \frac{1}{3}[a_\eta^4]_{bu^2} - \frac{1}{3}H_\nu^2(\nu\eta x)^2 \\ &= \frac{4}{9}ia_\nu^2 - \frac{1}{6}s_\vartheta^2(\theta su)^2 - \frac{5}{18}s_\nu^2 + \frac{1}{3}(atu)^2 + \frac{1}{54}Ju_x^2.\end{aligned}$$

$$\begin{aligned}\text{By } \theta_\nu^2(\vartheta\nu x)^2 : \quad \theta_\eta^2 H_\vartheta^2 &= [\theta_\nu^2(\vartheta\nu x)^2]_{\nu_1} + \frac{1}{3}[\theta_\nu^4]_{\nu_1}\hat{\nu}_1x^2 - \frac{1}{3}s_\vartheta^2(\theta su)^2 \\ &= \frac{4}{9}ia_\nu^2 - \frac{1}{6}s_\nu^2 - \frac{1}{3}s_\vartheta^2(\theta su)^2 + \frac{1}{3}(\nu\varrho x)^2.\end{aligned}$$

$$\begin{aligned}\text{According to a)} \quad \theta_\eta^3 H_\vartheta &= -[a_\eta^2(aHu)]_{bu}b^2 - \frac{1}{2}[a_\eta^2(aHu)^2]b^2 - \frac{1}{3}[a_\eta^3]b + \frac{1}{12}[a_\eta^4]_{bu^2} \\ &\quad + \frac{1}{2}u_x[a_\eta^2(aHu)^2]b^3 \\ &= -\frac{2}{9}ia_\nu^2 + \frac{1}{12}s_\nu^2 + \frac{4}{15}\theta_\varrho^2 - \frac{7}{90}(\nu\varrho x)^2 + \frac{1}{30}(atu)^2 + \frac{2}{27}i^2u_x^2 - \frac{1}{36}Ju_x^2.\end{aligned}$$

$$\begin{aligned}\text{According to c)} \quad \theta_\eta^3 H_\vartheta : a_\eta^3(Hab)(bHu) &= \frac{1}{2}(atu)^2 = \theta_\eta^2 H_\vartheta(\theta Hu)(\vartheta\eta x) + \frac{1}{4}H_\nu^2(\nu\eta x)^2 \\ &\quad + \frac{1}{2}\theta_\eta^2 H_\vartheta^2 = \frac{3}{2}\theta_\eta^2 H_\vartheta^2 + \theta_\eta^3 H_\vartheta - u_x[a_\eta^2(aHu)^2]b^3 \\ &\quad + \frac{1}{4}H_\nu^2(\nu\eta x)^2,\end{aligned}$$

and therefore

$$\begin{aligned}\theta_\eta^3 H_\vartheta &= -\frac{2}{3}ia_\nu^2 + \frac{5}{12}s_\nu^2 + \frac{1}{2}(\nu\varrho x)^2 - \frac{1}{2}(atu)^2 \\ &\quad + \frac{4}{27}i^2u_x^2 - \frac{1}{12}Ju_x^2.\end{aligned}$$

According to c)

$$\begin{aligned}\theta_\eta^4 &= \frac{4}{9}ia_\nu^2 - \frac{1}{6}s_\nu^2 + \frac{16}{15}\theta_\varrho^2 \\ &\quad - \frac{14}{45}(\nu\varrho x)^2 + \frac{2}{15}(atu)^2.\end{aligned}$$

According to g), from the two values for $\theta_\eta^2 H_\vartheta^2$:

$$\begin{aligned}s_\vartheta^2(\theta su)^2 &= \frac{2}{3}s_\nu^2 - 2(atu)^2 + 2(\nu\varrho x)^2 - \frac{1}{9}Ju_x^2 \\ &= \frac{8}{9}ia_\nu^2 + \frac{8}{15}\theta_\varrho^2 - \frac{14}{15}(atu)^2 + \frac{38}{45}(\nu\varrho x)^2 - \frac{4}{27}i^2u_x^2.\end{aligned}\tag{15}$$

According to g), from the two values for $\theta_\eta^3 H_\vartheta$:

$$s_\nu^2 = \frac{4}{3}ia_\nu^2 + \frac{4}{5}\theta_\varrho^2 + \frac{8}{5}(atu)^2 - \frac{26}{15}(\nu\varrho x)^2 + \frac{1}{6}Ju_x^2 - \frac{2}{9}i^2u_x^2.\tag{16}$$

Formula (16.) gives the basis for the reduction of the modulus $(ss'u)^4$, and formula (13.) gives the basis for the reduction of the system of s . (§ 17.)

Consequences:

- 1) $a_\nu(j\nu x)^3$, $\theta_\eta H_\vartheta$, $\theta_\eta^2(\vartheta\eta x)$, $\theta_\eta^2(\theta Hu)^2$ are reducers, $a_\nu(j\nu x)^2$, H_ϑ and H_ϑ^2 decomposable forms.
- 2) The form of the system II: $j(\nu) = g = (\nu\eta x)^4 H_x^2$ is reducible.
- 3) Since the only contragredient form g becomes reducible, ν does not enter at all in powers or in products with forms of the same system in folding with System I.

θ enters into foldings only as a contravariant in products or powers; correspondingly, H enters only as a covariant (cf. § 13 B).

§ 12. Forms of the seventh order.

$$\text{Folding of } \begin{cases} \theta \cdot K \text{ with } \nu, \\ K, j, \Delta \text{ with } H, \\ f \text{ with } L, \sigma. \end{cases}$$

Irreducible forms:

$$\begin{array}{l} \text{B)} \quad \begin{array}{c} \cdot \\ \cdot \\ (k\eta x)^2 \\ (k\eta x)^3 \\ \cdot \end{array} \left| \begin{array}{c} \cdot \\ K_\eta \\ \cdot \\ H_k(k\eta x)^2, K_\eta(k\eta x)^2 \\ K_\eta(k\eta x)^3 \end{array} \right| \begin{array}{c} (KHu)^2 \\ \cdot \\ K_\eta^2 \\ \cdot \\ \cdot \end{array} \left| \begin{array}{c} \cdot \\ K_\eta(KHu)^2 \\ \cdot \\ \cdot \\ \cdot \end{array} \right| \cdot \\ \\ \text{C)} \quad \begin{array}{c} (j\eta x) \\ \cdot \\ \cdot \end{array} \begin{array}{c} H_j \\ H_j(j\eta x) \\ H_j^2 \end{array} \quad \text{D)} \quad \begin{array}{c} (\Delta Hu) \\ \Delta_\eta \end{array} \\ \\ \text{E)} \quad \begin{array}{c} a_i \\ \cdot \\ \cdot \end{array} \begin{array}{c} (aLu)^2 \\ \cdot \\ a_i^2 \end{array} \begin{array}{c} (aLu)^3 \\ a_i(aLu)^2 \\ \cdot \end{array} \begin{array}{c} \cdot \\ a_i(aLu)^3 \\ \cdot \end{array} \end{array}$$

reductions:

A) $(\vartheta \cdot k, \nu, x)^4$ decomposes as single Überschiebung over the decomposable form $(\vartheta^2 \nu x)^4$.

B) form sequence $H_k K_\eta$: reducer $H_\vartheta \theta_\eta$.

form sequence $K_\eta^2(KHu)$: reducer $a_\eta^2(aHu)$.

form sequence $K_\eta^2(k\eta x)$: reducer $\theta_\eta^2(\vartheta\eta x)$.

The from this arising double reduction of the form sequence K_η^3 gives rise to the reduction of the higher form sequence $(j\eta x)^3$.

(KHu) , $(k\eta x)$, $K_\eta(k\eta x)$ decomposable forms according to § 3.

$H_k(KHu)$, $K_\eta(KHu)$ decompose as arising by single folding with the decomposable form $K_\nu(k\nu x)$ and the functional determinant

$$K_\nu^2 = \theta_\nu^2(a\theta u) \equiv a_\nu^2(a\theta u) \mod \nu^2.$$

A relation between decomposable forms corresponds to the double representation of K_ν^2 . One obtains:

$$H_k(KHu) = 2K_\nu(\nu\nu_1 k) = 2\theta_{\nu_1}(\nu\nu_1 a\theta) = 2\theta_\nu^2 a_\nu - a_\nu^2 \theta_\nu + f\theta_\eta(\theta Hu)^2 - f\nu\theta_\eta^3 \mod \nu^3,$$

$$\begin{aligned} K_\eta(KHu) &= -K_\nu^2(\nu\nu_1 x) = -a_\nu^2(a\theta u)(\nu\nu_1 x) \\ &= a_\eta(aHu)^2 \theta + a_\nu^2 \theta_\nu = -\theta_\nu(\theta Hu)^2 f - \theta_\nu^2 a_\nu. \end{aligned}$$

Therefore

$$0 = a_\nu^2 \theta_\nu + \theta_\nu^2 a_{\nu_1} + f\theta_\eta(\theta Hu)^2 + \theta a_\eta(aHu)^2 \mod \nu^3. \quad (17)$$

The forms H_k , $H_k(k\eta x)$, H_k^2 , $H_k^2(k\eta x)$ decompose as arising by single folding with the decomposable forms H_ϑ and H_ϑ^2 (cf. § 3. 2), a) and b)).

Es is

$$H_k = H_\vartheta(a\theta u) \sim [H_\vartheta]_{ua} - fH_\vartheta(\theta Hu)$$

(specialization $y = uv$ of 2)a).

$$H_k(k\eta x) = H_\vartheta a_\eta - H_\vartheta \theta_\eta f, \quad H_\vartheta a_\eta = \frac{5}{4}[H_\vartheta]a - \frac{1}{4}H_\vartheta a_\vartheta \quad (\S 3.2)b),$$

$$H_{\vartheta}^2(a\theta u) = [H_{\vartheta}^2]_{ua}, \quad H_{\vartheta}^2 a_{\eta} = [H_{\vartheta}^2]a = H_k^2(k\eta x) + H_{\vartheta}^2 \theta_{\eta} f.$$

C) form sequence $(j\eta x)^3$: Reducible by double reduction of the form sequence K_{η}^3 according to B); according to the ansatz:

$$0 = a_{\eta}^3(a\theta u) = \theta_{\eta}^3(a\theta u) = \theta_{\eta} + (a\theta\eta x)^3(a\theta u) - \theta_{\eta}^3(a\theta u) = \frac{1}{2}(j\eta x)^3 \bmod(\nu^3, s, \varrho, t, i, J).$$

The analogous ansatz holds up to the terminal form of the form sequence:

$$0 = H_{\vartheta}^2(a\theta\eta x)a_{\eta}^3 = H_{\vartheta}^2(a\theta\eta x)\theta_{\eta}^3 - \frac{1}{2}H_{\vartheta}^2(j\eta x)^4 \bmod(\nu^3, s, \varrho, t, i, J).$$

form $(j\eta x)^2$: decomposes 1) by twofold polarization of the relation for the decomposable form H_{ϑ}^2 ((12.) b);

2) by direct calculation of the products $a_{\nu}\theta_{\nu}^3$; thereby Decomposition of the higher form $(\Delta Hu)^2$. One obtains:

$$\left. \begin{array}{l} \text{a) } \frac{1}{3}(\Delta Hu)^2 + \frac{1}{3}(j\eta x)^2 = \theta_{\nu}^2 a_{\nu_1}^2, \\ \text{b) } (j\eta x)^2 = \frac{4}{3}\theta_{\nu}^3 a_{\nu_1}, \end{array} \right\} \bmod(\nu^3, s, \varrho, t, i, J). \quad (18)$$

according to the ansatz:

$$\begin{aligned} H_{\vartheta}^2(a\theta u)^2 - \frac{1}{3}(\Delta Hu)^2 &= 2\theta_{\eta}^2(a\theta u)(aHu) + \theta_{\nu}^2 a_{\nu_1}^2 = (a\theta u)(aHu)\{a_{\eta} - (a\theta\eta x)^2\} + \theta_{\nu}^2 a_{\nu_1}^2, \\ a_{\nu_1}\theta_{\nu}^3 &= -(a\widehat{\nu}\nu_1 u)\theta_{\nu}^2(\theta\widehat{\nu}\nu_1 u) - \frac{1}{2}(u\nu\nu_1 u)\theta_{\nu}\theta_{\nu_1}(\vartheta\nu\nu_1 u) \\ &= -\frac{3}{2}\theta_{\eta}^2(\theta Hu)(aHu) = -\frac{3}{2}\theta_{\eta}^2(\theta Hu)(a\theta u) \\ &= -\frac{3}{2}\{a_{\eta} - (a\theta\eta x)^2\}\{(aHu) - (a\theta u)\}(a\theta u). \end{aligned}$$

D) form sequence $\Delta_{\eta}(\Delta Hu)$: reducer Δ_{ν}^2 .

$(\Delta Hu)^2$ decomposable form according to C).

E) form sequence $a_{\eta}^2(aLu)$: reducer $a_{\eta}^2(aHu)$. forms (aLu) and $a_i(aLu)$: Decomposition as a transvectionen over functional determinants according to § 3.

F) form sequence a_{σ}^3 : reducer a_{σ}^3 .

forms a_{σ}^2 and a_{σ}^3 : reducer a_{ν}^3 and a_{η}^3 by double reduction of the corresponding expressions

$$H_{\nu}a_{\nu}^3 \quad \text{and} \quad H_{\nu}a_{\eta}^3, \quad H_{\nu}a_{\nu}^3 a_{\nu} \quad \text{and} \quad H_{\nu}a_{\eta}^4$$

(cf. § 10. C) reduction of Δ_{ν}^2 and Δ_{ν}^3 .

From this reduction for the higher forms $a_{\nu}s_{\nu}(asu)^2$, $a_{\nu}^2 s_{\nu}(asu)^2$ and for forms in symbolsn $\varrho, t(a_{\nu}, a_{\eta}^2)$ taking account of (16.):

$$\left. \begin{array}{l} a_{\nu}s_{\nu}(asu)^2 = \frac{1}{3}i\theta_{\nu}^2 - \frac{1}{18}iH, \\ a_{\nu}^2 s_{\nu}(asu)^2 = 0, \end{array} \right\} \bmod(\varrho, t). \quad (19)$$

form a_{σ} : decomposes by polarization of (12.)b or, more briefly, by direct expansion of the products $a_{\nu}^2\theta_{\nu}^3$ according to the ansatz:

$$a_{\nu}^2\theta_{\nu_1}^3 = (a_{\nu}^2\theta_{\nu_1})a_{\nu_1} - (a\theta\nu_1 x)^2 = -2a_{\eta}(aHu)(a\theta H)(a\theta\eta x) + (a\theta\nu_1 x)^2 a_{\nu}^2\theta_{\nu_1}$$

and taking account of the reduction of $H_j(j\eta x)^2$ (C):

$$a_{\sigma} = 2a_{\nu}^2\theta_{\nu_1}^3 \bmod(s, \varrho, t, i). \quad (20)$$

Consequences:

1) $(j\eta x)^3$, $\Delta_{\eta}(\Delta Hu)$, a_{σ}^2 are reducers, $(j\eta x)^2$, $(\Delta Hu)^2$, a_{σ} decomposable forms.

2) f enters into folding with L only in products with contragredient forms; therefore f , and likewise K and N , do not enter at all into folding with σ and consequently with $(\nu\sigma x)$. The form j enters only in products with contragredient forms; Δ does not enter at all in products in folding with H and L (this follows from $\Delta_{\eta}(\Delta Hu)$ and $(\Delta Hu)^2$). Likewise σ , and consequently $(\nu\sigma x)$, do not enter in products in folding with any form of System I. Thus, among products in System II, taking account of the consequences of § 11, there remain only powers of H and products of H with L .

§ 13. Forms of the eighth order (System III).

$$\text{Folding of } \begin{cases} \Delta \cdot j \text{ with } \nu, \\ \theta^2, f \cdot j, N \text{ with } H, \\ \theta \text{ with } L, \sigma. \end{cases}$$

Irreducible forms:

$$\begin{array}{ll} \text{A)} & \Delta_\nu(j\nu x) \\ \text{B)} & \frac{(\vartheta^2 \eta x)^3}{(\vartheta^2 \eta x)^4} \quad H_\vartheta(\vartheta^2 \eta x)^3, \theta_\eta(\vartheta^2 \eta x)^3, \\ \text{C)} & (aHu)H_j, \quad a_\eta(aHu)H_j \\ \text{D)} & H_n, \quad N_\eta, \\ \text{E)} & \begin{array}{c} \cdot \\ \theta_i \\ (\vartheta l x)^2 \\ \cdot \end{array} \left| \begin{array}{c} (\theta Lu)^2 \\ \cdot \\ \theta_i^2 \\ \theta_i(\vartheta l x)^2 \end{array} \right| \begin{array}{c} (\theta Lu)^3 \\ L_\vartheta(\theta Lu)^2, \theta_i(\theta Lu)^2 \\ \cdot \\ \cdot \end{array} \left| \begin{array}{c} \cdot \\ \theta_i(\theta Lu)^3 \\ \cdot \\ \cdot \end{array} \right. \end{array}$$

reductions:

A) $\Delta_\nu(j\nu x)^3$: reducer $a_\nu(j\nu x)^3$.

$\Delta_\nu(j\nu x)^2$ is reducible by the decomposable form $a_\nu(j\nu x)^2$ according to § 3. 2)b, taking account of the reducer $\theta_{\vartheta j}$. Indeed, according to § 11 B), one obtains:

$$\frac{3}{10}\Delta_\nu(j\nu x)^2 + \frac{3}{5}a_\nu(j\nu x)a_\vartheta(j\nu x) + \frac{1}{10}a_\nu(j\nu x)^2 = \frac{3}{5}\theta_\nu^3\theta_{\vartheta j} + \frac{2}{5}\theta_\nu^3\theta_{\vartheta j}\theta_{\vartheta j},$$

(reducer a_j and $\theta_{\vartheta j}$).

B) The forms $H_\vartheta(\vartheta' \eta x)^2$, $H_\vartheta^2(\vartheta' \eta x)$, $H_\vartheta^2(\vartheta' \eta x)^2$ decompose as arising by successive single folding with the decomposable forms H_ϑ and H_ϑ^2 , respectively $H_\vartheta a_\eta$ and $H_\vartheta^2 a_\eta$ according to § 3. 2)b) and according to the relation:

$$H_\vartheta(\vartheta' \eta x)^2 = 2H_\vartheta a_\eta^2 f - 2H_\vartheta a_\eta b_\eta.$$

C) form sequence $a_\eta(j\eta x)^2$: reducers a_j and $(j\eta x)^3$.

form $a_\eta(j\eta x)$ decomposes: reducer a_j and single folding with the decomposable form $(j\eta x)^2$ according to § 3, 2)a) (specialization $y = uv$).

form $a_\eta H_j$ decomposes: 1) by single folding with the decomposable form $a_\nu(j\nu x)^2$;

2) by a special twofold folding with the decomposable form $(j\eta x)^2$; from this one obtains a relation between decomposable forms.

From $a_\nu(j\nu x)^2$:

$$0 = \theta'_\nu \theta_\nu + 2a_\eta H_j \pmod{s, \varrho, t, i, J}.$$

From $(j\eta x)^2$:

$$0 = \theta'_\nu \theta_\nu - \frac{3}{2}a_\eta H_j \pmod{s, \varrho, t, i, J}$$

(products with contravariants are omitted), according to the ansatz:

$$0 = \frac{1}{2}a_\nu(abu)^2\theta_\nu^3 + \frac{1}{2}u_x(ubu)\theta_{\nu_1}^3(\theta bu) - \frac{3}{4}(\hat{j}\eta bu)^2 + \frac{3}{4}H_j(\hat{j}\eta bu) - \frac{1}{8}fH_j^2$$

and by interchanging the symbols in $a_\nu(ubu)(\theta bu)\theta_{\nu_1}^3$.

D) form sequence $N_\eta(NHu)$: reducer Δ_ν^2 .

(NHu) , $(n\eta x)$, $N_\eta(n\eta x)$, $H_\eta(NHu)$ decomposable forms (cf. § 12 B), $(NHu)^2$ decomposes by single folding with the form $(\Delta Hu)^2$.

E) form sequences $\theta_i L_\vartheta$, $\theta_i^2(\theta Lu)^2$, $\theta_i^2(\vartheta l x)$:

reducers: $\theta_\eta H_\vartheta$, $\theta_\eta^2(\theta Hu)^2$, $\theta_\eta^2(\vartheta \eta x)$.

forms (θLu) , $(\vartheta l x)$, L_ϑ , $L_\vartheta(\theta Lu)$, $\theta_i(\theta Lu)$, $L_\vartheta(\vartheta l x)$, $\theta_i(\vartheta l x)$, L_ϑ^2 , $L_\vartheta^2(\theta Lu)$, $\theta_i^2(\theta Lu)$ decompose. (Dualistically opposite to the decomposable forms of § 12 B).

F) form sequence $\theta_\sigma(\vartheta \sigma x)$: reducer a_σ^2 .

forms $(\vartheta \sigma x)$ and $(\vartheta \sigma x)^2$ decompose, as arising by single folding with the decomposable form a_σ ; θ_σ , by twofold folding arising, decomposes, since the form sequence

$$a_{\nu_2}^2(a\theta u)\theta_{\nu_1}^3 \equiv H_j^2(j\eta x) \equiv 0 \pmod{u_x(s, \varrho, t, i, J)} :$$

$$\theta_\sigma = a_\sigma(abu)^2 = a_\nu^2(abu)^2\theta_\nu^3.$$

Consequences:

θ^3 or $\theta^2 K$ does not enter into folding with H ; θ enters into folding with L only as a contravariant in powers or products, and not at all into folding with σ and $(\nu\sigma x)$.

N does not enter in products in folding with irgend einer form of System II enters, ebensowenig wie with powers or products of forms of System II. There remain an products in System I, taking account of the consequences of the last paragraphs, only products $f \cdot j$, $\Delta \cdot j$, powers of θ and products of $\theta \cdot K$.

§ 14. Forms of the ninth order (System III).

$$\text{Folding of } \begin{cases} \theta \cdot K \text{ with } H, \\ K, j, \Delta \text{ with } L, \\ j, \Delta \text{ with } \sigma, \\ f \text{ with } H^2. \end{cases}$$

Irreducible forms:

$$\begin{array}{ll} \text{A)} & (\vartheta \cdot k, \eta, x)^4, H_k(\vartheta \cdot k, \eta, x)^4 \\ \text{B)} & (KLu)^3, K_i(KLu)^3, K_i^2, (klx)^3, K_i(Klx)^3, \\ \text{C)} & L_j, L_j^2 \\ \text{D)} & \Delta_i, \\ \text{E)} & (j\sigma x) \\ \text{G)} & (aH^2u)^3, (aH^2u)^4, a_\eta(aH^2u)^3. \end{array}$$

reductions:

A) the forms $H_\vartheta(k\eta x)^3$, $H_\vartheta^2(k\eta x)^2$, $H_\vartheta^3(k\eta x)$ decompose as arising by successive single folding with decomposable forms.

B) form sequences $K_i L_k$, $K_i^2(KLu)$, $K_i^2(klx)$:

reducers: $\theta_\eta H_\vartheta$, $a_\eta^2(aHu)$, $\theta_\eta^2(\vartheta\eta x)$.

forms L_i , $K_i(KLu)$, $K_i(klx)$, $(KLu)^2$, $(klx)^2$ decompose as a transvectionen over functional determinants (§ 3).

forms L_k , $L_k(KLu)$, $L_k(KLu)^2$, $L_k(klx)$, $L_k(klx)^2$, L_k^2 , $L_k^2(KLu)$, $L_k^2(klx)$, L_k^3 , $K_i(KLu)^2$, $K_i(klx)^2$ decompose as arising by single folding with the decomposable forms:

$$L_\vartheta, H_k(KHu), L_\vartheta(\vartheta lx), H_k^2, L_\vartheta^2, L_\vartheta^2(\theta Lu), K_\eta(KHu), \theta_i(\vartheta lx)$$

according to § 3. 2), a) and b).

C) form sequence $(jlx)^3$: reducer $(j\eta x)^3$.

forms $(jlx)^2$ and $L_j(jlx)$: arising by single folding with the decomposable form $(j\eta x)^2$.

D) form sequence $\Delta_i(\Delta Lu)$: reducer Δ_ν^2 .

forms $(\Delta Lu)^2$, $(\Delta Lu)^3$: Single folding with the decomposable form $(\Delta Hu)^2$.

E) form sequence $(j\sigma x)^2$: reducer a_σ^2 by replacing j by lower forms of the form sequence (§ 5):

$$a_\sigma^2(a\theta u)^2 = \frac{1}{3}(j\sigma x)^2, \quad a_\sigma^2(a\theta\sigma x)^2 = \frac{1}{3}(j\sigma x)^4.$$

F) form sequence Δ_σ^2 : reducer a_σ^2 .

form Δ_σ : reducer a_σ^2 by replacing Δ by lower forms of the form sequence:

$$a_\sigma a_\sigma^2 = \frac{2}{3}\Delta_\sigma \text{ mod } \nu.$$

Consequences: Only the form j enters in folding with σ and $(\nu\sigma x)$ enters.

N does not enter in folding with L enters, since the Δ_i correspondinglye form N_i decomposes.

§ 15. Forms of the tenth order (System III).

$$\text{Folding of } \begin{cases} \theta^2, f \cdot j \text{ with } L, \\ \theta \text{ with } H^2. \end{cases}$$

Irreducible forms:

$$\begin{array}{ll} \text{A)} & (\vartheta^2 lx)^4, \theta_i(\vartheta^2 lx)^4 \\ \text{B)} & (aLu)^2 L_j, a_i(aLu)^2 L_j, \\ \text{C)} & (\theta H^2 u)^3, (\theta H^2 u)^4, H_\vartheta(\theta H^2 u), \theta_\eta(\theta H^2 u)^3. \end{array}$$

reductions:

A) and B): Decomposition of the remaining forms by single folding with decomposable forms.

C) Decomposition of the remaining forms according to § 13 B) by the dualistically opposite consideration.

Consequences:

$\theta^2 K$ does not enter into folding with L ; correspondingly, $H^2 L$ does not enter into folding with K . The forms H^3 and $H^2 L$ do not enter into folding with θ . The folding scheme of § 7 is therefore proved.

§ 16. Forms of the 11th, 12th, 13th, and 15th orders (System III).

11th order.

$$\text{Folding of } \begin{cases} \theta \cdot K \text{ with } L, \\ K, j \text{ with } H^2, \\ j \text{ with } (\nu \sigma x), \\ f \text{ with } H \cdot L. \end{cases}$$

Irreducible forms:

$$\begin{array}{ll} \text{A) } (\vartheta \cdot k, l, x)^5, \theta_i(\vartheta \cdot k, l, x)^5 & \text{B) } (KH^2 u)^4, K_\eta(KH^2 u)^4, \\ \text{C) } (\nu \sigma j), & \text{D) } (a_\eta H \cdot L, u)^4. \end{array}$$

12th order.

$$\text{Folding of } \begin{cases} \theta^3 \text{ with } L, \\ \theta \text{ with } H \cdot L. \end{cases}$$

Irreducible forms:

$$\text{E) } (\vartheta^3 l x)^6, \quad \text{F) } (\theta, H \cdot L, u)^4, \quad L_\vartheta(\theta, H \cdot L, u^4).$$

13th order.

Folding of K with $H \cdot L$.

Irreducible forms:

$$\text{G) } (K, H \cdot L, u)^5, \quad K_\eta(K, H \cdot L, u)^5.$$

15th order.

Folding of K with H^3 .

Irreducible form:

$$\text{H) } (KH^3 u)^6.$$

reductions:

The forms H_j^2 , H_j^2 have reducer L_j^3 , from the relation:

$$(\nu l x)L_x^3 = -\frac{1}{2}H^2 \bmod(\sigma, s).$$

Decomposition or reducibility of the remaining forms follows from the considerations of the earlier paragraphs.

Consequences:

According to the folding scheme of § 7 and the consequences of §§ 8–15, the irreducible forms of System III (117 forms) are thereby exhausted.

Chapter IV. The relatively complete system mod (ϱ, t) .

§ 17. Reduction of the modulus $(ss'u)^4$ and of the system of s .

For the formation of the next higher relatively complete system, one must, in analogy with § 7, form the system of s with respect to the next modulus

$$\nu(s) = (ss'u)^4$$

and fold it with the forms of the relatively complete system mod s . It will be shown that, after introduction of the modulus (ϱ, t) as the higher modulus,

- A) the modulus $(ss'u)^4$ becomes reducible,
- B) the system of s consists of the single form s .

A) The reduction of the modulus $(ss'u)^4$ follows at once from the reducer s_ν^2 found by formula (16.), § 11. From

$$s_\nu^2 = \frac{4}{3}ia_\nu^2 + \frac{4}{5}\theta_\varrho^2 + \frac{8}{5}(atu)^2 - \frac{26}{15}(\nu\varrho x)^2 + \frac{1}{6}J \cdot u_x^2 - \frac{2}{9}i^2 \cdot u_x^2$$

there follows, by polarization from x to ν_1 ,

$$\begin{aligned} (ss'u)^4 &= -\frac{2}{3}J \cdot \nu + 6s_\nu^2 s_{\nu_1}^2 \\ &= \frac{24}{5}\theta_\nu^2 \theta_\varrho^2 + \frac{48}{5}a_\nu^2 (atu)^2 - \frac{52}{5}H_\varrho^2 + \frac{4}{3}i \cdot (asu)^4 + \frac{1}{3}J \cdot \nu - \frac{4}{9}i^2 \cdot \nu, \end{aligned} \quad (21)$$

and with this the reduction of the modulus $(ss'u)^4$.

B) The reduction of

$$Z = (ss'u)^2 = \theta(s)$$

and hence the reduction of the system of s is obtained by replacing the form Z by the lower form g_ν^2 , according to formula (8.)b, § 7, taking into account the relation

$$s_\eta^2 = s_\nu^2 (\nu\nu_1 x)^2 - \frac{1}{3}Z.$$

The form g_ν^2 has, by formula (13.), § 11, the reducer s_ν^2 as a factor. By formula (8.) and (16.) one obtains

$$g_\nu^2 = -Z + \frac{14}{5}ia_\eta^2 + \frac{14}{15}i(asu)^2 \text{ mod}(\varrho, t),$$

and by formulas (13.), (14.), (15.), (16.),

$$\begin{aligned} g_\nu^2 &= 2[s_\vartheta^2]_{\nu^2} - \frac{4}{3}i\Delta_\nu^2 = 2s_\vartheta^2 s_\nu^2 - \frac{6}{5}[s_\vartheta^2(\theta su)^2]\widehat{\nu x}^2 - \frac{4}{3}i\Delta_\nu^2 \\ &= \frac{4}{5}i \cdot a_\eta^2 - \frac{2}{5}i \cdot (asu)^2 + \frac{1}{3}J \cdot \theta \text{ mod}(\varrho, t). \end{aligned}$$

Thus

$$Z = 2i \cdot a_\eta^2 + \frac{4}{3}i \cdot (asu)^2 - \frac{1}{3}J \cdot \theta \text{ mod}(\varrho, t). \quad (23)$$

The relatively complete system mod $((ss'u)^4, \varrho, t)$ therefore passes into a relatively complete system mod (ϱ, t) , and from this, by transvection over the known system of two quadratic forms, the absolutely complete system arises. In order to form the relatively complete system mod (ϱ, t) , since by formula (23.) the system of s reduces to the form s , we must transvect the relatively complete system mod (s, ϱ, t) over s and over powers of s . It will be shown that powers of s enter into folding only with System I.

The announced relation is

$$\begin{aligned} (ass')^4 &= \frac{24}{5}\theta_\nu^2 \theta_\varrho^2 a_\nu^2 a_\varrho^2 + \frac{48}{5}a_\nu^2 b_\nu^2 (tab)^2 - \frac{52}{5}H_\varrho^2 a_\nu^4 + \frac{5}{3}J \cdot i - \frac{4}{9}i^3, \\ (ass')^4 &= \frac{24}{5}a_\varrho^2 a_{\varrho'}^2 - \frac{12}{5}t_\varrho^2 + \frac{5}{3}J \cdot i - \frac{4}{9}i^3. \end{aligned} \quad (22)$$

²⁷From formula (21.) there follows the relation, mentioned in the note to the introduction, between the three invariants of order 9, with use of formula (10.)c.

Among forms of the same order and the same degree, we define as “higher forms” those which have arisen from higher forms of the relatively complete system mod (s, ϱ, t) by folding with s , regarding the folding with s merely as a polarization of the original forms. Thus the order of the individual forms is uniquely determined (cf. § 1, form sequences 2). For instance,

$$\begin{aligned}(\theta_\eta(\theta Hu), s, u)^2 &\text{ is higher than } s_\eta((\theta Hu)^2, s, u), \\ (\theta_\eta(\theta Hu)^2, s, u) &\text{ is higher than } (\theta_\eta(\theta Hu), s, u)^2.\end{aligned}$$

We divide the resulting system into the four subsystems

- System s_I : obtained by folding s and powers of s with System I;
- System s_{II} : obtained by folding s with System II;
- System s_{III} : obtained by folding s with the individual forms (without products) of System III and with products of one form each from Systems I and II;
- System s_{IV} : obtained by folding s with products of one form each from Systems I and III, II and III, III and III.

It should also be noted that from now on all formulas are to be understood mod (ϱ, t) ; from formula (27.) on, they are to be understood mod $(\varrho, t, i, J, \text{ higher forms})$, without this being expressly stated each time.

For the reduction we collect the formulas obtained earlier:

$$\begin{aligned}s_\nu^2 &= \frac{4}{3}ia_\nu^2 + \frac{1}{6}J \cdot u_x^2 - \frac{2}{9}i^2 \cdot u_x^2, & s_\nu^3 &= \frac{1}{3}J \cdot u_x, \quad s_\nu^4 = J, \\ g &= 2s_\vartheta^2 - \frac{4}{3}i\Delta, & s_\vartheta^2(\theta su) &= 0, \\ s_\vartheta^2(\theta su)^2 &= \frac{8}{9}i \cdot a_\nu^3 - \frac{4}{27}i^2 \cdot u_x^2, \\ Z &= 2i \cdot a_\eta^2 + \frac{4}{3}i(asu)^2 - \frac{1}{3}J \cdot \theta, \\ g_\nu &= -s_\eta + u_x \left\{ 2ia_\eta^2 - \frac{2}{3}i(asu)^2 + \frac{2}{3}J \cdot \theta \right\}, \\ g_\nu^2 &= \frac{4}{5}ia_\eta^2 - \frac{2}{5}i(asu)^2 + \frac{1}{3}J \cdot \theta, & g_\nu^3 &= 0, \quad g_\nu^4 = 0.\end{aligned} \tag{24}$$

Consequences:

$$s_\nu^3, \quad s_\vartheta^2(\theta su), \quad s_\vartheta^2 s_\nu, \quad s_\vartheta^2 \theta_\nu$$

are reducers.

§ 18. System s_I .

$$\text{Folding of } \begin{cases} s \text{ with } f; \theta; K, j, \Delta; f \cdot j, N; \Delta \cdot j, \\ s^3 \text{ with } \theta, K. \end{cases}$$

Irreducible forms.

Order 5:

$$(asu), \quad (asu)^2, \quad (asu)^3, \quad (asu)^4.$$

Order 6:

$$\begin{array}{cccc}(\theta su) & (\theta su)^2 & (\theta su)^3 & (\theta su)^4 \\ s_\vartheta & s_\vartheta(\theta su) & s_\vartheta(\theta su)^2 & s_\vartheta(\theta su)^3 \\ \cdot & s_\vartheta^2 & \cdot & \cdot\end{array}$$

Order 7:

$$\begin{array}{cccccc} \cdot & (Ksu)^2 & (Ksu)^3 & (Ksu)^4 & & \\ \text{A) } s_k & s_k(Ksu) & s_k(Ksu)^2 & s_k(Ksu)^3 & \text{B) } s_j, & \\ \cdot & s_k^2 & \cdot & \cdot & \text{C) } (\Delta su), (\Delta su)^2. & \end{array}$$

Order 8:

$$\begin{aligned} \text{A)} \quad & s_j(asu), \quad s_j(asu)^2, \quad s_j(asu)^3, \\ \text{B)} \quad & (Nsu)^2, \quad s_n, \quad s_n(Nsu). \end{aligned}$$

Order 10:

$$\text{A)} \quad s_j(\Delta su), \quad \text{B)} \quad s_\vartheta(\theta s'u)^4.$$

Order 11:

$$(Ks^2u)^6, \quad s_k(Ks^2u)^5, \quad s_k(Ks^2u)^6.$$

Reductions:

The form sequences

$$s_\vartheta^2(\theta su), \quad s_k^2(Ksu), \quad s_j^2, \quad (\Delta su)^3, \quad (Nsu)^3$$

have the reducer $s_\vartheta^2(\theta su)$ as a factor; s_j^2 and $(\Delta su)^3$ are obtained by going back from j and Δ to lower forms of the form sequence:

$$\begin{aligned} s_\vartheta^2(\theta su)(a\theta u)^3 &= -\frac{1}{2}s_j^2, \\ s_\vartheta^2(\theta su)(a\theta u)^2 &= \frac{1}{3}(\Delta su)^3, \\ s_\vartheta^2(\theta su)^2(a\theta u)^2 &= \frac{1}{3}(\Delta su)^4 + \frac{1}{3}s_j^2. \end{aligned}$$

Forms $s_\vartheta^2 s_{\vartheta'}$ and $s_\vartheta^2 s_{\vartheta'}^2$: reducers $s_\vartheta^2(\theta su)$ and $\theta_{\vartheta'}$:

$$\begin{aligned} s_\vartheta^2 s_{\vartheta'} &= s_\vartheta^2 \theta_{\vartheta'} - s_\vartheta^2(\theta s\vartheta'x), \\ s_\vartheta^2 s_{\vartheta'}^2 &= s_\vartheta^2 s_{\vartheta'} \theta_{\vartheta'} - s_\vartheta^2 s_{\vartheta'}(\theta s\vartheta'x). \end{aligned}$$

Form sequence $s_j(\Delta su)^2$: reducers Δ_j and $(\Delta su)^3$.

Consequences:

s does not enter into powers in folding with f, j, Δ, N . The forms f, j, Δ, N enter into folding only in products with contragredient forms; however, in products with f those forms may still be quadratic in x , and in products with Δ they may still be linear in x . Thus the following products of forms are to be considered:

$$f \cdot (0, \lambda), \quad f \cdot (1, \lambda), \quad f \cdot (2, \lambda), \quad j \cdot (\mu, 0), \quad N, \quad \Delta \cdot (0, \lambda), \quad \Delta \cdot (1, \lambda) \quad (\lambda > 0),$$

where (μ, λ) denotes a form of degree μ in x and of degree λ in u .

Neither θ nor K enters into powers or products with arbitrary forms in folding with s or s^2 . For products with the functional determinant K , $K \cdot \varphi_x$ ($\varphi \neq f$ or θ), the relation between decomposing forms is

$$K \cdot \varphi_x = (a\theta u)\varphi_x = f \cdot (\varphi\theta u) - \theta \cdot (\varphi a u).$$

The above scheme for folding System s_I is thereby proved.

§ 19. System s_{II} .

Folding of s with ν ; H ; L ; H^2 .

Irreducible forms:

Order 6	Order 8	Order 10	Order 12
s_ν	$(Hsu), (Hsu)^2$	$(Lsu)^2, (Lsu)^3$	$(H^2su)^3$
.	s_η	s_l	.
.	$s_\nu^4 = J$	.	.

Reductions:

Form sequences $s_\eta(Hsu)$ and $s_l(Lsu)$: reducer s_ν^2 .

Form sequence s_σ : reducer s_ν^2 from H , $s_\nu^2 = \frac{2}{3}s_\sigma$.

$(H^2su)^4$ decomposes by formula (7.)a (cf. § 11 A). Hence $(H \cdot L, s, u)^4$ also decomposes.

Consequences:

s^2 does not enter into folding with System II. The form ν enters only in products with contragredient forms. The form H , regarded as a covariant, enters into folding in products with forms $(1, \lambda)$, $(2, \lambda)$, $(3, \lambda)$,

with arbitrary λ . The forms σ and $(\nu\sigma x)$ do not enter at all, and L , as a functional determinant, does not enter into products in folding; the full transvections over covariants of System III become reducible. The products of Systems I and II that remain are

$$f \cdot \nu, \quad \Delta \cdot \nu.$$

§ 20. Forms of order 7 (System s_{III}).

Folding of s with $f \cdot \nu$, a_ν , a_ν^2 .

Irreducible forms:

$$\begin{aligned} s_\nu(asu), \quad a_\nu(asu), \quad s_\nu(asu)^2, \quad a_\nu(asu)^2, \quad s_\nu(asu)^3, \quad a_\nu(asu)^3, \\ a_\nu s_\nu, \quad a_\nu s_\nu(asu), \quad a_\nu^2(asu), \quad a_\nu^2(asu)^2. \end{aligned}$$

Reductions:

The evident reductions that arise by direct folding with the reducers s_ν^2 and $s_\eta(Hsu)$ will not be mentioned, nor will the decomposition of transvections with functional determinants.

For the forms $a_\nu s_\nu(asu)^2$ and $a_\nu^2 s_\nu(asu)^2$, see formula (19.), § 12; moreover,

$$a_\nu s_\nu(asu)^3 = a_\nu(a\hat{\nu}_1\nu_2u)^3(\nu_1\nu_2\nu) = 0.$$

Form sequence $a_\nu^2 s_\nu$: reducer $s_\vartheta^2(\theta su)$, obtained by going back from a_ν^2 to lower forms, i.e. by double reduction of the expressions

$$s_\vartheta^2(a\theta s)(a\theta u), \quad s_\vartheta^2(a\theta s)(a\theta u)^2$$

according to the Ansatz

$$\begin{aligned} s_\vartheta^2(a\theta s)(a\theta u) &= [(a\theta u)^2]s^3 + \frac{1}{5}[a_\vartheta(a\theta u)^2]s^2 + \frac{1}{60}[a_\vartheta^2(a\theta u)^2]s \\ &= \frac{1}{2}a_\nu^2 s_\nu - \frac{2}{3}a_\nu^2 s_\nu^2, \\ s_\vartheta^2(a\theta s)(a\theta u)^2 &= \frac{4}{5}[a_\vartheta(a\theta u)^2]_{us}s^2 + \frac{23}{180}[a_\vartheta^2(a\theta u)^2]_{us}s \\ &= -\frac{1}{2}a_\nu^2 s_\nu(asu). \end{aligned}$$

Here $a_\nu^2 b_\nu(abu) = 0$.

The formulas obtained are

$$\begin{aligned} a_\nu^2 s_\nu &= \frac{4}{9}i a_\nu^2 b_\nu, & a_\nu s_\nu(asu)^2 &= \frac{1}{3}i\theta_\nu^2 - \frac{1}{18}iH, \\ a_\nu s_\nu(asu)^3 &= 0, & a_\nu^2 s_\nu(asu) &= 0, \\ a_\nu^2 s_\nu(asu)^2 &= 0. \end{aligned} \tag{25}$$

Consequences:

1) $a_\nu s_\nu(asu)^2$ and $a_\nu^2 s_\nu$ are reducers.

2) The values of the forms $(\Delta su)^3$, $(\Delta su)^4$, already recognized as reducible in § 19, are thereby obtained; likewise for the corresponding form sequence $(agu)^2$, which, when folded with ν , gives rise to double reduction through the reducer g_ν :

$$\begin{aligned} \text{a) } (\Delta su)^3 &= -\frac{6}{5}a_\nu^2(asu) + 3a_\nu s_\nu(asu) + 2i\theta_\nu(\theta\nu x), \\ \text{b) } (\Delta su)^4 &= -\frac{2}{5}a_\nu^2(asu)^2 + \frac{4}{3}i\theta_\nu^2 + \frac{1}{9}iH. \end{aligned} \tag{26}$$

From formula (24.) and (26.), mod (i, J) (cf. p. 71),

$$\begin{aligned} \text{a) } (agu)^2 &= \frac{2}{3}(\Delta su)^2 - \frac{4}{3}a_\nu s_\nu + \frac{3}{5}s \cdot a_\nu^2, \\ \text{b) } (agu)^3 &= 2a_\nu s_\nu(asu) - a_\nu^2(asu)^2, \\ \text{c) } (agu)^4 &= -a_\nu^2(asu)^2. \end{aligned} \tag{27}$$

§ 21. Forms of 8th order (System s_{III}).

Folding of s with the forms of 4th order (System III). (§ 9.)

Irreducible forms:

$$\begin{aligned} & (\vartheta\nu x)_{s_\nu}, \quad ((\vartheta\nu x), s, u)^2, \\ & ((\vartheta\nu x)^2, s, u), \quad ((\vartheta\nu x)^2, s, u)^2, \quad \theta s_\nu, \\ & (\vartheta\nu x)^2_{s_\nu}, \quad ((\vartheta\nu x)^2, s, u)_{s_\nu}, \\ & ((\vartheta\nu x), s, u)^3, \quad (\theta_\nu s u)^2, \quad ((\vartheta\nu x), s, u)^4, \quad (\theta_\nu s u)^3, \\ & ((\vartheta\nu x)^2, s, u)^3, \quad (\theta_\nu(\vartheta\nu x), s, u)^2, \quad (\theta_\nu^2 s u), \quad ((\vartheta\nu x), s, u)^3, \quad (\theta_\nu^2 s u)^2, \quad (\theta_\nu(\vartheta\nu x), s, u)^4. \end{aligned}$$

Reductions:

$((\vartheta\nu x), s, u)_{s_\nu}$ decomposes as a transvection over functional determinants, by § 3.

Form sequence $((\vartheta\nu x), s, u)^2_{s_\nu}$: reducers s_ν^2 and $s_\nu^2\theta_\nu$:

$$(\widehat{\vartheta\nu s u})_{s_\nu}(\theta s u) = s_\nu s_\vartheta(\theta s u) = -\frac{1}{2}(\widehat{\vartheta\nu s u})^2(\theta s u),$$

$$\theta_\nu(\widehat{\vartheta\nu s u})_{s_\nu} = \theta_\nu s_\nu s_\vartheta = -\frac{1}{2}\theta_\nu(\widehat{\vartheta\nu s u})^2.$$

Form sequence $\theta_\nu s_\nu(\theta s u)$: reducer $a_\nu s_\nu(asu)^2$ by double reduction of the expressions

$$a_\nu s_\nu(asu)^2(abu) \quad \text{and} \quad a_\nu s_\nu(asu)^2(abu)(sbu);$$

$\theta_\nu s_\nu(\theta s u)^3$ by double reduction of $(agu)^3(abu)(bgu)^3$.

Form sequence $\theta_\nu(\vartheta\nu x)_{s_\nu}$: reducer $a_\nu^2 s_\nu$ from $\theta_\nu(\vartheta\nu x)_{s_\nu} = a_\nu^2(abu)_{s_\nu}$.

Form sequence $(\theta_\nu(\vartheta\nu x)^2, s, u)$: double reduction of the forms of the sequence $\theta_\nu(\widehat{\vartheta\nu s u})_{s_\nu}$, which at the same time belong to the form sequences $((\vartheta\nu x)su)^2_{s_\nu}$ and $\theta_\nu(\vartheta\nu x)_{s_\nu}$.

The form $(\theta_\nu(\vartheta\nu x), s, u)$ decomposes as a single transvection over the functional determinant $\theta_\nu(\vartheta\nu x) = a_\nu^2(abu)$.

The form $((\vartheta\nu x)^2, s, u)^4$: double reduction of the expression $(a\Delta u)^2(\Delta s u)^4$, or also of $(\theta g u)^4$, from formulas (27.) and analogously to the reduction of $(j\eta x)^4$ (§ 12 C)).

Double reduction gives the formulas:

$$\begin{aligned} 0 &= \theta_\nu s_\nu(\theta s u) - \frac{1}{6}(\widehat{\vartheta\nu s u})^2(\theta s u) - \frac{1}{12}(H s u), \\ 0 &= \theta_\nu s(\theta s u)^2 - \frac{1}{4}(\widehat{\vartheta\nu s u})^2(\theta s u)^2 - \frac{1}{24}(H s u)^2, \\ 0 &= (\widehat{\vartheta\nu s u})^2(\theta s u)^2 + 2\theta_\nu(\widehat{\vartheta\nu s u})(\theta s u)^2 + 3\theta_\nu^2(\theta s u)^2 - \frac{1}{3}H(su)^2, \\ 0 &= \theta_\nu s_\nu(\theta s u)^3 + \frac{1}{2}\theta_\nu(\widehat{\vartheta\nu s u})(\theta s u)^3, \\ 0 &= \theta_\nu s_\nu s_\vartheta - \frac{1}{4}s_\eta, \quad 0 = \theta_\nu s_\nu s_\vartheta(\theta s u), \quad 0 = \theta_\nu s_\nu s_\vartheta(\theta s u)^2. \end{aligned} \tag{28}$$

The last group corresponds to $((\theta_\nu(\vartheta\nu x)^2, s, u)^2, \dots)$.

Consequences:

- 1) $\theta(\vartheta\nu x)_{s_\nu}$, $(\widehat{\vartheta\nu s u})(\theta s u)_{s_\nu}$, $\theta_\nu(\widehat{\vartheta\nu s u})^2$, $(\widehat{\vartheta\nu s u})^2(\theta s u)^2$ are reducers.
- 2) The forms of 4th order $(5, \lambda)$, $(6, \lambda)$ etc. do not enter into folding with s^2 .
- 3) For the form sequence $(\theta g u)^2$, formulas (27.), with the reductions of this paragraph taken into account, give the following formulas:

$$\begin{aligned} \text{a)} \quad (\theta g u)^2 &= \frac{4}{3}\theta_\nu s_\nu + \frac{2}{3}s_\nu s_\vartheta - \frac{1}{3}s\theta_\nu^2 - \frac{1}{9}sH - \frac{1}{3}f a_\nu^2(asu)^2 + \frac{1}{3}a_\nu^2(asu)^2, \\ \text{b)} \quad (\theta g u)^3 &= -\frac{1}{3}(\widehat{\vartheta\nu s u})^2(\theta s u) - \theta_\nu(\widehat{\vartheta\nu s u})(\theta s u) - \theta_\nu^2(\theta s u), \\ \text{c)} \quad (\theta g u)^4 &= 2\theta_\nu^2(\theta s u)^2 - \frac{1}{3}(H s u)^2, \\ g\vartheta(\theta g u) &= (\widehat{\vartheta\nu s u})(\vartheta\nu x)_{s_\nu} - \theta_\nu(\vartheta\nu x)(\widehat{\vartheta\nu s u}), \\ g\vartheta(\theta g u)^2 &= \frac{2}{3}s\theta_\nu^3, \quad g\vartheta(\theta g u)^3 = \frac{1}{3}\theta_\nu^3(\theta s u), \quad g\vartheta(\theta g u)^4 = 0. \end{aligned} \tag{29}$$

§ 22. Forms of 9th order (System s_{III}).

Folding of s with the forms of 5th order (System III) and the product $\Delta \cdot \nu$ (§ 10).

Irreducible forms:

- A) $(k\nu x)^2 s_\nu, ((k\nu x)^3, s, u), (k\nu x)^3 s_\nu, ((k\nu x)^2, s, u)^2, K_\nu s_\nu,$
 $((k\nu x)^3, s, u)^2, ((k\nu x)^2, s, u) s_\nu, (K_\nu (k\nu x)^3, s, u),$
 $(K_\nu s u)^2, (K_\nu s u)^3, (K_\nu s u)^4, ((k\nu x)^2, s, u)^3, (K_\nu^2 s u)^4,$
- B) $(j\nu x) s_\nu, ((j\nu x)^2, s, u), (j\nu x)^3, s, u, ((j\nu x)^2, s, u)^2,$
- C) $s_\nu(\Delta s u), (\Delta_\nu s u),$
- D) $(aHu) s_\eta, (a_\eta s u), ((aHu), s, u)^2, ((aHu)^2, s, u),$
 $(aHu)^2 s_\eta, (a_\eta s u)^2, (a_\eta^2 s u), ((aHu), s, u)^3, ((aHu)^2, s, u)^2,$
 $((aHu), s, u)^4, (a_\eta s u)^3, (a_\eta(aHu), s, u)^2, (a_\eta(aHu)^2, s, u), (a_\eta(aHu), s, u)^3.$

Reductions:

A) The reductions follow directly from the reductions of § 21 by single folding. The form sequence $K_\nu s_\nu (Ksu)^2$ has the reducers $a_\nu s_\nu (asu)^2$ and $\theta_\nu s_\nu (\theta su)^2$ as factors. Hence the higher forms $K_\nu^2 (Ksu)^2$, $K_\nu^2 (Ksu)^3$ decompose according to the Ansatz:

$$0 = a_\nu s_\nu (asu)^2 (a\theta u) = K_\nu s_\nu (Ksu)^2 - \theta_\nu s_\nu (\theta su)^2 (a\theta u).$$

B) Form sequence $(j\nu x)^2 s_\nu$: reducer $a_\nu^2 s_\nu$ from

$$(j\nu x)^2 s_\nu = 3a_\nu^2 s_\nu (a\theta u)^2.$$

$((j\nu x)^3, s, u)^2$: reducer $a_\nu^2 s_\nu$ from

$$((j\nu x)^3, s, u)^2 = -6a_\nu^2 s_\nu (a\theta u)^2 (\theta su).$$

C) Forms $\Delta_\nu (\Delta s u)^2$ and $s_\nu (\Delta s u)^2$: reducer g_ν from formula (27.)a.

Form $\Delta_\nu s_\nu$: 1) reducer $a_\nu^2 s_\nu$ from $a_g a_\nu^2 s_\nu = \frac{2}{3} \Delta_\nu s_\nu$; 2) decomposes by formula (27.)a through double reduction of the expression $(\Delta s \hat{\nu} x)^2$.

Hence the higher form $a_\eta s_\eta$ decomposes.

D) Form sequence $(aHu)^2 s_\eta (asu)$: reducer $a_\nu s_\nu (asu)^2$ by double reduction of the form sequence $[a_\nu s_\nu (asu)^2]_{\nu_1 x^2}$; reduction of $(aHu)^2 s_\eta (asu)^2$ from $a_\nu s_\nu (asu)^2$ and $a_\nu s_\nu (asu)^3$. Hence the reduction of $(a_\eta, s, u)^4$.

Form sequence $a_\eta (aHu) s_\eta$: reducers $a_\nu^2 s_\nu$, $a_\eta^2 s_\eta$ by double reduction of the expression $(\Delta s \hat{\nu} x)^3$, since $a_\eta^2 s_\eta$ does not arise by folding from the reducible term $a_\nu^2 s_\nu (\nu \nu_1 x)$ of the form $a_\eta (aHu) s_\eta$.

Form sequence $(a_\eta^2 s u)^2$: reducer $a_\nu^2 s_\nu$ from

$$a_\nu^2 s_\nu s_{\nu_1} = -\frac{1}{2} (a_\eta^2 (Hsu))^2.$$

The form $(a_\eta (aHu), s, u)$ decomposes as a transvection over a functional determinant.

The reduction formulas obtained are:

$$\begin{aligned} \text{a)} \quad & 0 = (aHu)^2 (asu) s_\eta - a_\eta (asu) (Hsu)^2 - a_\eta (aHu) (asu) (Hsu), \\ \text{b)} \quad & 0 = (aHu)^2 (asu)^2 s_\eta - a_\eta (aHu) (asu)^2 (Hsu), \\ \text{c)} \quad & 0 = a_\eta (asu)^2 (Hsu)^2, \\ & 0 = a_\eta s_\eta + \frac{1}{4} a_\nu^2 g - \frac{1}{4} s \cdot a_\eta^2, \\ & 0 = a_\eta (aHu) s_\eta - \frac{1}{2} a_\eta^2 (Hsu), \quad 0 = a_\eta^2 (Hsu)^2, \dots, \quad 0 = a_\eta^2 s_\eta. \end{aligned} \tag{30}$$

Consequences:

1) $(j\nu x)^2 s_\nu$, $\Delta_\nu s_\nu$, $\Delta_\nu (\Delta s u)^2$, and consequently $N_\nu (Nsu)^2$, $(aHu)^2 (asu) s_\eta$, $a_\eta (aHu) s_\eta$, $a_\eta^2 (Hsu)^2$, are reducers; $a_\eta s_\eta$ is a decomposable form that passes into decomposable forms under all single and double foldings.

2) The forms of 5th order $(5, \lambda)$, $(6, \lambda)$ etc. do not enter into folding with s^2 .

§ 23. Forms of the tenth order (System s_{III}).

Folding of s with the forms of the sixth order (System III) (§ 11).

Irreducible forms:

- A) $(\vartheta^2\nu x)^3 s_\nu, ((\vartheta^2\nu x)^3, s, u), ((\vartheta^2\nu x)^3, s, u)^2, (\vartheta^2\nu x)^3 s_\nu s_\vartheta, \theta_\nu(\vartheta^2\nu x)^3 s_\vartheta,$
- B) $a_\nu(j\nu x) s_\nu, (a_\nu(j\nu x), s, u)^2, (a_\nu(j\nu x), s, u)^3, (a_\nu(j\nu x), s, u)^4, (a_\nu^2(j\nu x), s, u)^2, (a_\nu^2(j\nu x), s, u)^3,$
- C) $((\vartheta\eta x)^2, s, u), ((\vartheta\eta x), s, u)^2, (\theta Hu) s_\eta, (\theta_\eta su), ((\theta Hu), s, u)^2, ((\theta Hu)^2, s, u),$
 $((\theta\eta x), s, u)^3, (\theta_\eta su)^2, (\theta_\eta^2 su), ((\theta Hu), s, u)^3, ((\theta Hu)^2, s, u)^2, (H_\vartheta(\theta Hu), s, u)^2,$
 $(\theta_\eta(\theta Hu), s, u)^2, (\theta_\eta(\theta Hu)^2, s, u), ((\theta Hu), s, u)^4, (\theta_\eta su)^4, (\theta_\eta(\theta Hu), s, u)^3, (\theta_\eta^2(\theta Hu), s, u)^2.$

Reductions: A) The forms $((\vartheta^2\nu x)^3, s, u)^2, ((\vartheta^2\nu x)^3, s, u)^3$ decompose:

$$(\vartheta\nu x)(\widehat{\vartheta\nu su})(\widehat{\vartheta\nu su}) = (\vartheta\nu x) s_\vartheta s_\nu - (\vartheta\nu x) s_\nu s_\vartheta = -2\theta(\vartheta\nu x) s_\nu s_\vartheta + (\vartheta\nu x) s_\vartheta^2.$$

The remaining forms either have reducers as factors or are single foldings with decomposed forms.

B) Reducer $a_\nu^2 s_\nu$ and $(\widehat{j\nu su}) s_\nu$.

C) 1. The form sequence $(\theta Hu)^2 s_\eta$: a) reducer $(aHu)^2(asu) s_\eta$; b) double reduction of the form sequences $(\theta gu)^3 g_\nu$ and $g_\nu(agu)^3(bgu)^3(abu)$. Hence reduction of the higher forms $(\theta, s, u)^3, (H_\vartheta(\theta Hu), s, u)^3$ and $(a_\nu b_{\nu_1}^2, s, u)^4$ [the latter form replacing $(H_\vartheta su)^4$].

2. $(\vartheta\eta x, s, u)^4$: reducer $(a_\eta su)^4$.

3. $((\vartheta\eta x)^2, s, u)^3$: reducer $s_\vartheta^2 s_\nu$ from $(\widehat{\vartheta\eta su})^2(Hsu)$. The forms $(\vartheta\eta x) s_\eta, (\vartheta\eta x)^2 s_\eta, ((\vartheta\eta x)^2, s, u)^2, \theta_\eta s_\eta$ decompose by folding with the decomposed form $a_\eta s_\eta$.

4. Form sequences $H_\vartheta(\theta Hu) s_\eta, \theta_\eta(\theta Hu) s_\eta, H_\vartheta(\vartheta\eta x) s_\eta, \theta_\eta(\vartheta\eta x) s_\eta$: reducers $a_\eta(aHu) s_\eta$ and $a_\eta^2 s_\eta$.

The form sequence $(\theta_\eta(\vartheta\eta x), s, u)^2$: reducer $a_\eta^2(Hsu)^2$ [except for $(\theta_\eta^2(\theta Hu), s, u)^2$, which arises from the term $a_\eta^2(\theta su)(Hsu)$ by folding].

5. $(H_\vartheta(\vartheta\eta x), s, u), (H_\vartheta(\vartheta\eta x), s, u)^2$: double reduction of $a_\eta(aHu) s_\eta(abu)$ and $g_\vartheta(\theta gu) g_\nu$.

The form sequence $(H_\vartheta(\vartheta\eta x), s, u)^3$: reducer $((\vartheta\nu x)^2, s, u)^2 s_\nu$.

6. $(H_\vartheta(\theta Hu), s, u)$ decomposes by single folding with the decomposed form $(\theta_\nu(\vartheta\nu x), s, u)$ according to § 3. 2), c), that is, by double reduction of the lower decomposed form $(H_\vartheta su)^2$:²⁸

$$\begin{aligned} 0 &= \theta_\nu(\vartheta\nu_1)(\theta su) + \frac{2}{3}\theta_\nu^2(\vartheta\nu_1 x)(\theta su) + \theta_\nu(\vartheta\nu x)(\theta su) s_{\nu_1} \\ &\equiv -\frac{1}{2}H_\vartheta(\theta Hu)(\theta su) + \theta_\eta(\theta su)(Hsu) + \frac{1}{2}H_\vartheta(\theta su)(Hsu) \\ &\equiv -\frac{1}{2}H_\vartheta(\theta Hu)(\theta su) + \theta_\eta(\theta su)(Hsu) + \frac{1}{2}[H_\vartheta]_{sx}^2 - \frac{1}{2}\frac{3}{5}H_\vartheta(\theta Hu)(\theta su) \\ &\equiv -\frac{3}{5}H_\vartheta(\theta Hu)(\theta su). \end{aligned}$$

By double reduction, according to 1. and 5., the following formulae arise:

$$\begin{aligned} 0 &= \theta_\eta(\theta su)(Hsu)^2 + \frac{1}{4}H_\vartheta(\theta Hu)(\theta su)^2 \\ &\quad - \frac{3}{4}\theta_\eta(\theta Hu)(\theta su)(Hsu) - \frac{5}{4}\theta_\eta(\theta Hu)^2(\theta su)^3 \\ &\quad - (asu)^3 b_\eta(\theta Hu)^2 - a_\nu(asu)^3 b_\nu^2, \\ 0 &= H_\vartheta(\theta Hu)(\theta su)^3 - 7\theta_\eta(\theta Hu)(\theta su)^2(Hsu), \\ 0 &= a_\nu(asu)^2 b_{\nu_1}^2(bsu)^2 - \theta_\eta(\theta su)^2(Hsu)^2 \\ &\quad + 6\theta_\eta(\theta Hu)(\theta su)^2(Hsu), \\ 0 &\equiv (H_\vartheta(\vartheta\eta x), s, u), \end{aligned} \tag{31}$$

$$0 \equiv H_\vartheta(\widehat{\vartheta\eta su})(Hsu) - 4\theta_\eta^2(Hsu).$$

Consequences:

1. $(\theta Hu)^2 s_\eta, H_\vartheta(\theta Hu) s_\eta, \theta_\eta(\theta Hu) s_\eta, H_\vartheta(\vartheta\eta x) s_\eta, \theta_\eta(\vartheta\eta x) s_\eta, (H_\vartheta(\vartheta\eta x), s, u), (\theta_\eta(\vartheta\eta x), s, u)^2$ are reducers.

²⁸The sign $\equiv$ means that products of forms of lower degree have been omitted.

2. s^2 does not enter into folding with the forms $(5, \lambda)$, etc., of order 6.

§ 24. Forms of the 11th order (System s_{III}).

Folding of s with the forms of the 7th order (System III) (§ 12).

Irreducible forms:

- A) $((k\eta x)^3, s, u), (K_\eta(k\eta x)^3, s, u), (K_\eta su)^2, ((KHu)^2 su)^2,$
 $((KHu)^2, s, u)^3, ((KHu)^2, s, u)^4, (K_\eta(KHu)^2, s, u)^3,$
- B) $((j\eta x), s, u)^2, (H_j su), ((j\eta x), s, u)^3,$
- C) $(\Delta_\eta su),$
- D) $(aLu)^2 s_\eta, (a_\eta su)^2, ((aLu)^2, s, u)^2, ((aLu)^3, s, u)^2, ((aLu)^3, s, u)^3, (a_l(aLu)^2, s, u)^2.$

Reductions:

- A) Reduction or decomposition by single folding with the corresponding forms in the symbols $\theta - H$.
 $(K_\eta(KHu)^2, s, u)$ and $(K_\eta(KHu)^2, s, u)^2$: decomposition according to § 3. 2), a), by folding with $K_\nu^2(Ksu), K_\nu^2(Ksu)^2$.
 $(K_\eta(KHu), s, u)^4 \equiv (a_\nu^2 \theta_{\nu_1}, s, u)^4$: decomposition according to § 3. 2), b), from the decomposed form $K_\nu^2(Ksu)^3$.
- B) Form sequence $(j\eta x)_{s_\eta}$: reducer $(j\nu x)^2 s_\nu$.
Form $H_j(\widehat{j\eta su})(Hsu)$: reducer $(j\nu x)(\widehat{j\nu su})^2$.
Form $H_j(j\eta x)(Hsu)$: decomposes by reducing the decomposed form $((j\eta x)^2, s, u)^2$ by the reducer $(j\nu x)^2 s_\nu$ (formula (18.)a):

$$0 = -2(j\nu x)^2 s_\nu s_{\nu_1} = [(j\eta x)^2]_{su^2} + H_j(j\eta x)(Hsu).$$

- C) Reducers $\Delta_\nu s_\nu$ and $\Delta_\nu(\Delta su)^2$.

D) Reduction and decomposition by single folding with the corresponding forms in the symbols $f - H$.

E) From (30.)c one obtains the reduction of the decomposed form sequence $(a_\sigma su)^2$:

$$0 = a_\eta(Hsu)^2(as\widehat{\nu x})^2 = a_\eta(Hsu)^2(a\widehat{\nu\eta u})^2 + 2a_\eta(a\widehat{\nu\eta u})(\widehat{\eta\sigma u})(Hsu)^2 = \frac{1}{3}a_\sigma(asu)^2,$$

$$0 = a_\eta a_\nu(Hsu)^2(as\widehat{\nu x})(asu) = a_\eta(a\widehat{\nu\eta u})^2(Hsu)^2(asu) + a_\eta(a\widehat{\nu\eta u})(\widehat{\eta\sigma u})(Hsu)^2(asu) = \frac{1}{3}a_\sigma(asu)^3.$$

Consequence: s^2 does not enter into folding with the forms of order 7.

§ 25. Forms of the 12th, 13th, 14th, and 15th orders (System s_{III}).

Folding of s with the forms of orders 8, 9, 10, 11 (System III) (§§ 13–16).

Irreducible forms:

12th order.

- A) $((\vartheta^2 \eta x)^4, s, u), \theta_\eta(\vartheta^2 \eta x)^3 s_\vartheta,$
- B) $((aHu)H_j, s, u)^2, ((aH \cdot u)H_j, s, u)^3,$
- C) $(\theta_\nu su)^2, ((\theta Lu)^2, s, u)^2, ((\theta Lu)^3, s, u), ((\theta Lu)^2, s, u)^3, (\theta_l(\theta Lu)^2, s, u)^2.$

Reductions: The reductions arising by direct folding with reducers or decomposed forms are no longer to be mentioned.

B) Decomposition of $((aHu)H_j, s, u)$ and $(a_\eta(aHu)H_j, s, u)$ as transvections over functional determinants.

Decomposition of $(a_\eta(aHu)H_j, s, u)^2$: double reduction of the decomposed form $[\theta_\nu \theta_{\nu_1}^3]_{su^3}$ (§ 13. C):

$$0 \equiv [\theta_{\nu_1}^3 \theta_\nu]_{su^3} \equiv -\frac{3}{4}\theta_\nu(\theta su)^2(\theta\theta' u)\theta_{\nu_1}^3$$

$$\equiv -\frac{9}{8}(\theta\theta' \eta su)(\theta\theta' u)(\theta Hu)(\theta' Hu)\theta'_\nu(\theta su) \equiv -\frac{3}{8}a_\eta(aHu)H_j(asu)^2.$$

C) From formulae (31.) follows the reduction of the decomposed forms

$$[L_\vartheta(\theta Lu)]_{su^3} \quad \text{and} \quad [\theta_l(\theta Lu)]_{s^2 u^3},$$

that is, of the products $[\theta_\nu^2 H]_{su^3}$ and $[a_\nu^2 (bHu)^2]_{su^3}$:

$$\begin{aligned}
0 &\equiv \theta_\nu^2 (\theta su)^2 (Hsu), 0 && \equiv [L_\vartheta(\theta Lu)]_{su^4} - 7[\theta_l(\theta Lu)]_{su^4}, \\
0 &\equiv a_\nu^2 (asu)^2 (bHu)^2 (bsu) \\
&\quad + 6\theta_l(\theta Lu)^2 (\theta su)(Lsu), \\
0 &\equiv \theta_\nu^3 (\theta su)(Hsu)^2 && (32) \\
&\quad \text{from } 0 \equiv \theta_l^2 (\theta Lu)(\theta su)(Lsu)^2 \\
&\quad (\text{reducer } a_\eta^2 (Hsu)^2).
\end{aligned}$$

13th order.

$$\text{A) } ((KLu)^3, s, u)^2, ((KLu)^3, s, u)^3, \quad \text{B) } (L_j su)^2, \quad \text{C) } ((aH^2 u)^3, s, u)^2.$$

14th order.

$$\text{A) } ((aLu)^2 L_j, s, u)^2, \quad \text{B) } ((\theta H^2 u)^3, s, u)^2.$$

15th order.

$$((KH^2 u)^4, s, u)^2.$$

Reductions: Decomposition of $(K_l(KLu)^3, s, u)^2$, $(H_\vartheta(\theta H^2 u)^3, s, u)$, $((K, H \cdot L, u)^5, s, u)$ according to § 3. 2), c); that is, with simultaneous consideration of the decomposed forms $(K_l(KLu)^2, s, u)^3$, $(H_\vartheta(\theta H' u)^2, s, u)^2$, $((K, H \cdot L, u)^4, s, u)^2$.

Consequence: s^2 does not enter into folding with individual forms of System III.

§ 26. System s_{IV} .

1) *Folding of s with System I–III.*

Reductions: According to the reductions of the last paragraphs ($a_\nu^2 s_\nu (aHu)^2 (asu)s_\eta$, etc.), the forms $(0, \lambda)$, $(1, \lambda)$, $(2, \lambda)$ do not permit foldings I and III simultaneously; hence, by § 18, the forms f, Δ, N do not enter in products in folding to form System s_{IV} .

The form j does not enter into folding, since, by the same reductions, the foldings contragredient to j ,

$$(K_\nu(k\nu x)^3, s, u), \quad ((\vartheta^2 \eta x)^4, s, u) \quad \text{etc.},$$

correspond to the foldings cogredient to j :

$$K_\nu(k\nu x)^2 s_k, \quad (\vartheta^2 \eta x)^3 s_\vartheta \quad \text{etc.}$$

2) *Folding of s with System II–III*, that is, of s with $H \cdot (1, \lambda)$, $H \cdot (2, \lambda)$, $H \cdot (3, \lambda)$.

Irreducible forms:

$$\begin{aligned}
\text{A) } & (a_\nu H, s, u)^4, ((aHu)^2 H, s, u)^3, ((aHu)^2 H, s, u)^4, \\
& ((aLu)^2 H, s, u)^4, ((aLu)^3 H, s, u)^3, ((aH^2 u)^3 H, s, u)^4, \\
\text{B) } & (a_\nu^2 H, s, u)^3, (a_\eta (aHu)^2 H, s, u)^3, (a_l (aLu)^2 H, s, u)^4, \\
\text{C) } & (\theta_\nu H, s, u)^4, ((\theta Hu)^2 H, s, u)^3, ((\theta Hu)^2 H, s, u)^4, \\
& ((\theta Lu)^2 H, s, u)^4, ((\theta Lu)^3 H, s, u)^3, ((\theta H^2 u)^3 H, s, u)^4, \\
\text{D) } & (\theta_\eta (\theta Hu)^2 H, s, u)^3, (\theta_l (\theta Lu)^2 H, s, u)^4, \\
\text{E) } & ((KLu)^3 H, s, u)^4, ((KH^2 u)^4 H, s, u)^4, \\
\text{F) } & ((j\nu x)^2 H, s, u)^3, ((j\nu x)^2 H, s, u)^4, ((j\eta x)H, s, u)^4, \\
& (H_j H, s, u)^3, (L_j H, s, u)^4.
\end{aligned}$$

Reductions: ν does not enter into folding, analogously to j .

B) and D). $(a_\nu^2 H, s, u)^4$ and $(\theta_\nu^2 H, s, u)^4$ decompose by formula (27.)c) and (29.)c), taking into account $(gHu)^2 = -\frac{1}{3}s\sigma$:

$$(a_\nu^2 H, s, u)^4 = \frac{1}{3}(asu)^4 \sigma, \quad (\theta_\nu^2 H, s, u)^4 = -\frac{1}{6}(\theta su)^4 \sigma;$$

hence decomposition of $(a_\eta (aHu)H, s, u)^4$, $(\theta_\eta (\theta Hu)H, s, u)^4$.

$(\theta_\nu^2 H, s, u)^3$, $(\theta_\nu^3 H, s, u)^3$, and hence $(\theta_\eta^2 (\theta Hu)H, s, u)^4$, decompose according to § 25, forms of order 12, C).

3) *Folding of s with System III–III*, that is, of s with forms of System III:

$$(3, \lambda)(3, \lambda), \quad (3, \lambda)(2, \lambda), \quad (3, \lambda)(1, \lambda), \quad (2, \lambda)(2, \lambda), \quad (2, \lambda)(1, \lambda), \quad (1, \lambda)(1, \lambda).$$

Irreducible forms:

$$\begin{aligned} \text{A)} \quad & (a_\nu^2 a_\nu^2, s, u)^3, (a_\nu^2 a_\nu^2, s, u)^4, (a^2 a_\eta (aHu)^2, s, u)^3, \\ \text{B)} \quad & (a_\nu H_j, s, u)^4, ((aHu)^2 H_j, s, u)^3, ((aLu)^2 H_j, s, u)^4, ((aLu)^3 H_j, s, u)^2, ((aH^2 u)^3 H_j, s, u)^3. \end{aligned}$$

Reductions:

Products II–III, for the same order in the symbols ν and f , are to be considered higher than products III–III.

The reductions arise partly from relations between products (syzygies), partly by replacing the products by the corresponding decomposed forms and reducing the folding of these forms with s . Products containing contravariants are always omitted, since they pass into products.

A) f quadratic, symbols ν of arbitrary order. By § 11, formulae (12.), and by analogous computation, one has:

$$\begin{aligned} 1) \quad & 0 = a_\nu b_{\nu_1} + \frac{1}{2} f(aHu)^2 - \frac{1}{4} \theta H + \frac{1}{2} u_x H_\theta - \frac{1}{4} u_x^2 H_\theta^2, \\ 2) \quad & 0 = a_\nu b_{\nu_1}^2 + f a_\eta (aHu)^2 - \theta_\eta - \frac{1}{2} H_\theta, \\ 3) \quad & 0 = a_\nu^2 b_{\nu_1}^2 - H_\theta^2 + 2\theta_\eta^2. \end{aligned}$$

It follows that

$$\begin{aligned} 4) \quad & 0 = a_\nu^2 b_{\nu_1}^2 (j\nu_1 x), \\ 5) \quad & L_\theta(\theta Lu) = -a_\nu b_\eta (bHu)^2 + \frac{3}{4} a_\nu^2 (bHu)^2 + \frac{3}{4} \theta_\nu^2 H, \\ 6) \quad & L_\theta(\theta Lu) = -6a_\nu b_\eta (bHu)^2 + 2a_\nu^2 (bHu)^2 - \frac{1}{2} \theta_\nu^2 H, \\ 7) \quad & 0 = a_\nu (bHu)^2 + f(aLu)^3 + \theta_\nu H, \\ 8) \quad & 0 = a_\nu (bLu)^3 + \frac{3}{4} (\theta Hu)^2 H, \\ 9) \quad & 0 = (aHu)^2 (bHu)^2 + 2(\theta Hu)^2 H, \\ 10) \quad & 0 = a_\eta (aHu)^2 b_\eta (bHu)^2, \\ 11) \quad & 0 \equiv [a_\nu^2 b_\eta (bHu)]_{su^4}. \end{aligned}$$

The reductions of

$$(H_\theta su)^4, \quad (L_\theta(\theta Lu), s, u)^3, \quad (L_\theta(\theta Lu), s, u)^4$$

(formulae (31.) and (32.)) together with formulae 1) to 11) give the reduction of all foldings of s with products that are quadratic in the symbols f and arbitrary in ν .

B) Products of two of the symbols $f, \theta, j; \nu$ in arbitrary order. By formulae (17.), (18.), (20.), and by direct computation, the relations are:

$$\begin{aligned} 1) \quad & 0 = a_\nu \theta_{\nu_1} + \frac{1}{4} \theta (aHu)^2 + \frac{1}{4} f(\theta Hu)^2, \\ 2) \quad & 0 = \theta_\nu \theta_{\nu_1} + \frac{1}{2} \theta (\theta Hu)^2. \end{aligned}$$

Therefore:

$$\begin{aligned} 3) \quad & 0 = 3a_\nu (\theta Hu)^2 + \theta (aLu)^3 + 2f(\theta Lu)^3, \\ 4) \quad & 0 = 3\theta_\nu (aHu)^2 + f(\theta Lu)^3 + 2\theta (aLu)^3, \\ 5) \quad & 0 = a_\nu (\theta Lu)^3, \quad 0 = \theta_\nu (aLu)^3, \quad 0 = (aHu)^2 (\theta Hu)^2, \\ 6) \quad & 0 = a_\nu \theta_\nu^2 + \theta_\nu a_\nu^2 + f\theta_\eta (\theta Hu)^2 + \theta a_\eta (aHu)^2, \\ 7) \quad & 0 = \theta_\nu \theta_\nu^2 + \theta \theta_\eta (\theta Hu)^2 + \frac{1}{6} f H_j, \end{aligned}$$

$$\begin{aligned}
8) \quad 0 &= a_\nu(j\nu x)^2 + fH_j, \\
9) \quad 0 &= (aHu)^2(j\nu x)^2 - 2a_\nu H_j, \\
10) \quad 0 &= \theta_\nu(j\nu x)^2, & 0 &= \theta H_j, \\
11) \quad 0 &= a_\nu \theta_\nu^3 - \frac{3}{4}(j\eta x)^2, \\
12) \quad 0 &= a_\nu^2 \theta_\nu^2 - \frac{1}{3}(j\eta x)^2 - \frac{1}{3}(\Delta Hu)^2, \\
13) \quad 0 &= a_\nu^2 \theta_\nu^3 - \frac{1}{2}a_\sigma, \\
14) \quad 0 &= \theta_\nu \theta_\nu^3, & 0 &= a_\eta H_j.
\end{aligned}$$

The formulae have been carried far enough for the products to become polarizable: that is, by folding only one new product arises, because a) the folding of the product into itself becomes reducible, and b) the remaining products arising by folding become reducible or lead to contravariants (cf. § 3. 2), a) and b)).

Formulae 1) to 5) give the reduction of all foldings of s with products arising from $a_\nu \theta_\nu$ by folding. Formulae 6) to 10) give the reduction of those products arising from $a_\nu \theta_\nu^2$ by folding. By § 24 A), taking formulae 6) to 10) into account, the foldings of s with products arising from $a_\nu^2 \theta_\nu$ decompose.

One has:

$$\begin{aligned}
0 &\equiv a_\nu^2(asu)^2\theta_{\nu_1}(\theta su)^2, & 0 &\equiv a_\nu^2(asu)\theta_{\nu_1}(\theta su)^3 - K_\eta(KHu)^2(Ksu)^3, \\
0 &\equiv a_\nu^2(asu)^2\theta_{\nu_1}(\theta su), & 0 &\equiv a_\nu^2(asu)\theta_{\nu_1}(\theta su)^2, \\
0 &\equiv a_\nu^2(asu)\theta_{\nu_1}(\theta su).
\end{aligned}$$

The reducers

$$((j\eta x)^2, s, u)^3 \text{ [from } (\widehat{\vartheta\eta su})^2(Hsu), \text{ § 23. C3],}$$

$$(H_j(j\eta x), s, u)^2 \text{ [§ 24, B], } ((\Delta Hu)^2, s, u)^2 \text{ [§ 24, C], } (a_\sigma su)^2 \text{ [§ 24, E]}$$

together with formulae 11) to 14) give the reduction of all products arising from $a_\nu \theta_{\nu_1}^3$, $a_\nu^2 \theta_{\nu_1}^2$, $a_\nu^2 \theta_{\nu_1}^3$.

Our preceding computations can be summarized in the following *conclusion*:

The relatively complete system modulo (ϱ, t) consists of the following 331 forms and subsystems, which are also reproduced below in tabular order.

Table I (cf. p. 90)

$\begin{smallmatrix} x \\ u \end{smallmatrix}$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
0	i				f		Δ		$K_\nu(k\nu x)^3$		$K_\eta(k\eta x)^3$		$H_\theta(\theta^2\eta x)^3$		$(\theta^3\eta x)^4$	$H_k(k\theta, \eta x)^4$		$\theta_l(\theta k, Lx)^5$				$(\theta^2 Lx)^6$
1		u_x			$\theta_\nu(\theta\nu x)^2$		$\theta_\eta(\theta\eta x)^2$	N	$(k\nu x)^3$	$\theta_\nu(\theta^2\nu x)^3$	$(k\eta x)^3$		$\theta_\eta(\theta^2\eta x)^3$		$\theta_l(\theta^2 Lx)^4$		$(\theta k, \eta, x)^4$		$(\theta k, L, x)^5$			
2			a_ν^2		$\frac{\theta}{a_\eta^2}$		$(\theta\nu x)^2$	$K_\nu(k\nu x)^2$	$(\theta\eta x)^2$	$H_k(k\eta x)^4$	$K_\eta(k\eta x)^2$		$(\theta^2\nu x)^3$	$K_l(kLx)^3$		$(\theta^2\eta x)^3$		$(\theta^2 Lx)^4$				
3		θ_ν^3		a_ν	$\theta_\nu(\theta\nu x)$	$\frac{\Delta_\nu}{a_\eta}$	$H_\theta(\theta\eta x)$	Δ_η	$(k\nu x)^2$		$(k\eta x)^2$		$(\theta^2\nu x)^3$	$(kLx)^3$								
4	ν		$\frac{H}{\theta_\nu^2}$	$(j\nu x)^3$	$\frac{\theta_\nu^2}{a_\eta(aHu)}$	$\frac{\Delta_\nu}{a_l^2}$	$(\theta\nu x)$	N_ν	$\theta_l(\theta Lx)^2$	$\frac{H_\eta}{N_\eta}$	N_η	$(\theta Lx)^2$										
5		$a_\eta(aHu)^2$	$\theta_\eta^2(\theta Hu)$	θ_ν	$\frac{K_\nu^2}{(aHu)}$	θ_η	$\frac{K_\eta^2}{(aHu)}$	Δ_l														
6	$\begin{smallmatrix} j \\ \sigma \end{smallmatrix}$		$(j\nu x)^2$	$\frac{L}{a_\nu^2(j\nu x)}$	$\frac{K_\nu}{\theta_l^2}$			K_η														
7		$\theta_\eta(\theta Hu)^2$	$\frac{H_j(j\eta x)}{a_l(aLu)^2}$	$\theta_\nu(j\nu x)$	$\Delta_\nu(j\nu x)$	θ_l	K_l^2															
8	$\begin{smallmatrix} H_j^2 \\ a_l(aLu)^3 \end{smallmatrix}$	$(\nu\sigma x)$	$(\theta Hu)^2$	$K_\eta(KHu)^2$																		
9		$\frac{H_j}{(aLu)^3}$	$\frac{a_\eta(aHu)H_j}{L_\theta(\theta Lu)^2}$	$\theta_l(\theta Lu)^2$	(KHu)																	
10	$\theta_l(\theta Lu)^3$	$\frac{L_j^2}{(j\sigma x)}$	$a_\eta(aH^2u)^3$	$H_j(aHu)$	$(\theta Lu)^2$																	
11		$(\theta Lu)^3$	$K_l(KLu)^3$	L_j	$(aH^3u)^3$																	
12	$(aH^2u)^4$	$\frac{a_\eta(aLu)^2 L_j}{H_\theta(\theta H^2u)^3}$	$\theta_\eta(\theta H^2u)^3$	$(KLu)^3$																		
13	$(\nu\sigma j)$		$(aLu)^2 L_j$	$(\theta H^3u)^3$																		
14	$(\theta H^2u)^4$	$\frac{K_\eta(KH^2u)^4}{(a_\eta H \cdot L, u)^4}$																				
15	$L_\theta(\theta \cdot H \cdot L, u)^4$		$(KH^2u)^4$																			
16	$(\theta \cdot H \cdot L, u)^5$																					
17	$K_\eta(K \cdot H \cdot L, u)^5$																					
18		$(K \cdot H \cdot L, u)^5$																				
19																						
20																						
21	$(KH^3u)^6$																					

(The numbers in the uppermost horizontal row denote the degree in the x ; the numbers in the first vertical row denote the degree in the u .)

Table II (cf. p. 90)

$\frac{x}{u}$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	J				s		s_j^3					s_ν	$(k\nu x)^3 s_\nu$	$(\theta^3 \nu x)^2 s_\theta$ $(\theta^2 \nu x)^2 s_\theta$		$\theta_\eta (\theta^2 \eta x)^2 s_\eta$	
1							(asu)	s_η		s_η^2 (Δsu)	$s_\eta (Nsu)$ $(\theta \nu x)^2 s, u$	$((k\nu x)^3, s, u)$ $(K_\nu (k\nu x)^3, s, u)$	$K_\eta (k\eta x)^3, s, u$		$(\theta^2 \nu x)^2 s_\nu$		$(\theta^2 \eta x)^4, s, u$
2					$(asu)^2$	$s_\theta (\theta su)$	$(\Delta su)^2 / a_\nu s_\nu$	$(\theta \nu x)^2, s, u$ $(\theta_\ell (\theta \nu x)^2, s, u)$		$\theta_\eta (\theta^2 \eta x)^2, s, u$		$(k\eta x)^2 s_\nu$ $((k\nu x)^2, s, u)$		$(k\eta x)^3, s, u$			
3			$(asu)^3$	$s_\theta (\theta su)^2$ s_ν	$a_\nu s_\nu (asu)$ $a_\ell^2 (asu)$	s_η	(θsu) $a_\ell^2 su$	(θsu)		$(Nsu)^2$ $(\theta \nu x)^2 s_\nu$ $((\theta \nu x)^2, s, u)$	$((k\nu x)^2, s, u)^2$	$(\theta^2 \eta x)^2, s, u$					
4	$(asu)^4$	$s_\theta (\theta su)^3$	$a_\nu^2 (asu)^2$	$(\theta_\ell^2 su)$	$(\theta su)^2$	$s_k (Ksu)^3$ $a_\nu (asu)$ $s_\nu (asu)$	$((\theta \nu x)^2, s, u)^2$				$s_\nu (\Delta su)$ (Δsu) (aHu) $(a_\nu su)$		(Δsu)				
5			$(\theta su)^3$	$s_k (Ksu)^3, s_j$ $s_\theta (\theta' s' u)^4$ $s_\nu (asu)^2$ $a_\nu (asu)^2$	(Hsu) $((\theta \nu x)^2, s, u)$ $(\theta_\nu (\theta \nu x)^2, s, u)$ $(\theta_\ell^2 su)$	$((j\nu x)^2, s, u)$ $(aHu)^2 s_\nu$ $(a_\eta su)^2$	$(Ksu)^2$			$((k\nu x)^2, s, u)^2$							
6	$(\theta su)^4$	$s_\nu (asu)^3$ $a_\nu (asu)^3$	$(Hsu)^2$ $(\theta_\nu (\theta \nu x), s, u)^3$ $(\theta_\ell^2 su)$	$a_\eta s_\nu^2$ $(a_\nu (aHu), s, u)^2$ $(a_\nu (aH^2 u), s, u)$	$(Ksu)^3$	$s_\eta (asu)$ $((\theta \nu x), s, u)^3$ $(\theta_\eta su)$	$((k\nu x)^2, s, u)^3$			$s_\nu (\Delta su)$ $a_\nu (j\nu x) s_\nu$ $(\theta Hu) s_\eta$ $(\theta_\eta su)$							
7	$\theta_\nu (\theta \nu x), s, u)^4$	$(a_\nu (aHu), s, u)^3$	$(Ksu)^4$ $(\theta_\ell^2 (\theta Hu), s, u)^2$ $(a_\eta^2 - a_\ell^2, s, u)^2$	$s_j (asu)^4$ $s_k (K^2 u)^3$ $((j\nu x), s, u)^2$ $(aHu)^2, s, u)^2$ $(\theta_\eta su)^3$	$((j\nu x), s, u)$ $(\langle j\eta x \rangle, s, u)$ $(aHu), s, u)$ $(aH^2 u), s, u)$	$((\theta \nu x), s, u)^3$				$(aLu)^3 s_\ell / (asu)^3$							
8	$(a^2 - a_\ell^2, s, u)^4$	$s_j (asu)^2$ $s_k (K^2 u)^3$ $((j\nu x), s, u)^4$ $(\theta_\eta su)^3$	$((j\nu x)^2, s, u)^2$ $(aHu), s, u)^3$ $((aHu)^2, s, u)^3$ $(\theta_\ell^2 su)^3$	$(Lsu)^2$ $(a_\ell^2 (j\nu x), s, u)^3$ $H_\theta (\theta Hu), s, u)^3$ $\theta_\ell (\theta Hu), s, u)^3$	$(asu)^3$	$(K, su)^3$				$(K_\theta su)^2$							
9	$K_\ell^2 su^4$ $((aHu), s, u)^4$	$(Lsu)^4$ $(a_\ell^2 (j\nu x), s, u)^4$ $(\theta_\nu (\theta Hu), s, u)^2$	$(K_\nu u)^4$ $(a_\nu (aLu), s, u)^4$ $(a_\ell^2 H, s, u)^3$	$(K, su)^3$	$(a_\nu (j\nu x), s, u)^3$ $(\theta Hu), s, u$ $(\theta Hu)^2, s, u$		$(\theta_\ell su)^3$										
10			$(K, su)^4$ $(a_\ell^2 a_\eta (aHu)^2, s, u)^3$	$((j\eta x), s, u)^3$ (H_j, su) $((aLu)^2, s, u)^2$ $((aLu)^3, s, u)$													
11	$(a_\nu (j\nu x), s, u)^4$ $((\theta Hu), s, u)^4$	$(K_\ell (KHu))^2, s, u)^3$ $((j\eta x), s, u)^3$ $((aLu)^2, s, u)^4$ $(a_\theta H, s, u)^3$	$H^2 su^3$ $\theta_\ell (\theta Lu)^2, s, u)^3$	$((KHu)^2, s, u)^3$													
12		$(a_\eta (aHu)^2 H, s, u)^3$	$((KHu)^3, s, u)^3$ $(L, su)^2$	$((aHu) H_j, s, u)^2$ $((\theta Lu)^2, s, u)^3$													
13	$((KHu)^3, s, u)^4$	$((aHu) H_j, s, u)^3$ $((\theta Lu)^2, s, u)^4$ $(\nu \cdot H, s, u)^3$	$((aH^2 u)^3, s, u)^3$ $((j\eta x) H, s, u)^3$ $((aHu)^2 H, s, u)^3$														
14	$((j\nu x)^2 H, s, u)^4$ $((aHu)^2 H, s, u)^4$	$(\theta_\eta (\theta Hu)^2 H, s, u)^3$		$((KLu)^3, s, u)^2$													
15	$(a_\ell (aLu)^3 H, s, u)^4$	$((KLu)^3, s, u)^3$		$((aLu)^2 L_j, s, u)^2$ $((\theta Hu)^3 H, s, u)^3$ $((\theta Hu)^2 H, s, u)^2$													
16	$((\theta Hu)^2 H, s, u)^4$ $(\sigma \cdot H_j, s, u)^4$	$((j\eta x) H, s, u)^4$ $(H_j H, s, u)^3$ $((aLu)^2 H, s, u)^4$ $((aLu)^3 H, s, u)^3$															
17	$(\theta (\theta Lu)^2 H, s, u)^4$		$((KH^3 u)^3, s, u)^3$														
18		$((\theta Lu)^2 H, s, u)^4$ $((\theta Lu)^3 H, s, u)^3$ $(aHu)^2 H, s, u)^4$															
19	$((aH^3 u)^3 H, s, u)^4$ $(L_j H, s, u)^2$																
20		$((KLu)^3 H, s, u)^5$	$((aLu)^3 H, s, u)^5$														
21	$((\theta H^2 u)^3 H, s, u)^4$ $((aLu)^2 H_j, s, u)^4$																
22																	
23	$((KH^3 u)^3 H, s, u)^3$	$((aH^3 u)^3 H, s, u)^3$															

(The numbers in the uppermost horizontal row denote the degree in the x ; the numbers in the first vertical row denote the degree in the u .)