

Supporting proofs for the consumed group-scheme foundations

GPT-6.1 Sol (OpenAI)
Codex, Ultra setting

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Course sources: <https://github.com/KokunoYumeto/open-math-courses/tree/main/docs/courses/AG-RG>.

Contents

Supporting proofs for the consumed group-scheme foundations	1
G.0. The component structure used in the field-group proof	1
G.1. Dimension at a point after field extension	7
G.2. Smoothness descends under field extension	8
G.3. The local regularity step in Cartier’s theorem	9
G.4. The reduced products used in the closed-orbit argument	11
G.5. The semilinear descent used for character modules	15
Free provenance and rights	17
History	17

Supporting proofs for the consumed group-scheme foundations

This lesson supplies the additional arguments consumed from the earlier group-scheme lessons, before their use in the following reductive-group lessons. The statements retain their field, characteristic, finiteness and base hypotheses. Definitions of a group scheme, a character, an fpqc form and a quasi-coherent representation are definitions, rather than theorem obligations.

The earlier algebra proofs used below are the current programme lessons, with their actual locators specified at the point of use. Those locators refer to written proofs. External free sources at the end explain the origin of the arguments; none replaces a proof below or the stated earlier proof. The separate general flat quotient proof is [Flat quotient bootstrap](#), Theorem 8.2 and Corollary 8.3; the further field scheme-quotient argument remains in the earlier quotient lesson.

G.0. The component structure used in the field-group proof

Lemma G.0.A (the algebraic overfields used here). Every field has an algebraic closure. Any two extensions of a field embed, over that field, in a common algebraically closed overfield.

Proof. For a field F , put one indeterminate X_f for every monic nonconstant polynomial $f \in F[T]$, and form the ideal generated by all $f(X_f)$. This ideal is proper. Indeed, a putative expression for one uses only finitely many of these relations and indeterminates. For those finitely many polynomials, adjoining a root of an irreducible factor successively gives a field in which they all have roots; the irreducible-factor quotient is a field by the division and Bezout proof of AG-RG-S02, Lemma 4.A. Evaluate the specified indeterminates at these roots and any remaining finitely many indeterminates at zero. The purported expression becomes $1 = 0$, a contradiction. Choose a maximal ideal containing the relation ideal: a union of a chain of proper ideals is proper, so Zorn applies. Its quotient is a field (each nonzero class generates the unit ideal) and contains F , since a proper ideal cannot contain a nonzero element of a field. Each indeterminate class is algebraic over F ; every element of this quotient uses finitely many classes and is algebraic by the integral-element proof of AG-RG-S04, Lemma P0.2. Thus we have an algebraic extension in which every polynomial of F has a root.

Repeat this construction with the resulting field, and then countably many times, always identifying the original field with its injected image. The union is algebraic over F , by transitivity of integrality in Lemma P0.2. Each polynomial in the union has finitely many coefficients, all at one stage, and has a root at the next stage. Dividing by its leading coefficient reduces to the monic case. Repeated division by linear factors therefore splits every nonconstant polynomial. The union is an algebraic closure.

For extensions K, L of k , the tensor ring $K \otimes_k L$ is nonzero: choose vector-space bases containing one, whose pairs give a tensor basis with nonzero $1 \otimes 1$. Lemma P0.1 supplies a prime of this ring. Its quotient domain, then its fraction field, contains both K and L , since a nonzero unital map from a field is injective. Applying the preceding construction gives a common algebraically closed

overfield. $\square$

Lemma G.0.1 (canonical flat closed structure). Let X be any scheme and let $Y \subset |X|$ be closed and stable under generalization. There is exactly one closed subscheme $X_Y \subset X$ with underlying space Y whose inclusion is flat. Every scheme morphism $f : T \rightarrow X$ with $f(|T|) \subset Y$ factors uniquely through X_Y . This structure commutes with base change, when the underlying subset is replaced by its inverse image. In particular it applies to every connected component of any scheme and retains its ambient local rings, including their nilpotents.

Proof. Begin with $X = \operatorname{Spec} A$ and define

$$I_Y = \ker \left(A \longrightarrow \prod_{\mathfrak{q} \in Y} A_{\mathfrak{q}} \right).$$

If Y is empty, this means $I_Y = A$. For every $\mathfrak{q} \in Y$, $(I_Y)_{\mathfrak{q}} = 0$. We show that $(I_Y)_{\mathfrak{p}} = A_{\mathfrak{p}}$ for $\mathfrak{p} \notin Y$. Write $Y = V(K)$, and choose $f \in K \setminus \mathfrak{p}$. Every prime of $A_{\mathfrak{q}}$, for $\mathfrak{q} \in Y$, comes from a prime contained in $\mathfrak{q}$, hence still in Y by generalization stability. Thus $f/1$ lies in every prime of $A_{\mathfrak{q}}$. The prime-intersection proof in AG-RG-S04, Lemma P0.1 makes it nilpotent. Consequently there are $n_{\mathfrak{q}} > 0$ and $s_{\mathfrak{q}} \notin \mathfrak{q}$ with $s_{\mathfrak{q}} f^{n_{\mathfrak{q}}} = 0$ in A .

The affine scheme is quasi-compact by the same prime-separation argument: if principal opens $D(a_i)$ cover its spectrum, no prime contains the ideal generated by all a_i ; Lemma P0.1 makes that ideal the unit ideal. An expression for one uses finitely many a_i , and those opens already cover. An arbitrary open cover refines to principal opens. A closed subset is quasi-compact by adding its open complement to any cover. Thus the opens $D(s_{\mathfrak{q}})$ covering Y have a finite subcover. Let N be the largest of its exponents. At every $\mathfrak{r} \in Y$, some chosen $s_{\mathfrak{q}}$ is a unit in $A_{\mathfrak{r}}$, so $f^N/1 = 0$ there. Hence $f^N \in I_Y$, while $f^N \notin \mathfrak{p}$. This proves the assertion outside Y . It also proves $V(I_Y) = Y$.

Each localization of A/I_Y , as an A -module, is therefore $A_{\mathfrak{p}}$ if $\mathfrak{p} \in Y$ and zero otherwise. It is flat at every prime, hence flat over A by the earlier localization criterion in AG-CA, *Tor and flat modules*, Theorem 3.3. That criterion follows by localizing every tensor-injection kernel and using the zero-module localization test. Thus $\operatorname{Spec}(A/I_Y)$ is the required flat closed subscheme.

For uniqueness, suppose A/J is flat over A and $V(J) = Y$. For $\mathfrak{p} \in Y$, the quotient map $A_{\mathfrak{p}} \rightarrow A_{\mathfrak{p}}/J_{\mathfrak{p}}$ is a flat local map of nonzero local rings, hence faithfully flat by AG-CA, *Faithful flatness and the local criterion for flatness*, Theorem 2.1. Flatness tensors the injection $J_{\mathfrak{p}} \hookrightarrow A_{\mathfrak{p}}$ to an injection into $A_{\mathfrak{p}}/J_{\mathfrak{p}}$, whose image is zero. Faithful flatness makes $J_{\mathfrak{p}} = 0$. For $\mathfrak{p} \notin Y$, $J_{\mathfrak{p}} = A_{\mathfrak{p}}$. Thus J and I_Y have the same localizations at all primes and are equal, by the earlier zero-module localization test applied to their sums and quotients.

If $f : T \rightarrow \operatorname{Spec} A$ has image in Y , any $a \in I_Y$ is zero in $A_{f(t)}$ at each $t \in T$. Its image is therefore zero in $\mathcal{O}_{T,t}$. A section whose every stalk is zero is zero, so f kills I_Y and factors through the quotient. Uniqueness follows from the closed immersion being a monomorphism.

On an affine open of a general X , use this construction for its intersection with Y . Restrictions to further affine opens remain flat closed subschemes with that same underlying subset, so affine uniqueness identifies the restrictions. Their quotient ideals glue and give X_Y . The same local uniqueness proves global uniqueness and the factorization assertion. Under any base change, the pulled-back closed subscheme is flat and has the inverse-image underlying subset; that subset is closed and stable under generalization. Uniqueness proves the asserted compatibility.

A connected component is closed: the closure of a connected set is connected, since any separation of the closure by two nonempty relatively open sets would meet the dense original set on both sides and separate it. Maximality thus makes the closure the component itself. It is stable under generalization: if y is in the component and x generalizes y , every nonempty open in the closure of $\{x\}$ contains x , so that closure is irreducible and connected and contains y . Its union with the component is connected: a separation of the union would put both intersecting connected sets on the side containing their common point. Maximality therefore puts x in the component. The

construction thus applies. The localized quotient at a point of the component is exactly the ambient local ring, as already computed. $\square$

Lemma G.0.2 (the algebraic-extension connectedness step). Let X be a connected scheme over a field k with a k -rational point. Then X_L is connected for every algebraic field extension L/k , including infinite extensions and imperfect fields. No finiteness or reducedness assumption on X is needed.

Proof. The projection $p : X_L \rightarrow X$ is integral and closed. On an affine chart its algebra is $A \otimes_k L$, the filtered union of the finite free A -algebras $A \otimes_k E$, where E/k ranges over finite subextensions. Every element is therefore integral over A , by AG-RG-S04, Lemma P0.2. Lemma P0.4 proves closedness, also for the quotients defining closed subsets.

Here is the needed openness calculation over an arbitrary affine chart, without a finite-presentation or reducedness assumption on A . First let E/k be finite of degree $d \geq 1$, and put $B = A \otimes_k E$. Choose its free basis $e_1, \dots, e_d$. For $b \in B$, write $be_j = \sum_i m_{ij}e_i$, and set

$$P_b(T) = \det(TI - (m_{ij})) = T^d + c_{d-1}T^{d-1} + \dots + c_0 \in A[T].$$

The row of basis elements annihilates $bI - (m_{ij})$. Multiplying by its adjugate, using the cofactor identity proved in AG-RG-S01, Lemma 1.2, shows that $P_b(b)e_j = 0$ for every j , hence $P_b(b) = 0$, since the basis spans one.

We also record the determinant multiplication calculation used in the next sentence. Expanding $\det(PQ)$ by permutations and then expanding each entry of PQ , group terms by the chosen intermediate indices. If two indices coincide, the inner alternating sum is a determinant with two equal rows of Q , and is zero by the paired-permutation cancellation of Lemma 1.2. If the indices form a permutation τ , that inner sum is $\text{sgn}(\tau) \det Q$. Summing the remaining coefficients gives exactly $(\det P)(\det Q)$. Thus this identity holds over every commutative ring.

At a prime $\mathfrak{p} \subset A$, the fibre of $D(b)$ is empty precisely when the image $\bar{b}$ in $\kappa(\mathfrak{p}) \otimes_k E$ is nilpotent, by the prime-intersection statement of Lemma P0.1. If $\bar{b}$ is nilpotent, its multiplication matrix $\bar{M}$ is nilpotent. The finite geometric sum $\sum_{j < N} z^j \bar{M}^j$ inverts $I - z\bar{M}$ when $\bar{M}^N = 0$. Determinant multiplication makes $\det(I - z\bar{M})$ a unit in the polynomial ring over $\kappa(\mathfrak{p})$. Degrees add for nonzero polynomials over a field, so every such unit is constant; its value at zero is one. Therefore all c_i vanish in $\kappa(\mathfrak{p})$. Conversely, if all these coefficients vanish there, the already proved equation $P_b(\bar{b}) = 0$ gives $\bar{b}^d = 0$. Consequently

$$p_E(D(b)) = \bigcup_{0 \leq i < d} D(c_i),$$

which is open. Principal opens form a basis, proving openness for every finite E/k , including inseparable extensions.

For general algebraic L/k , the finitely many coefficients of any $b \in A \otimes_k L$ lie in one finite subextension E . Its principal open is the pullback of the corresponding principal open over E . The projection from the L -level to the E -level is surjective: field extension is faithfully flat by the vector-space basis calculation, and the prime-fibre argument is proved in AG-CA, *Faithful flatness and the local criterion for flatness*, Theorem 2.2. Its image in $\text{Spec } A$ therefore equals the open finite-level image just calculated. This proves openness of p .

If $X_L = U \amalg V$ were a decomposition into nonempty clopen subsets, both images $p(U)$ and $p(V)$ would be nonempty clopen subsets of the connected X , hence both would equal X . The fibre at the rational point is $\text{Spec } L$, which has one point. It cannot meet both disjoint U and V . This contradiction proves the assertion. $\square$

Lemma G.0.3 (the consumed field-group identity component). Let G/k be a group scheme locally of finite type over any field. Its connected components are open, closed and irreducible. The component G^0 through the identity, with its open subscheme structure, is a normal, geometrically

irreducible, quasi-compact subgroup; its inclusion is a flat closed immersion and its formation commutes with every field extension. No reducedness or characteristic assumption is imposed.

Proof. Begin over an algebraically closed field. Every affine chart is Noetherian by AG-RG-S01, Lemma 1.3. Irreducible components therefore have reduced closed structures locally of finite type. The product of two such irreducible schemes is irreducible: cover each by finite-type integral affine opens and apply Lemma G.4.1 below to each product chart. Any two charts meet, since nonempty opens of each irreducible factor meet, and the product of these intersections is nonempty (each intersection has a rational point by the earlier Nullstellensatz). The union of irreducible open charts with pairwise nonempty intersections is irreducible: any two nonempty opens of the union meet by first meeting the charts and then their overlap.

If Z_1, Z_2 are two irreducible components through the identity, the closure of the multiplication image of $Z_1 \times Z_2$ is irreducible and contains both, using the identity in the other factor. Maximality of the components forces $Z_1 = Z_2$. Translation proves uniqueness of the component through every rational point. On a Noetherian affine chart there are finitely many irreducible components, by AG-CA, *Noetherian and Artinian rings*, Proposition 2.2. The closure of a chart component is a global component: any larger irreducible closed subset intersects that chart in an irreducible closed subset containing the chart component, hence equal to it; this nonempty open is dense in the larger subset, so the closures coincide. Conversely a global component meeting the chart intersects it in a chart component, by the same maximality and density argument. If two distinct global components met at a point, their intersections with such a chart would be distinct closed chart components with nonempty intersection. That intersection has a rational point by the Nullstellensatz, contradicting uniqueness there. Thus all components are pairwise disjoint. Their finite disjoint intersections with every Noetherian chart are open and closed there, so the global components are open and closed. They are exactly the connected components.

Return to the original field and let C be its connected identity component with the canonical flat closed structure of Lemma G.0.1. The identity factors through this structure. For an algebraic closure $\bar{k}$, Lemma G.0.2 makes $C_{\bar{k}}$ connected. Its underlying space is the inverse image of $|C|$. The preceding algebraically closed argument puts it inside the irreducible component Z through the geometric identity. Conversely the connected image of Z in G contains the identity and is contained in $|C|$, so $Z \subset |C_{\bar{k}}|$. These subsets are equal. In particular $C_{\bar{k}}$ is irreducible and open in $G_{\bar{k}}$. The projection is open by the calculation in Lemma G.0.2 and surjective by faithful flatness, so its image $|C|$ is open in G . It is already closed. Its open subscheme structure is a flat closed subscheme, and uniqueness in Lemma G.0.1 identifies that structure with C , retaining every ambient nilpotent.

We spell out why irreducibility persists under arbitrary field extensions, rather than only algebraic ones. If A is a finite-type domain over the algebraically closed $\bar{k}$ and $F/\bar{k}$ is any field extension, write any two nonzero tensors in $A \otimes_{\bar{k}} F$ with independent coefficients from F . For each tensor a nonzero coefficient from A defines a nonempty principal open where its evaluation is nonzero in F . These opens meet because A is a domain, and their intersection has a $\bar{k}$ -point by the Nullstellensatz. Evaluating there gives two nonzero elements of the field F , whose product is nonzero. Thus the tensor algebra is a domain; this is the coefficient proof of Lemma G.4.1 with the second factor allowed to be any field. In a Noetherian affine chart of $C_{\bar{k}}$, its nilradical has finitely many nilpotent generators; a power with exponent larger than the sum of their nilpotence exponents is zero. Its extension remains nilpotent. Quotienting by it therefore leaves the underlying space unchanged after field extension, and the domain argument proves irreducibility of each extended chart. The original nonempty chart intersections remain nonempty by faithful flatness. Hence C_F is irreducible for every $F/\bar{k}$.

For an arbitrary K/k , Lemma G.0.A gives a common algebraically closed overfield Ω of K and $\bar{k}$. Then C_{Ω} is irreducible by the preceding argument, and its surjective projection to C_K makes C_K irreducible. This proves geometric irreducibility. The same affine product proof over $\bar{k}$, with nilpotent chart thickenings retained, makes $C \times C$ geometrically irreducible. Its multiplication image is connected and contains the identity, hence lies in $|C|$. Inversion has the same property.

Lemma G.0.1 factors both entire morphisms through C ; their group identities follow from its inclusion being a monomorphism.

For any K/k , let D be the connected identity component of G_K with its canonical flat closed structure. The connected C_K contains the identity and is contained in $|D|$. Conversely the connected image of D in G contains the identity, so lies in $|C|$. Lemma G.0.1 factors its projection through C , hence puts $|D|$ inside $|C_K|$. Equality of underlying subsets and flat closed uniqueness give $D = C_K$, as schemes. Conjugation by every $g \in G(K)$ preserves that identity component. Every point of $G \times C$ lifts to a pair over its residue field. Thus conjugation has set-theoretic image in $|C|$, and Lemma G.0.1 factors its entire morphism through C . This proves normality on all test schemes, including nonreduced ones.

Every other connected component D of G is open, closed and irreducible as well. Choose a finite-type affine chart meeting its closed subset and give that subset its reduced closed structure. A maximal ideal supplies a point $x \in D$ with finite residue field K/k , by the earlier weak Nullstellensatz. The rational lift of x to G_K translates C_K to an open, closed, irreducible subset. Its image U in G is open and closed by the finite-extension projection calculation of Lemma G.0.2 and the integral closed-image proof. It is connected, irreducible and contains x , so $U \subset D$. A nonempty clopen subset of the connected D is all of D . Hence $D = U$, proving these assertions without losing its original scheme structure.

Finally work over $\bar{k}$ and choose a nonempty affine open $U \subset C_{\bar{k}}$. For every rational point g , irreducibility makes $gU \cap U$ nonempty, and the Nullstellensatz supplies a rational point in that intersection. Thus $g = u_1 u_2^{-1}$ for $u_1, u_2 \in U(\bar{k})$. The map $C_{\bar{k}} \times U \rightarrow C_{\bar{k}}$, $(a, b) \mapsto ab^{-1}$, is open: the shear $(a, b) \mapsto (ab^{-1}, b)$ is an automorphism of $C_{\bar{k}} \times U$, and projection is open on every finite-type affine target chart V , since $V \times U \rightarrow V$ is flat and of finite type and Lemma G.4.F below proves openness in exactly that setting. Restriction to the open $U \times U$ is still open. Its image contains every rational point. Its closed complement, if nonempty, has a rational point on a finite-type affine chart by the Nullstellensatz, a contradiction. The affine $U \times U$ is quasi-compact and surjects onto $C_{\bar{k}}$, so that component is quasi-compact. The surjective projection to C makes C quasi-compact as well. All assertions are proved. $\square$

Lemmas G.0.1 and G.0.3 supply the precise identity-component inputs used in the following reductive-group lessons. Their geometric connectedness and product connectedness follow from the proved geometric irreducibility, rather than an additional arbitrary-scheme theorem. Lemma G.0.2 independently supplies the algebraic-extension connectedness step.

For free comparison, the corresponding canonical identity-component statement is [Stacks, Proposition 39.7.11, Tag 0B7R](#), and the geometric irreducibility input is [Tag 047M](#). The complete arguments used here are the proofs above and the exact earlier programme proofs they specify.

Lemma G.0.T (finite separable splitting of the consumed field groups). Let G be a finite-type group scheme over a field k . If it becomes diagonalizable after any field extension, then it is diagonalizable over a separable closure and over some finite separable extension of k . In particular a k -torus splits over a finite separable extension, including over imperfect fields.

Proof. Choose an algebraic closure using Lemma G.0.A. The elements separable over k form a subfield k_s . Here are the needed field details. Finitely many such elements generate a finite separable extension: adjoining them one at a time uses minimal polynomials which divide their original distinct-root polynomials, so every step is separable; the tower embedding-count proof of AG-RG-S02, Lemma 4.A gives separability over k . Every element of that generated field is separable by the same embedding-count proof. Thus sums, products and inverses stay in this set. If an element of the algebraic closure is separable over k_s , take its minimal polynomial and a Bezout identity between that polynomial and its derivative. Their finitely many coefficients lie in one finite separable extension E/k . The same Bezout identity makes the polynomial separable over E ; its root is separable over E , hence over k by the tower proof, and lies in k_s . Therefore k_s is separably closed. A separable irreducible polynomial over k_s has a root in the fixed algebraic

closure of k . The preceding argument puts that root in k_s , so this polynomial has degree one; the same conclusion holds in every algebraically closed overfield of k_s . Finite collections of elements of k_s lie in a finite separable extension of k .

Let K/k diagonalize G . Its affine base change descends to an affine G by the full affine-effectivity and comparison proof in AG-RG-S04, Corollary A3.2; the field cover is faithfully flat by its vector-space basis. Write $A = \mathcal{O}(G)$. Lemma G.0.A gives an algebraically closed common overfield Ω of K and k_s , with

$$A \otimes_k \Omega \simeq \Omega[M]$$

for an abelian group M . We will descend each basis character using a finite system of equations, before descending a finite choice of coordinates.

Every finite subset of A lies in a finite-dimensional subcoalgebra. To see this, use the complete comodule proof of AG-GS, *Group schemes, actions and Hopf algebras*, Lemma 5.1 on the regular comodule, obtaining a finite-dimensional subcomodule W containing that subset. For a basis w_j , write $\Delta(w_j) = \sum_i w_i \otimes a_{ij}$. Coassociativity and the counit give

$$\Delta(a_{ij}) = \sum_{\ell} a_{i\ell} \otimes a_{\ell j}, \quad w_j = \sum_i \epsilon(w_i) a_{ij}.$$

The span V of all a_{ij} is therefore a finite-dimensional subcoalgebra containing W . These V cover A .

For one such V , coordinates of $c \in V \otimes_k R$ satisfying

$$\Delta(c) = c \otimes c, \quad \epsilon(c) = 1$$

define an affine finite-type parameter scheme C_V over k . There are finitely many quadratic equations from the first equality and a linear equation from the second, using bases of V and $V \otimes V$. This represents the group-like elements in V on every algebra R , and the equations commute with scalar extension.

In $\Omega[M]$, the subcoalgebra V_Ω is the span of a finite set F of basis monomials. Indeed, if $v = \sum_m d_m e^m$ lies in it, applying the coefficient functional for e^m in the second factor of $\Delta(v)$ puts $d_m e^m$ in V_Ω . Each nonzero coefficient therefore puts that monomial in V_Ω . Finite dimensionality makes the resulting set finite. Over an arbitrary Ω -algebra, comparing the coefficients of $\Delta(c) = c \otimes c$ gives

$$c = \sum_{m \in F} x_m e^m, \quad x_m^2 = x_m, \quad x_m x_n = 0 \ (m \neq n), \quad \sum_{m \in F} x_m = 1.$$

Their coordinate algebra is precisely Ω^F : the standard orthogonal idempotents map to the x_m ; every monomial in the x_m is either zero or one of these idempotents, so they span the algebra; evaluation at the standard basis idempotents gives the inverse. If F is empty both algebras are zero. Thus

$$(C_V)_\Omega = \coprod_{m \in F} \text{Spec } \Omega$$

on every test algebra, not just on field-valued points.

Put $E_V = \mathcal{O}(C_V)$. It is finite dimensional over k , since any independent family remains independent after extension to Ω , and its extension is Ω^F . It is reduced, since it injects into that reduced algebra. The Artinian decomposition and Chinese remainder proofs of AG-CA, *Noetherian and Artinian rings*, Lemma 4.1 and Theorem 4.2 make it a product of finite field extensions. Each factor is separable: an element with inseparable irreducible polynomial would give, after extension to the algebraically closed Ω , a nonzero nilpotent in its polynomial quotient. That quotient injects into the extended field factor by exact field tensor, contradicting reducedness of $E_V \otimes_k \Omega$. The repeated-root criterion is proved in AG-RG-S02, Lemma 4.A.

After extension to the separably closed k_s , each of these finite separable factors becomes a product of copies of k_s . Explicitly, take a finite list of generators for a factor and adjoin them in a tower.

At every step its separable minimal polynomial, in each existing k_s -factor, splits into distinct linear factors; the Chinese remainder calculation gives one k_s for each root. Iterating proves the assertion without a primitive-element assumption. The scalar extension from this product to Ω gives a bijection of its points. Consequently each group-like element of A_Ω lying in V_Ω descends uniquely to A_{k_s} . Every monomial lies in one such V_Ω , since its coefficient expression uses finitely many elements of A .

The group N of group-like elements of A_{k_s} therefore maps bijectively, compatibly with multiplication and inverse, to M . The Hopf map $k_s[N] \rightarrow A_{k_s}$ becomes the basis isomorphism over Ω . Exact faithful field tensor detects both its kernel and cokernel, hence this map is an isomorphism already over k_s .

Finally this isomorphism descends to a finite separable extension. The group M , hence N , is finitely generated: a finite list of algebra generators of $\Omega[M]$ has finite monomial support, and polynomial expressions in that list have support in the subgroup generated by that finite union. Since every e^m occurs, that subgroup is all of M . Thus both A and $k[N]$ are finite-type algebras and have finite polynomial presentations by the Hilbert basis proof of AG-RG-S01, Lemma 1.3. Express the isomorphism and inverse on their finitely many algebra generators. Their images use only finitely many coefficients of k_s ; put these in one finite separable L/k . The finite relation lists vanish after extension to k_s , hence already over L by injectivity of field tensor, so the two maps are defined there. Their compositions, counits, antipodes and comultiplications agree on the generators after extension to k_s , hence already over L , using the same injectivity on the algebras and their tensor squares. They are inverse Hopf isomorphisms over L , giving $G_L = D_L(N)$. For a torus the diagonal group has $N \simeq \mathbf{Z}^r$, so this is exactly $\mathbf{G}_{m,L}^r$. $\square$

G.1. Dimension at a point after field extension

We use the earlier proofs in AG-CA, *Krull dimension and Noether normalization*, Corollary 3.2, Theorems 4.2 and 6.1: normalization over a polynomial ring, dimension as transcendence degree for a finite-type domain, and dimension at a point as the largest dimension of a component through it. We also use *Associated primes and primary decomposition*, Theorem 3.2 and Theorem 1.2, for minimal primes being associated and the associated-prime description of zero divisors. For an integral inclusion, going up lifts every finite prime chain and incomparability makes the contraction of every strict chain strict, so the two dimensions agree. These two assertions have their complete earlier proofs in *Descent and Zariski Main*, Lemma P0.4; its normal going-down proof P0.8 supplies the further input used by the earlier normalization dimension argument.

Lemma G.1.1 (flat going down). If $A \rightarrow B$ is flat, $\mathfrak{q} \subset B$ contracts to $\mathfrak{p} \subset A$, and $\mathfrak{p}_0 \subset \mathfrak{p}$, there is $\mathfrak{q}_0 \subset \mathfrak{q}$ contracting to $\mathfrak{p}_0$.

Proof. The local map $A_{\mathfrak{p}} \rightarrow B_{\mathfrak{q}}$ is flat and faithfully flat. The latter assertion is proved in AG-CA, *Faithful flatness and the local criterion for flatness*, Theorem 2.1: a proper ideal of the source lies in its maximal ideal and its extension lies in the target maximal ideal, so the residue-field test for faithful flatness applies. Consequently

$$B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} \kappa(\mathfrak{p}_0)$$

is a nonzero ring. A maximal ideal of it gives a prime of $B_{\mathfrak{q}}$ over $\mathfrak{p}_0 A_{\mathfrak{p}}$, by the fibre prime correspondence of AG-CA, *Localization, local properties and support*, Theorem 3.2. Contracting this prime to B proves the assertion. This is also the complete earlier proof in *Tor and flat modules*, Theorem 6.3. $\square$

Lemma G.1.2. Let X be locally of finite type over a field k , let K/k be any field extension, and let $y \in X_K$ lie over $x \in X$. Then

$$\dim_y X_K = \dim_x X.$$

Proof. Work on an affine finite-type neighbourhood, with algebra A . First suppose A is a domain of dimension d . Choose a normalization $P = k[t_1, \dots, t_d] \subset A$, finite over P . Put $F = \text{Frac } P$. The inclusion

$$A \hookrightarrow A \otimes_P F \simeq F^r$$

is an injection of P -modules for some finite r . Tensoring over k preserves injection and embeds A_K into

$$(F \otimes_k K)^r.$$

Here $F \otimes_k K$ is a localization of the domain $P_K = K[t_1, \dots, t_d]$. Thus A_K is torsion-free over P_K , as well as finite over it. A minimal prime $\mathfrak{q}$ of the Noetherian ring A_K is associated. It cannot contain a nonzero element of P_K , since such an element acts injectively on A_K . Therefore $P_K \rightarrow A_K/\mathfrak{q}$ is an integral inclusion and every component of $\text{Spec } A_K$ has dimension d .

For general A , flat going down shows that every minimal prime of A_K contracts to a minimal prime of A . Indeed, a smaller prime downstairs could be lifted below it, contrary to minimality. If $\mathfrak{q}$ is a minimal prime upstairs contracting to $\mathfrak{p}$, it is minimal in the quotient $(A/\mathfrak{p})_K$, so the preceding domain argument gives component dimension $\dim(A/\mathfrak{p})$.

Conversely, for every minimal $\mathfrak{p} \subset \mathfrak{p}_x$, Lemma G.1.1 lifts $\mathfrak{p}$ below the prime of y ; choosing a minimal prime below that lift produces an upstairs component through y contracting to $\mathfrak{p}$. Thus the component dimensions through y are exactly those through x . The point-dimension formula of AG-CA, *Krull dimension and Noether normalization*, Theorem 6.1 takes their maximum on each side and proves the equality. $\square$

This supplies the field-extension step in AG-GS-02, Lemma 3.1 and Theorem 3.2, without identifying point dimension with local-ring dimension at a nonclosed point.

G.2. Smoothness descends under field extension

Lemma G.2.1. For a locally finite-type k -scheme X and a field extension K/k , if X_K/K is smooth, then X/k is smooth.

Proof. It suffices to treat an affine chart $S = k[X_1, \dots, X_n]/I$. The polynomial ring is Noetherian by the earlier Hilbert basis theorem, so I has a finite generating list and S is finitely presented. Its conormal sequence is

$$I/I^2 \longrightarrow S^n \longrightarrow \Omega_{S/k} \longrightarrow 0.$$

Formation of every term and map commutes with the flat scalar extension $k \rightarrow K$, by the polynomial presentation and conormal calculation in AG-CA, *Kähler differentials*, Theorems 3.3 and 5.1. Upstairs smoothness makes the conormal arrow injective and $\Omega_{S_K/K}$ finite projective: this is proved by the lifting argument in AG-CA, *Formally smooth, unramified and étale ring maps*, Theorem 3.1. Smoothness is local on the affine chart, so these properties hold on the whole upstairs module by localization and the earlier local criterion for finite projectivity.

Faithful scalar extension detects the kernel of the conormal arrow. Finite projectivity of $\Omega_{S/k}$ descends by AG-CA, *Faithful flatness and the local criterion for flatness*, Corollary 3.2. The conormal sequence downstairs is consequently split exact. The converse lifting construction in the same Theorem 3.1 then makes S formally smooth: for a square-zero test, the error on I/I^2 extends to the free polynomial differential module using this splitting, and subtracting that derivation corrects all defining equations. Finite presentation now makes S smooth. Apply this to each affine chart. $\square$

No perfectness hypothesis occurs in this descent argument.

G.3. The local regularity step in Cartier's theorem

The local assertions needed here have complete earlier programme proofs: Nakayama in AG-CA, *Localization, local properties and support*, Theorem 4.2; Krull intersection in *Noetherian and Artinian rings*, Theorem 6.1; the one-equation dimension theorem in *Dimension theory of Noetherian local rings*, Theorem 3.2; the dimension bound by a maximal-ideal generating list in that lesson, Theorem 2.1; and rational closed-point smoothness in *Smooth algebras over a field and the Jacobian criterion*, Proposition 3.1 and Theorem 2.1.

Lemma G.3.1 (lifting regularity). If $(R, \mathfrak{m})$ is Noetherian local, $f \in \mathfrak{m}$ is a nonzerodivisor, and R/fR is regular local, then R is regular local.

Proof. Put $D = \dim R$. AG-CA, *Dimension theory of Noetherian local rings*, Theorem 3.2 gives $\dim(R/fR) = D - 1$, since f is a nonzerodivisor. Regularity of the quotient supplies $D - 1$ generators of its maximal ideal. Lift them to R . Every element of $\mathfrak{m}$, modulo their generated ideal, belongs to fR ; thus the lifts together with f generate $\mathfrak{m}$. The embedding dimension of R is at most D . Conversely, the local dimension theorem, AG-CA Theorem 2.1, bounds $\dim R$ by the number of generators of any ideal of definition, in particular any minimal generating list of $\mathfrak{m}$. Hence embedding dimension equals dimension, which is precisely regularity. $\square$

Lemma G.3.2. Suppose R is a Noetherian local algebra over a characteristic-zero field, $f \in \mathfrak{m}$, and a derivation $\delta : R \rightarrow R$ satisfies $\delta(f) = 1$. Then f is a nonzerodivisor.

Proof. If $fa = 0$, the iterated product rule gives

$$f\delta^n(a) + n\delta^{n-1}(a) = 0 \quad (n \geq 1).$$

Since $n!$ is invertible,

$$a = (-1)^n \frac{f^n \delta^n(a)}{n!} \in f^n R.$$

Krull intersection gives $a = 0$. $\square$

Lemma G.3.3. Let k be algebraically closed of characteristic zero, let $\mathfrak{m}$ be a closed point of a finite-type k -algebra, and let R be its local ring. If $\Omega_{R/k}$ is free, then R is regular.

Proof. Its residue field is k , by the earlier Nullstellensatz. The rational-point cotangent calculation identifies

$$\mathfrak{m}/\mathfrak{m}^2 \simeq \Omega_{R/k} \otimes_R k.$$

Let the free rank be r , and induct on r . It is finite: if the original finite-type algebra is generated by $a_1, \dots, a_n$, the universal product rule makes its differentials generated by $da_1, \dots, da_n$, and the differential localization formula, AG-CA Theorem 2.2, gives the same finiteness for R . If $r = 0$, Nakayama, AG-RG-S01 Lemma 1.2, gives $\mathfrak{m} = 0$, so R is a field and is regular of dimension zero. If $r > 0$, choose $f \in \mathfrak{m}$ with df nonzero modulo $\mathfrak{m}$. One coordinate of df in a free basis is a unit. Reorder it to the first position and replace the first basis vector by df . The change-of-basis matrix has determinant that unit, so the adjugate identity makes it invertible. The functional taking value 1 on df and zero on the remaining basis vectors corresponds, by the universal differential property, to a derivation with $\delta(f) = 1$. Lemma G.3.2 makes f a nonzerodivisor.

The quotient conormal sequence gives

$$\Omega_{(R/fR)/k} = (\Omega_{R/k} \otimes_R R/fR) / (R/fR) df,$$

which is free of rank $r - 1$. This is the full conormal sequence of AG-CA, *Kähler differentials*, Theorem 3.3: the image of $(f)/(f^2)$ is exactly the span of df , by the product rule. The quotient remains a rational closed-point local ring of a finite-type k -algebra. Indeed write $R = A_{\mathfrak{m}_A}$ and $f = a/s$, with $s \notin \mathfrak{m}_A$. Then $R/fR = (A/(a))_{\mathfrak{m}_A/(a)}$, and its residue field is still k . The induction hypothesis therefore makes R/fR regular, and Lemma G.3.1 makes R regular. $\square$

Theorem G.3.4 (Cartier, with the consumed scope). A group scheme locally of finite type over a characteristic-zero field is smooth.

Proof. Extend the field to its algebraic closure. The cotangent sheaf of any group scheme is

$$\Omega_{G/k} \simeq \mathcal{O}_G \otimes_k e^* \Omega_{G/k}.$$

Here is the complete identification, valid without affineness or smoothness. Write $p_1, p_2 : G \times_k G \rightarrow G$ for the projections and $s(g) = (g, e)$, $\Delta(g) = (g, g)$ for the two sections of p_1 . The automorphism $a(g, x) = (g, gx)$, with inverse $(g, y) \mapsto (g, g^{-1}y)$, is over the first factor and satisfies $a \circ s = \Delta$. The differential base-change identity, AG-CA, *Kähler differentials*, Theorem 2.1, gives $\Omega_{(G \times_k G)/G} \simeq p_2^* \Omega_{G/k}$. Since a is an automorphism over G , its relative differential is an isomorphism $a^* \Omega_{(G \times_k G)/G} \simeq \Omega_{(G \times_k G)/G}$. Pulling this isomorphism back along s identifies $\Delta^* p_2^* \Omega_{G/k} = \Omega_{G/k}$ with $s^* p_2^* \Omega_{G/k} = \mathcal{O}_G \otimes_k e^* \Omega_{G/k}$, proving the display. These calculations on affine charts glue because the universal differential and pullback maps agree on intersections. In the present locally finite-type scope, a finite-type chart at the identity makes $e^* \Omega$ finite-dimensional, so the cotangent sheaf is free.

Every closed point of each finite-type affine chart is rational. Lemma G.3.3 makes its local ring regular; the earlier rational-point Jacobian criterion makes that point smooth. The nonsmooth locus is closed on the chart by the earlier open-locus proof. If nonempty it has a rational closed point by the Nullstellensatz, a contradiction. Hence the group is smooth after algebraic closure. Lemma G.2.1 descends smoothness to the original field. This proof permits infinitely many components and makes no global affineness or quasi-compactness assumption. $\square$

Lemma G.3.5 (universal schematic density). Let $p : X \rightarrow S$ be smooth, with geometrically irreducible nonempty fibres, and let $U \subset X$ be an open whose intersection with every fibre is dense. After every base change $S' \rightarrow S$, the restriction $\mathcal{O}_{X_{S'}} \rightarrow (j_{S'})_* \mathcal{O}_{U_{S'}}$ is injective, where $j_{S'} : U_{S'} \hookrightarrow X_{S'}$. Consequently two maps from $X_{S'}$ to a separated S' -scheme which agree on $U_{S'}$ agree everywhere.

Proof. Schematic density is local on the source and the base. Choose a smooth affine chart $V \rightarrow \text{Spec } R$ of finite presentation. Its image in the base is open. Over each principal base open contained in that image, replace R, V by their restrictions; then every fibre of V is nonempty. It is geometrically integral: after a geometric field extension it is a nonempty open of the smooth irreducible fibre of X , hence reduced and irreducible. An empty chart contributes no condition.

Write $V = \text{Spec } A$. The principal opens $D(f) \subset U \cap V$ cover $U \cap V$. Their base images are open because their morphisms to the base are smooth. They cover $\text{Spec } R$: the dense U_s meets each nonempty V_s . Quasi-compactness of this affine base therefore selects finitely many $f_1, \dots, f_m$ whose base images cover it. No quasi-compactness of U has been assumed.

Use AG-RG-S02 Theorem 4.5 to choose a smooth affine model $V_0 = \text{Spec } A_0$ over a finitely generated $\mathbf{Z}$ -algebra R_0 , with geometrically connected nonempty fibres. These fibres are geometrically integral by the smooth component argument in its Lemma 4.1. Enlarge the coefficient ring to retain the finitely many f_i , using the finite-presentation map and equality proof. All these enlargements are base changes of that model. In the Noetherian model the base images W_i of $D(f_i)$ are open and constructible; their union E pulls back to the whole final base. AG-RG-S02 Lemma 4.4 therefore gives a finite coefficient stage at which their pullbacks cover the base. We retain this finite stage and its specified functions.

On each W_i every fibre of V_0 is integral and f_i is nonzero in it, because $D(f_i)$ has a point in that fibre. At a local ring of V_0 where f_i is not a unit, the local map from the corresponding Noetherian base local ring is flat and f_i is regular in its closed fibre. The written slicing proof in [Faithful flatness and the local criterion for flatness](#), Theorem 5.2, shows that multiplication by f_i is injective and that its cokernel is flat over that base local ring. Where f_i is a unit the same conclusions are immediate, with zero cokernel. Thus, locally over W_i , the sequence

$$0 \longrightarrow \mathcal{O}_{V_0} \xrightarrow{f_i} \mathcal{O}_{V_0} \longrightarrow \mathcal{O}_{V_0}/f_i \mathcal{O}_{V_0} \longrightarrow 0$$

remains exact after every base change: the last module is base-flat, so its first Tor vanishes. This assertion is made on affine base charts and their affine inverse images, and then glues.

It follows that, after base change first to R and then to any R -algebra, restriction to $D(f_i)$ is injective on every open of the inverse image of W_i . Indeed an element killed by localization is killed by a power of f_i , and injectivity of multiplication by f_i and of all its localizations makes that element zero. A function vanishing on U vanishes on every such $D(f_i)$. The W_i cover the base, so it vanishes on V . The argument applies after every further base change and on all source charts, proving the sheaf injection. Finally the equalizer of two maps to a separated target is closed; its defining ideal restricts to zero on $U_{S'}$, so the injection makes that ideal zero everywhere. $\square$

G.4. The reduced products used in the closed-orbit argument

Lemma G.4.1. Over an algebraically closed field, the product of two reduced finite-type schemes is reduced, and the product of two integral finite-type schemes is integral.

Proof. On affine charts let the algebras be A, B . The earlier Nullstellensatz says that an element of A vanishing at every k -point is zero when A is reduced. For a tensor $u = \sum_i a_i \otimes b_i$, choose the b_i linearly independent over k . If all evaluations of u at k -points of $\text{Spec } A$ are zero in B , coefficient independence makes each a_i vanish at every such point, so $u = 0$. If u is nilpotent and B is reduced, each of those evaluations is zero. This proves reducedness.

If A, B are domains and u, v are nonzero tensors, the locus in $\text{Spec } A$ where the evaluation of u is nonzero contains a nonempty open: choose one nonzero coefficient in an independent expression. The analogous open for v meets it, since $\text{Spec } A$ is irreducible. Their intersection has a k -point. At this point both evaluations are nonzero in the domain B , so their product is nonzero. Thus $uv \neq 0$. Finite affine covers give the stated scheme assertions. $\square$

Lemma G.4.A (finite-type images). The image of a morphism between finite-type schemes over an algebraically closed field is constructible. Every nonempty constructible subset of such a scheme contains a k -point.

Proof. Reduction preserves the underlying spaces and images, so first reduce the source and target. They are Noetherian by the Hilbert basis proof. They have finitely many irreducible components: a minimal closed subset which is not a finite union of irreducible closed subsets would split into two proper closed subsets, contradicting minimality.

Here is the dense-open calculation. Suppose an integral source dominates an integral target. Choose nonempty affine opens at their generic points, with coordinate domains $A \subset B$, and put $F = \text{Frac } A$. The generic algebra B_F is a finite-type domain over F . Noether normalization supplies algebraically independent $t_1, \dots, t_r \in B_F$ such that B_F is finite over $F[t_1, \dots, t_r]$. Clear denominators in these elements and in the finitely many monic equations for algebra generators of B . After inverting one nonzero $a \in A$, we have an inclusion $A_a[t_1, \dots, t_r] \subset B_a$ which is finite and integral, by the complete finite-integral calculation. Lying over makes its spectrum map surjective. Projection from polynomial affine space is surjective too: a prime $\mathfrak{p}$ of A_a lifts to $\mathfrak{p}A_a[t_1, \dots, t_r]$, whose quotient is a polynomial domain over $A_a/\mathfrak{p}$. Thus the original image contains the nonempty target open $D(a)$.

Use Noetherian induction on closed target subsets, with the assertion quantified over every finite-type source mapping to that subset. If the closure of an image is proper, the induction hypothesis applies to it. If the target is reducible, apply induction to the inverse images of its finitely many proper irreducible components and take their finite union. Otherwise the target is integral and the image is dense. One of the finitely many source components dominates it, so the preceding calculation gives a nonempty open U in the image. The remaining image is the image of the source's inverse image of the proper closed subset $Y \setminus U$, and is constructible there by induction. Its union with U is constructible in Y . This proves the assertion. Affine charts used in the dense-open calculation suffice because every nonempty open of an integral scheme contains its generic point.

A dense constructible subset contains a dense open. Write it as a finite union of locally closed pieces. On each irreducible component, one piece has dense closure, since a finite union of proper closed subsets cannot contain that component. Its locally closed description then contains an open neighbourhood of the component's generic point. Remove the other components and take the union of these neighbourhoods over the finitely many components; this is a dense open of the whole scheme contained in the subset.

A nonempty constructible subset has a nonempty locally closed piece, hence a nonempty affine open in its reduced locally closed scheme. This is a finite-type k -scheme. A maximal ideal in its nonzero coordinate algebra has residue field k by the weak Nullstellensatz. The resulting point is a k -point of the original scheme and is closed there, again by that theorem. $\square$

Lemma G.4.B (generic flatness by monic division). If A is a Noetherian domain and B a finite-type A -algebra, then B_a is a free A_a -module for some nonzero $a \in A$. The basis may be infinite or empty.

Proof. Write $B = A[X_1, \dots, X_n]/J$, where J is finite by the Hilbert basis theorem, and put $F = \text{Frac } A$. Order monomials first by total degree and then lexicographically. Each bounded total degree has only finitely many monomials, so there is no infinite decreasing sequence; multiplication preserves the order. The ideal of leading monomials of J_F has a finite generating subset of those monomials, since $F[X_1, \dots, X_n]$ is Noetherian. Choose polynomials $f_i \in J_F$ realizing them and normalize their leading coefficients to one.

Division by these monic polynomials terminates by the chosen order. The remainder of an element of J_F is zero: a nonzero remainder would still belong to J_F , while its leading monomial would be divisible by one of the chosen generators, contradicting the remainder condition. Hence the monomials divisible by none of those generators form an F -basis of B_F : division spans them and the same leading-monomial argument proves independence.

Invert a single nonzero $a \in A$ which clears the coefficients of the f_i , their expressions in the original generators of J , and the expressions of those original generators in the ideal of the f_i . The latter expressions are supplied by the terminating division just proved. There are only finitely many coefficients and expressions. Thus $J_a = (f_i)$, and monic division over A_a makes the same monomials span B_a . They are independent because a relation becomes a relation over F , and $A_a \hookrightarrow F$ is injective. They are therefore an A_a -basis. This includes $J_F = (1)$, where the basis is empty and $B_a = 0$. A free module is flat by the elementary tensor calculation in S01 Lemma 1.0. $\square$

Theorem G.4.2 (the closed-orbit argument). Let a smooth finite-type group G over an algebraically closed field k act on a nonempty reduced separated finite-type k -scheme X . The reduced orbits are locally closed, and X contains a closed orbit.

Proof. Choose $x \in X(k)$. Let O be the topological image of $G \rightarrow X$, $g \mapsto gx$, and let Y be its reduced closure. Lemma G.4.A gives constructibility, including its complete dense-open and Noetherian-induction arguments.

Consequently O is constructible and contains a dense open U of Y . The union $V = \bigcup_{g \in G(k)} gU$ is open in Y and contained in O . It contains all rational points of O : a nonempty orbit-map fibre is finite type and has a rational point, so $G(k)$ is transitive on those points. The constructible subset $O \setminus V$, if nonempty, has a rational point; hence it is empty. Thus O is open in Y , proving local closedness.

Translations preserve Y . The action restricts schematically to Y and to O . Indeed Lemma G.4.1 makes $G \times Y$ reduced. Each defining equation of Y pulls back to a function vanishing at every rational point, so the Nullstellensatz makes it zero. The inverse image of the boundary $Y \setminus O$ under the restricted action on $G \times O$ is a closed subset with no rational point and is therefore empty.

For completeness, the components of smooth G are open and disjoint: its local rings are regular domains by the earlier smoothness and regular-local proofs, so two distinct irreducible components cannot meet. There are finitely many components. Its identity component G^0 is therefore irreducible.

Lemma G.4.1 makes $G^0 \times G^0$ irreducible; multiplication and inversion preserve the component containing the identity, so G^0 is a subgroup. Every component contains a rational point and is a translate of G^0 .

The closures of the images of those finitely many translated components under the orbit map are translates of one irreducible closed set, and have a common dimension d . They cover Y ; after repetitions are removed they are precisely its irreducible components. The boundary meets each component in a proper closed subset, so every boundary component has dimension less than d . This strict inequality is the earlier finite-type domain dimension theorem: a nonzero prime in a finite-type domain lowers the transcendence degree and hence the dimension.

Choose an orbit of minimum dimension; this is possible among the nonnegative integers since $X(k) \neq \emptyset$. If its boundary were nonempty, that boundary would contain a rational point, whose orbit lies in it by invariance and has smaller dimension. This contradicts minimality. The chosen orbit is closed. The same argument inside any orbit closure or nonempty invariant closed reduced subscheme gives a closed orbit there. $\square$

Theorem G.4.C (smooth connected group orbit quotients). Let a smooth connected affine finite-type group G over algebraically closed k act on a separated finite-type k -scheme X , and let $x \in X(k)$. Its reduced locally closed orbit O is smooth, and the orbit map $q : G \rightarrow O$ is faithfully flat and of finite presentation. If K is the full schematic stabilizer of x , then q is a K -torsor and O represents the fppf quotient sheaf G/K . In particular this full quotient maps to X by a locally closed immersion; no reduction of K is made.

Proof. Smoothness and connectedness make G integral: smooth local rings are regular domains, so its finitely many irreducible components are disjoint, and connectedness leaves one. The action restricts to X_{red} , since $G \times X_{\text{red}}$ is reduced by Lemma G.4.1. Theorem G.4.2 gives the locally closed orbit, with its reduced structure and the restricted action. The orbit map factors through it: equations of its reduced closure pull back to zero on the reduced G by rational-point detection, and the inverse image of its boundary has no k -point and is empty. The closure and O are integral because they are dominated by integral G .

We prove flatness without assuming a homogeneous quotient exists. Take a nonempty affine open $U = \text{Spec } A$ of O and a finite affine cover of $q^{-1}(U)$. Each nonempty chart contains the generic point of G , and its coordinate map from the domain A is injective because q is dominant. Its algebra is finite type over A . Apply Lemma G.4.B to each of the finitely many charts and invert the product of its nonzero chart elements. This gives a nonempty open $U_0 \subset U$ over which q is flat. Choose $u \in U_0(k)$. The orbit map is surjective onto its topological orbit; each nonempty closed-point fibre is finite type and has a k -point by the weak Nullstellensatz. Hence $G(k)$ acts transitively on $O(k)$. The translates gU_0 cover O : their open union contains every closed point and its closed complement must be empty. Equivariance identifies the orbit maps over these opens by translation on both source and target. Flatness consequently holds everywhere. The map is surjective and of finite presentation: both schemes are finite type over k , and on Noetherian affine charts a finite-type algebra has a finite presentation by the Hilbert basis theorem. Thus q is fppf.

The stabilizer K is the closed fibre over x . For every test scheme, $gx = hx$ is equivalent to $g^{-1}h \in K$. Multiplication and inversion therefore give inverse scheme maps

$$G \times K \longrightarrow G \times_O G, \quad (g, n) \longmapsto (g, gn), \quad (g, h) \longmapsto (g, g^{-1}h).$$

After the fppf cover obtained by pulling back q , every section of O has a lift in G , and any two lifts differ by a unique section of K . This proves the torsor assertion and identifies O with the fppf quotient sheaf. Its immersion in X is the original reduced locally closed orbit immersion, now with the full quotient interpretation.

Finally O is smooth. At its generic field, Noether normalization and S02 Lemma 3.F give differential dimension equal to transcendence degree. AG-CA-18 Theorem 2.1 and Corollary 2.2 give a nonempty open smooth locus. It contains a k -point, and the transitive translations just proved give a smooth

neighbourhood of every k -point. The remaining closed nonsmooth locus has no closed point and is empty. $\square$

Lemma G.4.D (orbit dimension, including nonreduced stabilizers). In Theorem G.4.C,

$$\dim O = \dim G - \dim K.$$

Proof. First, dimension of a finite-type algebra A over a field F is unchanged by extending F to any field E . For a domain, Noether normalization makes A finite integral over an injected $F[T_1, \dots, T_d]$. Tensoring with E preserves that injection and finiteness, so the integral-extension dimension theorem gives dimension d after extension as well. For general A , its finitely many minimal primes give its irreducible components. Their product ideal is contained in the nilradical; that nilradical is nilpotent, because its finitely many nilpotent generators give a uniform monomial bound. Thus, after field extension, the closed subschemes $(A/\mathfrak{p})_E$ still cover the spectrum. Dimension is the maximum of the dimensions of a finite closed cover, as proved in the earlier dimension lesson. Apply the domain calculation to each component to obtain the assertion.

Put $F = k(O)$ and $L = k(G)$. Each nonempty affine chart of the generic fibre of q is an integral finite-type F -algebra with fraction field L . Noether normalization gives its dimension $\text{trdeg}_F L$. This is therefore the dimension of the generic fibre. After extending F to an algebraic closure, that nonempty finite-type fibre has a point. Translation by the point identifies it, on every test algebra, with K after the same extension, by the torsor identity in Theorem G.4.C. The field-extension calculation above gives $\text{trdeg}_F L = \dim K$.

Finite transcendence bases in the tower $k \subset F \subset L$ add: a basis for F/k followed by one for L/F is algebraically independent over k , and L is algebraic over the field they generate, by transitivity of algebraicity. Consequently $\text{trdeg}_k L = \text{trdeg}_k F + \text{trdeg}_F L$. The finite-type domain dimension formula for G and O now gives the claimed equality. $\square$

Lemma G.4.E (generic recognition for affine monomorphisms). Let $f : Y \rightarrow X$ be an affine finite-type monomorphism between affine finite-type schemes over the algebraically closed field k . Let $W \subset X$ be its schematic image, defined by the kernel of the coordinate-ring map. There is an open $V \subset W$ containing all generic points of W for which $f^{-1}(V) \rightarrow V$ is an isomorphism. The schemes may be nonreduced.

Proof. A nonempty finite-type monomorphism $D \rightarrow \text{Spec } F$, for any field F , is an isomorphism. Choose a closed point of an affine chart of D ; the field-algebra lemma makes its residue field L/F finite. After extending to L , the monomorphism has a section. A monomorphism with a section is an isomorphism: its section followed by its projection equals the identity because both maps have the same composite with the monomorphism. The inverse descends along the faithfully flat field extension, by the morphism descent of S04 Lemma A1.5. Thus $D = \text{Spec } F$, with its full scheme structure.

A finite monomorphism is a closed immersion. On an affine target write it as $C \rightarrow B$, with B finite as a C -module. At every prime of C , the fibre is empty or is the residue field itself by the preceding argument. Hence the cokernel of $C \rightarrow B$ has zero tensor product with every residue field. It is a finite module, so localization and the determinant Nakayama argument make it zero. The ring map is surjective.

Now write $W = \text{Spec } C$ and $Y = \text{Spec } B$. The map $C \hookrightarrow B$ is injective and of finite type. Its image on spectra is dense: an element of C whose principal open misses the image becomes nilpotent in B , and hence in C . Lemma G.4.A makes that image constructible, so it contains every generic point of W . Fix the minimal prime $\mathfrak{p}$ of one such point. The Noetherian local ring $R = C_{\mathfrak{p}}$ has just one prime; its maximal ideal $\mathfrak{m}$ is its nilradical. This ideal is finitely generated by nilpotent elements, so $\mathfrak{m}^N = 0$ for some N . The preceding field calculation gives

$$(B \otimes_C R) / \mathfrak{m}(B \otimes_C R) = \kappa(\mathfrak{p}).$$

Take finitely many algebra generators b_j of $B \otimes_C R$. Lift their residues to $r_j \in R$. Then $(b_j - r_j)^N = 0$, so each generator satisfies a monic equation over R . The integral-generator calculation makes this algebra a finite R -module. The coefficients and the finitely many equalities in these monic equations involve only finitely many denominators outside $\mathfrak{p}$. Their product gives $a \in C \setminus \mathfrak{p}$ such that B_a is finite over C_a . The restricted map remains a monomorphism, so $C_a \rightarrow B_a$ is surjective by the finite calculation. Localization preserves its original injection; it is therefore an isomorphism. The union of these principal neighbourhoods for the finitely many minimal primes is the required V . Their inverses agree on overlaps because they invert the same morphism. $\square$

Lemma G.4.F (openness of flat finite-type maps). A flat morphism between finite-type schemes over the algebraically closed field k is open. In particular this applies to smooth morphisms, since their flatness was proved in AG-CA-18 Theorem 6.1.

Proof. For a point y of the source over x , the local map $R = \mathcal{O}_{X,x} \rightarrow S = \mathcal{O}_{Y,y}$ is flat and local. It is faithfully flat. Indeed, for any nonzero R -module M , choose a nonzero element and its cyclic submodule R/I , with I proper and hence contained in the maximal ideal. Its tensor product is $S/IS \neq 0$, since IS is contained in the maximal ideal of the local S . Flatness embeds this tensor product in $S \otimes_R M$, so the latter is nonzero. For every prime $\mathfrak{p}$ of R , it follows that $S \otimes_R \kappa(\mathfrak{p})$ is a nonzero ring. A prime of that ring gives a prime of S contracting to $\mathfrak{p}$. Interpreting local spectra, every generalization of x therefore lifts to a generalization of y .

For an open $U \subset Y$, all these generalizations of $y \in U$ stay in U . Thus its image is stable under generalization. It is constructible by Lemma G.4.A: U is quasi-compact in the Noetherian Y , and its restriction is still of finite type. A constructible subset stable under generalization is open. To check this last assertion, its complement is constructible and stable under specialization. In each irreducible component of the closure of that complement, its dense constructible part contains the generic point, by Lemma G.4.A. Stability under specialization then includes every point of that component. The complement therefore equals its closure and is closed. This proves openness. $\square$

For arbitrary finite-type G , without smoothness or reducedness, retain AG-GS-03, Lemma 7.12 and Theorem 7.13 with their original scheme structures. Their generic monomorphism argument is compatible with nilpotents: at a minimal prime the target ring is Artinian; a finite-type algebra with the same residue field becomes finite because its lifted generators differ from scalars by nilpotents; a finite monomorphism is a closed immersion by the residue-field test and Nakayama. Schematic dominance then makes it an isomorphism on a generic open. Translating that open proves the orbit immersion once the scheme quotient G/H exists. The full quotient and schematic-orbit arguments retain their separate exact earlier proofs; the smooth connected group result above keeps its stabilizer's full scheme structure.

G.5. The semilinear descent used for character modules

Lemma G.5.1 (independence of automorphisms). Distinct automorphisms $\sigma_1, \dots, \sigma_s$ of a field K are linearly independent over K as functions $K \rightarrow K$.

Proof. In a nonzero relation with the fewest terms, normalize the first coefficient to 1. A relation with one term is impossible by evaluation at 1. Choose b with $\sigma_1(b) \neq \sigma_j(b)$ for another participating automorphism. Evaluate the relation at bx and subtract $\sigma_1(b)$ times the relation at x . The first term disappears and the j -th term remains. This is a shorter nonzero relation, a contradiction. $\square$

Lemma G.5.2 (finite Galois descent). Let K/k be finite Galois with group Δ , and let V be any K -vector space with a semilinear Δ -action. Then

$$K \otimes_k V^\Delta \longrightarrow V, \quad a \otimes v \longmapsto av$$

is an isomorphism. Equivariant maps descend uniquely. Tensor products, algebras, Hopf operations and their identities descend through this isomorphism.

Proof. Lemma G.5.1 shows that the vectors $(\sigma(b))_{\sigma \in \Delta}$, $b \in K$, span K^Δ as a K -vector space: otherwise a nonzero linear functional annihilating their span would give a relation among the automorphisms. Consequently there are finite lists $a_i, b_i \in K$ such that

$$\sum_i a_i \sigma(b_i) = \begin{cases} 1 & \sigma = 1, \\ 0 & \sigma \neq 1. \end{cases}$$

For $v \in V$ set $w_i = \sum_{\sigma \in \Delta} \sigma(b_i v)$. These vectors are invariant, and

$$\sum_i a_i w_i = v.$$

This proves surjectivity, with no finite-dimensionality hypothesis on V .

Vectors in V^Δ independent over k remain independent over K . Otherwise choose a shortest relation among them and normalize its first coefficient to 1. Applying any σ and subtracting gives a shorter relation, so all its coefficients are fixed by Δ , hence belong to k . This contradicts the original independence. A k -basis of V^Δ therefore proves injectivity.

An equivariant map preserves invariant vectors, and the displayed isomorphisms recover it from its restriction. They also identify the scalar extension of $V^\Delta \otimes_k W^\Delta$ with $V \otimes_K W$, compatibly with descent. An equivariant unit, multiplication, comultiplication, counit or antipode thus descends as a map on invariant spaces. The algebra and Hopf identities can be checked after the injective scalar extension to K .

If the K -algebra V is finitely generated, write a finite generating list as finite linear combinations of invariant vectors. Let B_0 be the k -algebra generated by those finitely many invariant vectors in $B = V^\Delta$. Then $K \otimes B_0 = V = K \otimes B$, so faithful scalar extension detects $B/B_0 = 0$. Thus $B = B_0$ is finite type and is finitely presented by the earlier Hilbert basis theorem. $\square$

The usual field facts in this argument can be checked without an unstated descent theorem. In a finite normal separable extension, count embeddings by adjoining generators one at a time: each embedding of the preceding field extends by choosing one of the distinct roots of the next minimal polynomial. Normality puts all these embeddings in the field itself, giving $|\Delta| = [K : k]$. If $F = K^\Delta$, Lemma G.5.1 shows $|\Delta| \leq [K : F]$: apply its spanning argument to an F -basis of K . The degree tower then gives $F = k$. A finite separable extension embeds in a finite normal separable extension by adjoining the finitely many roots of the minimal polynomials of its generators; every embedding permutes those root sets. These arguments justify the fixed-field and normal-closure facts used here.

Corollary G.5.3. For a finitely generated abelian group M with a continuous action of $\text{Gal}(k_s/k)$, choose a finite Galois splitting quotient $\Delta = \text{Gal}(K/k)$. The semilinear action

$$\sigma(ae^m) = \sigma(a)e^{\sigma m}$$

on $K[M]$ descends all its Hopf operations to

$$B = K[M]^\Delta, \quad K \otimes_k B = K[M].$$

The descended group has geometric character group M , with the prescribed action. Equivariant character homomorphisms descend uniquely in the reverse direction.

Proof. Intersect the open stabilizers of a finite generating list of M . This fixes all of M and lies in the action kernel. By the definition of the Krull topology, this neighbourhood contains the subgroup fixing some finite separable subextension of k_s/k . Adjoin all the finitely many conjugate roots of its generators. The normal-closure calculation of Lemma G.5.2, with the finite separability proof of S02 Lemma 4.A, gives a finite normal separable extension K/k inside k_s , whose fixing subgroup still acts trivially on M .

We also prove that restriction onto $\Delta = \text{Gal}(K/k)$ is surjective. For $\sigma \in \Delta$, order the extensions of its k -embedding to intermediate subfields of k_s by extension. The union along a chain is again an embedding, so Zorn gives a maximal one, $\tau : E \rightarrow k_s$. If $E \neq k_s$, choose $b \in k_s \setminus E$. Its minimal polynomial over E divides its separable minimal polynomial over k . The polynomial obtained by applying τ to its coefficients still divides that same split separable k -polynomial, so has a root in k_s . Evaluation extends τ to $E(b)$, a contradiction. Thus its domain is all of k_s . It is onto: every element of k_s lies in a finite normal separable extension L/k by the same normal-closure construction, and a k -embedding sends L into L because its generator images are roots of their split k -polynomials. The injective endomorphism of this finite-dimensional k -space is surjective by S06 Section 1. Hence its image contains each such L and all of k_s . This gives an automorphism extending σ .

The action therefore factors through the full finite quotient Δ : two lifts have quotient fixing K and hence act identically on M . Apply Lemma G.5.2 to $K[M]$ and its displayed operations. The character calculation is the coefficient proof already written in AG-GS-05, Lemma 2.1 and Theorem 2.2: a group-like element over a connected field is one monomial, so a Hopf map is precisely its homomorphism on exponents. Transport of that monomial is exactly $e^m \mapsto e^{\sigma m}$. This proves the assertion and supplies the descent step of AG-GS-05, Theorem 5.2. $\square$

AG-GS-05, Theorem 5.0 has its full separable-splitting argument: finite-dimensional subcoalgebras cover the Hopf algebra; the scheme of group-like elements in one is finite étale because its scalar extension is a finite disjoint union of points; over k_s each such point descends; the resulting monomials form the entire Hopf basis after faithful extension. Its finite splitting-field step is Lemma 5.1, where finitely many expressions for an algebra isomorphism and its inverse descend to a finite separable field and then a finite Galois field. These arguments do not assume perfectness of k . They remain required earlier written proofs, together with the affine descent argument of AG-GS-04.

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The source comparisons for point dimension and field descent are the freely accessible official Stacks Project, [02FX](#), [02FY](#) and [0CDR](#). The proof arguments actually needed in this draft have been supplied above, with the named earlier programme algebra proofs.

For the character classification the verified free author version is J. S. Milne, *Algebraic Groups*, version 2.00, 20 December 2015, [author PDF](#), Chapter 14, especially Theorem 14.17. Its numbering is specific to that free version. The later corrected author edition is also freely accessible: [5 October 2021 text, published in 2022](#). That edition uses Chapter 12 for character classification. Edition-specific locators must be matched to the cited free PDF; both free versions remain valid comparison sources.

History

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