

Completing a smooth affine curve and extending its group action

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Course sources: <https://github.com/KokunoYumeto/open-math-courses/tree/main/docs/courses/AG-RG>.

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Completing a smooth affine curve and extending its group action

Original proof draft, October 2026. CC0. This lesson precedes the reductive-group lessons. Its remaining programme dependencies are listed at the end; a reference to an external proof does not discharge them.

Let k be algebraically closed. A **curve** here is an integral scheme of finite type over k of dimension one. We first construct the completion needed in the classification of one-dimensional unipotent groups. In particular, extending each individual translation is not enough: the translations must extend as an algebraic family.

1. Finite normalization for curves

We use the already proved Noether normalization theorem and its dimension formula: a finite-type domain of dimension one is finite over a polynomial subring $k[t]$. The exact earlier proof is AG-CA, *Krull dimension and Noether normalization*, Lemma 2.1, Theorem 3.1, Corollary 3.2 and Theorem 4.2, with the integral-extension proofs specified there. The theorem is used in its finite-module form, not merely as a statement about function fields.

Lemma 1.1 (a separating finite parameter). If A is a finite-type k -domain of dimension one and $K = \text{Frac}(A)$, some $u \in A$ makes A finite over $k[u]$ and $K/k(u)$ finite separable.

Proof. Choose t by Noether normalization. In characteristic zero use $u = t$. Suppose the characteristic is $p > 0$. Write $n = [K : k(t)]$. Frobenius identifies the extension $K/k(t)$ with $K^p/k(t^p)$, since k is perfect. Thus

$$[K : k(t^p)] = np, \quad [K^p : k(t^p)] = n, \quad [K : K^p] = p.$$

The tower formula gives the last equality. Some $z \in A$ lies outside K^p , because otherwise its fraction field would lie in K^p . Hence $K = K^p(z)$ and $1, z, \dots, z^{p-1}$ is a basis over K^p . A k -derivation annihilates K^p , and arbitrary assignment of its value on z defines a derivation: use the presentation $K^p[Z]/(Z^p - z^p)$. Consequently $\Omega_{K/k}$ is one-dimensional with basis dz .

The element z satisfies a monic equation

$$z^d + c_1(t)z^{d-1} + \dots + c_d(t) = 0, \quad c_i(t) \in k[t].$$

Choose a p -power P so large that $\deg c_i < iP$ for every nonzero c_i , and put $u = t^P + z$. Substitution of $z = u - t^P$ gives a polynomial equation in t over $k[u]$. Its term of highest t -degree is $(-t^P)^d$, of degree Pd and coefficient $(-1)^d$; every other term has smaller degree by the stated inequalities. Dividing by that unit makes a monic equation for t . Thus $k[t, u]$ is finite over $k[u]$. A finite $k[t]$ -module generating list for A also generates it over $k[t, u] \subset A$, so A is finite over $k[u]$.

We have $du = dz \neq 0$, hence $\Omega_{K/k(u)} = 0$: a $k(u)$ -derivation kills du , which is a basis of $\Omega_{K/k}$, so every such derivation vanishes. Here u is transcendental, since otherwise the finite $k[u]$ -module A would be finite-dimensional over k . A finite-dimensional domain is a field, because multiplication by a nonzero element is injective and hence surjective; its dimension is zero, a contradiction.

For clarity, for a finite field extension E/F the condition $\Omega_{E/F} = 0$ is equivalent to separability. A separable generator has a minimal polynomial with nonzero derivative, whose differential forces its differential to vanish; successively adjoining separable generators proves one direction. For the converse in positive characteristic, the elements of E separable over F form a subfield F_s : a finite list of such elements generates a separable extension by the complete tower-and-embedding calculation in S02 Lemma 4.A, so its sums, products and inverses are separable too. Every element of E has a p -power in F_s . Indeed its irreducible polynomial is $g(T^{p^r})$ with $g' \neq 0$, obtained by repeatedly removing zero derivatives. The polynomial g remains irreducible, since a factorization would give one of the original polynomial after substitution; thus the p^r th power of the element is separable. Consequently E/F_s is purely inseparable.

If this extension is nontrivial, choose a maximal proper intermediate field E' . Such a choice exists because the intermediate degrees are integers bounded by $[E : F_s]$. Adjoining any $\alpha \notin E'$ generates E by maximality. Its minimal polynomial is $T^{p^e} - a$, where $e \geq 1$ is least with $\alpha^{p^e} \in E'$. Indeed that polynomial is $(T - \alpha)^{p^e}$ over E , so its monic minimal factor is $(T - \alpha)^d$. If $d = p^r m < p^e$ with $p \nmid m$, its coefficient at degree $d - p^r$ is $-m\alpha^{p^r}$; that coefficient in E' would contradict the choice of e . Thus $d = p^e$. If $e > 1$, then $E'(\alpha^p)$ has degree p^{e-1} over E' : a lower degree would give an annihilating polynomial for α of degree smaller than p^e after substituting T^p . It would therefore be a proper intermediate field, a contradiction. Hence $e = 1$, and $E = E'[Z]/(Z^p - a)$. Sending Z to 1 and E' to zero defines a nonzero F -derivation of this quotient, since it kills the defining polynomial. This contradicts $\Omega_{E/F} = 0$. In characteristic zero every irreducible polynomial is separable by the nonzero-derivative and Bezout calculation of S02 Lemma 4.A. The criterion and the lemma follow in all characteristics. $\square$

Lemma 1.2 (finite integral closure). The integral closure of A in K is a finite A -module.

Proof. Lemma 1.1 gives a subring $R = k[u] \subset A$ such that A is a finite R -module and K/F , with $F = k(u)$, is finite separable. The parameter u is transcendental: otherwise algebraic closedness would give $R = k$, and a finite-dimensional domain over k would be a field, of dimension zero, contradicting $\dim A = 1$.

We check the hypotheses of the earlier S02 Lemma 4.B, whose complete proof constructs the splitting product, both interpolation inverses, the trace formula and the integral-closure bound. Polynomial division, proved in S02 Lemma 4.A, makes every nonzero ideal of $k[u]$ principal: choose a nonzero member of least degree and divide every other member by it. The remainder must be zero by minimality. The zero ideal is principal too. Thus R is Noetherian by the ideal finite-generation criterion proved in S01 Lemma 1.3.

The ring R is normal. Write a fraction in lowest terms as a/b , with $a, b \in R$, $b \neq 0$, and $ra + sb = 1$ for some $r, s \in R$, using the same polynomial Bezout proof. If it satisfies a monic equation of degree m , multiplication by b^m shows $b \mid a^m$. Expanding $(ra + sb)^m = 1$ shows that b then divides 1: the first term is divisible by b , as are all the other terms. Hence b is a unit and the fraction lies in R .

Apply S02 Lemma 4.B to this Noetherian normal domain and the finite separable extension K/F . Its integral closure B in K is a finite R -module. In particular the trace pairing and its nondegeneracy are supplied by an earlier programme proof, including the case when the characteristic divides $[K : F]$; they are not assumed from a Vandermonde assertion.

The integral closure of A in K is precisely B . Every element integral over R is integral over A , since the same monic equation has coefficients in A . Conversely, A is integral over R , and an element integral over A is integral over R by the complete transitivity proof in the earlier AG-CA *Integral extensions*, Theorem 1.2. Its finite-coefficient argument applies here: the finitely many coefficients of the monic equation lie in a finite R -algebra, adjoining the element gives a finite module over that

algebra and hence over R , and the determinant argument gives a monic equation over R . Finally an R -module generating list for B also generates it as an A -module, since $R \subset A \subset B$. This proves the assertion. $\square$

Integral closure in the fraction field commutes with localization. One inclusion follows by localizing a monic equation; sums and products of integral elements remain integral by the earlier AG-CA Theorem 1.2. Conversely, let x satisfy a monic equation of degree m over A_s , whose i th coefficient is a_i/s^{h_i} . Choose N with $Ni \geq h_i$ for every coefficient. Multiplying the equation by s^{Nm} gives the monic equation for $s^N x$ with i th coefficient $s^{Ni-h_i} a_i \in A$. Thus $s^N x$ belongs to the integral closure of A and x belongs to its localization. For a general multiplicative set, use the product of the finitely many denominators appearing in the equation.

2. Normal local curves and the smooth completion

Lemma 2.1. A one-dimensional Noetherian normal local domain is a discrete valuation ring.

Proof. Let $(R, \mathfrak{m})$ be such a ring and choose $0 \neq a \in \mathfrak{m}$. Every prime containing a is $\mathfrak{m}$, since any other nonzero prime would give a chain of length at least two. Thus the radical of (a) is $\mathfrak{m}$. Finite generation of $\mathfrak{m}$ gives $\mathfrak{m}^n \subset (a)$ for some n : choose a power of each generator lying in (a) and use the pigeonhole principle on monomials. Choose the least n and $b \in \mathfrak{m}^{n-1} \setminus (a)$. Then $c = b/a \notin R$ and $c\mathfrak{m} \subset R$.

If $c\mathfrak{m} \subset \mathfrak{m}$, apply the determinant trick to multiplication by c on the finite module $\mathfrak{m}$. Its resulting monic polynomial annihilates this module, hence is zero in $\text{Frac}(R)$ because the domain has a nonzero element in $\mathfrak{m}$. Normality would imply $c \in R$, a contradiction. Consequently cx is a unit for some $x \in \mathfrak{m}$. It follows that $\pi = c^{-1} = x/(cx) \in \mathfrak{m}$ and $\mathfrak{m} = \pi R$.

The intersection $N = \bigcap_j \pi^j R$ is zero. In fact N is a finite ideal, since R is Noetherian, and cancellation of $\pi \neq 0$ shows $N = \pi N$: if $x = \pi y$ lies in every $\pi^{j+1} R$, then y lies in every $\pi^j R$. Nakayama gives $N = 0$. Repeated division of a nonzero nonunit by π must therefore stop. Every nonzero element is uniquely a unit times π^j , and every nonzero ideal is generated by the element of smallest such exponent. This is the discrete valuation ring assertion. $\square$

Lemma 2.A (normal curves are smooth over the algebraically closed field). Every normal integral finite-type curve over k is smooth over k .

Proof. Let Z be such a curve and z a closed point. Its local ring is a one-dimensional Noetherian normal local domain. Indeed, on an affine neighborhood $\text{Spec } B$, the maximal prime of the closed point is nonzero: if it were zero, the domain B would be a field finite over k by the earlier Nullstellensatz, contradicting dimension one. Every chain below this nonzero prime has length at most one. The denominator calculation in the next paragraph gives normality of the localized ring. Lemma 2.1 therefore makes the local ring a DVR. Its maximal ideal is generated by one parameter, so its embedding dimension and its dimension are both one. Its residue field is k by the Nullstellensatz.

The coordinate domain B of an affine chart of a normal scheme is normal: an integral fraction belongs to every maximal localization by stalk normality. If it were absent from B , its proper denominator ideal would lie in a maximal ideal; membership in that localization would then give a denominator outside that ideal, a contradiction. Localization normality also follows directly. If a fraction y is integral over $B_{\mathfrak{p}}$, choose one $s \notin \mathfrak{p}$ clearing the finitely many coefficients of its monic equation. Multiplication of that equation by s^m , where m is its degree, makes sy integral over B : its i th coefficient is s^i times the original coefficient and belongs to B . Normality of B gives $sy \in B$, hence $y \in B_{\mathfrak{p}}$.

The earlier AG-CA *Smooth algebras over a field and the Jacobian criterion*, Proposition 3.1, proves the isomorphism

$$\mathfrak{m}_z/\mathfrak{m}_z^2 \simeq \Omega_{B/k} \otimes_B k$$

at this rational point, using the split conormal sequence of the structural field section. The right side consequently has dimension one. In the notation of its Theorem 2.1, $a_z = d_z = 1$, so its complete Jacobian proof gives a principal smooth neighborhood of z . This application needs neither an unproved geometric-regularity converse nor an identification of maps between completed local rings.

Corollary 2.2 of that earlier lesson proves that the smooth locus is open. If its closed complement in Z were nonempty, intersect it with an affine neighborhood of one of its points. A nonzero finite-type quotient algebra has a maximal ideal; the Nullstellensatz makes its residue field k . Such a rational point is closed also in Z : on any affine neighborhood of a specialization its residue map to k is surjective, hence has maximal kernel, so there is no proper specialization. The complement would thus contain a closed point, contrary to the preceding argument. It is empty, proving smoothness everywhere. $\square$

Theorem 2.2 (smooth projective completion). Every smooth connected affine curve X/k is an open subscheme of a smooth projective curve C/k , and $D = C \setminus X$ is a nonempty finite reduced set of closed points.

Proof. Smooth local rings are regular by the earlier AG-CA Jacobian Theorem 2.1, and regular local rings are domains by AG-CA *Regular local rings*, Theorem 1.1. Thus different irreducible components of X cannot meet: at a point of intersection the local ring would have distinct minimal primes. A finite-type scheme is Noetherian, so it has finitely many components. To justify this topological step, a closed subset failing to be a finite union of irreducible closed subsets has a minimal such subset by the descending chain condition; splitting it into two proper closed subsets contradicts minimality. Disjoint components are then both closed and open. Connectedness leaves one component, and local domains give reducedness, so X is integral. Its closed-point local rings have dimension one and are regular domains; the earlier [S02 DVR Lemma 6.D](#) proves directly that they are normal DVRs. Its generic local ring is its function field and is normal as well. These are all points of an integral curve, so X is normal. Its affine coordinate domain is normal too: an integral fraction belongs to every maximal localization by the just proved local normality. If it were absent from the coordinate ring, its proper denominator ideal would lie in a maximal ideal, whereas membership in that localization supplies a denominator outside that maximal ideal, a contradiction. No normality hypothesis is used to establish this conclusion.

Choose finitely many generators of its affine coordinate algebra, obtaining a closed immersion $X \hookrightarrow \mathbb{A}_k^n$. Let Y be its reduced projective closure in $\mathbb{P}_k^n$. The closure of the irreducible X is irreducible; reducedness makes Y integral. Its dense open $Y \cap \mathbb{A}_k^n = X$ has the same function field and dimension one, by the earlier finite-type dimension theorem. Normalize its affine charts in $k(X)$. Lemma 1.2 makes their closures finite modules, and the localization argument following that lemma identifies the closures on all principal overlaps. They therefore glue to a finite birational morphism $\nu : C \rightarrow Y$. The integral-closure construction is normal: an element integral over a closure is integral over the original ring by transitivity and hence already lies in the closure. The source is integral of dimension one by the integral-extension dimension theorem, and $\nu^{-1}(X) \rightarrow X$ is an isomorphism because X is normal. Lemma 2.A makes C smooth.

The morphism ν is proper. On an affine chart its diagonal is the closed immersion induced by the surjection $B \otimes_A B \rightarrow B$, $b \otimes b' \mapsto bb'$; this proves separatedness. It is finite type. For universal closedness, every base change is still finite. On any affine chart after that base change the image of a closed set $V(J) \subset \text{Spec } B$ is $V(J \cap A)$: apply the earlier lying-over theorem to the injective integral map $A/(J \cap A) \hookrightarrow B/J$. These affine assertions prove that every base change is closed. The projective-properness and composition proofs in [S01 Lemma 7.A](#) and [Corollary 7.B](#) now show that C is proper over k .

We also prove projectivity. Pull back the standard coordinate sections of $\mathcal{O}_Y(1)$ to sections of the invertible sheaf $L = \nu^* \mathcal{O}_Y(1)$. Their nonvanishing opens cover C and are affine, since ν is finite. Principal opens in each of these affine charts form a basis. [S01 Lemma 8.1](#) extends a function on

such a chart to a section of a power of L , after multiplication by a sufficiently high power of the chart's section; if necessary one further multiplication makes it vanish off the chart. Its nonvanishing locus is exactly the chosen principal open. Thus the nonvanishing loci of sections of powers of L form a basis: L is ample in the sense used by S01 Lemma 8.3. That lemma's complete construction gives an immersion $C \rightarrow \mathbb{P}_k^N$. This immersion is proper: its graph is closed because projective space is separated, and projection from $C \times_k \mathbb{P}_k^N$ is proper by base change from $C \rightarrow \operatorname{Spec} k$. The proper-immersion proof in S01 Corollary 7.B makes the immersion closed. Therefore C is projective.

The complement $C \setminus X$ is a proper closed subset of an integral Noetherian curve. Every point in it is closed, since the generic point belongs to X ; a zero-dimensional Noetherian closed subset has finitely many irreducible components, each a single point. Endow this finite set with its reduced closed-subscheme structure and call it D .

We retain the extension calculation needed for the action theorem below. A rational map from a smooth integral curve to a closed projective scheme extends at every missing closed point. In that point's DVR choose homogeneous coordinates in the function field and divide all coordinates by a common power of its parameter so that their minimum valuation is zero. All coordinates then belong to the local ring and at least one is a unit. Represent them by regular functions on a neighborhood and shrink it so that this coordinate stays invertible. They define a morphism to projective space there. The homogeneous equations of the target remain zero, because they vanish in the function field and the neighborhood is integral. Two extensions agree on overlaps: the inverse image of the target's closed diagonal contains their dense common domain, and its defining ideal is zero in a reduced integral source. This proves uniqueness and gluing.

Finally D is nonempty. A regular function f on the proper integral C defines a morphism $C \rightarrow \mathbb{A}_k^1 \subset \mathbb{P}_k^1$. This morphism to projective space is proper by the same graph and base-change argument, so its image is closed. If f were nonconstant, it would be transcendental over the algebraically closed field k : an element algebraic over k satisfies a polynomial split into linear factors and hence lies in k . Thus the map on generic points is dominant and its image contains the generic point of $\mathbb{P}_k^1$. A closed image containing that point is all of $\mathbb{P}_k^1$, impossible for a map factoring through $\mathbb{A}_k^1$. Every regular function on C is consequently constant. If D were empty, $C = X$ would be affine with coordinate ring k , so $C = \operatorname{Spec} k$, contradicting dimension one. This proves every assertion. $\square$

The curve C is geometrically integral. Its global functions are k by the final argument of Theorem 2.2. Here is the precise base-change calculation required for this assertion. Choose a finite affine cover V_i of the separated C . Each intersection $V_i \cap V_j$ is affine: it is the inverse image of $V_i \times_k V_j$ under the closed diagonal. The sheaf axiom identifies global functions with the kernel of the difference-of-restrictions map from $\prod_i \Gamma(V_i, \mathcal{O})$ to $\prod_{i,j} \Gamma(V_i \cap V_j, \mathcal{O})$. For a field extension L/k , tensoring with the vector space L preserves this kernel and these finite products. The affine coordinate and localization calculation gives the base-changed coordinate rings, and hence identifies the tensored kernel with $H^0(C_L, \mathcal{O})$. Thus $H^0(C_L, \mathcal{O}) = L$. Consequently C_L is connected: a nontrivial partition into open-and-closed subsets would give a nontrivial idempotent among its global functions. Smoothness persists after base change. Regular local domains make its finitely many irreducible components disjoint and open, by the same Noetherian argument as in Theorem 2.2; connectedness leaves one, and reducedness gives integrality. Its nonempty open X_L is integral too. Products of these geometrically integral schemes are integral: on affine charts the tensor product injects into the tensor product with the function field of either integral factor, because tensoring a vector space over k preserves injections; the latter ring is a domain by geometric integrality of the other factor. This proves the density assertions used next.

3. Extending translations as a family

Theorem 3.1. Let H/k be a smooth connected affine group of finite type and dimension one. In its smooth projective completion C , multiplication extends uniquely to a morphism $H \times C \rightarrow C$ defining an action. Every point of $D = C \setminus H$ is fixed.

Proof. Each translation on H is a birational map of C . The rational-map argument in Theorem 2.2 extends it to a morphism of C , and does the same for its inverse. Their composites agree with the identity on H , hence on C . Thus each translation is an automorphism of C .

We now prove regularity in the group variable. Multiplication gives a rational map $H \times C \dashrightarrow C$ defined on $H \times H$. Over the generic point η of H it is a birational map of the smooth projective curve $C_{k(H)}$, with inverse the generic inverse translation. The same DVR argument over that field extends it everywhere on the generic fibre. Each extension chart is given by finitely many rational functions and finitely many equations, so clearing their denominators extends it to an open neighbourhood in $H \times C$. These finitely many charts agree where they overlap: their maps to the separated target agree on the dense open $H \times H$, and the source is integral. Their union contains the whole generic fibre. The largest domain of this regular rational map is therefore an open W containing that fibre.

Since C is proper, projection sends the closed complement of W to a closed subset of H . It misses the generic point. Consequently there is a nonempty open $U \subset H$ with $U \times C \subset W$. Choose $a \in U(k)$, which exists by the Nullstellensatz. Let τ_a denote the already extended fixed translation. On the neighbourhood $V = a^{-1}U$ of the identity the formula

$$\beta(g, x) = \tau_a^{-1}(\alpha(ag, x))$$

is regular and equals multiplication on $V \times H$. For every $g_0 \in H(k)$, the formula

$$(g, x) \longmapsto \tau_{g_0}(\beta(g_0^{-1}g, x))$$

is regular on $g_0V \times C$. These opens cover H : they contain every closed point, and a nonempty closed complement in a finite-type scheme over k has a closed point. They agree on overlaps by density and separatedness, so glue to $\alpha : H \times C \rightarrow C$. The unit and associativity identities hold on the dense open with all curve variables in H , and hence everywhere. Smoothness and connectedness make H , C and their products geometrically integral, so this density argument applies. Uniqueness follows the same way.

Every translation preserves H , thus its complement D . The orbit map $H \rightarrow C$ of a point of D has image in the finite set D on closed points. Its scheme-theoretic factorization through that reduced set follows from reducedness of H and detection by its rational points. Explicitly, a nonzero function on a reduced affine finite-type k -scheme is not nilpotent, so its localization is a nonzero finite-type algebra. A maximal ideal of that localization has residue field k by the weak Nullstellensatz, giving a rational point where the function is nonzero. Apply this on affine charts to the equations of D . Connectedness makes the orbit map constant. Thus each boundary point is fixed on all test rings. Differentiating the action at the identity gives a regular vector field on C . At a boundary point its derivation carries a local parameter into its own ideal, because the orbit of that reduced point is constant; the field therefore vanishes there. It is consequently a section of $T_C(-D)$. Its restriction to H is the translation field whose value at the identity is any chosen nonzero element of the one-dimensional $\text{Lie}(H)$, so it is nonzero. $\square$

4. How this enters the one-dimensional unipotent proof

Lemma 4.1 (characters of a unipotent group). Let H be a reduced affine group of finite type over the algebraically closed k , with a closed embedding into GL_n whose closed-point matrices are unipotent. Every group homomorphism $\chi : H \rightarrow \mathbb{G}_m$ is trivial.

Proof. In characteristic $p > 0$, choose $p^e \geq n$. For a matrix $h = I + N$ with $N^n = 0$, the binomial identity gives $h^{p^e} = I + N^{p^e} = I$. Hence $\chi(h)^{p^e} = 1$. A field of characteristic p has no nonidentity root of that equation, so $\chi(h) = 1$.

In characteristic zero fix such an h and define the polynomial matrix

$$\gamma(t) = \sum_{j=0}^{n-1} \binom{t}{j} N^j.$$

For every nonnegative integer m it equals h^m . Every defining polynomial of the closed subgroup H therefore vanishes on $\gamma(t)$ at infinitely many values of t , and vanishes identically. Its determinant is one, since N is nilpotent, so this gives a morphism $\mathbb{A}^1 \rightarrow H$. The identity $\gamma(s)\gamma(t) = \gamma(s+t)$ follows from the usual binomial formula at all pairs of nonnegative integers; a polynomial in two variables vanishing on that grid is zero, by first fixing one integer variable and then the other. Thus γ is a group homomorphism $\mathbb{G}_a \rightarrow H$, and $\gamma(1) = h$. The pullback of a character is an invertible element of $k[t]$, hence a nonzero constant. The identity at zero makes it one, so again $\chi(h) = 1$.

In either characteristic $\chi - 1$ vanishes at every closed point of H . The Nullstellensatz and reducedness make that regular function zero, proving triviality as a scheme morphism. This proof uses no triangularization or solvable-group classification. $\square$

Theorem 4.2 (one-dimensional unipotent groups). Let H/k be a smooth connected affine group of finite type and dimension one, with a closed embedding into GL_n whose closed-point matrices are unipotent. Then H is isomorphic to $\mathbf{G}_a$ as a group scheme.

Proof. Theorems 2.2 and 3.1 give its smooth projective completion C , a nonempty finite reduced boundary D , and a nonzero section of $T_C(-D)$. The complete curve-duality and degree proofs in S02 Theorem 6.2 and Lemma 6.3 give $\deg T_C(-D) = 2 - 2g - |D|$ and the vanishing of sections of negative-degree line bundles. Thus $g \geq 1$ would contradict the nonzero section. Hence $g = 0$, and S02 Proposition 6.5 identifies C with $\mathbf{P}^1$.

For completeness, an automorphism of $\mathbb{P}_k^1$ is fractional linear. Its map on function fields sends t to a nonconstant rational function $r(t)$ generating $k(t)$. Write $r = a(t)/b(t)$ in reduced form. The polynomial $a(T) - Ub(T)$ is irreducible in $k[T, U]$: since its degree in U is one, a factorization would have a nonconstant factor depending only on T , dividing both coprime polynomials a and b . It is primitive as a polynomial over $k[U]$ too. Indeed a common nonconstant factor of its coefficients would have a root $u \in k$, forcing $a(T) = ub(T)$ and making r constant. The complete Gauss calculation in the earlier AG-CA *Integral extensions*, Proposition 2.3, therefore makes $a(T) - rb(T)$ irreducible in $k(r)[T]$. Thus $[k(t) : k(r)] = \max(\deg a, \deg b)$. Since that degree is one, $r = (at + b)/(ct + d)$ with $ad - bc \neq 0$. After putting a fixed point at infinity its stabilizer consists of $t \mapsto at + b$; after putting two individually fixed points at zero and infinity the transformations are $t \mapsto at$. Fixing a third distinct point then forces $a = 1$. The curve action is faithful, since on its open orbit H it is left translation. With three or more boundary points its action would be trivial, an impossibility. With two individually fixed boundary points it acts by scalings, and with one it acts by affine transformations.

Here these restrictions on the automorphism family are scheme-theoretic. With three boundary points every translation on a closed point of H is the identity. The family is consequently the identity: on affine charts its coordinate differences vanish on all closed points of the reduced finite-type source, so the Nullstellensatz makes them zero. With two fixed boundary points put them at zero and infinity. On their complementary $\mathbb{G}_m$ write the pullback of t as a Laurent

polynomial in t with coefficients in $k[H]$. For every closed point of H it is at ; all other coefficients vanish on all closed points and hence are zero. The inverse action makes a a unit. The action law makes $a : H \rightarrow \mathbb{G}_m$ a group homomorphism. With one boundary point put it at infinity. The pullback of t on $H \times \mathbb{A}^1$ is a polynomial in t . Each closed-point automorphism has degree one, so the same coefficient argument leaves $at + b$, with a a unit. Again a is a regular group homomorphism, not merely a character on rational points.

For a unipotent H as in Lemma 4.1, every homomorphism $H \rightarrow \mathbb{G}_m$ is trivial. Thus two boundary points would make the action trivial and are impossible. With one boundary point the slope character is trivial, so the action is $t \mapsto t + b$. The action identity says $b(gh) = b(g) + b(h)$ over every test algebra. After choosing the coordinate to give the identity of H coordinate zero, its orbit map is precisely $b : H \rightarrow \mathbb{A}^1$ and is the original open embedding $H = C \setminus D = \mathbb{A}^1$. It therefore identifies H with the additive group $\mathbb{G}_a$, as schemes and with its group law. In particular the result applies to the one-dimensional quotients embedded in upper unitriangular groups in the later solvable-group proof. $\square$

Internal proof dependencies and free comparison

The construction above consumes the complete earlier AG-CA proofs of Noether normalization, integral extensions, the Nullstellensatz, regular local rings, and the smooth Jacobian criterion. It consumes [AG-RG-S01 Lemma 7.A and Corollary 7.B](#) for projective-space properness, projections and graphs, and Sections 8.1–8.3 of that supporting lesson for denominator extension and ample immersion. Section 4 additionally consumes the earlier curve-duality and genus-zero proofs; its character-vanishing proof is Lemma 4.1 above and does not appeal to the later solvable-group lesson. These are internal proof dependencies to be included in the programme order before use.

For freely readable comparison of the completion statement, see [Stacks, Tag 0H1F](#). Its citation is additional reading; Theorems 2.2 and 3.1 above provide the completion and family-extension arguments themselves.