

Roots and reductive groups of rank one

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Codex, Ultra setting

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Course sources: <https://github.com/KokunoYumeto/open-math-courses/tree/main/docs/courses/AG-RG>.

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Lesson 3

Roots and reductive groups of rank one

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The diagonal entries of a matrix act on its off-diagonal entries through ratios. Those ratios are the roots of a general linear group. For an arbitrary reductive group, roots are still characters of a maximal torus, but their one-dimensional spaces need not be globally trivial over the base. This lesson constructs the corresponding groups without using a power-series exponential, and shows why every root carries a copy of the rank-one geometry of SL_2 .

We use torus conjugacy, torus deformation, and the centralizer theorem from the preceding two lessons. A reductive S -group is smooth affine of finite presentation with connected reductive geometric fibres. A semisimple S -group has semisimple geometric fibres. Statements about line bundles and subgroup schemes are made over arbitrary schemes, including nonreduced ones.

1. The centre and the two ranks

Let T be a split maximal torus of G , and, on an open set where the ranks are constant, write

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_{\alpha}.$$

For now Φ means the nonzero weights; we will prove that they form a reduced root system. Put

$$Z = \bigcap_{\alpha \in \Phi} \ker(\alpha : T \rightarrow \mathbf{G}_m).$$

Lemma 1.A (the torus weight kernel is central). Let G/R be smooth affine of finite presentation with connected geometric fibres, let $T = D_R(X) \subset G$ be a split torus, and let Φ be the finite set of nonzero T -weights occurring in $\mathrm{Lie}(G)$. Put $L = \mathbf{Z}\Phi \subset X$. Then $D_R(X/L) \subset T$ is central in G , is of finite presentation, and commutes with arbitrary base change as a specified subgroup of T .

Proof. The ideal of the displayed intersection of character kernels in $R[X]$ is generated by the finitely many elements $e^{\alpha} - 1$. Its quotient is $R[X/L]$: these relations identify exactly the exponent basis elements in the same coset of L , and the distinct cosets remain an R -basis. The Hopf operations preserve the relations. This gives the indicated diagonalizable subgroup Z , including any torsion in X/L ; no smoothness of Z is assumed. Finite presentation follows by adjoining these

finitely many equations to the Laurent presentation of T . This description also proves arbitrary base-change compatibility.

For conjugation by Z , put $H = G^Z = C_G(Z)$. AG-GS-05, Proposition 3.2 and Theorem 3.4, prove that this is a closed smooth subgroup of finite presentation and that $\mathrm{Lie}(H) = \mathrm{Lie}(G)^Z$, even for a nonreduced diagonalizable Z . Every nonzero T -weight restricts trivially to Z by construction, and the zero-weight part does too. Thus this Lie inclusion is an isomorphism. Translation by points of H , after each residue-field extension, makes the relative tangent map of $H \hookrightarrow G$ an isomorphism at every point. S02 Lemma 3.A consequently makes this closed immersion étale.

Here is the scheme argument turning that conclusion into equality. A closed immersion of finite presentation which is étale is open. Indeed, locally write it as $\mathrm{Spec}(A/J) \hookrightarrow \mathrm{Spec} A$ with J finitely generated. Formal smoothness applied to $A/J^2 \rightarrow A/J$ lifts the identity of A/J as an A -algebra map. Compatibility with A forces every element of J to vanish in A/J^2 , so $J = J^2$. At any prime containing J , Nakayama's lemma gives $J_{\mathfrak{p}} = 0$; finite generation makes J zero on a neighbourhood of that prime. Hence the closed image is also open. Apply this to $H \hookrightarrow G$. On every geometric fibre its image is open and closed and contains the identity. Connectedness of that fibre makes its image the whole fibre. Its open complement in G therefore has no point, so $H = G$ as schemes. This says exactly that Z is central on every test algebra. $\square$

For a maximal T , the displayed intersection Z is precisely the scheme-theoretic centre. Lemma 1.A proves its centrality. Conversely, on every test algebra a central point belongs to $C_G(T) = T$ by the preceding lesson, Theorem 2.2. Its action on each $\mathfrak{g}_{\alpha}$ is multiplication by α . On the present rank-constant open this summand is locally free of positive rank; choose a local basis to see that a scalar acting as the identity is one. Thus $\alpha = 1$ for each $\alpha \in \Phi$, and the point lies in Z . This proves the equality of functors, including nilpotent test algebras. On splitting-cover overlaps these subgroup schemes represent the same centrality condition, so their ideals and maps descend by S04 Lemmas A1.2 and A1.5. The resulting equality and the coefficient presentations commute with every base change.

The centre need not be smooth: $Z(\mathrm{SL}_n) = \mu_n$, including when the characteristic divides n . Its largest subtorus is called the **radical**, denoted $\mathrm{Rad}(G)$. For a group of multiplicative type with finitely generated character sheaf M , it is $D(M/M_{\mathrm{tors}}) \subset D(M)$. Indeed the quotient is a finite free lattice by S04 Lemma P0.10, and the group-algebra quotient gives this closed torus inclusion. Any homomorphism from a torus into $D(M)$ induces a character map from M into its torsion-free lattice, so kills M_{tors} and factors through this inclusion, by the anti-equivalence proved in AG-GS-05 Theorem 2.2. These character maps agree on splitting-cover overlaps. Thus the construction descends, its rank is locally constant, and it commutes with every base change. The **rank** of G is the dimension of a maximal torus; its **semisimple rank** is

$$\mathrm{rank}(G) - \dim \mathrm{Rad}(G).$$

Over an algebraically closed field, every smooth connected solvable normal subgroup P of G is a central torus. RG01 Theorem 2.K writes $P = U \rtimes D$ and makes U characteristic in P . Rational conjugations by G preserve U ; the reducedness of $G \times U$ and the Nullstellensatz make the whole conjugation morphism preserve it. Thus U is normal in G and is trivial by reductivity. Hence P is a torus. The integer-matrix description of torus automorphisms in RG01 Lemma D.1 makes the conjugation map from connected G locally constant, with identity value at the identity; it is the identity everywhere. Therefore P is central and belongs to the radical constructed above. Conversely that radical is smooth connected solvable and normal. Its quotient Q is reductive by the independent central-quotient construction in Theorem 1.1 below. To prove semisimplicity, let $N \subset Q$ be smooth connected solvable and normal and put $P = q^{-1}(N)$. The torsor $P \rightarrow N$ has the split radical torus as kernel. The diagonalizable-torsor algebra calculation in AG-GS-05 Theorem 4.7 and Corollary 4.8 gives local trivializations: trivialize the finitely many degree modules for a basis of its character lattice, whose homogeneous units identify the torsor algebra with a Laurent

algebra. Thus P is smooth and this map is open with connected geometric fibres. The finitely many disjoint open components of smooth P have disjoint open images: a connected fibre lies in only one component. These images partition connected N , so P is connected. It is solvable on all test algebras: a sufficiently long derived commutator word maps to one in N and hence into the central radical, and the next commutator is one. It is normal as the inverse image of N . The preceding argument gives $P \subset \text{Rad}(G)$, while the reverse inclusion holds by construction; the quotient torsor therefore gives $N = 1$. This proves that Q is semisimple. The same inverse-image argument shows that a torus in Q lifts to a torus in G . Its inverse image is smooth connected and solvable by the preceding torsor and commutator arguments. Write it $U \rtimes D$ using RG01 Theorem 2.K. Every character of unitriangular U is trivial by RG01 Lemma 2.A, so its map into the quotient torus is trivial and U lies in the radical torus. Its inclusion into that torus is itself trivial by the same character calculation, giving $U = 1$. The inverse image is therefore a torus. Conversely the image of a torus in G is a torus, by the character-lattice quotient calculation of AG-GS-05 Theorems 4.1–4.2. Hence maximal tori correspond under the radical quotient, and their dimensions decrease by the dimension of its torus kernel. In particular the rank of Q is the stated semisimple rank of G . The derived group will be constructed for arbitrary bases after the pinned existence theorem in [Pinnings and the classification of split reductive groups](#). It is semisimple and the multiplication map

$$\text{Rad}(G) \times G_{\text{der}} \longrightarrow G$$

is a central isogeny. Its construction uses the root groups established here. We do not assume a relative derived group in constructing these groups. Central quotients are constructed by graded invariant algebras in Theorem 1.1 below, without using coroots. In rank one we will construct the simply connected cover explicitly below.

Central quotients before the rank-one argument

The rank-one argument below needs the affine quotient even when the centre is not smooth. The invariant-algebra construction is independent of coroots and root groups. We first supply the field-differential calculation used to check its geometric fibres.

Lemma 1.B (differentials over a perfect field). Let k be perfect, let $t_1, \dots, t_d$ be algebraically independent over k , and let K be a finite extension of $F = k(t_1, \dots, t_d)$. Then $\dim_K \Omega_{K/k} = d$. No separability assumption on K/F is required.

Proof. This is S02 Lemma 3.F. Its complete field-differential calculation proves the assertion in both characteristics, including finite inseparable extensions. $\square$

Lemma 1.C (reduced groups over an algebraically closed field). A reduced affine group scheme E of finite type over an algebraically closed field k is smooth. Its finitely many irreducible components are open and closed translates of its identity component. That component is a smooth connected normal subgroup.

Proof. First prove uniqueness of the component through the identity, without presupposing smoothness. If Z_1, Z_2 are two such components with their reduced structures, S08 Lemma G.4.1 makes $Z_1 \times Z_2$ irreducible. The closure of its multiplication image is contained in a component of E and contains both Z_1 and Z_2 , because the identity belongs to both. Maximality of components forces them to be equal. If two components met elsewhere, their nonempty closed intersection would have a k -point by the Nullstellensatz. Translation of that point to the identity would give two components there. Thus components are disjoint. They are finite in number by Noetherianity, so each is open as well as closed. Each contains a k -point, whose translation identifies it with the identity component. In particular they have one dimension d . The identity component is a subgroup: its product with itself is irreducible, and its multiplication image contains the identity;

inversion also preserves the component. Both maps factor through this open and closed component. The restriction of E to each open component has its reduced closed structure. Indeed, at any point of that component all other minimal primes disappear under localization. The sole remaining minimal prime is the nilradical, which is zero because E is reduced. Thus the component's defining prime localizes to zero there. AG-GS-03, *Lie algebras and smoothness*, Proposition 2.7, gives $\Omega_{E/k} \simeq \mathcal{O}_E \otimes_k \omega_E$, of rank $r = \dim_k \omega_E$. At a component's generic point the local ring is a field K , since it is reduced with only one prime. Noether normalization and Theorem 4.2 of AG-CA, *Krull dimension and Noether normalization*, make K finite over a rational function field on d variables. Localization of differentials and Lemma 1.B give $r = \dim_K \Omega_{K/k} = d$. At the identity, Corollary 4.4 of the dimension lesson gives local dimension d . AG-CA-18, Proposition 3.1, identifies the identity cotangent space with ω_E , so its embedding dimension is also d and its local ring is regular. The same proposition makes E smooth at that rational point. Translations give smoothness at every k -point. The smooth locus is open by AG-CA-18, Corollary 2.2; its closed complement, if nonempty, would have a k -point. Thus E/k is smooth everywhere.

Finally conjugation by every k -point preserves the identity component. The source of the conjugation map from E times that component is reduced by S08 Lemma G.4.1. The equations of the component therefore pull back to functions vanishing at every k -point, and hence to zero by the Nullstellensatz. This proves normality of the entire group-scheme morphism. $\square$

Theorem 1.1 (affine central quotient). Let G be a reductive group over a scheme S , and let $H \subset Z(G)$ be a closed subgroup of multiplicative type of finite presentation. The fppf quotient G/H is an affine reductive S -group. Formation of the quotient commutes with arbitrary base change.

Proof. Work étale locally on $S = \operatorname{Spec} R$, with $H = D_R(M)$. Such splitting neighbourhoods are proved in AG-GS-05, Theorem 7.7. On a nonempty affine chart M is finitely generated: a finite algebra generating set for $R[M]$ contains only finitely many exponents; its polynomial expressions contain only the subgroup generated by those exponents, and independence of the exponent basis forces that subgroup to be M . Right translation gives the grading $A = \Gamma(G, \mathcal{O}_G) = \bigoplus_{m \in M} A_m$. Put $B = A_0$ and $Q = \operatorname{Spec} B$. These degree projectors prove exactness of invariants and arbitrary base-change compatibility. Since smooth algebras are flat over arbitrary rings by AG-CA-18, Theorem 6.1, the direct summand B is flat over R too.

Here is finite presentation directly over R . Choose finitely many homogeneous algebra generators of A , of weights $m_1, \dots, m_n$, and a graded surjection $P = R[z_1, \dots, z_n] \rightarrow A$. For $d \in M$ let $E_d = \{v \in \mathbf{N}^n : \sum_i v_i m_i = d\}$. Every such subset has finitely many minimal elements under coordinatewise order. Indeed, every infinite sequence in $\mathbf{N}$ has an infinite nondecreasing subsequence: either a value repeats infinitely, or the tails are unbounded and one chooses successive larger values. Extract such a subsequence successively in each of the finitely many coordinates. An infinite antichain is impossible. A minimal element below a prescribed vector exists in the finite box below it. The minimal nonzero vectors in E_0 generate that monoid by repeated subtraction, which decreases the coordinate sum. For any d , its finitely many minimal vectors generate P_d as a P_0 -module, because subtraction from a comparable vector of the same weight leaves a vector of weight zero.

The monoid algebra $P_0 = R[E_0]$ is finitely presented over R . Choose the just-proved finite monoid generators. They give a surjection from a finite polynomial ring onto $\mathbf{Z}[E_0]$; its kernel ideal is finitely generated because the polynomial ring over $\mathbf{Z}$ is Noetherian, by S02 Lemma 3.N. Tensor this presentation with R . Tensor product preserves its cokernel, and the monomial basis identifies $R \otimes_{\mathbf{Z}} \mathbf{Z}[E_0]$ with $R[E_0]$, giving the required finite presentation.

The kernel I of $P \rightarrow A$ is finitely generated: compare this generating set with a fixed finite presentation of A by adjoining both finite sets of variables and the equations for their mutual expressions, then eliminate the old variables. This is the affine presentation calculation of AG-MO-02, Theorem 3.1. Since I is graded, replace its finite generators by all their homogeneous

components, obtaining homogeneous generators r_j of weights d_j . Taking weight zero in an expression in these generators gives $I_0 = \sum_j P_{-d_j} r_j$. Each coefficient module is finite over P_0 by the preceding minimal-vector calculation. Thus I_0 is a finite ideal of P_0 , and $B = P_0/I_0$ is finitely presented over R . This also supplies the whole invariant-algebra finiteness argument of AG-GS-05, Theorem 3.5 in the present application.

Centrality first makes Q a group, directly on its algebra. For $b \in B$, the function $b(g_1 g_2)$ is invariant under right translation by H in either variable: for the first variable move the central element past g_2 , and for the second use the original invariance. The double grading of $A \otimes_R A$ therefore puts $\Delta(b)$ in $B \otimes_R B$. Likewise $(gh)^{-1} = g^{-1} h^{-1}$ proves that inversion preserves B , and the counit restricts to B . The Hopf identities restrict because all the relevant inclusions are split degree-summand inclusions, even after tensoring. Thus Q is an affine group and $q : G \rightarrow Q$ a homomorphism. It is finitely presented by the affine-pair presentation calculation, since A and B are finitely presented over R . It is surjective: the degree-zero projector is a B -linear retraction of $B \hookrightarrow A$, so $A \otimes_B \kappa(y)$ is nonzero for every prime y of B , and prime existence gives a point above y .

The closed embedding $H \hookrightarrow G$ gives a graded surjection $A \rightarrow R[M]$. For each of finitely many generators m_i of M , choose $a_i \in A_{m_i}$ and $b_i \in A_{-m_i}$ restricting to e^{m_i} and e^{-m_i} . Put $f = \prod_i a_i b_i \in B$. Its restriction to H , and hence its value at the identity, is one. In A_f all these factors are units. Products of their powers provide a unit u_m in every weight m . Multiplication by u_m identifies $(A_f)_m$ with the free rank-one B_f -module $B_f u_m$. The torsor algebra map

$$A_f \otimes_{B_f} A_f \longrightarrow A_f \otimes_R R[M], \quad a \otimes c_m \longmapsto a c_m \otimes e^m$$

is an isomorphism: its inverse on each exponent basis element sends $a \otimes e^m$ to $a u_m^{-1} \otimes u_m$. The forward map is an algebra map and the displayed inverse makes it bijective, so its inverse is an algebra map as well. Also $A_f = \bigoplus_m B_f u_m$ is faithfully flat over B_f . Hence q is an H -torsor over the principal open $U = D(f) \subset Q$, which contains the identity.

To cover the entire quotient over the original ring, form the principal open $W = \{(g, y) \in G \times_R Q : q(g)^{-1} y \in U\}$. Its projection to Q is smooth of finite presentation, being a principal open in the base change of G/R . It is surjective. For a geometric point $y : \text{Spec } \Omega \rightarrow Q$, the same module retraction makes $A \otimes_B \Omega$ nonzero. This is a finite-type Ω -algebra, so the proved weak Nullstellensatz supplies $g \in G(\Omega)$ above y . Then $q(g)^{-1} y = 1 \in U$, giving a point of W above y . Thus $W \rightarrow Q$ is an fppf cover, over arbitrary nonreduced bases as well.

On W , its first coordinate is an actual section g of G_W . Left translation $x \mapsto g^{-1} x$ identifies the pullback of q with the pullback of the already constructed torsor over U , through the map $(g, y) \mapsto q(g)^{-1} y$. This translation commutes with the right H -action. S04 Lemma A1.3 descends flatness through the affine faithfully flat cover $W \rightarrow Q$, and finite presentation of q was already proved. The torsor isomorphism $G \times H \rightarrow G \times_Q G$ and its inverse descend by S04 Lemma A1.5. Consequently q is an fppf H -torsor globally. Every section of Q lifts fppf locally through q , and two lifts differ by a unique section of H by this isomorphism. This identifies Q with the fppf quotient sheaf G/H .

On overlaps of the splitting cover the quotient sheaf is the same, so these affine representing groups are canonically isomorphic, with the cocycle identity forced by their quotient maps. The affine effectivity proof in S04 Appendix A1 descends their algebras, and Lemma A1.5 descends the group laws and torsor isomorphisms. Lemmas A1.3 and A1.3a descend flatness and finite presentation. The degree-zero projector on each splitting chart commutes with every scalar extension, so this descended quotient and its torsor identity commute with arbitrary base change.

We check geometric smoothness directly. Over an algebraically closed field k , smoothness of G makes A reduced: AG-CA-18, Theorem 2.1, makes its local rings regular, and the regular-local-domain proof makes them reduced. The inclusion $B \hookrightarrow A$ therefore makes B reduced. Lemma 1.C proves Q/k smooth. It is connected because $G \rightarrow Q$ is surjective.

Let $U \subset Q$ be any smooth connected normal unipotent subgroup, and put $P = q^{-1}(U)$. This is a closed normal subgroup of finite type, and $P \rightarrow U$ is the pullback of the surjective H -torsor. Over k , the product of reduced finite-type schemes is reduced, by S08 Lemma G.4.1. Multiplication and inversion therefore factor through P_{red} : nilpotent functions pull back to zero on their reduced sources. Thus P_{red} is a group. Lemma 1.C makes it smooth with open and closed irreducible components. Put $K = (P_{\text{red}})^0$. In particular K is smooth connected and closed in G .

Here normality is a statement about morphisms, including their infinitesimal points. For each $g \in G(k)$, conjugation preserves P , its reduction and its identity component. Every defining equation of $K \subset G$ consequently pulls back under conjugation $G \times K \rightarrow G$ to a function vanishing at all k -points. The source is reduced by S08 Lemma G.4.1, and the Nullstellensatz makes that function zero. Hence the whole conjugation morphism factors through K , proving $K \triangleleft G$ schematically. Also K is solvable. Indeed $U(k)$ is unitriangular and hence nilpotent: commuting matrices with vanishing entries below distances a and b gives vanishing entries below distance $a + b$, as follows by multiplying their strictly upper triangular parts. A sufficiently long derived commutator word in $P(k)$ therefore lands in $H(k)$; the next derived commutator is one because H is commutative. The corresponding word map on a power of the reduced K is identically one by the same Nullstellensatz argument. This proves solvability as an algebraic group.

The preceding lesson, Theorem 2.K, writes $K = V \rtimes T$ with V smooth connected unipotent and T a torus and proves that V is the largest smooth connected normal unipotent subgroup. Thus every conjugation automorphism of K preserves V . Reducedness of $G \times V$ and the preceding equation argument then give $V \triangleleft G$. Reductivity of G forces $V = 1$. Consequently K is a torus.

There are only finitely many components of P_{red} . Each contains a k -point and is a translate $p_i K$: translation carries the identity component to the component containing p_i . Let J be the topological closure of the image of $K \rightarrow U$. Surjectivity of $P \rightarrow U$, and equality of the underlying spaces of P and P_{red} , imply $U = \bigcup_i q(p_i)J$. The smooth connected U is reduced and hence irreducible by Lemma 1.C. One of these finitely many closed translates is all of U , hence $J = U$. But the homomorphism $K \rightarrow U$ is trivial. To see this directly, pull back a faithful unitriangular representation of U . Its coordinate flag has trivial successive quotients. Exact torus weight projections split every term of this flag, so the whole representation of K is trivial. Faithfulness of the representation of U then makes $K \rightarrow U$ the identity-valued morphism. Thus $J = \{e\}$ and U is the trivial smooth group. This proves that every geometric fibre of Q is reductive. Flatness, finite presentation and these smooth geometric fibres prove smoothness of Q/S by AG-FSE-05, Theorem 3.1.

□

2. Why rank one has only two moving directions

We first prove the geometric rank-one statement, rather than assuming a root classification. Work over an algebraically closed field k .

Lemma 2.1. If a torus acts linearly on an irreducible projective variety X , it has at least $\dim X + 1$ fixed points.

Proof. Put $d = \dim X$ and let T be the acting torus. Since k is algebraically closed, an étale splitting cover of T has a k -point: take a nonempty finite-type affine chart and apply the earlier Nullstellensatz. Pulling the splitting back along this point gives $T \simeq \mathbf{G}_m^r$. The weight decomposition proved in the preceding lesson, *Diagonal actions and their descent*, applies to the finite-dimensional representation defining the linear action. Choose homogeneous coordinates in which the action on points has the form

$$t \cdot [c_0 : \cdots : c_N] = [\chi_0(t)c_0 : \cdots : \chi_N(t)c_N], \quad \chi_j \in X^*(T) = \mathbf{Z}^r.$$

Delete any coordinates vanishing identically on X , thereby restricting to an invariant projective linear subspace.

Here is an explicit cocharacter separating all distinct characters in this finite list. If $r > 0$, write $\chi = (a_1, \dots, a_r)$. For distinct characters χ, ψ , the integer polynomial

$$\sum_{i=1}^r (a_i - b_i) M^{i-1}$$

is nonzero. The polynomial root bound proved in S02 Lemma 4.A, applied over $\mathbf{Q}$, makes its set of integer roots finite. Choose a positive integer M outside the union of these finitely many sets and put

$$\lambda(u) = (u, u^M, \dots, u^{M^{r-1}}) : \mathbf{G}_m \longrightarrow T.$$

The integers $w_j = \langle \chi_j, \lambda \rangle$ are equal only for equal characters. For $r = 0$, the unique cocharacter has this property since there is only the trivial character. Reorder the coordinates so that w_0 is minimal. By the deletion step, c_0 does not vanish identically on X . The coordinate hyperplane $H = V(c_0)$ is T -invariant.

We induct on d . For $d = 0$, the reduced irreducible finite-type X has one point. An affine neighborhood is a zero-dimensional domain, hence a field; the Nullstellensatz makes it k . Thus $X = \operatorname{Spec} k$, and its point is fixed.

Suppose $d > 0$. The section $X \cap H$ is nonempty. Otherwise X would be closed in the affine complement of H and hence affine. S01 Theorem 8.4 and its closed-immersion cohomology calculation prove that $H^0(X, \mathcal{O}_X)$ is finite-dimensional over k . It is a domain because X is integral. Multiplication by a nonzero element is an injective endomorphism of a finite-dimensional vector space and therefore surjective, so this algebra is a field. Algebraic closedness makes it k . Affineness would then give $X = \operatorname{Spec} k$, contradicting $d > 0$.

Every irreducible component of $X \cap H$, with its reduced structure, has dimension $d - 1$. To check the dimension rather than assume it, take a standard projective affine chart meeting the component and intersect it with X . Its algebra A is a finite-type domain of dimension d , by the function-field dimension formula in AG-CA *Krull dimension and Noether normalization*, Theorem 4.2. In a local frame the hyperplane section is defined by a nonzero element $f \in A$; its component corresponds to a prime $\mathfrak{p}$ minimal over (f) . This prime is nonzero. The principal ideal theorem in AG-CA *Dimension theory of Noetherian local rings*, Theorem 3.1, gives $\operatorname{ht} \mathfrak{p} \leq 1$, while $(0) \subsetneq \mathfrak{p}$ gives the reverse inequality. The height formula, Theorem 4.3 of the former lesson, yields

$$\dim(A/\mathfrak{p}) = \dim A - \operatorname{ht} \mathfrak{p} = d - 1.$$

Passing between nonempty affine opens of this integral component preserves its function field and hence its dimension by that same Theorem 4.2.

Choose one such reduced component Y . It is T -stable. Indeed, $T \times Y$ is integral: on affine charts of Y its algebra is the domain $A[t_1^{\pm 1}, \dots, t_r^{\pm 1}]$, and these charts have nonempty intersections. The closure of its image under the action is therefore irreducible, lies in $X \cap H$, and contains Y through the identity of T . Maximality of the irreducible component forces this closure to be Y . The action factors through its reduced structure, since pullbacks of its defining ideal vanish at every point of the reduced source and therefore vanish. The restriction is a linear action on the projective variety Y . Induction supplies at least d distinct T -fixed points in $Y \subset H$.

There is one more fixed point outside H . Choose $a \in (X \setminus H)(k)$, using the Nullstellensatz on this nonempty affine chart, and normalize its coordinates so that $c_0(a) = 1$. Its λ -orbit on the same chart has coordinates

$$c_j(\lambda(u)a)/c_0(\lambda(u)a) = u^{w_j - w_0} c_j(a), \quad j > 0.$$

All exponents are nonnegative integers, so these polynomials define a morphism $\mathbf{A}_k^1 \rightarrow \mathbf{P}_k^N \setminus H$. It lands in X : the equations of the closed affine subscheme $X \setminus H$ vanish after substitution in $k[u, u^{-1}]$, hence in its subring $k[u]$. At $u = 0$, every coordinate of weight greater than w_0 is zero;

the remaining coordinates have character χ_0 by the separating property. Every element of T , over every k -algebra, scales these remaining coordinates by the same unit $\chi_0(t)$. Thus the resulting projective point is T -fixed. Its zeroth coordinate is still one, so it lies outside H and differs from all d points obtained inside H . This proves the bound $d + 1$. $\square$

The curve duality used in rank one

Lemma (curve duality). If C is a smooth projective pure one-dimensional curve over a field k and L is invertible, then naturally

$$H^1(C, L)^* \simeq H^0(C, \Omega_{C/k} \otimes L^{-1}).$$

Proof. Embed $i : C \hookrightarrow P = \mathbf{P}_k^N$, and put $c = N - 1$. The regular-immersion calculation in [Projective cohomology and smooth affine models](#), Lemma 6.1, proves over the original field, including imperfect fields, that I/I^2 is locally free of rank c , that the ideal is locally generated by a regular sequence, and that

$$0 \longrightarrow I/I^2 \longrightarrow i^* \Omega_{P/k} \longrightarrow \Omega_{C/k} \longrightarrow 0$$

is exact. Its proof first calculates at geometric closed points, then descends the conormal module and the exact Koszul complex by faithful field extension.

Dualizing that Koszul resolution, with a local frame of L , gives

$$\mathcal{E}xt_P^q(i_* L, \omega_P) = 0 \quad (q \neq c), \quad \mathcal{E}xt_P^c(i_* L, \omega_P) = i_*(L^{-1} \otimes i^* \omega_P \otimes \det(I/I^2)^*).$$

Here $\omega_P = \det \Omega_{P/k} = \mathcal{O}_P(-N - 1)$. Taking determinants of the conormal sequence identifies the sheaf in degree c with $i_*(L^{-1} \otimes \Omega_{C/k})$. The finite-cover Hom double complex written in S02 Theorem 6.2 has only this nonzero sheaf-Ext row, and therefore gives

$$\mathrm{Ext}_P^c(i_* L, \omega_P) = H^0(C, L^{-1} \otimes \Omega_{C/k}).$$

S02 Ambient Theorem 6.B proves projective-space duality by the monomial trace, a two-stage twist presentation and dimension shifting; it identifies the same Ext group with $H^{N-c}(P, i_* L)^* = H^1(C, L)^*$. Its trace sends the class of $T_0^{-1} \cdots T_N^{-1}$ to one. All the Koszul, determinant, finite-cover and trace comparisons are natural. This proves the asserted isomorphism. $\square$

Lemma 2.2 (the curve argument). Over an algebraically closed field, a smooth projective curve dominated by $\mathbf{P}^1$ is isomorphic to $\mathbf{P}^1$.

Proof. Write C for the dominated curve. The dense image of the irreducible $\mathbf{P}_k^1$ is irreducible. Smoothness gives reducedness, so C is integral and connected. If domination is initially given by a rational map, the homogeneous-coordinate DVR extension in the earlier S03 Theorem 2.2 makes it a morphism. Thus the hypotheses of [S02 Proposition 6.5](#) hold: its complete separable differential calculation and inseparable-subfield reduction prove $g = \dim_k H^1(C, \mathcal{O}_C) = 0$ in every characteristic. Its proof is earlier in the programme; no unproved external form of Lüroth's theorem is used.

We supply explicitly the concluding rational-function calculation from that proposition. Choose a closed point $q \in C(k)$, which exists by the earlier Nullstellensatz. S02 Lemma 6.3 proves $H^0(C, \mathcal{O}_C) = k$, divisor Euler characteristics, the canonical degree $\deg \Omega_{C/k} = 2g - 2 = -2$, and vanishing for negative-degree line bundles. Its earlier Theorem 6.2 gives duality. For every integer $m \geq 0$, these results yield

$$H^1(C, \mathcal{O}_C(mq))^* = H^0(C, \Omega_{C/k}(-mq)) = 0, \quad h^0(C, \mathcal{O}_C(mq)) = m + 1.$$

The first vanishing uses degree $-2 - m < 0$; the second is the proved Euler characteristic formula with $g = 0$.

Take x in $H^0(C, \mathcal{O}_C(q))$ independent of the constants. In the divisor description of this sheaf, it is a rational function regular away from q with pole order at most one at q . It has a pole: otherwise it would be a global regular function and belong to k . Its pole order is therefore exactly one. The functions $1, x, \dots, x^m$ belong to $H^0(C, \mathcal{O}_C(mq))$ and have distinct pole orders. A nonzero linear combination has the pole order of its highest nonzero power, since all coefficients lie in k and that term cannot be canceled by terms of smaller order. They are independent, and their number $m + 1$ is the dimension just computed. They form a basis.

Let $h \in k(C)$ be any nonzero rational function. The earlier divisor proof shows that it has finitely many poles. List its poles $a_1, \dots, a_r$ outside q . The function x is regular at each of them, and $x(a_i) \in k$. Each nonzero function $x - x(a_i)$ has positive valuation at a_i and no pole outside q . Choose integers $N_i \geq 0$ large enough that

$$P(x)h, \quad P(T) = \prod_{i=1}^r (T - x(a_i))^{N_i},$$

has no pole outside q . Multiplying cannot introduce a pole at another such point. Its remaining pole order at q is finite, so the product is a section of $\mathcal{O}_C(mq)$ for some $m \geq 0$. By the basis calculation, it equals a polynomial $Q(x)$. The polynomial P is nonzero and x is transcendental over the algebraically closed k , so $P(x) \neq 0$. Consequently $h = Q(x)/P(x) \in k(x)$. The empty pole list gives $P = 1$, and the zero function is already in $k(x)$. Hence $k(C) = k(x)$.

This equality gives inverse rational maps between C and $\mathbf{P}_k^1$: on a nonempty affine chart, send its finitely many coordinate generators to their rational functions in the other function field and clear their denominators; their equations remain satisfied. Both targets are projective and both sources are smooth curves. The earlier S03 Theorem 2.2 extends these maps at every DVR by homogeneous coordinates. Their composites are the identity on the common function field and hence on a dense open. The closed-diagonal equalizer argument for the reduced integral sources makes the composites the identity everywhere. The resulting morphisms are inverse isomorphisms. This argument needs no degree formula for an unchecked inseparable map. $\square$

Proposition 2.3. A semisimple group of rank one has exactly two Borel subgroups containing a maximal torus, its Borel quotient is $\mathbf{P}^1$, and the action on that quotient has image PGL_2 and kernel its centre.

Proof. Fix a maximal torus $T \subset B$ and put $X = G/B$. RG01 Lemma 2.1 constructs it as a closed orbit with a linear equivariant projective embedding and the full fppf B -torsor $G \rightarrow X$. The complete orbit proof S08 Theorem G.4.C makes this actual quotient smooth. It is integral, being reduced and the surjective image of the irreducible smooth connected G . RG02's complete torus-fixed-coset proof makes X^T a finite disjoint union of reduced k -points. Its following transitivity argument identifies this set with $N_G(T)/C_G(T)$. RG02 Theorem 2.1 gives $C_G(T) = T$, so this quotient acts faithfully on the rank-one character lattice $\mathbf{Z}$, whose automorphisms are precisely multiplication by 1 or -1 . Hence X^T has at most two points. It has at least one, the coset B . If $\dim X = 0$, integrality and the Nullstellensatz make $X = \mathrm{Spec} k$, so the torsor stabilizer forces $G = B$, contradicting nontrivial semisimplicity. Lemma 2.1 now gives $|X^T| \geq \dim X + 1 \geq 2$. Therefore there are exactly two fixed points and $\dim X = 1$. The Borel-normalizer theorem already proved in RG02 identifies these with exactly two Borels containing T .

Let K be the schematic kernel of the action on X . It is the intersection of the conjugates of B . This retains the full kernel on every test algebra: in the equivariant projective embedding the fixed-point equations are the coordinate two-by-two minors. If they vanish on every constant k -point of the reduced X , each of their finitely many linearly independent scalar coefficients vanishes on every such point, hence vanishes identically by the Nullstellensatz. Thus these equations vanish after arbitrary scalar extension as well. Its reduced identity component is a smooth connected normal solvable subgroup, hence is trivial by semisimplicity. Normality here holds schematically: conjugation by the smooth connected G preserves that component, and its reduced source factors through its reduced structure, as in S08 Lemma G.4.1. Thus K is zero-dimensional. It is finite:

its affine algebra is Noetherian, has finitely many minimal primes, and in dimension zero they are maximal, with residue fields k . The nilradical is generated by finitely many nilpotent elements and hence is nilpotent, by the bound on monomial exponents. The reduced algebra is a finite product of copies of k ; the successive nilradical quotients are finite modules over it. This gives a finite-dimensional algebra for K . A positive-dimensional Borel consequently has a nonconstant orbit map to X . Otherwise all its point orbits would be points, and density of rational points in the reduced $B \times X$ would make the action trivial schematically, contradicting $B \subset K$.

We supply the rational coordinates required for restricting such an orbit map to a line. For a smooth connected solvable group $B = U \rtimes T$, RG01 Lemma 2.B gives

$$U = U_0 \supset U_1 \supset \cdots \supset U_a = 1, \quad U_i/U_{i+1} \simeq \mathbf{G}_a.$$

The quotients U/U_i are smooth connected and affine by the complete earlier RG01 Lemma 2.G: each U_i is a closed normal unitriangular subgroup of the smooth connected U . The resulting quotient maps are fppf torsors. The map

$$U/U_{i+1} \longrightarrow U/U_i$$

is explicitly a $U_i/U_{i+1} \simeq \mathbf{G}_a$ -torsor. Pull it back along the fppf cover $U \rightarrow U/U_i$. If two local representatives have the same U_i -coset, their ratio belongs to U_i ; its U_{i+1} -coset is unchanged on replacing either representative by its U_{i+1} -equivalent lift. Multiplication and division therefore give inverse maps between this pullback and $(U_i/U_{i+1}) \times U$. This is a local trivialization with the translation action, and its equality relation gives the asserted torsor on every test scheme. All these subgroups are normal, so either coset convention gives these quotient groups. Every such torsor over an affine algebra has a section by the explicit vector-line module-descent calculation in RG01's solvable-structure proof: the translation cocycle descends $0 \rightarrow A \rightarrow E \rightarrow A \rightarrow 0$, and a lift of 1 gives the section. This calculation works over any affine A . Translation by a section identifies the torsor with its base times $\mathbf{A}^1$. Starting with $U/U_0 = 1$ and repeating gives $U \simeq \mathbf{A}^a$ as varieties. The torus is split over the algebraically closed field, by the splitting argument in Lemma 2.1. Thus

$$B \simeq \mathbf{A}^a \times \mathbf{G}_m^r, \quad k(B) = k(z_1, \dots, z_d), \quad d = a + r.$$

These are scheme coordinates; no assertion about abstract group coordinates is needed.

Choose a nonconstant orbit map $b \mapsto bx$ and a nonconstant $h \in k(X)$. The map is dominant because its image has positive dimension in the integral curve, so its pullback $f \in k(B)$ is nonconstant. Choose the least $j \geq 1$ for which $f \in k(z_1, \dots, z_j)$. Then $f \notin k(z_1, \dots, z_{j-1})$. Write, after clearing finitely many coefficient denominators,

$$f = P(z_j)/Q(z_j), \quad P = \sum_i A_i z_j^i, \quad Q = \sum_i B_i z_j^i, \quad A_i, B_i \in k[z_1, \dots, z_{j-1}].$$

Some coefficient minor $A_i B_l - A_l B_i$ is nonzero. If they all vanished, choosing $B_l \neq 0$ would give $P = (A_l/B_l)Q$, contradicting the choice of j .

The inverse image of the domain where h is regular contains a nonempty principal open of our coordinate chart. Choose a nonzero polynomial F defining such an open, multiplying by torus coordinates to clear Laurent denominators if necessary. Viewed as a polynomial in z_j , it has a nonzero coefficient in the other variables. Specialize every other coordinate so that this coefficient, the selected minor, and all fixed torus coordinates stay nonzero. Such a specialization exists: their product is a nonzero polynomial over the infinite field k , and repeated use of the univariate root bound in S02 Lemma 4.A supplies a point where it does not vanish. For no remaining variables the product is already a nonzero scalar. On the resulting coordinate line, P/Q stays nonconstant and F does not vanish identically. If z_j is a torus coordinate, restrict also to $z_j \neq 0$. We have obtained a nonconstant rational map $\mathbf{A}^1 \dashrightarrow X$, since its composition with h is this nonconstant rational function. The explicit DVR extension calculation in S03 Theorem 2.2 extends it to $\mathbf{P}^1 \rightarrow X$. Lemma 2.2, which includes inseparable domination, identifies X with $\mathbf{P}^1$.

The natural map $\mathrm{PGL}_2 \rightarrow \mathrm{Aut}(\mathbf{P}^1)$ is an isomorphism on all test algebras. An automorphism sends $0, 1, \infty$ to three sections that remain pairwise disjoint on every fibre. Locally trivialize their quotient lines. A change of basis sends $0, \infty$ to the first and third; the two nonzero coordinates of the middle section are units, so diagonal scaling also sends 1 to it. For uniqueness, an automorphism fixing 0 and ∞ preserves both Cartier divisors. The ratio of its pulled-back coordinate to that coordinate is therefore an invertible regular function on all of $\mathbf{P}^1$. S01 Theorem 7.2 gives $H^0(\mathbf{P}_R^1, \mathcal{O}) = R$, so that ratio is a base unit. Fixing 1 makes it one. The local projective matrices consequently agree on overlaps and glue in the represented projective linear group. This proves the assertion even over rings with nilpotents.

Construct the schematic image H directly from the image of the coordinate Hopf algebra of PGL_2 in the domain $k[G]$. Its finitely generated image algebra is a domain; the kernel is a Hopf ideal because tensoring inclusions over a field is injective. Thus H is a closed integral affine group. The full integral-target monic-division and translation proof in AG-GS-04 Lemma 11.1c makes $G \rightarrow H$ faithfully flat and of finite presentation. The kernel identity gives its torsor relation. Lemma 1.C above makes reduced H smooth, and integrality makes it connected. The orbit-dimension calculation of S08 Lemma G.4.D applied to the left action on H gives $\dim H = \dim G - \dim K$. Its kernel K is finite. Its torus image has dimension one, and maximal-torus conjugacy in RG01 conjugates it to the diagonal torus of PGL_2 . If H were proper, it would have dimension at most two. In every characteristic, $\mathrm{Lie} \mathrm{PGL}_2 = \mathfrak{gl}_2/kI$ has the diagonal line of weight zero and two distinct root lines of weights 1, -1 for the cocharacter represented by $\mathrm{diag}(t, 1)$. The T -stable Lie algebra of H contains the diagonal line and at most one root line. Choose that cocharacter or its inverse so that all its weights on $\mathrm{Lie} H$ are nonnegative. The general limit-subgroup proof already given in RG01 Lemma 3.3 makes $P_H(\lambda)$ smooth with this entire Lie algebra, hence of dimension $\dim H$. A closed subgroup of that dimension in the connected smooth H is all of H . But

$$P_H(\lambda) = H \cap P_{\mathrm{PGL}_2}(\lambda),$$

and the ambient limit subgroup is triangular. Thus H would be solvable. The finite group $K(k)$ is central in $G(k)$: each conjugation orbit is the image of the connected reduced G in a finite set and is the identity translate. Every $H(k)$ -point lifts through the faithfully flat finite-type image map by the Nullstellensatz. Hence $G(k)$ is a central extension of a solvable group and is solvable. Its finite derived-word identities hold on the dense rational points of each reduced finite Cartesian power of G , so they hold schematically, making G solvable. This contradicts semisimplicity. Therefore $H = \mathrm{PGL}_2$.

In particular $\dim G = 3$. Since $\dim X = 1$, each of the two Borels B_+, B_- has dimension two and unipotent radical of dimension one. The one-dimensional unipotent classification proved in S03 Section 4 identifies each radical with $\mathbf{G}_a$. Its image has dimension one because the kernel is finite. Projection of the corresponding point stabilizer onto its diagonal torus gives a character of the radical, trivial by RG01 Lemma 2.A. Thus its reduced closed image lies in the ambient unipotent line; dimension one makes it that entire line. Its torus character α_+ or α_- is nonzero, since the induced diagonal-torus map is an isogeny and the ambient root character is nonzero. The normalizer element interchanging the two fixed points acts by inversion on T , so $\alpha_- = -\alpha_+$.

The Lie algebra of K lies in both $\mathfrak{b}_+$ and $\mathfrak{b}_-$. Their nonzero-weight lines have these distinct opposite characters, while the shared torus has weight zero. Thus $\mathrm{Lie} K$ has no vector in either unipotent root line. Each finite intersection $K \cap R_u(B_\pm)$ is torus-stable. Since $\alpha_\pm \neq 0$, its polynomial ideal is homogeneous for the distinct weights $i\alpha_\pm$, hence is (x^n) for some $n \geq 1$. The Hopf identity requires every intermediate binomial coefficient in $(X + Y)^n$ to vanish. In characteristic zero the coefficient n excludes $n > 1$. In characteristic p , write $n = p^e m$ with $p \nmid m$. If $m > 1$, the coefficient m of $X^{p^e} Y^{n-p^e}$ in $(X^{p^e} + Y^{p^e})^m$ is nonzero, a contradiction. Thus $n = p^e$. Any $n > 1$ gives nonzero tangent space at the identity, already excluded. Both intersections are therefore trivial. The image maps of the two radicals are faithfully flat with trivial kernel, hence isomorphisms onto the ambient unipotent subgroups, by their full torsor relations.

For a point of $B_+ \cap B_-$ over any test algebra, write it tu_+ by the solvable decomposition. Its image lies in the diagonal intersection of the two ambient point stabilizers, so the image of u_+ is trivial. The just-proved isomorphism gives $u_+ = 1$. Therefore $B_+ \cap B_- = T$ schematically and $K \subset T$. A finite normal multiplicative-type subgroup of a smooth connected group is central: over k it has a finite character group, and the earlier coefficient proof for diagonalizable groups represents its automorphisms by that finite constant étale character-automorphism group. The conjugation map from connected G to this discrete group has value the identity automorphism at the identity of G , hence is constant with that value as a scheme morphism. Thus $K \subset Z(G)$. Conversely $Z(G)$ maps to the centre of PGL_2 , which is trivial schematically. Indeed a central automorphism of $\mathbf{P}^1$ commutes with the diagonal torus and preserves its fixed sections $0, \infty$; interchanging them would invert the universal torus parameter and cannot commute with it. It is therefore a scaling by a base unit, and commuting with every translation forces that unit to be one. Thus $Z(G) \subset K$, proving $K = Z(G)$ and every assertion. $\square$

For a reductive group of semisimple rank one, the radical-quotient proof in Section 1 produces a semisimple group Q of rank one. The inverse image of a Borel of Q is smooth connected and solvable by that same central-torus torsor argument; maximality and the inverse-image comparison make it a Borel of G . It has the same projective-line quotient, by the equality of its fppf cosets. Conversely a Borel of G contains the radical: their product is smooth connected and solvable, since the radical is central, and maximality forces equality with the Borel. Its image in Q is a Borel by the inverse-image comparison. In its decomposition $B = U \rtimes T$, the radical belongs to T by RG01 Lemma 2.N and centrality, and $U \cap \mathrm{Rad}(G) = 1$ schematically by the unitriangular character calculation in RG01 Lemma 2.A. Quotienting the product gives $B/\mathrm{Rad}(G) = U \rtimes (T/\mathrm{Rad}(G))$ as fppf sheaves, hence as the represented groups. Thus its two unipotent radicals map isomorphically to the additive radicals of Q , with their opposite characters pulled back to T . The action kernel is $q^{-1}(Z(Q))$. A point of this kernel lies in both opposite Borel inverse images, whose intersection is T by the displayed product descriptions and $B_+ \cap B_- = T$ for Q . It acts trivially on both root lines. The centre formula in Section 1, now with those two nonzero weights, makes it central. The reverse containment follows from the trivial centre of the projective linear image. Thus its centre is precisely the kernel of the action on that same projective line.

Now let α be any nonzero weight of a maximal torus of an arbitrary reductive k -group. Define the codimension-one torus

$$T_\alpha = (\ker \alpha)_{\mathrm{red}}^0.$$

The centralizer $G_\alpha = C_G(T_\alpha)$ is reductive by the preceding lesson. Its weights are precisely the weights of G that are rational multiples of α , since these and only these vanish on T_α . Its maximal central torus has codimension one in T : the nonzero characters on T/T_α span a rank-one lattice. Thus its semisimple rank is one. Proposition 2.3 proves

$$\Phi \cap \mathbf{Q}\alpha = \{\alpha, -\alpha\}, \quad \dim \mathfrak{g}_\alpha = 1.$$

This proves both the reducedness along each root line and the one-dimensional root-space assertion without appealing to a general root-system classification.

3. Subgroups obtained by taking a limit

The definitions and full grading, smoothness, connectedness and open-cell proof for these subgroups are in [Tori, maximal tori and their conjugacy](#), Lemma 3.3, an earlier lesson. In this lesson Lemma 3.1 denotes that already proved general lemma; it applies to every smooth affine group of finite presentation, without reductivity or a root classification.

For SL_2 and $\lambda(t) = \mathrm{diag}(t, t^{-1})$, conjugation sends $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to $\begin{pmatrix} a & t^2 b \\ t^{-2} c & d \end{pmatrix}$. Thus $P(\lambda)$ is upper triangular, $U(\lambda)$ is upper unipotent, and $L(\lambda)$ is diagonal. This calculation works in characteristic two because the group character t^2 remains nontrivial.

4. Root groups over the base

Let $T = D_S(M)$ be a split maximal torus. Locally the nonzero weights are a fixed finite set $\Phi \subset M$, and each $\mathfrak{g}_\alpha$ is a line bundle by Proposition 2.3 on geometric fibres. Define T_α by the torsion-free quotient of $M/\mathbf{Z}\alpha$. Then

$$G_\alpha = C_G(T_\alpha), \quad \mathrm{Lie}(G_\alpha) = \mathfrak{t} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}.$$

Choose a cocharacter λ of T with $n = \langle \alpha, \lambda \rangle > 0$, and set $U_\alpha = U_{G_\alpha}(\lambda)$. It is a smooth connected one-dimensional unipotent group, with Lie algebra $\mathfrak{g}_\alpha$.

For a line bundle L , write $\mathbf{V}(L) = \mathrm{Spec}_S \mathrm{Sym}(L^*)$ for its additive group.

Theorem 4.1. There is a unique T -equivariant homomorphism

$$\exp_\alpha : \mathbf{V}(\mathfrak{g}_\alpha) \longrightarrow G$$

whose differential is the inclusion of the root line. It is a closed immersion, with image U_α , and commutes with every base change. Moreover

$$U_{-\alpha} \times T \times U_\alpha \longrightarrow G_\alpha$$

is an open immersion.

Proof. The earlier RG01 Lemma 3.3, used as Lemma 3.1 here, gives the smooth affine subgroup $U_\alpha = U_{G_\alpha}(\lambda)$ of finite presentation, its action extending to $\mathbf{A}^1$ with zero equal to the identity, and its Lie module $\mathfrak{g}_\alpha$. On that line $\lambda(t)$ acts by t^n , with the positive integer $n = \langle \alpha, \lambda \rangle$. The earlier RG01 Lemma 3.D therefore constructs a unique normalized equivariant group isomorphism

$$\mathbf{V}(\mathfrak{g}_\alpha) \xrightarrow{\sim} U_\alpha.$$

Its entire graded-algebra, differential and gluing proof works over the original base, even if n is zero in some residue fields. Since T commutes with λ , the last assertion of that lemma makes this isomorphism T -equivariant on every test algebra. Compose it with the closed immersion $U_\alpha \subset G_\alpha \subset G$ to obtain $\exp_\alpha$.

For uniqueness as a homomorphism into G , let h be any T -equivariant homomorphism with the required differential. The subtorus T_α acts trivially on its source, so the equality expressing equivariance makes every value of h commute with T_α , including its universal points. Hence h factors through the represented centralizer G_α . Moreover

$$\lambda(t)h(x)\lambda(t)^{-1} = h(t^n x).$$

The right side extends regularly to $t = 0$ and there equals $h(0) = 1$. The defining orbit-limit equations from RG01 Lemma 3.3 therefore force h to factor through $U_{G_\alpha}(\lambda)$. In a local Lie-line frame, compose with the inverse of the preceding isomorphism. The result is a weight- n equivariant endomorphism of an affine line with identity differential. The Laurent-coefficient calculation in RG01 Lemma 3.D makes it the identity. These equalities hold on overlaps, proving global uniqueness.

We also verify the equality needed for the open cell over the original base. The centralizer theorem of the preceding lesson makes the geometric fibres of $L = L_{G_\alpha}(\lambda)$ connected. The two opposite nonzero weights established in Section 2 and RG01 Lemma 3.3 give $\mathrm{Lie}(L) = \mathfrak{t}$. On a geometric fibre the closed torus T thus has the same dimension as the smooth connected L , and is all of it: a proper closed subset of the irreducible smooth connected group has smaller dimension. Over S , both groups are smooth, and the inclusion $T \rightarrow L$ has invertible differential at the identity and hence everywhere by translation. S02 Lemma 3.A makes it étale; S04 Lemma A1.9 makes this monomorphism an open immersion. Its image contains every geometric fibre by the equality just proved, so it is all of L . Thus $L = T$ as schemes over S . Applying the open-cell assertion of RG01 Lemma 3.3 inside G_α , with the cocharacters λ and $-\lambda$, gives the displayed open immersion. Formation of these centralizers and limit subgroups commutes with every base change by their defining equations. So do their Lie modules and the uniquely normalized isomorphism of RG01 Lemma 3.D. All the asserted constructions are therefore compatible with every base change. $\square$

The notation $\exp_\alpha$ denotes this algebraic map, not the analytic exponential series. No division by factorials occurs. A root-group parameterization $x_\alpha : \mathbf{G}_a \simeq U_\alpha$ is exactly a trivializing section of the root line.

Example 4.2. In GL_n , conjugation by $\mathrm{diag}(t_i)$ sends E_{ij} to $(t_i/t_j)E_{ij}$. Thus

$$\alpha_{ij} = e_i - e_j, \quad \mathfrak{g}_{\alpha_{ij}} = \mathcal{O}_S E_{ij}, \quad x_{ij}(u) = I + uE_{ij}.$$

Example 4.3. Let $E = \mathcal{O}_S \oplus L$ for a line bundle L . Define $\mathrm{SL}(E)$ by the automorphisms whose action on $\wedge^2 E = L$ is the identity. On opens trivializing L this is exactly the determinant-one matrix group of RG01 Lemma 1.M. If two frames differ by a unit ℓ , their matrix coordinates differ by conjugation with $\mathrm{diag}(1, \ell)$; these conjugations satisfy the same cocycle as the frames and preserve determinants. The complete affine algebra/morphism descent of S04 Lemmas A1.2, A1.3a and A1.5 therefore glues these local affine smooth finite-presentation groups into $G = \mathrm{SL}(E)$. Each geometric fibre is the already proved reductive SL_2 . The local diagonal maps (t, t^{-1}) commute with the transitions, so glue to a closed split maximal torus, maximality being checked on those fibres.

The off-diagonal endomorphism modules are $\mathrm{Hom}(L, \mathcal{O}) = L^{-1}$ above and $\mathrm{Hom}(\mathcal{O}, L) = L$ below. Conjugation gives their full weights t^2 and t^{-2} , and the trace-zero diagonal tangent module has weight zero. For an upper entry $u : L \rightarrow \mathcal{O}$, its square as an endomorphism of E is zero. Consequently $I + u$ has determinant one, inverse $I - u$, and $(I + u)(I + v) = I + (u + v)$ on every test scheme. This identifies the upper subgroup with $\mathbf{V}(L^{-1})$; the same calculation gives $\mathbf{V}(L)$ below. They are the root groups of Theorem 4.1 because their equivariant tangent maps are the displayed identities, whose uniqueness that theorem proves. Here $\mathbf{V}(M)$ means the additive group whose local sections are sections of the line module M . Its Lie module is M , by the same square-zero calculation. A group-scheme isomorphism with $\mathbf{G}_a$ would induce an isomorphism $M \simeq \mathcal{O}_S$ on Lie modules. Thus when L is nontrivial neither displayed root group is globally isomorphic to $\mathbf{G}_a$. Over a field every line is trivial, by choosing a nonzero vector. Having a split maximal torus is therefore weaker than having globally parameterized root groups.

5. A projective quotient and its rank-one form

We use the general algebraic-space tool proved before this lesson in [Affine descent, Zariski Main and recognition of spaces](#), Theorems E8.1 and E9.1. The proof there includes the finite flat affine quotient, the off-diagonal induction, separated locally quasi-finite scheme effectivity, and relative integral closure with its étale base change. It applies over every base and on every test scheme.

Quasi-finite normalization lemma. Let S be a scheme and let $f : X \rightarrow Y$ be a quasi-finite separated morphism of algebraic spaces over S . Let Y' be the relative integral closure of Y in X . Then the factorization $X \xrightarrow{f'} Y' \xrightarrow{\nu} Y$ has f' a quasi-compact open immersion and ν integral. In

particular f is quasi-affine.

Proof. Theorem E9.1 of the preceding supporting lesson proves the full factorization, after Theorem E8.1 establishes recognition and Corollary E8.2 establishes representability on every scheme test. Its integral algebras are descended using the proved affine faithfully flat equalizer, and Corollary D4.3 supplies their étale base-change compatibility. Thus for every affine scheme $Z \rightarrow Y$, the pullback $Z \times_Y X$ is a quasi-compact open subscheme of the affine integral scheme $Z \times_Y Y'$. This is quasi-affineness. All these input proofs precede the present application in the programme. $\square$

The relative integral closure retains nilpotents; a reduced normalization is not substituted for it. The full argument is in the preceding supporting lesson, rather than being delegated to a source citation.

The following construction will also be used for flag varieties. It makes explicit why a fibrewise projective quotient is a scheme, even before we have classified the group over the base.

A flat-source fibre criterion. A finitely presented closed immersion $Y \hookrightarrow Z$ over S , with Y flat over S , is an isomorphism if it is so on every geometric fibre. Locally write its ideal as $I \subset C$. Flatness of C/I makes $I \otimes k(s) \rightarrow C \otimes k(s)$ injective. Fibre equality makes this module zero. Since I is finitely generated as an ideal, Nakayama at every prime of C gives $I = 0$. No flatness assumption on Z is needed.

Formal separation and normalizers

We first justify identity-neighbourhood detection over bases with nilpotents, and then prove the full finite-jet normalizer assertion.

Formal separation lemma. Let R be Noetherian and let H/R be a smooth affine group of finite presentation with geometrically connected fibres. Write $C = \mathcal{O}(H)$ and let J be its augmentation ideal. Then

$$\bigcap_{n \geq 1} J^n = 0.$$

Proof. The entire affine smooth-field component proof of AG-RG-S02 Lemma 4.1 makes every geometric fibre integral. For a prime $\mathfrak{p} \subset R$, flatness embeds $C/\mathfrak{p}C$ into its generic fibre over the fraction field of $R/\mathfrak{p}$: multiplication by any nonzero base element stays injective. That generic fibre is integral by geometric connectedness and smoothness, so $C/\mathfrak{p}C$ is a domain. Its augmentation onto $R/\mathfrak{p}$ also shows that it is nonzero.

We give the prime-filtration argument used here. In any nonzero finite module over a Noetherian ring choose a maximal annihilator of a nonzero element. It is prime: if ab annihilates that element and b does not, the nonzero element obtained by multiplying by b has a larger annihilator unless a already belongs to the first annihilator. Insert this cyclic prime-annihilator submodule and repeat in the quotient. The ascending chain condition on submodules makes the process stop. Apply it to the finite R -module R and tensor its filtration with the flat C . Its factors are $C/\mathfrak{p}C$. For any $j \in J$, the element $1 - j$ has augmentation one in each factor, so is nonzero in that domain and acts injectively. Induction through the finite filtration makes it a nonzerodivisor on C .

The algebra C is Noetherian by the full Hilbert-basis calculation of AG-RG-S01 Lemma 1.3. Put $I = \bigcap_n J^n$. The complete Artin–Rees proof of AG-RG-S02 Lemma 1.1 gives $I = JI$ by applying it to the finite submodule $I \subset C$ and taking $n = c + 1$. With finite generators of I , write this equality as $v = Mv$, with entries of M in J . The adjugate identity gives $\det(1 - M)I = 0$, with $\det(1 - M) = 1 - j$ for a $j \in J$. The just-proved nonzerodivisor property gives $I = 0$. This proves the stated separation, including when R is nonreduced. $\square$

Normalizer lemma. If H is a smooth closed subgroup with geometrically connected fibres of a smooth affine group K of finite presentation over any scheme S , then its scheme-theoretic normalizer

is a closed subgroup of finite presentation in K . Its formation commutes with every base change.

Proof. First suppose $S = \operatorname{Spec} R$ is Noetherian. We justify the finite locally free identity neighbourhoods used for the equations. For a smooth affine group F/R with augmentation ideal J_F , the identity conormal bundle $J_F/J_F^2 = \operatorname{Lie}(F)^*$ is finite locally free. On a base neighbourhood where it is free of rank d , lift a basis to functions $y_1, \dots, y_d \in J_F$. They define $F \rightarrow \mathbf{A}_R^d$, sending the identity to zero and with invertible differential there. The complete differential proof of AG-RG-S02 Lemma 3.A makes it etale on a neighbourhood of that section. The full formal-etale lifting theorem of AG-FSE, *Infinitesimal lifting and invariance under thickenings*, Theorem 4.1, constructs its unique inverse on successive zero thickenings. At each step the kernel $(y)^n/(y)^{n+1}$ is square-zero; both composites are the identity by lifting uniqueness. The thickenings are supported on the identity and stay in the chosen etale neighbourhood. Hence, locally on the base,

$$\mathcal{O}(F)/J_F^{n+1} \simeq R[y_1, \dots, y_d]/(y_1, \dots, y_d)^{n+1}.$$

This proves finite local freeness of every identity neighbourhood and the intrinsic associated graded identity

$$\operatorname{gr}_{J_F} \mathcal{O}(F) = \operatorname{Sym}(\operatorname{Lie}(F)^*).$$

The monomial descriptions agree through the natural graded maps, so the latter identity is independent of coordinates and glues.

Apply this to K_n, H_n , the n th identity neighbourhoods of K, H . They are finite locally free schemes, without a claim that these thickenings are subgroup schemes. The closed immersion $H \subset K$ induces a surjection on augmentation ideals: lift a zero-augmentation function using the coordinate-algebra surjection. Thus its augmentation powers are also the induced powers, and $H_n = H \times_K K_n$. Conjugation preserves the identity and all these powers. It acts by invertible linear maps on the finite locally free coordinate module of K_n .

Write $D = \mathcal{O}(K)$ and choose finite generators f_i of the ideal of H . Restrict their universal conjugates to H_n , for conjugation by both g and g^{-1} . A section in $D \otimes_R \mathcal{O}(H_n)$ vanishes exactly when all its coefficients vanish: locally use a free basis, or globally a finite generating list of its projective dual. The dual and its evaluation commute with arbitrary base change by finite local freeness. These are finitely many equations in D and define the closed finite-presentation stabilizer $N_n \subset K$ of the closed subscheme $H_n \subset K_n$. They represent the stabilizer on every test ring, including nonflat and nonreduced tests. Its two conjugation conditions make it a subgroup.

Its ideals $L_n \subset D$ increase: preserving the $(n+1)$ st thickening preserves the n th. Since D is Noetherian, their union is L_N at one finite index N . Over the universal Noetherian quotient $A = D/L_N$, both conjugation restrictions vanish on every identity neighbourhood of H_A . The formal separation proof above applies to the smooth geometrically connected H_A , so each restricted defining equation is zero in its whole coordinate algebra. Both conjugations therefore preserve the entire H_A . Pulling back these actual zero equations proves sufficiency over every R -algebra. Conversely an automorphism preserving H preserves its augmentation powers, giving necessity. Thus $N_K(H) = N_N$ as full scheme functors. This finite universal description proves arbitrary base change, rather than assuming that an infinite intersection commutes with it.

For an arbitrary affine base, first descend K, H simultaneously to a finitely generated $\mathbf{Z}$ -algebra, retaining smoothness, the closed immersion and group-law identities by the complete smooth-model proof AG-RG-S02 Proposition 3.2. Then the entire connected-fibre model argument of its Theorem 4.5 enlarges that coefficient stage so that H has geometrically connected fibres; the already retained smoothness and finite identities of K survive that base change. No connectedness hypothesis is imposed on K . Apply the preceding universal Noetherian construction there. Its pullback represents the original normalizer on every test scheme. The unique represented subgroup identifications agree on overlaps and glue. This proves closedness, finite presentation and every asserted base change over the original S . $\square$

Cohomology of a projective completion

The preceding supporting lesson [Projective cohomology and smooth affine models](#), Theorem 2.3, proves the precise formal-functions statement used here. If A is Noetherian, X/A projective, F coherent and $I \subset A$, then

$$H^q(X, F)_I^\wedge \simeq \varprojlim_n H^q(X, F/I^n F).$$

Here is the mechanism of that written proof. Over the Rees algebra $B = \bigoplus I^n t^n$, the sheaf $\bigoplus I^n F$ is finite on the projective base change X_B ; projective Serre finiteness therefore makes $\bigoplus H^q(X, I^n F)t^n$ finite over B . If $M^q = H^q(X, F)$, $J_n^q = \text{im}(H^q(X, I^n F) \rightarrow M^q)$ and K_n^q is its kernel, finite homogeneous generators give constants a_q, b_q with

$$I^n M^q \subset J_n^q \subset I^{n-a_q} M^q, \quad K_m^q \longrightarrow K_n^q = 0 \quad (m \geq n + b_q).$$

The sheaf exact sequence consequently gives

$$0 \longrightarrow M^q/J_n^q \longrightarrow H^q(X, F/I^n F) \longrightarrow K_n^{q+1} \longrightarrow 0.$$

Its compatible kernel coordinates vanish by the displayed transition bound; the remaining quotient filtration is cofinal with $I^n M^q$. This proves the inverse-limit comparison, without an unexplained exchange of cohomology and inverse limit.

S02 Lemmas 1.2–1.3 prove exact finite-module completion, Noetherianity and faithful flatness of local completion. Thus a finite module over a complete local Noetherian ring is complete. S02 Lemma 1.4 proves unique lifting of idempotents through nilpotent ideals, and Corollary 2.4 turns the compatible thickened idempotents into an actual section of a projective scheme. These are precisely the two completion operations in the quotient proof below.

Lemma 5.A (Grassmannians). The scheme $\text{Gr}_d(V)$ of locally free rank- d quotients of a finite locally free V/S is smooth and projective, commutes with arbitrary base change, and its Plücker line bundle is $\det Q$, for its universal quotient Q .

Proof. Trivialize $V = \mathcal{O}^n$. On the open where columns I give a basis of a quotient, its unique normalized matrix has those columns equal to the identity and $d(n-d)$ free entries. A second basis J gives the transition $M_J = (M_I|_J)^{-1} M_I$. These open affine charts represent the quotient functor on all tests and glue. The determinant quotient $\bigwedge^d V \twoheadrightarrow \det Q$ defines the Plücker map to $\mathbf{P}(\bigwedge^d V)$, with the quotient convention. Over the target chart $p_I \neq 0$, the ratios of minors containing a single replacement of an identity column recover, with their determinant signs, every free matrix entry. The corresponding map of coordinate rings is surjective. These target charts cover the projective bundle, so the Plücker map is a closed immersion, and its $\mathcal{O}(1)$ pulls back to $\det Q$. The construction glues for general V and uses only invertible minors, hence works over nonreduced bases. The subbundle convention follows by dualizing. $\square$

Projective homogeneous quotients

Lemma 5.1. Let H be a smooth closed subgroup with geometrically connected fibres of a smooth affine S -group K with geometrically connected fibres. Suppose $N_K(H) = H$, and suppose every geometric quotient $K_{\bar{s}}/H_{\bar{s}}$ is proper. Then the fppf quotient $X = K/H$ is a smooth projective S -scheme, locally on S . Its universal conjugate subgroup $\mathcal{H} \subset K_X$ supplies a canonical relatively ample line bundle $\det(\text{Lie } \mathcal{H})^*$.

Proof. The claim is local on the base, so work on affine pieces with fixed smooth relative dimensions. If $\dim H = 0$, its connected smooth geometric fibres are the identity points. The identity section into H is an étale monomorphism by the entire differential and open-monomorphism proofs of AG-RG-S02 Lemma 3.A and AG-RG-S04 Lemma E7.2. It is surjective on geometric fibres and hence an isomorphism. Thus $H = 1$. Its normalizer is K , so the hypothesis gives $K = 1$ and

$X = S$. The asserted line bundle is then relatively ample through $\mathbf{P}(\mathcal{O}_S) = S$. We may treat the positive-dimensional pieces below.

The entire finite-model theorem of AG-RG-S02 Theorem 4.5 descends K, H and their finite equations to a finite-type Noetherian base S_0 , retaining smoothness and geometrically connected fibres. The full normalizer theorem just proved commutes with every base change. The equality $N_K(H) = H$ descends at a later coefficient stage: both are finitely presented closed subschemes of the affine group, and equality of their finite ideal generators uses finitely many relation witnesses. Work on the open-and-closed model rank pieces for the finite locally free modules below. Their pullbacks cover the original base, so this introduces no additional hypothesis.

The complete free-action proof of AG-RG-S07 Corollary 8.3 constructs $X_0 = K_0/H_0$ as an algebraic space and the full fppf H_0 -torsor $K_0 \rightarrow X_0$, with every base change. Flatness and local finite presentation of X_0/S_0 descend on étale scheme charts by the complete AG-RG-S07 Lemma 5.2. Here is the smoothness argument in full. Choose an étale scheme chart $W \rightarrow X_0$. For any $s \in S_0$ and any field extension $L/\kappa(s)$, the scheme $K_L \times_{X_L} W_L$ is étale over the smooth L -group K_L , hence smooth over L by the square-zero composition proof of AG-CA-17 Theorem 1.2 and finite-presentation permanence. Its local rings are regular by AG-CA-18 Theorem 2.1. Its map to W_L is surjective and flat, being the base change of the torsor. At each point of W_L , a point above it gives a faithfully flat local map from its local ring into a regular local ring. The full regularity descent in *Descending properties*, Proposition 1.1, applies: the residue-field resolution acquires a projective syzygy upstairs by AG-CA-14 Theorem 2.2, that finite projectivity descends by AG-RG-S04 Lemma A1.3 and AG-CA-07 Theorem 5.3, and the same residue-field criterion proves regularity downstairs. Thus every W_L is regular. To turn this into fibre smoothness without an omitted geometric-regularity converse, fix $w \in W_s$ and choose an algebraically closed field L containing $\kappa(w)$ by AG-RG-S07 Lemma 2.3. Evaluation gives a rational closed point $z \in W_L$ above w . The cotangent calculation of AG-CA-18 Proposition 3.1 identifies its differential fibre with its maximal ideal modulo its square. Regularity makes its dimension equal to $\dim \mathcal{O}_{W_L, z}$, which is $\dim_z W_L$ by AG-CA-09 Theorem 6.1. AG-RG-S07 Lemma 3.1 identifies this point dimension with $\dim_w W_s$. Differential base change, AG-CA-15 Theorem 2.2, gives the same cotangent dimension at w . The full field Jacobian criterion AG-CA-18 Theorem 2.1 now makes W_s smooth at w . Flatness, finite presentation and the complete arbitrary-base fibrewise criterion of *Smooth morphisms*, Theorem 3.1, make W/S_0 smooth. This proves smoothness of X_0 on its étale charts. Geometric connectedness follows from the surjective image of each connected $K_{\bar{s}}$. Quasi-compactness follows from the quasi-compact surjective source. The quotient is separated: the pullback of its diagonal to $K_0 \times K_0$ is the inverse image of the closed H_0 under $(g, g') \mapsto g^{-1}g'$. Apply the whole ideal/closed-immersion descent of AG-RG-S04 Corollary A3.2 on étale scheme charts. Thus X_0 is a separated finite-presentation space over S_0 .

Choose the finite jet index $n \geq 1$ that detects the full normalizer on this model. The whole preceding coordinate proof makes $\mathcal{O}(K_n)$ and its quotient $\mathcal{O}(H_n)$ finite locally free. Their ranks are fixed on the pieces chosen above. Conjugation gives a section of the Grassmannian of quotients of $\mathcal{O}(K_n)$ of rank $\text{rank } \mathcal{O}(H_n)$, with stabilizer exactly H . Its orbit map factors through the quotient sheaf. It is a monomorphism on all tests: lift two cosets locally to g_1, g_2 ; equality of their Grassmannian quotients says that $g_2^{-1}g_1$ stabilizes H_n , and the full stabilizer theorem puts this element in H . The cosets are equal. Thus we have a finite-type monomorphism

$$\theta : X_0 \longrightarrow \text{Gr}.$$

It is separated and quasi-finite. The full space recognition and normalization proofs of AG-RG-S04 Theorems E8.1–E9.1, as stated in the preceding quasi-finite normalization lemma, make X_0 a scheme. On each affine Grassmannian chart its pullback is an open subscheme of the represented affine integral closure; these scheme opens cover it.

We now construct the compactification needed below. The Grassmannian is qcqs and projective by the complete Lemma 5.A. The full scheme finite-factorization theorem AG-RG-S04 Theorem D7.1

applies to θ and gives

$$X_0 \hookrightarrow \overline{X}_0 \xrightarrow{\pi} \text{Gr},$$

with first map a quasi-compact open immersion and π finite. We supply the full universal closed-image calculation for projective space used here. Over an arbitrary affine base $\text{Spec } R$, a closed subset $C \subset \mathbf{P}_R^N$ is cut out by a homogeneous ideal $I \subset R[T_0, \dots, T_N]$. Indeed on the chart $T_i \neq 0$ its closed subset is cut out by an ideal J_i in the ratio coordinates. Homogenize every $f \in J_i$ and multiply its homogenization by T_i . These homogeneous polynomials vanish on C also outside this chart because of that last factor. If a point is outside C , choose a chart containing it and an $f \in J_i$ nonzero at that point; its homogenization times T_i is still nonzero there. The ideal generated by all these polynomials consequently cuts out exactly C . Put $D = R[T]/I$. A fibre over $p \in \text{Spec } R$ is empty precisely when $\text{Proj}(D \otimes_R \kappa(p))$ is empty. For such an empty fibre each T_i is nilpotent in this graded algebra: otherwise its nonzero localization has a nonzero degree-zero subring, whose spectrum gives a point in the $T_i \neq 0$ chart by the prime-existence proof. A sufficiently large degree m therefore has $(D_m) \otimes_R \kappa(p) = 0$, since every degree- m monomial contains one of those vanishing powers. The module D_m is finite over R , being a quotient of the finite free degree- m polynomial module. Local Nakayama gives $(D_m)_p = 0$. Its finitely many generator images are killed by denominators outside p , so one $a \notin p$ gives $(D_m)_a = 0$. Every higher graded piece is generated from degree m , since D is generated in degree one; hence all those pieces also vanish over R_a . Every projective fibre over $D(a)$ is empty. The complement of the image of C is thus open. This argument works for every ring and every closed subset, so it proves universal closedness after every base change, affine locally on the new base. The finite affine projective charts give finite type and the diagonal calculation of AG-RG-S01 Section 7 gives separatedness. Therefore projective space, and every closed subscheme of it, is proper. Apply this to the actual Plucker closed immersion of the Grassmannian from Lemma 5.A. The scheme $\overline{X}_0$ is now proper over S_0 : finite maps are finite type and separated on affine charts, and the integral closed-image calculation of AG-RG-S04 Lemma P0.4 proves universal closedness after every base change. Properness is preserved by composition. We use this constructed finite-over-projective compactification, and require no assertion that it is itself projective.

Let p be in the image of the original base in S_0 . Its quotient fibre is proper: after an algebraically closed extension of a residue field of a point above p , it is the given proper geometric quotient. Properness descends under a field extension here directly. Finite type and separatedness are already known; for any base-change test and closed subset, its image becomes closed after the faithfully flat field cover. The images commute with that cover because nonempty field fibres remain nonempty under extension. The full fpqc quotient-topology proof of AG-RG-S04 Lemma A1.4 therefore makes the original image closed. This proves universal closedness on every test.

Localize S_0 at p and pass to its complete local Noetherian ring A , faithfully flat by the entire AG-RG-S02 Lemma 1.3. Write $X, \overline{X}, \text{Gr}$ for these base changes. Its proper special fibre X_s is open and closed in $\overline{X}_s$: it is open by the constructed immersion, and its map to the separated $\overline{X}_s$ is proper by the closed-graph/projection argument, so its image is closed. It corresponds to an idempotent. The full nilpotent idempotent-lifting proof AG-RG-S02 Lemma 1.4 lifts it uniquely through all maximal-ideal thickenings of $\overline{X}$; local lifts glue by uniqueness.

Here formal functions is applied to an actual projective scheme. Put $F = \pi_* \mathcal{O}_{\overline{X}}$ on Gr_A . This is coherent since π is finite and the target is Noetherian. On every affine target chart, the finite module quotient calculation gives

$$F/\mathfrak{m}_A^r F = \pi_{r*} \mathcal{O}_{\overline{X}_r}.$$

Finite pushforward also identifies their global sections. To apply projective-space finiteness precisely, take the Plucker closed immersion $i : \text{Gr}_A \hookrightarrow \mathbf{P}_A^M$ supplied by Lemma 5.A. The sheaf $i_* F$ is coherent: on an affine target chart it is the same finite module, now over the ambient ring through its quotient. By the definition of pushforward, its global sections are $H^0(\text{Gr}_A, F)$. Thus AG-RG-S01 Theorem 8.4, applied to $i_* F$ on $\mathbf{P}_A^M$, makes this module finite over A . The full completion and

formal-functions proofs AG-RG-S02 Lemma 1.2 and Theorem 2.3 then give the canonical ring isomorphism

$$\Gamma(\overline{X}, \mathcal{O}_{\overline{X}}) = H^0(\mathrm{Gr}_A, F) \simeq \varprojlim_r \Gamma(\overline{X}_r, \mathcal{O}_{\overline{X}_r}).$$

It is multiplicative because each comparison is the restriction map of the finite algebra. Thus the compatible formal idempotents give an actual idempotent and a proper open-and-closed component $Z \subset \overline{X}$ with $Z_s = X_s$.

The closed subset $Z \setminus X$ has empty special fibre. Its image in the local base is closed by properness; any nonempty closed subset of a local spectrum contains its closed point, so this subset is empty. Hence $Z \subset X$. The image of $Z \rightarrow \mathrm{Spec} A$ is open by the whole flat-openness proof for its smooth restriction from X , and it contains the closed point. Any such open is the entire local spectrum. In every geometric fibre Z is now a nonempty open-and-closed subset of the connected $X_{\bar{s}}$, and is all of that fibre. The open immersion $Z \subset X$ is therefore surjective and is an isomorphism. This proves that X is proper over A .

The monomorphism θ_A to the separated Grassmannian is proper by its closed graph and the proper source projection. It is quasi-finite, so the whole AG-RG-S04 Corollary D7.3 makes it finite. A finite monomorphism is closed: on an affine target chart $R' \rightarrow B$ its field fibre algebra has dimension e , and its monomorphism diagonal gives $e^2 = e$, so it is zero or the residue field. The coordinate map is therefore fibrewise surjective. Its finite cokernel is zero by the complete local Nakayama proof in AG-RG-S01 Lemma 1.2. Thus the ring map is surjective. Consequently θ_A is a closed immersion.

Closed-immersion descent AG-RG-S04 Corollary A3.2 gives the same assertion over $\mathcal{O}_{S_0, p}$. The whole eventual-property proof of AG-MO-03 Theorem 4.2 spreads it to an open neighbourhood of p : the local spectrum is the inverse limit of principal neighbourhoods and these are the given finite-presentation models. The union of these neighbourhoods contains the entire image of the original base. Pulling back proves that its orbit map is a closed Grassmannian immersion locally on S . This proves the original scheme and projectivity assertion; no properness condition has been imposed on unrelated model fibres.

The canonical Plücker line and its exponent

The full Grassmannian proof in Lemma 5.A makes its Plücker line the determinant of the universal quotient. Its pullback here is $\det \mathcal{O}(\mathcal{H}_n)$, where $\mathcal{H}$ is the universal conjugate subgroup. The entire finite-jet coordinate argument above identifies the augmentation filtration quotients with $\mathrm{Sym}^i((\mathrm{Lie} \mathcal{H})^*)$ for $0 \leq i \leq n$. These are finite locally free; their short exact sequences split locally because the quotient is projective. An adapted basis proves multiplicativity of the determinant in such a sequence: the matrix is block triangular, so its signed permutation determinant is the product of the two diagonal-block determinants. More generally the top exterior power of a composition is the composition of its top exterior powers; its coefficient in a wedge basis is the signed permutation determinant. This proves the determinant composition identity, including the conjugation calculation below, over every ring. Hence this line is

$$\bigotimes_{i=0}^n \det(\mathrm{Sym}^i((\mathrm{Lie} \mathcal{H})^*)).$$

We prove its exponent over every ring. For a rank- d free module the monomials of total degree i form a basis of its symmetric power, as follows directly from its presentation. Tensor words in a chosen basis form a free basis before the symmetric relations are imposed. Those relations exchange adjacent letters, so they identify exactly the words having the same multiplicity list. Within each such class, the coefficient-sum map has kernel generated by word differences: subtract one chosen representative from each other word. Its quotient is one free rank-one summand. The multiplicity classes consequently give precisely the asserted monomial basis over every ring. A diagonal matrix

with entries $t_1, \dots, t_d$ has determinant on that power equal to

$$\prod_{a_1 + \dots + a_d = i} t_1^{a_1} \dots t_d^{a_d} = (t_1 \dots t_d)^{c_i}, \quad c_i = \binom{i + d - 1}{d}.$$

All coordinate exponent sums are equal by permuting the coordinates; their total is $i \binom{i+d-1}{d-1}$, since the separator-position bijection counts the weak compositions. Division of this integer identity by d gives the displayed binomial coefficient. For $i = 0$ the exponent is zero.

This determinant identity holds for every matrix, including in positive characteristic. Both sides are universal integer polynomials in its entries. We prove the density calculation used to compare them. For the monic degree- d characteristic polynomial χ , the Sylvester determinant is the determinant of

$$(u, v) \mapsto u\chi + v\chi', \quad \deg u \leq d - 2, \quad \deg v \leq d - 1,$$

with target the polynomials of degree at most $2d - 2$. Both spaces have dimension $2d - 1$; for $d = 1$ the first summand is zero. Over a characteristic-zero field this determinant is nonzero exactly when χ and χ' are coprime. If they are coprime, their Bezout identity makes $\chi \mid v$, and the degree bound gives $v = u = 0$. If a nonconstant h divides both, the pair $(\chi'/h, -\chi/h)$ is a nonzero kernel vector within those bounds. These are the full earlier polynomial division and Bezout calculations of AG-RG-S04 Lemma P0.4a. In an algebraic closure the factored derivative at a root is the product of its differences from the other roots, so coprimality is exactly distinct eigenvalues. A diagonal matrix with distinct integer entries makes this determinant nonzero, proving it is a nonzero polynomial. On that open, the eigenbasis calculation is explicit. For each of the d distinct roots λ of χ , the matrix $X - \lambda I$ has determinant zero; the finite Gaussian reduction of AG-RG-S06 Section 1 gives a nonzero vector in its kernel. Eigenvectors for distinct roots are independent: applying $\prod_{\mu \neq \lambda} (X - \mu I)$ to a relation kills every other term and multiplies the chosen vector by the nonzero scalar $\prod_{\mu \neq \lambda} (\lambda - \mu)$. Thus all coefficients are zero. The d independent vectors are a basis by that same finite matrix calculation. Conjugation now reduces the determinant identity to the diagonal one. If its difference polynomial is g and this nonzero Sylvester polynomial is δ , then $g\delta$ vanishes at every field-valued matrix. The multivariable root bound follows by induction on the number of matrix entries: view the polynomial as a polynomial in the last entry; for every tuple of the other entries, the univariate root bound over the infinite characteristic-zero field makes each coefficient zero, and induction kills those coefficient polynomials. It makes $g\delta$ the zero polynomial; the polynomial domain and $\delta \neq 0$ give $g = 0$. Its integer coefficients are zero since $\mathbf{Z}$ injects into the characteristic-zero field. Specializing proves the universal identity over arbitrary rings. Applying it to the transition matrices gives the identity of line bundles

$$\det \mathcal{O}(\mathcal{H}_n) = (\det(\text{Lie } \mathcal{H})^*)^c, \quad c = \sum_{i=0}^n \binom{i + d - 1}{d} > 0.$$

The inequality follows from $n \geq 1$ and its $i = 1$ summand equal to one, for $d > 0$.

This positive power is relatively very ample through the actual closed Plücker immersion. The line itself is relatively ample, with the defining section criterion verified directly. Work over an affine base piece trivializing the finite Plücker bundle, so L^c is the pullback of $\mathcal{O}(1)$ from a closed immersion into $\mathbf{P}^N$. Its coordinate nonvanishing opens are affine. At any point and inside any chosen open neighbourhood, choose a principal open $D(f)$ in one such affine chart contained in that neighbourhood, by the full affine principal-basis proof of AG-RG-S01 Section 2. Express f as a polynomial in the coordinate ratios and homogenize to a polynomial h of degree a . Multiplying h by that chart coordinate gives a global section of $L^{c(a+1)}$ whose nonvanishing locus is exactly this affine principal open. These loci form a basis, and all their exponents are positive. This is the entire ample-section criterion proved in AG-RG-S01 Sections 8–9, so L is relatively ample. The zero-dimensional case was treated at the start. All finite-jet quotients, character actions,

determinants and quotient-sheaf identities commute with arbitrary base change. Their unique comparisons descend the canonical line and the stated quotient to the original base. $\square$

The quotient proof is the complete earlier AG-RG-S07 Theorem 8.2 and Corollary 8.3. Recognition and the finite compactification use AG-RG-S04 Theorems E8.1–E9.1 and D7.1. Formal functions is used on the projective Grassmannian with its coherent finite algebra, by AG-RG-S02 Theorem 2.3. The scheme and ample-line arguments are written here under the original hypotheses.

A universal finite cohomology complex

Lemma (finite projective replacement). Let A be Noetherian and let C be a complex of flat A -modules concentrated in degrees $[a, b]$, with finite cohomology modules. There is a complex K of finite projective modules in $[a, b]$ and a quasi-isomorphism $K \rightarrow C$ that remains a quasi-isomorphism after tensoring with any A -module.

Proof. Construct a bounded-above complex P of finite free modules and a map $f : P \rightarrow C$, descending from degree b . Once the terms above j are chosen, its cone has terms

$$D^i = C^i \oplus P^{i+1}, \quad d(c, p) = (d_C c + f(p), -d_P p).$$

Its cohomology is finite: the cone long exact sequence uses the finite cohomology of C and of the already chosen finite free portion of P . Choose finitely many cocycles (c_ℓ, p_ℓ) generating $H^j(D)$. Take P^j free on e_ℓ , and set $f(e_\ell) = c_\ell$, $d_P(e_\ell) = -p_\ell$. The cocycle equations give $d_P^2 = 0$ and $d_C f = f d_P$. In the new cone, $d(0, e_\ell) = (c_\ell, p_\ell)$, so this kills H^j without changing higher cohomology. Continue to the left. Each degree stabilizes, giving a quasi-isomorphism $P \rightarrow C$.

Its cone is acyclic, bounded above and flat termwise. Such a complex stays acyclic after every tensor: start at its last degree and descend through

$$0 \longrightarrow Z^i \longrightarrow D^i \longrightarrow Z^{i+1} \longrightarrow 0.$$

The quotient is flat; hence the kernel is flat and the sequence stays exact under tensor. Thus $H^i(P \otimes_A M) = H^i(C \otimes_A M)$ for every M .

Put $Q = \text{coker}(P^{a-1} \rightarrow P^a)$. The left tail is a free resolution of Q , and

$$\text{Tor}_1^A(Q, M) = H^{a-1}(P \otimes_A M) = 0$$

for every M , since $C \otimes_A M$ starts in degree a . The module Q is finitely presented, so it is flat and finite projective. Take $K^a = Q$ and $K^i = P^i$ for $a < i \leq b$. The map to C factors through Q because $C^{a-1} = 0$. Truncation preserves cohomology, and the same acyclic flat-cone argument proves the assertion after every tensor. $\square$

Apply this to the finite section complex of a line bundle L_0 on a smooth projective scheme over a Noetherian ring A_0 . Choose a finite affine cover. Separatedness makes its intersections affine, and S01 Sections 4–5 prove that its ordered section complex computes quasi-coherent cohomology. On each intersection the line bundle is a projective module over its chart algebra; smoothness makes that algebra A_0 -flat, so every term is A_0 -flat. S01 Theorem 8.4 and its closed-immersion passage prove that all cohomology modules are finite. Under every algebra base change, affine module-sheaf correspondence and localization, proved in S01 Section 2, identify the new section complex with its tensor product over A_0 . The finite projective replacement just proved therefore computes the line bundle cohomology after every base change.

Sections in a vanishing family

Lemma (sections in a vanishing family). Let $f : X \rightarrow S$ be smooth and projective of finite presentation, and let L be an invertible sheaf on X . Suppose that $H^q(X_{\bar{s}}, L_{\bar{s}}) = 0$ for every geometric point $\bar{s}$ and every $q > 0$. Then f_*L is finite locally free, all its positive higher direct images vanish, and these statements commute with every base change. In particular the canonical map $g^*f_*L \rightarrow f'_*L'$ is an isomorphism for any $g : S' \rightarrow S$. The base may be nonreduced.

Proof. Work with $S = \operatorname{Spec} A$. Choose a projective embedding, a finite affine cover trivializing L , and finite affine covers of the pairwise and triple overlaps. Write A as the filtered union of its finitely generated $\mathbf{Z}$ -subalgebras. The embedding, chart opens and overlap maps descend at a finite stage by [finite-presentation descent](#), Theorems 4.1–4.2. Retain the closed immersion, so the model X_0 is projective.

Retain smoothness at a finite stage by S02 Lemma 3.1 and Proposition 3.2. In a finite polynomial presentation, Lemma 3.0 identifies formal smoothness with a left inverse of the conormal differential map. The left inverse uses finitely many coefficients; its equations on the finitely many relation generators have finite polynomial witnesses. Descend these coefficients and witnesses to one stage. The same left inverse then proves formal smoothness there, and the retained finite presentation proves smoothness. Apply this on a finite affine cover, retaining the cover and the projective closed immersion by the finite-presentation limit proof. The line-bundle transition units, their inverses and the finitely many overlap and cocycle witnesses descend at the same stage, as in S02 Proposition 3.2. Gluing these descended transition units defines the invertible sheaf L_0 on the smooth projective model X_0/A_0 .

The preceding finite-projective-replacement lemma gives a bounded finite projective complex K_0 such that

$$H^q(K_0 \otimes_{A_0} B) \simeq H^q(X_0 \times_{A_0} \operatorname{Spec} B, L_0 \otimes_{A_0} B)$$

for every A_0 -algebra B , compatibly with the canonical comparison maps. Its proof uses the finite affine-cover complex and projective Serre finiteness; ordinary tensor of K_0 computes its derived tensor. This finite argument supplies the universal comparison before pullback to the original base.

Put $K = K_0 \otimes_{A_0} A$. Near any $s \in \operatorname{Spec} A$, trivialize its terms. Whenever a differential has an entry invertible at s , shrink the neighbourhood and perform row and column operations to put that entry in an identity block. The equation $d^2 = 0$ makes the adjacent differentials zero on that block, so it splits off a contractible pair $A \xrightarrow{1} A$. After finitely many cancellations, all remaining differential entries vanish in $\kappa(s)$. The remaining complex therefore has

$$K^q \otimes_A \kappa(s) = H^q(X_s, L_s).$$

For $q > 0$ the right side is zero by the hypothesis and field extension; for $q < 0$ it is zero because L_s is a sheaf. Thus every remaining free term except degree zero has rank zero and disappears on a neighbourhood of s . The degree-zero term is finite free. Every cancelled identity pair stays contractible after any tensor. The universal comparison consequently proves the direct-image assertions and every base-change assertion, including nonflat changes involving nilpotents. The canonical comparisons glue. Notice that vanishing was imposed only on the original family, not on unrelated fibres of its Noetherian model. $\square$

Lemma 5.B (the automorphism sheaf of a projective line). Over every base scheme T , the sheaf of T -automorphisms of $\mathbf{P}_T^1$ is represented by $\operatorname{PGL}_{2,T}$. This identification holds on all schemes over T and includes nonreduced tests.

Proof. Work locally on a test base, and write $p : \mathbf{P}_T^1 \rightarrow T$. For an automorphism a , put $M' = a^*\mathcal{O}(1) \otimes \mathcal{O}(-1)$. On every geometric fibre this line is trivial. Indeed $\mathcal{O}(1)$ is the divisor

line of any rational point: a linear homogeneous form cuts that point simply, and the two affine coordinate frames identify its divisor line with $\mathcal{O}(1)$. Pullback by an automorphism cuts the inverse-image point simply and gives the same line. The whole sections-in-a-vanishing-family lemma applies to M' , since the explicit projective-line cohomology of AG-RG-S01 Theorem 7.2 gives $H^q(\mathcal{O}) = 0$ for $q > 0$ and $H^0(\mathcal{O})$ of rank one. Hence $M = p_*M'$ is a line bundle with every base change. The evaluation $p^*M \rightarrow M'$ is an isomorphism: on every geometric fibre it is the constant-section isomorphism, and on local line-bundle frames its coefficient is therefore outside every maximal ideal of the local ring, hence a unit. Thus $a^*\mathcal{O}(1) = \mathcal{O}(1) \otimes p^*M$. Trivialize M locally. Pullback of the two coordinate sections, together with this trivialization, gives a linear map on the rank-two module of sections of $\mathcal{O}(1)$, whose standard linear-form basis is again the explicit AG-RG-S01 Theorem 7.2 calculation. Its geometric-fibre maps are invertible because a is an automorphism. Its determinant is a unit on the base, by the same local residue test, so the matrix belongs to GL_2 . The quotient-functor construction of projective space identifies the corresponding projective transformation with a : its coordinate quotient sections are precisely these pulled-back coordinate sections. A different trivialization of M multiplies the matrix by a unit. The kernel of the projective action consists exactly of scalar matrices on all tests. A matrix acting identically preserves the points $[1 : 0]$ and $[0 : 1]$, so its off-diagonal entries are zero and its diagonal entries are units; preserving $[1 : 1]$ makes the two entries equal. Conversely scalar matrices act identically. Finally the fppf quotient $\mathrm{GL}_2/\mathbf{G}_m$ is the actual scheme open $D_+(\det) \subset \mathbf{P}_T^3$: on each projective coordinate chart a line of four entries is locally a matrix, and its determinant section being nowhere vanishing means an invertible determinant after this frame is chosen. Changing the frame is exactly scalar multiplication. These local coordinates, their unique scalar comparisons and line-bundle frame gluing represent the quotient sheaf on every test, by the full local-coordinate sheaf construction of AG-RG-S04 Lemma A1.7. Matrix multiplication and inversion descend and give the group scheme PGL_2 . The local matrix descriptions just proved therefore identify its functor with the automorphism sheaf, compatibly with composition and every base change. $\square$

Proposition 5.2. If G has trivial centre and geometric semisimple rank one, and has a split maximal torus T , then Zariski locally (G, T) is isomorphic to (PGL_2, D) .

Proof. Its geometric fibres are PGL_2 by Proposition 2.3: the action on the projective line has trivial kernel. Choose a root α locally. It is an isomorphism $T \rightarrow \mathbf{G}_m$, with the following full base-ring check. Locally identify the rank-one split torus with $\mathbf{G}_m$. A character is a finite Laurent polynomial $\sum_n a_n t^n$. Its coproduct and identity equations give $a_n^2 = a_n$, $a_m a_n = 0$ for $m \neq n$, and $\sum_n a_n = 1$, by comparing the independent Laurent coefficients. Thus its finitely many nonzero coefficient idempotents split the affine base into open-and-closed pieces on which it is t^n . Its geometric-fibre character is primitive, so on every nonempty piece $n = 1$ or $n = -1$. The inverse there is respectively t or t^{-1} , and these inverses glue over the idempotent pieces. Put $B = P_G(\alpha^{-1})$. It is smooth and on each geometric fibre is an upper triangular Borel subgroup of PGL_2 .

Its scheme-theoretic normalizer equals B . The normalizer lemma makes it closed of finite presentation and proves that its formation commutes with arbitrary base change. On geometric fibres the point stabilizer in $\mathbf{P}^1$ is self-normalizing. Infinitesimally, use the diagonal class of $\mathrm{diag}(1, 0)$ in $\mathfrak{pgl}_2$. Its bracket with the lower off-diagonal class is the negative of that class, nonzero in every characteristic. Thus the normalizer has exactly the two-dimensional tangent space of B at the identity. Its local dimension is at least two because it contains the smooth closed B , and is at most the dimension of that tangent space by the embedding-dimension bound. It is therefore a regular local ring of dimension two. The rational-point cotangent calculation and field Jacobian criterion of AG-CA-18 Proposition 3.1 and Theorem 2.1 make it smooth at the identity. Every closed point of the geometric normalizer is in B : a normalizing element must preserve the unique fixed point of the upper triangular group on $\mathbf{P}^1$, as is checked using its translations on the other affine chart. Translation carries the smooth identity neighbourhood through every such point. The open smooth locus consequently covers the fibre, since any nonempty closed complement in a finite-type scheme over an algebraically closed field contains a closed point by the full Zariski lemma.

The closed inclusion $B \rightarrow N$ is now a map between smooth groups with identical tangent spaces, so AG-RG-S02 Lemma 3.A makes it étale. The full étale-monomorphism proof of AG-RG-S04 Lemma E7.2 makes it an open immersion; all geometric points belong to it, so it is an isomorphism. This proves equality on the entire geometric fibres, including nilpotent tests. The flat-source criterion above, applied with source B , proves the equality over the base without assuming the normalizer flat.

Lemma 5.1 makes $f : X = G/B \rightarrow S$ a smooth projective curve whose geometric fibres are $\mathbf{P}^1$, and it has the section $e = 1B$. A section of a separated smooth relative curve is an effective Cartier divisor, with the following local calculation. Near the section over a chosen base point, take an étale coordinate q by AG-RG-S02 Lemma 3.A and subtract its pullback along the section, so it vanishes there. It is a nonzerodivisor: multiplication by the affine-line coordinate is injective over every base ring, and the étale coordinate algebra is flat. The zero scheme $Z(q)$ is étale over the base and the section factors through it. The base and $Z(q)$ are both smooth of relative dimension zero; AG-RG-S02 Lemma 3.A therefore makes the section étale. It is a monomorphism because it is a section, so the full étale-monomorphism proof makes it an open immersion. It is also closed because the curve is separated. Its complement in $Z(q)$ is consequently closed also in the ambient coordinate neighbourhood. Remove that complement; on the resulting neighbourhood the section is cut out exactly by q . Its ideal is thus locally free of rank one. Its reciprocal defines the divisor line bundle, and on each geometric curve fibre this coordinate cuts its point simply, giving degree one. Put $L = \mathcal{O}_X(e)$; it has degree one on every geometric fibre. The sections-in-a-vanishing-family lemma applies to L , since $H^1(\mathbf{P}^1, \mathcal{O}(1)) = 0$, and gives $E = f_*L$ locally free of rank two, with arbitrary base change. The evaluation $f^*E \rightarrow L$ is surjective: its cokernel is finite locally, its geometric fibres vanish by generation of $\mathcal{O}(1)$, and local Nakayama kills that cokernel. With the quotient convention it defines $j : X \rightarrow \mathbf{P}(E)$, and $j_{\bar{s}}$ is the usual isomorphism given by the two sections of $\mathcal{O}(1)$. Both schemes are smooth over S , and the differential of j is an isomorphism on all geometric fibres, hence locally an isomorphism of their cotangent bundles. The étale differential criterion makes j étale. Its geometric fibres are isomorphisms, so it is universally injective and surjective; an étale universally injective morphism is an open immersion, and surjectivity makes it an isomorphism. Trivializing E Zariski locally now gives $G/B \simeq \mathbf{P}_S^1$.

Lemma 5.B identifies the automorphism sheaf of this projective line with the actual group scheme PGL_2 , so the action gives a morphism $G \rightarrow \mathrm{PGL}_2$ on all base tests. It is an isomorphism on every geometric fibre. Both groups are smooth of finite presentation; their differential is an isomorphism on those fibres, hence locally an isomorphism of differential bundles by the finite matrix and Nakayama calculations. AG-RG-S02 Lemma 3.A makes the morphism étale. Its geometric fibres are single reduced points, so it is universally injective and surjective. AG-RG-S04 Lemma E7.2 then makes it a surjective open immersion, hence an isomorphism. It carries B to the point stabilizer. Here is the full Zariski-local pair calculation. In that stabilizer, every projective matrix has a unique scalar-normalized representative $\begin{pmatrix} v & u \\ 0 & 1 \end{pmatrix}$, with v invertible; this gives its group law $(v, u)(v', u') = (vv', u + vu')$. The diagonal character on T is an isomorphism to $\mathbf{G}_m$, since its geometric-fibre character is the primitive root already identified above. Use its value t as the torus coordinate. The homomorphism $T \rightarrow B$ has the form $\begin{pmatrix} t & u(t) \\ 0 & 1 \end{pmatrix}$ for a finite Laurent polynomial $u(t)$. Commutativity gives $(1 - s)u(t) = (1 - t)u(s)$ in $R[t^{\pm 1}, s^{\pm 1}]$. For each exponent $m \notin \{0, 1\}$, its coefficient in t says $(1 - s)a_m = 0$; independence of Laurent coefficients gives $a_m = 0$. The coefficient at t then gives $u(s) = a_1(s - 1)$. Conjugation by $\begin{pmatrix} 1 & a_1 \\ 0 & 1 \end{pmatrix}$ replaces $u(t)$ by $u(t) + a_1(1 - t) = 0$. Thus it takes T to the diagonal torus over the affine base itself. This calculation uses no division or reducedness and finishes the Zariski-local pair assertion. $\square$

6. Splitting a central extension of the rank-one cover

We need an integral result, including characteristic two. The following splitting argument is due to Gabber; we give the argument for a separated commutative group scheme Z , which includes the multiplicative-type centres used here.

Theorem 6.1. Every fppf central extension

$$1 \longrightarrow Z \longrightarrow E \longrightarrow \mathrm{SL}_{2,S} \longrightarrow 1$$

has a unique homomorphic section.

Proof. We first record the full scheme calculation behind the cell covers. The group SL_2 is cut out by $ad - bc = 1$. The opens $a \neq 0$ and $b \neq 0$ cover it. On the first, elimination of d gives the algebra $R[a, a^{-1}, b, c]$; on the second, elimination of c gives $R[b, b^{-1}, a, d]$. Both are polynomial localizations, so are smooth over every ring by the square-zero lifting and finite-presentation proofs. Over every field both charts are integral and their intersection is nonempty, containing $(\begin{smallmatrix} 1 & \\ 0 & 1 \end{smallmatrix})$. Their union is therefore reduced and irreducible, hence integral. The identity cell is the first chart: the matrix factorization $y(v)h(t)x(u)$ gives $a = t$, $b = tu$, $c = vt$ and $d = vt u + t^{-1}$, uniquely. Thus it is an open dense cell in every geometric fibre. In SL_2 use

$$x(u) = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \quad y(v) = \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix}, \quad h(t) = \mathrm{diag}(t, t^{-1}).$$

Let D', U', V' be their inverse images in E . A central extension of a commutative group has a commutator pairing on its quotient, on all tests. Indeed any two lifts differ by a central-kernel element, so replacing a lift leaves the commutator unchanged. Local lifts consequently glue the commutator through the represented fppf sheaf Z . Every such commutator lies in Z , hence is central; the identities $[ab, c] = a[b, c]a^{-1}[a, c]$ and $[a, bc] = [a, b]b[a, c]b^{-1}$ therefore make the pairing additive in each variable. This constructs the pairing as a natural map of sheaves, hence as a scheme morphism by its values on every test.

For D' , the pairing $\mathbf{G}_m \times \mathbf{G}_m \rightarrow Z$ is trivial. The n th-power map of $\mathbf{G}_m$, for every $n \geq 1$, is an fppf epimorphism over every ring: above a unit $u \in R^*$ its root algebra is $R[z]/(z^n - u)$, with free basis $1, z, \dots, z^{n-1}$. The element z is already invertible, with inverse z^{n-1}/u . This free positive-rank algebra is faithfully flat, and its monic equation is a finite presentation. Restrict the first pairing argument to μ_n . Bilinearity makes the pairing there killed by n in that argument. Locally write the second argument as an n th power using this cover, and bilinearity makes the pairing the identity. Faithful sheaf descent removes the cover. The union of the μ_n is universally schematically dense in $\mathbf{G}_m$, with the following actual ideal test. On an affine test base, write a Laurent polynomial as a finite sum with coefficients in the base ring (or in the Laurent ring of the second coordinate). Choose n greater than every difference of two exponents present. Reduction modulo $t^n - 1$ then leaves those exponents in distinct residue classes, whose images are independent in the free basis $1, t, \dots, t^{n-1}$. A polynomial vanishing on every μ_n therefore has all coefficients zero. Since Z is separated, equality of the pairing with the identity is a closed subscheme of the product; this ideal test kills its entire ideal on every affine base chart. Thus the pairing is the identity and D' is commutative.

The conjugation action on E factors through SL_2 : two local lifts differ by Z , which is central, so give identical conjugation maps; their maps glue by the sheaf property. The commutator pairing on U' is invariant under simultaneous multiplication of its arguments by t^2 . The square-power cover just proved makes it invariant under simultaneous multiplication by every unit. For given u, v over any test, the morphism $a \mapsto c(au, av)$ from $\mathbf{A}^1$ to Z is constant on $\mathbf{G}_m$. This equality extends to zero on nonreduced tests too. Its equalizer with the constant morphism is closed by separatedness; on an affine test chart the equalizer ideal becomes zero in $R[a, a^{-1}]$. The map $R[a] \rightarrow R[a, a^{-1}]$ is

injective because multiplication by a is injective on the polynomial basis, so that ideal was zero. At zero the pairing is the identity, and at one it is $c(u, v)$. Hence $c(u, v) = 1$. Thus U' is commutative; the same argument proves it for V' .

We construct a unique D -equivariant section of $U' \rightarrow U$. Fppf locally choose a unit t with $t^2 - 1$ invertible and a lift $\tilde{h}$ of $h(t)$. The first choice is the actual finite-presentation cover $R[t, 1/(t(t^2 - 1))]$: it is a smooth polynomial localization and its geometric fibres are nonempty. An algebraically closed field is infinite, since a finite field's polynomial $\prod_a (T - a) + 1$ would have no root; the nonzero degree-three polynomial $t(t^2 - 1)$ therefore cannot vanish at all its elements by the univariate root bound. Smoothness gives flatness by the earlier arbitrary-base smooth-flatness proof. Thus this open is surjective, flat and finitely presented. The extension's fppf epimorphism supplies the second local choice. In the commutative group U' , the endomorphism

$$\varphi(a) = \tilde{h}a\tilde{h}^{-1}a^{-1}$$

kills Z and induces multiplication by $t^2 - 1$ on $U = \mathbf{G}_a$. The factorization through U is explicit: locally lift a section of U to U' and apply φ . Changing the lift multiplies by Z and leaves its value unchanged, so these values glue uniquely. This defines a homomorphism of represented sheaves $U \rightarrow U'$. Dividing its input by the unit $t^2 - 1$ gives a section. It is D -equivariant because all lifts in the commutative D' commute with $\tilde{h}$. Any two such equivariant sections differ by a D -invariant homomorphism $U \rightarrow Z$. Square-power surjectivity gives scalar invariance under every unit, and the same injective-polynomial equalizer argument used above extends that invariance to the zero scalar, where the homomorphism is the identity. The difference is consequently zero. This uniqueness descends the local sections, giving $x' : \mathbf{G}_a \rightarrow E$ and similarly y' .

Set

$$w'(t) = y'(-t^{-1})x'(t)y'(-t^{-1}), \quad h'(t) = w'(t)w'(1)^{-1}.$$

Their images are $\begin{pmatrix} 0 & t \\ -t^{-1} & 0 \end{pmatrix}$ and $h(t)$. Thus $h'(t) \in D'$. Equivariance gives

$$h'(s)w'(t)h'(s)^{-1} = w'(s^2t).$$

Divide by the instance $t = 1$. Since D' is commutative, this gives $h'(t) = h'(s^2t)h'(s^2)^{-1}$. The square map being fppf surjective, h' is a homomorphism. Its definition then gives $h'(s)w'(t) = w'(st)$.

Expand the identity $h'(s)w'(t) = w'(st)$ and move $h'(s)$ across the first y' using $h'(s)y'(v)h'(s)^{-1} = y'(s^{-2}v)$. Multiply on the left by $y'(1/(st))$ and on the right by $y'(1/t)$. Additivity of y' gives the actual intermediate equality $x'(st)y'((s-1)/(st)) = y'((s-1)/(s^2t))x'(s^2t)h'(s)$. Move the final $h'(s)$ to the left of this x' using its conjugation identity, obtaining $x'(st)y'((s-1)/(st)) = y'((s-1)/(s^2t))h'(s)x'(t)$. Set $u = st$ and $v = (s-1)/(st)$. Conversely every pair for which u and $1 + uv$ are units has this form with $s = 1 + uv$ and $t = u/s$. Thus we have

$$x'(u)y'(v) = y'\left(\frac{v}{1+uv}\right)h'(1+uv)x'\left(\frac{u}{1+uv}\right) \quad (1)$$

when u and $1 + uv$ are units. The restriction that u be a unit can be removed on all tests. Fppf locally write $u = a + b$ with $a, b, 1 + bv$ units, by adjoining b on the open $b(u - b)(1 + bv) \neq 0$. This is a smooth polynomial localization. In every geometric fibre its defining polynomial is nonzero (of degree two when $v = 0$, and degree three otherwise), so the infinite-field root bound makes its image surjective; the same smooth-flatness argument makes it an fppf cover. Apply (1) first to $x'(b)y'(v)$, then to $x'(a)y'(v/(1 + bv))$. Write $q = 1 + bv$ and $p = 1 + uv$. The second denominator is p/q , so its result is $y'(v/p)h'(p/q)x'(aq/p)h'(q)x'(b/q)$. The rule $x'(c)h'(q) = h'(q)x'(q^{-2}c)$ and multiplicativity of h' combine the two torus factors into $h'(p)$. The resulting x' parameter is

$$\frac{a}{(1+bv)(1+uv)} + \frac{b}{1+bv} = \frac{u}{1+uv}.$$

This proves (1) in its full domain.

These identities, the additivity of x', y' , the multiplicativity of h' , and its two conjugation identities are exactly the multiplication rules on the open cell $\Omega = U_-DU_+$ of SL_2 . Define a section there by

$$y(v)h(t)x(u) \longmapsto y'(v)h'(t)x'(u).$$

It is multiplicative whenever both factors and their product are in Ω : reorder their factors with (1). Write $s_\Omega : \Omega \rightarrow E$ for this cell section. For any test scheme $R \rightarrow S$ and finite list $g_1, \dots, g_m \in \mathrm{SL}_2(R)$, the open

$$\Omega_R \cap \bigcap_i \Omega_R g_i^{-1}$$

is nonempty in every geometric fibre: the two-chart calculation above makes that fibre integral, each translate of the identity cell is a nonempty dense open, and irreducibility makes every finite intersection of these opens nonempty. The intersection is a smooth open of SL_2 and is surjective over R . On an affine test base, the cell translates are principal opens, their finitely many defining functions giving a polynomial localization of the finite-presentation group algebra. Thus it is locally finitely presented; smoothness gives flatness, and affine refinements supply an actual fppf local point c with $c, cg_i \in \Omega$.

In particular, for one g , use the nonempty open

$$\Omega_R \cap \Omega_R^{-1} \cap \Omega_R g^{-1}.$$

Then $g = c^{-1}(cg)$ is a product of two points of Ω . Assign to a word in cell points the corresponding product of their s_Ω -values. This is independent of the word. For two words with the same endpoint g , choose c so that c and its translates by every partial product of both words lie in Ω . Successive cell multiplicativity says that multiplying either lifted word on the left by $s_\Omega(c)$ gives $s_\Omega(cg)$. Cancellation proves equality. In particular a word representing the identity has lift one; concatenation therefore respects multiplication and inverses. These word values agree after further fppf pullbacks, so descend through the sheaf property of E . They define a homomorphism of represented fppf sheaves, hence a scheme homomorphism $\mathrm{SL}_2 \rightarrow E$. Its composite with $E \rightarrow \mathrm{SL}_2$ is the identity on generating words. It is the required homomorphic section on all test schemes.

Finally any homomorphism $\mathrm{SL}_2 \rightarrow Z$ kills $x(u)$, since

$$[h(t), x(u)] = x((t^2 - 1)u),$$

and one may choose $t^2 - 1$ invertible fppf locally. It kills $y(v)$ similarly, and then $h(t) = w(t)w(1)^{-1}$. It kills the open cell, hence the group. The difference between two central sections is such a homomorphism, proving uniqueness. $\square$

7. Coroots from multiplication

Theorem 7.1. For every root α there is a canonical coroot $\alpha^\vee : \mathbf{G}_m \rightarrow T$, with $\langle \alpha, \alpha^\vee \rangle = 2$. There is a canonical perfect pairing $\mathfrak{g}_\alpha \otimes \mathfrak{g}_{-\alpha} \rightarrow \mathcal{O}_S$ such that, writing its value as XY ,

$$\exp_\alpha(X) \exp_{-\alpha}(Y) = \exp_{-\alpha}\left(\frac{Y}{1+XY}\right) \alpha^\vee(1+XY) \exp_\alpha\left(\frac{X}{1+XY}\right) \quad (2)$$

whenever $1+XY$ is a unit. This unit condition is precisely membership in the open cell $U_{-\alpha}TU_\alpha$.

Proof. Work in G_α , whose scheme-theoretic centre is $Z = \ker \alpha \subset T$, by Section 1. Theorem 1.1 constructs $q : G_\alpha \rightarrow Q = G_\alpha/Z$ as an affine reductive quotient and an fppf Z -torsor, compatible with every base change. The torus quotient has character group $\mathbf{Z}\alpha \subset X^*(T)$, so $T/Z \simeq \mathbf{G}_m$ via the descended character α . It is a closed subgroup of Q : its pullback under q is $T \subset G_\alpha$, and closed immersions descend through the fppf torsor. On geometric fibres Proposition 2.3 identifies this quotient with PGL_2 , so T/Z is a maximal torus.

The open cell $U_{-\alpha}TU_\alpha$ is right Z -stable; centrality makes Z act only on its torus coordinate. Its quotient is the open subscheme

$$U_{-\alpha} \times (T/Z) \times U_\alpha \subset Q,$$

as follows by invariant-open descent and the product torsor. At the identity this gives the two nonzero torus weights $\alpha, -\alpha$, with their root lines, in $\mathrm{Lie} Q$. The centre formula of Section 1, applied to $Q, T/Z$, is consequently $Z(Q) = \ker(\alpha : T/Z \rightarrow \mathbf{G}_m) = 1$, as a scheme over the original base. Proposition 5.2 therefore identifies the pair $Q, T/Z$, Zariski locally, with PGL_2 and its diagonal torus. Choose the identification so $\alpha(\mathrm{diag}(t, 1)) = t$.

The map $\mathrm{SL}_2 \rightarrow \mathrm{PGL}_2$ used next is an fppf epimorphism over the original base, with the following integral proof. Lemma 5.B represents PGL_2 by the invertible-matrix open in $\mathbf{P}^3$, so a section has a matrix representative M after locally trivializing its coordinate line. Its determinant d is a unit. The finite-free square-root cover $R[z]/(z^2 - d)$ from Theorem 6.1 has an invertible z ; the matrix $z^{-1}M$ has determinant one and lifts the section. The kernel consists on every test of scalar matrices zI with $z^2 = 1$, hence is μ_2 , even in characteristic two. On the upper and lower additive matrix groups the map is the identity on their parameters. Pull back $G_\alpha \rightarrow \mathrm{PGL}_2$ along this map. Theorem 6.1 splits the resulting central extension by Z , giving a homomorphism $\rho : \mathrm{SL}_2 \rightarrow G_\alpha$. We verify its full T -equivariance. For any test point $t \in T$, put $a = \alpha(t)$. Conjugation of matrices by $\mathrm{diag}(a, 1)$ defines an automorphism c_a of SL_2 , scaling the upper parameter by a and the lower by a^{-1} . Its adjoint-quotient action equals conjugation by $q(t)$. Thus $g \mapsto t\rho(c_a^{-1}(g))t^{-1}$ is another homomorphic lift of the same map $\mathrm{SL}_2 \rightarrow Q$. Together with its SL_2 coordinate it is a section of the same pullback extension; uniqueness in Theorem 6.1 makes it ρ . This proves equivariance on every test. The quotient-cell coordinates above make dq an isomorphism on both root lines, and the matrix cover has the same property. Consequently $d\rho$ is an isomorphism on those lines. Apply Theorem 4.1 to the upper map after identifying its tangent basis with $d\rho(e)$, and likewise to the lower map: their images are exactly $U_\alpha, U_{-\alpha}$ and they identify the respective additive lines. The diagonal cocharacter of SL_2 therefore defines $\alpha^\vee$.

Direct multiplication gives

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ y/(1+xy) & 1 \end{pmatrix} \begin{pmatrix} 1+xy & 0 \\ 0 & (1+xy)^{-1} \end{pmatrix} \begin{pmatrix} 1 & x/(1+xy) \\ 0 & 1 \end{pmatrix}.$$

The quotient cell is saturated: its pullback along the Z -torsor is exactly $U_{-\alpha}TU_\alpha$, since on that open the torsor acts only in the torus factor and the product torsor gives its whole inverse image. In the projective invertible-matrix model the cell is precisely the locus where the upper-left coordinate is invertible in a local coordinate-line frame. Indeed the lower-diagonal-upper matrix factorization determines its parameters uniquely; changing the frame multiplies that coordinate by a unit. For the displayed product that coordinate is $1+xy$. Thus the product lies in the cell exactly when $1+xy$ is a unit, on all tests. Transporting the matrix identity through ρ gives (2). The chosen root-line tangent bases have pairing one, which supplies a perfect pairing, and the diagonal image gives $\alpha \circ \alpha^\vee(t) = t^2$.

The pairing and coroot are unique, including over nonreduced bases. Work in paired root-line frames and let an alternative perfect pairing have coefficient $c \in R^*$. Its coroot still has adjoint image $\mathrm{diag}(t^2, 1)$, since its pairing with α is two. Set $Y = F$ and $X = xE$, and compare the upper-root coordinates in (2) after passing to the adjoint quotient. On $R[x, 1/((1+x)(1+cx))]$ the two coordinates are $x/(1+x)$ and $x/(1+cx)$. Equality gives $(c-1)x^2 = 0$. The factors $1+x$

and $1 + cx$ are monic up to an invertible leading coefficient, so multiplication by their product is injective on $R[x]$; the localization is injective. Independence of the polynomial basis therefore gives $c = 1$. With the pairing fixed, uniqueness of the three actual coordinates in the open cell of G_α fixes the actual torus value $\alpha^\vee(1 + x)$, not only its adjoint image. On any test ring every unit u occurs by taking $x = u - 1$ and $Y = F$. Thus the coroot itself is fixed on all tests. This also rules out a possible difference by a cocharacter into Z . The constructions on different local identifications consequently agree, so the morphisms and pairing glue; the same characterization proves arbitrary base-change compatibility. $\square$

For SL_2 , use $T \simeq \mathbf{G}_m$ via $\mathrm{diag}(t, t^{-1})$. Its root is $\alpha(t) = t^2$, and $\alpha^\vee(t) = \mathrm{diag}(t, t^{-1})$. For PGL_2 , parameterize its diagonal torus by the class of $\mathrm{diag}(t, 1)$. Its root is $t \mapsto t$, and its coroot is the class of $\mathrm{diag}(t^2, 1)$. In both cases the pairing is two, but the root and coroot occupy different sublattices.

For GL_2 , the roots are $e_1 - e_2$ and its negative, and the positive coroot is $t \mapsto \mathrm{diag}(t, t^{-1})$. These explicit formulas remain valid over $\mathbb{Z}$.

8. The three rank-one groups

A lattice calculation with pairing two

Lemma 8.A. Let X be a finite free abelian group of rank $r + 1$. Let $a \in X$ and $b \in X^* = \mathrm{Hom}(X, \mathbf{Z})$ be nonzero and satisfy $b(a) = 2$. Up to an isomorphism of X preserving this pair, exactly one of the following descriptions occurs:

1. $X = \mathbf{Z}e_0 \oplus \mathbf{Z}^r$, $a = 2e_0$, $b = e_0^*$.
2. $X = \mathbf{Z}e_0 \oplus \mathbf{Z}^r$, $a = e_0$, $b = 2e_0^*$.
3. $r \geq 1$, $X = \mathbf{Z}e_1 \oplus \mathbf{Z}e_2 \oplus \mathbf{Z}^{r-1}$, $a = e_1 - e_2$, $b = e_1^* - e_2^*$.

In these formulas b is zero on the displayed extra summand.

Proof. Choose a basis of X . For a nonzero vector the positive greatest common divisor of its coordinates is its divisibility; division by it gives a primitive vector. The same calculation applies to the coordinate list of a nonzero integral functional. Integer division, Bezout and the Smith basis operations are proved in S02 Lemma 3.M. In particular a primitive vector extends to an integral basis: the Euclidean row operations reduce its coordinate list to $(1, 0, \dots, 0)$ and their inverses give that basis. A kernel of an integral functional is finite and torsion-free, hence free by that lemma. Write $a = ma_0$ and $b = nb_0$, with a_0, b_0 primitive and $m, n > 0$. Then

$$mn b_0(a_0) = 2.$$

Consequently (m, n) is $(2, 1)$, $(1, 2)$, or $(1, 1)$.

In the first case $b(a_0) = 1$. Every $x \in X$ has the unique expression

$$x = b(x)a_0 + (x - b(x)a_0), \quad x - b(x)a_0 \in \ker b.$$

Thus $X = \mathbf{Z}a_0 \oplus \ker b$; choose a basis of the free kernel. This gives case 1. In the second case apply the same expression using $b_0(a) = 1$. It gives case 2.

Suppose now that both a and b are primitive. Extend a to a basis $a, c_1, \dots, c_r$. At least one $b(c_i)$ is odd: otherwise every value of b , including $b(a) = 2$, would be even, contradicting primitivity of b . Relabel so that $b(c_1) = 2q + 1$, and replace c_1 by $c_1 - qa$. This is a basis operation and makes its value under b equal to one. Call the new vector c . Replace each remaining c_i by $c_i - b(c_i)c$; their values become zero. Finally set

$$e_1 = c, \quad e_2 = c - a.$$

These replace the basis pair a, c by another unimodular basis pair. They satisfy $a = e_1 - e_2$, $b(e_1) = 1$ and $b(e_2) = -1$, giving case 3. If $r = 0$, both primitive elements would pair to ± 1 , so this case cannot occur.

The divisibilities (m, n) of a and b are invariant under lattice isomorphisms. They distinguish all three cases. $\square$

Extending a map from the rank-one cell

Lemma 8.B. Let K/S be a smooth group with geometrically connected fibres. Let $\Omega \subset K$ be a fibrewise dense open containing the identity. Let H/S be a group scheme. Suppose $f : \Omega \rightarrow H$ sends the identity to the identity and satisfies

$$f(xy) = f(x)f(y)$$

on the open where $x, y, xy \in \Omega$. Then f extends uniquely to a homomorphism $K \rightarrow H$.

Proof. We spell out irreducibility without a quasi-compactness assumption on K . In a geometric fibre, every local ring of a smooth scheme is a regular local domain, by AG-CA-18 Theorem 2.1 and the regular-local-domain proof of AG-CA-14 Theorem 1.1. Each affine smooth chart is Noetherian. Its finitely many irreducible components cannot meet: two distinct minimal primes at a meeting point would remain distinct in the local ring, contradicting that it is a domain. The components on that chart are consequently disjoint closed-and-open subsets, each reduced and irreducible. Thus every point of the fibre has an irreducible open neighbourhood. Form equivalence classes of such neighbourhoods under chains of nonempty intersections. A finite union along a chain is irreducible, since the union of two intersecting irreducible open subsets is irreducible; this follows by testing two nonempty opens in their union. Any two members in one class lie in such a finite union, so the whole class union is irreducible. These class unions are disjoint opens covering the fibre. Connectedness leaves only one, proving irreducibility. This is the local form of the calculation in S02 Lemma 4.1. For any test scheme $R \rightarrow S$ and any finite list $g_1, \dots, g_d \in K(R)$, the open

$$\Omega_R \cap \bigcap_{i=1}^d \Omega_R g_i^{-1}$$

has a nonempty open in every geometric fibre of K_R : each translate is a dense nonempty open in the irreducible fibre and finite intersections remain nonempty. Its map to R is a smooth surjection. Smoothness gives flatness over arbitrary rings by AG-CA-18 Theorem 6.1, and local finite presentation is part of the proved smooth-chart criterion. Refine by affine smooth charts above affine base opens; these are finitely presented, flat maps and jointly cover R . They give an actual fppf covering family and therefore a local point c with c and all cg_i in the cell. No rational-point choice over the original residue fields and no flatness of H are required.

In particular, for $g \in K(R)$, choose $c \in \Omega_R \cap \Omega_R^{-1} \cap \Omega_R g^{-1}$ locally. Then $g = c^{-1}(cg)$ is a product of two cell points. Assign to any word in cell points the corresponding product of f -values.

This assignment is independent of the word. For two words $x_1 \cdots x_d$ and $y_1 \cdots y_e$ with the same endpoint g , choose c so that it and every translated partial product of both words lie in the cell. Repeatedly apply the given identity. The first image word multiplied on the left by $f(c)$, and the second image word multiplied by the same element, both equal $f(cg)$. Cancellation proves equality. The assignment also respects inverses: for $x \in \Omega(R)$, choose $d, d^{-1}, dx^{-1} \in \Omega$. The identities $(dx^{-1})x = d$ and $d^{-1}d = 1$ then give $f(d^{-1})f(dx^{-1}) = f(x)^{-1}$. The word $d^{-1}(dx^{-1})$ represents x^{-1} , as required.

The assignments agree after fppf pullback, so the sheaf property of H descends them. Concatenation of words proves multiplicativity. Yoneda makes this map of represented fppf sheaves a scheme morphism. Every extension must have these values on generating words, proving uniqueness. $\square$

The three rank-one models

Theorem 8.1. A split reductive group of rank $r + 1$ and semisimple rank one over a field is isomorphic to exactly one of

$$\mathbf{G}_m^r \times \mathrm{SL}_2, \quad \mathbf{G}_m^r \times \mathrm{PGL}_2, \quad \mathbf{G}_m^{r-1} \times \mathrm{GL}_2 \quad (r \geq 1).$$

For a reductive group over a scheme with a split maximal torus and semisimple rank one, the same description holds Zariski locally on fixed-type loci. Trivial root lines with chosen frames give a global isomorphism to the corresponding base change from $\mathbf{Z}$.

Proof. The root and coroot construction in Theorem 7.1 gives two roots $\pm\alpha$, their additive root lines, the coroot $\eta = \alpha^\vee$, and $\langle \alpha, \eta \rangle = 2$. Apply Lemma 8.A to $X = X^*(T)$, $a = \alpha$, $b = \eta$. The three resulting pairs are precisely those computed for the three displayed matrix groups: in SL_2 the root is twice the primitive character and the coroot is primitive; in PGL_2 the root is primitive and the coroot twice primitive; in GL_2 they are respectively $e_1 - e_2$ and $e_1^* - e_2^*$.

Let G_0 be the indicated matrix model. The lattice isomorphism gives a torus isomorphism $f_T : T \rightarrow T_0$ carrying root characters and coroots to those of the model. Choose a nonzero positive-root frame $E \in \mathfrak{g}_\alpha$. The perfect pairing of Theorem 7.1 gives a unique negative-root frame F with pairing one. Define $x(u) = \exp_\alpha(uE)$ and $y(v) = \exp_{-\alpha}(vF)$, and match these to the upper and lower matrix parameters x_0, y_0 of G_0 .

The matrix model has a smooth geometrically connected underlying scheme over every base. For SL_2 the two polynomial-localization charts and their nonempty intersection were calculated in Theorem 6.1. For GL_2 it is the principal determinant open of affine four-space, smooth by the polynomial lifting proof and geometrically integral because a localization of a polynomial domain is a domain. For PGL_2 it is the determinant open of $\mathbf{P}^3$ from Lemma 5.B, smooth on its affine coordinate charts and geometrically integral as a nonempty open of integral projective space. The latter integrality follows from the polynomial domains on its affine charts, whose pairwise intersections are nonempty. A product with the extra split torus stays smooth; its affine fibre charts have Laurent polynomial algebras over these domains and remain integral, with nonempty chart intersections. Thus both the model and G satisfy the hypotheses of Lemma 8.B. The open cell theorem gives $\Omega = U_-TU_+$. On it define

$$f(y(v)tx(u)) = y_0(v)f_T(t)x_0(u).$$

This is an isomorphism of schemes from Ω to the model cell. The group law on the torus, the additivity of x, y , and the conjugation identities $tx(u)t^{-1} = x(\alpha(t)u)$, $ty(v)t^{-1} = y(\alpha(t)^{-1}v)$, agree under this map. The remaining multiplication rule is

$$x(u)y(v) = y\left(\frac{v}{1+uv}\right)\eta(1+uv)x\left(\frac{u}{1+uv}\right), \quad 1+uv \in \mathcal{O}^\times,$$

proved in Theorem 7.1 by the arbitrary-base central-extension argument and the explicit 2×2 matrix identity.

For two cell points $y(v)tx(u)$ and $y(v')t'x(u')$, their product is in the cell exactly when $x(u)y(v')$ is: left multiplication by U_-T and right multiplication by TU_+ preserve the cell, as do their inverses. The preceding rule and its unit criterion therefore give every multiplication of cell points whose product remains in the cell. It follows that f satisfies Lemma 8.B. The lemma extends it to a group

homomorphism $G \rightarrow G_0$. The inverse cell map satisfies the same identities and extends in the other direction. Uniqueness in the lemma makes their compositions the identity. Thus $G \simeq G_0$.

Any group isomorphism carries a maximal torus to a maximal torus. Over an algebraic closure the torus conjugacy theorem then identifies its root/coroot pair with the pair for the fixed torus. It can interchange the signs of both elements, but does not change either divisibility. The three invariant pairs $(2, 1), (1, 2), (1, 1)$ distinguish the models in all characteristics, including characteristic two. This proves uniqueness without a forward reference to the derived group.

Over a scheme, restrict to an open where the root/coroot lattice pair is constant and the positive root line is trivial. Its paired negative line is then also trivial. The same torus map, cell formula and fppf word argument work over every test scheme, so give the asserted local isomorphism. When the lattice pair and frames are fixed globally, these constructions are global. All maps commute with base change because the displayed formulas and their character pairings do. $\square$

9. Exercises

Exercise 9.1 (first steps). Compute all roots and root spaces of GL_n with respect to its diagonal torus. Determine the zero-weight space and the positive coroot for $e_i - e_j$.

Exercise 9.2 (rank-one multiplication). Compute $U_{\pm\alpha}$ and $\alpha^\vee$ for SL_2 . Verify the factorization (2), and explain why the character differential in characteristic two does not change this factorization.

Exercise 9.3 (infinitesimal information). In characteristic different from two, show that $\mathfrak{sl}_2 \simeq \mathfrak{pgl}_2$ as Lie algebras, while the groups are not isomorphic. What happens to the differential of $\mathrm{SL}_2 \rightarrow \mathrm{PGL}_2$ in characteristic two?

Exercise 9.4 (classification). List the rank-two split reductive groups of semisimple rank one. Distinguish them by their centres and derived groups, including characteristic two.

Exercise 9.5 (families). For $G = \mathrm{SL}(\mathcal{O} \oplus L)$, identify the two root groups and their pairing. Give a condition on L necessary for a pinning of this pair with its displayed torus.

Solutions

Solution 9.1

The diagonal matrix units have weight zero and form $\mathfrak{t}$. For $i \neq j$, E_{ij} has weight $e_i - e_j$, each once. The corresponding root group is $I + uE_{ij}$. Its coroot has diagonal entries t in position i , t^{-1} in position j , and 1 elsewhere. Evaluating $e_i - e_j$ gives t^2 , so the root–coroot pairing is two.

Solution 9.2

The two groups are the upper and lower unipotent matrices. Their parameters are x, y in the displayed matrix identity, and $\alpha^\vee(t) = \mathrm{diag}(t, t^{-1})$. Multiplication of the three factors gives $\begin{pmatrix} 1 + xy & x \\ y & 1 \end{pmatrix}$, equal to $x_\alpha(x)x_{-\alpha}(y)$. The character on the upper group is t^2 , so its pairing with this coroot is two. Although its differential is zero in characteristic two, the Laurent character and all polynomial matrix identities remain valid; no division by two was used.

Solution 9.3

The projective affine charts and scalar torsor constructed over $\mathbf{Z}$ in RG02 Lemma 4.P compute $\mathrm{Lie}(\mathrm{PGL}_2) = \mathfrak{gl}_2/kI$ in every characteristic: an infinitesimal scalar changes $I + \epsilon X$ to $I + \epsilon(X + qI)$.

The derivative of the determinant is the trace, so $\text{Lie}(\text{SL}_2) = \{X : \text{tr } X = 0\}$. Inclusion followed by quotient is bracket preserving, with scalar kernel defined by $2q = 0$. When two is invertible it has zero kernel and both spaces have dimension three, so it is an isomorphism. In characteristic two the scalar line lies in the trace-zero space, giving exactly a one-dimensional kernel.

To compute the group kernel on every test algebra, a scalar determinant-one matrix is qI with $q^2 = 1$, giving μ_2 . Every projective class lifts locally to an invertible matrix M . After the cover $R[z]/(z^2 - \det M)$ it has the determinant-one representative $z^{-1}M$. This is a finite free rank-two faithfully flat cover (basis $1, z$ and nonzero fibres); z is a unit because its square is a unit. Thus $\text{SL}_2 \rightarrow \text{PGL}_2$ is the fppf central quotient by μ_2 in every characteristic, and Theorem 1.1 supplies reductivity of this quotient as well. Its centre is trivial: a central element centralizes the diagonal torus and hence lies in it by RG02 Theorem 2.1; writing its ratio as q , commutation with the universal upper root matrix gives $(q - 1)u = 0$ in $R[u]$, whence $q = 1$. For SL_2 the same diagonal-centralizer calculation makes a central element $\text{diag}(q, q^{-1})$, and the universal root equation gives $q^2 = 1$. Hence its entire centre is μ_2 . This is nontrivial as a scheme even in characteristic two, whereas the projective centre is trivial, so the groups are not isomorphic.

Solution 9.4

Put $r = 1$ in Theorem 8.1. The list is $\mathbf{G}_m \times \text{SL}_2$, $\mathbf{G}_m \times \text{PGL}_2$, and GL_2 . Their centres are respectively $\mathbf{G}_m \times \mu_2$, $\mathbf{G}_m$, and $\mathbf{G}_m$. The first centre is not a torus: in characteristic two it is nonsmooth, and otherwise it has two geometric components. We compute the indicated derived groups directly. The extension $R \rightarrow R[t, t^{-1}, (t^2 - 1)^{-1}]$ is faithfully flat and finitely presented: it is a localization of the polynomial algebra, is R -flat, and every field fibre is a nonzero localization at a nonzero polynomial. Each such fibre has a prime by AG-CA-01 Lemma 1.1, so the map on spectra is surjective and flatness is faithful by AG-CA-07 Theorem 2.1. Thus fppf locally we can choose t with $t^2 - 1$ invertible, including in characteristic two. The identities $[h(t), x(u)] = x((t^2 - 1)u)$ and $[h(t), y(v)] = y((t^{-2} - 1)v)$ put both root groups of SL_2 in its commutator subgroup. The open cell word argument of Theorem 6.1 shows that these root groups generate SL_2 as an fppf sheaf, since $h(t) = w(t)w(1)^{-1}$. Their images generate PGL_2 , because $\text{SL}_2 \rightarrow \text{PGL}_2$ is fppf surjective; hence its derived group is itself. For GL_2 , determinant kills all commutators, and conjugation by $\text{diag}(t, 1)$, with $t - 1$ invertible fppf locally, puts both determinant-one root groups in the commutator subgroup. The same generation argument gives its derived group SL_2 . The central $\mathbf{G}_m$ factors contribute no commutators. Thus the last two models have derived groups PGL_2 and SL_2 ; the first and third have the same derived group but different centres. These computations are scheme-theoretic and remain valid in characteristic two.

Solution 9.5

An upper off-diagonal entry is a map $L \rightarrow \mathcal{O}$, so its line is L^{-1} ; a lower entry is a map $\mathcal{O} \rightarrow L$, so its line is L . Composition gives $L^{-1} \otimes L \rightarrow \mathcal{O}$, the perfect pairing in Theorem 7.1. A pinning chooses a nowhere-vanishing section in the simple root line. It therefore requires L^{-1} , equivalently L , to be trivial. This illustrates the global hypothesis absent from the field case.

The [course prerequisite guide](#) records the exact supporting statements, their full proof routes, and the hypotheses needed in their applications.

References and prerequisite proofs

- Brian Conrad, [Reductive group schemes](#), §§2.3 and 4.1–4.3, for the dynamic approach and Gabber’s central-extension splitting argument. The proofs above are written out independently.

- Milne’s freely accessible *Algebraic Groups*, version 2.00 (2015). This exact free author edition provides comparison material; its citations do not replace the programme proofs.
- SGA 3, *Schémas en groupes*, Exposés XIX–XX, for roots and the relative rank-one theorem.
- The [official Stacks project](#) and AI Integrated Stacks Project, an edition with AI-proposed corrections and AI-written additions, not reviewed by the Stacks project’s maintainers. The normalization lemma in Section 5 and its proof follow the Stacks project authors’ argument (Tag 0ABS) in this edition.

The following supporting statements have their exact proof routes recorded in the course prerequisite guide:

- [Algebra and sheaf cohomology before reductive groups](#), §§3–8: injectives, finite-cover comparisons, affine vanishing, projective-space twists, ample embeddings and projective Serre finiteness.
- [Projective cohomology and smooth affine models](#), §§1–6: exact and faithful local completion; projective formal functions and idempotents; smooth connected affine models; universal finite projective complexes and all-base-change direct images; regular curve immersions, curve duality, divisor degrees and the genus-zero calculation.
- [Completing a smooth affine curve and extending its group action](#), §§1–4: finite curve normalization, completion, the actual family action and one-dimensional unipotent identification used through the preceding solvable-group proofs.
- [Affine descent, Zariski Main and recognition of spaces](#), Theorems E8.1–E9.1: recognition on every scheme test, representability and the full relative integral-closure factorization.
- [Flat quotient bootstrap](#), Theorem 8.2 and Corollary 8.3: the algebraic-space quotient and torsor relation for the free smooth subgroup action, over arbitrary bases.
- [Lemma 5.A of this lesson](#): quotient Grassmannian charts, arbitrary base change, and the closed Plücker immersion with determinant quotient bundle; the subbundle convention follows by dualizing.
- The current written group-scheme proofs: *Diagonalizable groups*, Theorems 3.4–3.5 and §4, for smooth fixed loci, invariant algebras and torus quotients; *Group schemes over a field*, Proposition 5.8, for the reduced schematic image and faithful-flat image factorization; *Lie algebras and smoothness*, Theorem 2.9 and Theorem 3.2, for the universal normalizer tangent calculation and geometric smoothness criterion; and *Quotients and torsors*, its affine descent proof and Theorem 8.1, for effective algebra descent and line-bundle frame torsors. These are the actual written proofs, including positive-characteristic and nonreduced scope, rather than source citations.

Theorem 1.1 proves the affine central quotient before the rank-one argument; the following lesson adds the consequences for root-group coordinates. The arbitrary-base derived group is constructed after the existence theorem in the following lesson.

Source credit

The Stacks Project Authors, [Tag 0ABS](#), with the [consulted spaces-morphisms chapter](#).