

Regular elements and centralizers

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Codex, Ultra setting

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Course sources: <https://github.com/KokunoYumeto/open-math-courses/tree/main/docs/courses/AG-RG>.

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Lesson 2

Regular elements and centralizers

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A maximal torus describes a reductive group by diagonal symmetries. A regular semisimple element describes the same torus by one element: the torus is the identity component of its centralizer. This observation is useful in families, where the centralizer of a section is easier to specify than a choice of eigenvectors.

We use the torus deformation and conjugacy arguments of [Tori, maximal tori and their conjugacy](#). The classical geometric input concerning Borel subgroups is used below to prove the centralizer theorem, rather than importing that theorem from a reference. All centralizers are scheme-theoretic. Over a field, a geometric assertion means an assertion after algebraic closure.

1. The geometry behind the centralizer theorem

We first work over an algebraically closed field k . A Borel subgroup of a smooth connected affine group H is a maximal smooth connected solvable subgroup. Write $B = U \rtimes T$ for a Borel subgroup and one of its maximal tori. The solvable-group construction in the preceding lesson identifies U with the unipotent radical of B .

We shall need several consequences of that construction. Every smooth connected solvable subgroup lies in a Borel: among those containing it choose one of largest dimension, bounded by $\dim H$. A proper inclusion would strictly increase dimension, since both groups are reduced connected and hence integral by RG01 Lemma 2.L, and the earlier domain dimension theorem excludes a proper closed subset of the same dimension. Thus the chosen subgroup is maximal. RG01 Lemma 2.1 proves that H/B is projective and that all Borel subgroups are conjugate. The scheme-theoretic normalizer theorem is proved below, after the centralizer and Borel-intersection arguments needed for its induction.

Lemma 1.A (unipotent radicals). A smooth connected affine H/k has a unique largest smooth connected normal unipotent subgroup $R_u(H)$. It is preserved by every automorphism. Every smooth connected normal unipotent subgroup lies in every Borel. If K is a smooth connected normal solvable subgroup of H , then $R_u(K)$ is normal in H .

Proof. We first prove the precise affine image assertion used here. If $f : E \rightarrow H$ is a homomorphism with E smooth connected affine of finite type, then $k[E]$ is a domain: Lemma 1.0 below gives irreducibility, and smooth local rings are regular and therefore domains by AG-CA, *Smooth algebras over a field and the Jacobian criterion*, Theorem 2.1, and the regular-parameter theorem. Let C be the image of $k[H] \rightarrow k[E]$. This is a finitely generated domain. Comultiplication, inversion and counit restrict to it, because they commute with f and tensoring vector-space inclusions over k is injective. Their identities therefore restrict as well. Thus $J = \operatorname{Spec} C$ is a closed subgroup of H ,

and $E \rightarrow J$ is schematically dominant. The complete monic-division and translation argument of AG-GS, *Quotients and torsors*, Lemma 11.1c makes it faithfully flat and of finite presentation, in precisely its integral-target case.

The group J is smooth and connected. Put $r = \text{trdeg}_k \text{Frac } C$. Noether normalization, proved in AG-CA, *Krull dimension and Noether normalization*, Corollary 3.2, makes this field finite over a rational function field with r parameters. S02 Lemma 3.F gives its differential dimension r , since k is perfect. At the generic point the local dimension is zero and the residue transcendence degree is r , so the earlier Jacobian Theorem 2.1 makes that point smooth. Its Corollary 2.2 gives a nonempty smooth open. This open contains a k -point by the weak Nullstellensatz. Translation by the rational points of J makes every rational point smooth; a nonempty closed complement of the smooth locus would again have a rational point. Hence J is smooth everywhere. Its coordinate domain makes it irreducible and connected. This proves the entire finite-type affine image assertion needed here, without consuming an image theorem for arbitrary group schemes.

For smooth connected normal unipotent N_1, N_2 , the semidirect product $N_1 \rtimes N_2$ has underlying scheme $N_1 \times N_2$. It is smooth and integral by Lemma 1.0, the regular-local-domain calculation above and S08 Lemma G.4.1, so is connected. Apply the preceding image proof to multiplication into H ; denote its closed image by $N_1 N_2$. It contains both factors. Normality holds schematically: conjugation by any H -point preserves both factors, hence their multiplication image; pulling back equations along the faithfully flat cover $H \times (N_1 \rtimes N_2) \rightarrow H \times (N_1 N_2)$ detects their vanishing. Thus their product is a smooth connected normal closed subgroup.

This product is unipotent. In any nonzero finite-dimensional representation W , AG-RG-01, Lemma 2.A makes $W^{N_1} \neq 0$. Normality makes this subspace N_2 -stable, so AG-RG-01, Lemma 2.A supplies a nonzero vector fixed by both groups, and hence by their product: the latter equality follows faithfully flatly from the surjective multiplication map. Applying the same argument to successive quotients gives a trivial-quotient full flag. A faithful representation of the affine image is therefore unitriangular.

Choose a smooth connected normal unipotent subgroup of maximal dimension, possible since the dimensions are bounded by $\dim H$. Its product with any other such subgroup is again of this type. Maximal dimension forces that product to equal it: a proper integral closed subgroup of an integral finite-type group has strictly smaller dimension, by the strict-specialization and dimension proofs in AG-CA, *Krull dimension and Noether normalization*, Lemma 4.1 and Theorem 4.2. This proves greatestness and uniqueness. Automorphisms preserve its defining property, so preserve the subgroup.

For a Borel B and such a normal N , the product NB is similarly the smooth connected closed image of $N \rtimes B$. It is solvable directly on all test rings. Projection to the solvable B makes every sufficiently iterated commutator word in $N \rtimes B$ take values in N . The unitriangular entry-distance filtration, with its polynomial commutator identities proved in RG01 Lemma 2.J, makes finitely many further derived commutators equal to the identity. Every tuple of points in the image lifts fppf locally through the multiplication map; hence the same commutator identities hold for NB . Thus maximality of B gives $NB = B$, and N lies in B .

If K is normal in H , every conjugation by $H(k)$ gives an automorphism of K and preserves its unique $R_u(K)$. The whole conjugation morphism $H \times R_u(K) \rightarrow H$ factors through that closed subgroup: the source is reduced by AG-RG-S08, Lemma G.4.1, and its defining equations vanish at all closed points. To justify the required detection, a nonzero element a of a reduced finite-type k -algebra has a nonzero localization at a , since it is not nilpotent. That localization is still finite type and has a maximal ideal, whose residue field is k by the weak Nullstellensatz. It gives a rational point where a is invertible. Therefore an element vanishing at every rational point is zero. Apply this calculation to the affine source of conjugation. This proves scheme normality in H . In the solvable K , the decomposition constructed in AG-RG-01, Theorem 2.K, $K = U \rtimes T$, identifies its radical with U : every smooth connected unipotent subgroup, including one not assumed normal

in K , projects trivially to T by AG-RG-01, Lemma 2.A and is contained in U . $\square$

Lemma 1.0 (components of an affine group over an algebraically closed field). Let J be an affine group scheme of finite type over an algebraically closed field k . Its irreducible components are pairwise disjoint and open and closed. In particular a connected such group is irreducible. No smoothness or reducedness hypothesis is needed for this topological assertion.

Proof. The Hilbert basis theorem, proved in AG-CA, *Noetherian and Artinian rings*, Theorem 2.1, makes $k[J]$ Noetherian. Its spectrum has finitely many irreducible components: if a closed subset failed to be a finite union of irreducible closed subsets, choose a minimal such subset using the descending chain condition on closed sets. It is reducible, hence the union of two proper closed subsets, both finite unions by minimality, a contradiction.

If C_1, C_2 are components containing the identity, give them their reduced closed structures. They are integral finite-type schemes, so their product is integral by the complete coefficient-and-Nullstellensatz proof in the earlier S08 Lemma G.4.1. The closure of its multiplication image in J is irreducible and contains both C_1 and C_2 , by multiplication with the identity in either factor. It is contained in a component, and maximality of C_1 and C_2 forces them to be that same component. Thus exactly one component contains the identity.

If two different components met, their closed intersection would contain a k -point by the earlier Nullstellensatz, AG-CA Theorem 2.1 of *The Nullstellensatz and Jacobson rings*. Translation by the inverse of that point is a scheme automorphism of J . It would carry the two distinct components to two distinct components containing the identity, contradicting the preceding paragraph. Hence all components are disjoint. There are only finitely many, so each has closed complement and is open as well as closed. Connectedness leaves exactly one component, proving irreducibility. This is the finite-type algebraically closed case of the earlier AG-GS, *Group schemes over a field*, Section 1 and Theorem 2.2; the proof just given supplies precisely the case consumed here. $\square$

Lemma 1.1. Let H be a smooth connected affine group of finite type over an algebraically closed field k , and let $A \subset H$ be a torus. Its schematic centralizer $C_H(A)$ is smooth and connected.

Proof. The complete fixed-subgroup proof in AG-GS, *Diagonalizable groups and groups of multiplicative type*, Theorem 3.4 supplies smoothness. We give explicitly the cocharacter reduction needed for connectedness. Over the algebraically closed k , the torus A is split. Indeed its étale splitting cover, given by that lesson's Theorem 7.7, has a nonempty finite-type affine chart; the earlier Nullstellensatz supplies a k -point, and pulling the splitting back along that point splits A . Write $A = \mathbf{G}_m^r$ and its character lattice as $\mathbf{Z}^r$.

Choose finitely many algebra generators of $k[H]$ and replace them by all their finitely many homogeneous components for conjugation by A . The complete weight decomposition of AG-GS Theorem 3.1 shows that the resulting homogeneous list still generates. Let $\Xi \subset \mathbf{Z}^r$ be its finite set of nonzero weights. If $r > 0$, associate to $\chi = (a_1, \dots, a_r) \in \Xi$ the nonzero polynomial

$$p_\chi(Z) = a_1 + a_2 Z + \dots + a_r Z^{r-1} \in \mathbf{Q}[Z].$$

Each has only finitely many roots: division by $Z - c$ removes a root, and induction on the degree bounds the number of distinct roots by that degree. This is also the earlier explicit polynomial calculation in S02 Lemma 4.A. Choose a positive integer M outside the finite union of their root sets, and put $\lambda(t) = (t, t^M, \dots, t^{M^{r-1}})$. Its pairing with every member of Ξ is nonzero. For an empty weight list choose $M = 1$, and for $r = 0$ use the unique cocharacter of the trivial torus.

The centralizers $C_H(A)$ and $C_H(\lambda)$ are equal as schemes, including on nonreduced test algebras. To check this, use the universal conjugation action on each homogeneous algebra generator f of weight χ . A test point fixed by A must satisfy $f(h)(e^\chi - 1) = 0$ in its group algebra. The distinct basis monomials force $f(h) = 0$ when $\chi \neq 0$, and impose no condition when $\chi = 0$. The corresponding condition for λ is $f(h)(t^{\langle \chi, \lambda \rangle} - 1) = 0$ in the Laurent polynomial algebra. Our choice makes its exponent nonzero precisely for the nonzero weights on this generating list, so coefficient comparison

imposes exactly the same equations. Checking the algebra generators checks the full action on every function. Thus the two represented fixed subgroups have the same defining ideal over k and the same points on every test algebra.

Use the general cocharacter construction proved in the preceding lesson, Lemma 3.3. Its proof is a grading and smoothness argument for arbitrary smooth affine groups and uses no reductive centralizer theorem. Multiplication gives an open immersion

$$U_H(-\lambda) \times C_H(\lambda) \times U_H(\lambda) \longrightarrow H.$$

The domain contains the triple of identity elements. Lemma 1.0 makes the connected finite-type H irreducible, so the nonempty open image, and hence the domain, is irreducible. Its projection onto $C_H(\lambda)$ is surjective: insertion of the identity in the two unipotent factors gives a section. The centralizer is therefore irreducible and connected. Together with the fixed-subgroup smoothness proof and the schematic equality already established, this proves the lemma. $\square$

The cocharacter lemma and the classical fixed-point arguments are independent of the root classification and of the reductive centralizer theorem proved below.

Lemma 1.B (fixed unipotent groups are smooth connected). Let a diagonalizable D act on U as in AG-RG-01, Lemma 2.C. Then U^D is smooth connected. Moreover the sequence

$$1 \longrightarrow U_1^D \longrightarrow U^D \longrightarrow (U/U_1)^D \longrightarrow 1$$

is exact as fppf sheaves.

Proof. Fixed subschemes are affine finitely presented and smooth by the coordinate and infinitesimal proofs in AG-GS-05, Proposition 3.2 and Theorem 3.4, including a nonreduced D . Let x be a D -fixed quotient section on any test algebra. Choose a lift u fppf locally. The defect

$$c(d) = u^{-1}d(u)$$

lies in U_1 and is a regular crossed homomorphism. Write it fppf locally as $c(d) = vd(v)^{-1}$ by AG-RG-01, Lemma 2.C. Then $u \cdot v$ is fixed, since $d(uv) = uc(d)d(v) = uv$. This proves surjectivity of the last map of the displayed sequence, with the kernel equality evident on every test scheme. Differences of two fixed lifts belong uniquely to U_1^D ; consequently its action identifies $U_1^D \times U^D$ with $U^D \times_{(U/U_1)^D} U^D$ on every test scheme. A local fixed lift translates this torsor into the product with U_1^D .

For a vector line of character χ , its D -fixed subgroup is the whole line if $\chi = 0$, and is zero otherwise: comparison of the two distinct basis monomials e^χ and 1 forces the vector coordinate to vanish, including over nonreduced tests. We induct on the given filtration length, simultaneously after every algebraically closed extension of k ; the length-zero group is the identity. The fixed kernel in the displayed sequence is smooth and geometrically connected by induction, and the quotient Q is either $\mathbf{G}_a$ or 1 . On a geometric fibre the torsor has a point by the earlier Nullstellensatz, and translation identifies it with that connected kernel. Hence all geometric fibres of $q : U^D \rightarrow Q$ are connected.

Here is an explicit descent proof that the source is connected. Both source and target are affine. Write $A = k[Q]$, $B = k[U^D]$. On each affine fppf trivializing chart $\text{Spec } A' \rightarrow \text{Spec } A$, the algebra $B \otimes_A A'$ is flat over A' : it is the base change of the coordinate algebra of the smooth k -group U_1^D , and AG-CA, *Smooth algebras over a field and the Jacobian criterion*, Theorem 6.1 proves smooth flatness, with the flat base-change proof in the flat-module lesson, Proposition 3.2. For any prime $\mathfrak{p}$ of A , choose such a chart and a prime $\mathfrak{p}'$ above it. The flat local map $A_{\mathfrak{p}} \rightarrow A'_{\mathfrak{p}'}$ is faithfully flat by AG-CA, *Faithful flatness and the local criterion for flatness*, Theorem 2.1. For an injection of $A_{\mathfrak{p}}$ -modules, its tensor map with $B_{\mathfrak{p}}$ becomes injective after this faithfully flat extension, because the extended algebra is flat. Flatness of the extension and faithful detection make the

original tensor map injective. Thus B_p is flat, and local flatness detection in the flat-module lesson, Theorem 3.3, proves B flat over A . The local torsor sections give a point above every prime; hence $A \rightarrow B$ is faithfully flat by the same faithful-flatness Theorem 2.1.

A partition of the source into open and closed pieces is an idempotent $e \in B$: glue the function equal to one on one piece and zero on the other. Conversely an idempotent gives these pieces, since in every local ring one of e and $1 - e$ is a unit and their product is zero.

Let $e_1, e_2 \in B \otimes_A B$ be the pullbacks of e from the two factors. On each geometric fibre of q , connectedness makes that idempotent constant, so e_1 and e_2 have equal values at every geometric point of the fibre product. Their difference therefore belongs to every prime ideal and is nilpotent. Here is the required nilradical argument for any ring C . If $z \in C$ is not nilpotent, the powers $1, z, z^2, \dots$ avoid zero. An ideal maximal among those avoiding these powers exists by the union-of-chains argument. It is prime: if ab lies in it but neither a nor b does, both larger ideals obtained by adjoining a and b meet the powers; multiplying the two resulting equations puts a power of z in the original ideal, a contradiction. This prime avoids z . Conversely a nilpotent belongs to every prime. Apply this to $C = B \otimes_A B$. Two idempotents with nilpotent difference are equal: $e_1(1 - e_2)$ and $e_2(1 - e_1)$ are both idempotent and nilpotent, hence zero; thus $e_1 = e_1e_2 = e_2$.

The faithfully flat equalizer proved in the earlier S04 Lemma A1.1 now descends e to A . The map $A \rightarrow B$ is injective, so the descended element is also idempotent. But A is $k[x]$ or k , both domains, and has only the idempotents zero and one. Hence e was trivial. This proves connectedness, and the same argument after every algebraically closed extension proves geometric connectedness. The torsor and kernel equalities were identities on all test schemes, so the entire lemma retains the stated schematic exactness. $\square$

For $A \subset T \subset B = U \rtimes T$, these proofs give

$$C_B(A) = U^A \rtimes T,$$

so this centralizer is smooth connected and solvable.

Lemma 1.2. A unipotent group acting on an affine variety has closed orbits, with their full scheme structures. No smoothness or connectedness assumption on the group or the schematic stabilizer is needed.

Proof. We first prove closedness for a smooth connected unitriangular group L . Give the closure Y of one point orbit its reduced structure. It is integral. S08 Theorem G.4.2 makes the orbit open in Y , and makes both Y and its reduced boundary D invariant under L , using reducedness of the products and the strong Nullstellensatz. If $D \neq \emptyset$, choose a nonzero function in its ideal. AG-GS-01 Lemma 5.1 puts that function in a finite-dimensional L -stable subspace of the ideal: in the coefficient construction all coefficient functions stay in the ideal, since its coaction maps into $k[L] \otimes I_D$. RG01 Lemma 2.A gives a nonzero invariant function in this subspace. Its values are constant on the dense point orbit and hence on Y . It vanishes on the nonempty D , so the constant is zero, a contradiction. Thus the orbit is closed. S08 Theorem G.4.C represents the full quotient by its possibly nonreduced stabilizer as this reduced locally closed orbit. Since its support is closed, it is a closed immersion in the affine target and is affine.

Now embed the given unipotent group N as a closed subgroup of the smooth connected group $L = U_m$ of upper unitriangular matrices. Its coordinate ring is the polynomial ring in the entries above the diagonal, and the matrix formulas give its group operations. Fix x in the affine variety and retain the entire schematic stabilizer $K = N_x \subset N \subset L$. The RG01 detecting-line construction supplies an L -representation and a line kv with full stabilizer K . RG01 Lemma 2.A makes the character of K on this line trivial, including its infinitesimal points. The vector stabilizer is consequently K as well. The first paragraph and S08 Theorem G.4.C now construct an affine closed vector orbit P , with an fppf K -torsor

$$q : L \longrightarrow P, \quad L \times K \simeq L \times_P L.$$

All these identities hold on arbitrary test algebras.

The closed subscheme $N \subset L$ descends along this torsor. Indeed, in the displayed relation the two projections are $(l, k) \mapsto l$ and $(l, k) \mapsto lk$. The inverse images of N coincide as schemes: right multiplication by any section of $K \subset N$ preserves N , and its inverse does too. This is the descent datum for its defining ideal. The module, ideal and algebra effectivity proofs of S04 Lemmas A1.1, A1.2 and A1.5 give a closed subscheme $Z \subset P$ with

$$q^{-1}(Z) = N, \quad N \longrightarrow Z \text{ an fppf } K\text{-torsor.}$$

For clarity, ideal descent applies on the affine cover $L \rightarrow P$: the compatible ideal descends as a submodule of $\mathcal{O}_P$, multiplication preserves it after faithful pullback and hence before pullback, and its quotient algebra pulls back to $\mathcal{O}_N$. Thus Z retains any nilpotents of N/K ; no reduction has been taken here. The torsor identity is the restriction of that for L .

The action map $N \rightarrow X$, $n \mapsto nx$, descends to $i : Z \rightarrow X$, since its two pullbacks on $N \times K$ agree. This is a finite-type morphism between affine schemes and is a monomorphism. To check the latter on every test scheme, lift two sections of Z fppf locally to $n, n' \in N$. Equality of their images says exactly $n^{-1}n' \in K$, so the two quotient sections agree. Morphism descent then gives their equality before the cover. Left translation by N likewise descends to an action on Z , and i is equivariant.

Let $W \subset X$ be the schematic image of i . It is invariant under the entire N -action. In affine coordinates the ideal of the schematic image is a kernel, and tensoring with the k -vector space $k[N]$ preserves that kernel. Thus the schematic image of $1_N \times i$ is $N \times W$. The shear automorphisms $(n, z) \mapsto (n, nz)$ on $N \times Z$ and $N \times X$ commute with this map; the latter automorphism and its inverse consequently preserve $N \times W$. This proves invariance with its original scheme structure.

S08 Lemma G.4.E gives an open $V \subset W$ over which i is an isomorphism. Choose $z_0 \in i^{-1}(V)(k)$. Every point of $Z(k)$ lifts to $N(k)$: a fibre of its fppf torsor is nonempty and of finite type over the algebraically closed field, so has a rational point. Hence $N(k)$ acts transitively on $Z(k)$. Translating V by these elements gives opens $nV \subset W$ whose inverse images are isomorphic to them and cover all closed points of Z . They cover Z , since a nonempty closed complement in a finite-type scheme has a closed rational point. The inverses agree on overlaps. Therefore Z is isomorphic to the open union $O = \bigcup_{n \in N(k)} nV \subset W$.

Finally this open has closed support. RG01 Lemma 2.L gives the smooth connected group $J = (N_{\text{red}})^0$, finitely many component translates, and $N(k) = N_{\text{red}}(k)$. The first paragraph makes every J -point-orbit closed. The $N(k)$ -orbit of x is the finite union of translates of such an orbit, so it is the set of rational points of a closed reduced subscheme $D \subset X$. The locally closed image O has precisely these rational points. Its support equals $|D|$: the difference in either direction, if nonempty, is constructible and has a rational point by S08 Lemma G.4.A. The schematic image W has this same support, because the image of i is dense in it. Thus the open $O \subset W$ contains every point of W , so $O = W$ as an open subscheme. It follows that $i : Z \rightarrow X$ is a closed immersion. This constructs the full possibly nonreduced orbit and proves the assertion. $\square$

Put $X = H/B$, initially as a homogeneous space with stabilizer B . Choose a maximal torus T of H contained in B , and put $C = C_H(T)$. Such a torus exists: a maximal torus is contained in a Borel by the dimension argument of RG01 Theorem 2.2, and Borel conjugacy in RG01 Lemma 2.1 moves that Borel to B . We justify both finiteness of X^T and the assertion that C fixes it, instead of assuming either one.

Lemma 1.C (a central maximal torus). If a maximal torus T of a smooth connected affine group C is central, then

$$C = T \times U_C$$

with U_C smooth connected unipotent.

Proof. We construct the affine quotient by T , without presupposing solvability of C . Write $T = D_k(M)$, with M a finite free character group. Right translation grades $A = k[C]$ by M ; put

$B = A_0$. Choose finitely many homogeneous generators a_i of A , of weights m_i . The weight-zero monomials in these generators generate B . They have a finite generating list. Here is the needed integer argument: in $\mathbf{N}^r$ every infinite sequence has an infinite subsequence nondecreasing in every coordinate, by successive extraction in the finitely many coordinates. Thus an antichain is finite. The nonzero vectors v with $\sum v_i m_i = 0$ have finitely many minimal elements; subtracting a minimal element from a comparable vector of the same weight leaves weight zero and decreases the coordinate sum. Induction expresses every such vector as a sum of minimal ones. The corresponding finitely many monomials generate B . Hilbert's basis theorem makes B finitely presented over k .

Centrality makes B a Hopf subalgebra: for $b \in B$, the function $b(c_1 c_2)$ is invariant under right translation by T in either variable. The two exact weight projectors put its comultiplication in $B \otimes B$. Inversion and the counit restrict for the same reason. Their identities restrict along the injective degree-summand inclusions. We obtain an affine group $Q = \text{Spec } B$ and a homomorphism $q : C \rightarrow Q$. The projector $A \rightarrow B$ is B -linear and retracts its inclusion. Consequently $A \otimes_B \kappa(y) \neq 0$ for every $y \in Q$, so q is surjective.

The closed embedding $T \subset C$ gives a graded surjection $A \rightarrow k[M]$. For a basis m_j of M , lift e^{m_j}, e^{-m_j} to a_j, b_j of those weights and put $f = \prod_j a_j b_j \in B$. Its value at the identity is one. All these elements are units in A_f . Their products supply a unit u_m of each weight m , and

$$A_f = \bigoplus_{m \in M} B_f u_m.$$

This is a faithfully flat B_f -module. The torsor algebra map sends $a \otimes c_m$ to $ac_m \otimes e^m$, and its inverse sends $a \otimes e^m$ to $au_m^{-1} \otimes u_m$. Thus q is a T -torsor on $D(f)$. Every $Q(k)$ -point lifts to $C(k)$, by the nonzero finite-type fibre and the weak Nullstellensatz. Translates of $D(f)$ by these image points cover all closed points of Q , and hence Q . Translation in C gives the torsor calculation on each such open. Finite presentation follows from the finite-type Noetherian coordinate rings. It follows that q is an fppf T -torsor globally, representing C/T on all test schemes.

The smooth connected C is integral, and its subring B is an integral finite-type k -algebra. The perfect-field differential-rank calculation of S02 Lemma 3.F and the Jacobian criterion give a nonempty smooth open of Q ; translating it covers all closed points, so Q is smooth. Surjectivity makes Q connected. Also q is smooth: its geometric fibres are translates of the smooth T , and it is flat of finite presentation, so the proved geometric-fibre criterion applies.

Suppose $S \subset Q$ is a nontrivial torus and put $E = q^{-1}(S)$. Then $E \rightarrow S$ is a smooth surjective T -torsor. The group E is smooth and connected. Indeed its smooth components are disjoint, open and closed; its geometric fibres are translates of the connected T . Its ordinary fibres are connected as well, because field-extension projections are surjective. By S08 Lemma G.4.F the images of these open components are open; connectedness of the fibres makes the images of distinct components disjoint. They cover the connected S , so there is just one component. Its commutator maps land in the central T , hence its next derived commutator is trivial. Thus E is solvable, and RG01 Theorem 2.K writes it as $V \rtimes M_E$. The homomorphism $V \rightarrow S$ is trivial because a unitriangular group has no character, by RG01 Lemma 2.A. Hence $V \subset T$; a diagonalizable subgroup of a unitriangular group is trivial by that same lemma. Thus $V = 1$ and E is a torus. It properly contains T , since its quotient S is nontrivial, contradicting maximality. We conclude that Q has no nontrivial torus.

RG01 Lemma 2.D now makes Q unipotent. A sufficiently long derived commutator in C lands in T , since its image in the unitriangular Q is trivial; the next derived commutator is one. Thus C is solvable. RG01 Theorem 2.K gives $C = U_C \rtimes M_C$, with U_C smooth connected unipotent. By RG01 Lemma 2.N the maximal torus T is conjugate to M_C . Centrality makes every conjugate of T equal to T , so $M_C = T$. The action of this central torus on U_C is trivial, giving the asserted direct product. $\square$

Apply Lemma 1.C to $C = C_H(T)$, which is smooth and connected by Lemma 1.1. The torus T is central and remains maximal in this subgroup.

Torus-fixed cosets. The scheme X^T is a finite disjoint union of k -points, and C fixes it pointwise.

Proof. RG01 Lemma 2.1 constructs the projective homogeneous orbit $X = H/B$, and S08 Theorem G.4.C gives its full fppf B -torsor and its smooth scheme structure. The complete equivariant embedding of RG01 Lemma 2.1 realizes X as a closed subvariety of a projective space with a linear H -action. Choose a weight basis for the restricted T -representation, using the weight decomposition proved in RG01. Every standard projective chart given by a nonzero weight coordinate is T -stable. Its intersection with X is affine and smooth; these intersections cover X .

On each of these affine charts, AG-GS, *Diagonalizable groups*, Proposition 3.2 constructs the fixed scheme by the ideal of nonzero-weight elements and proves that it is closed, of finite presentation and compatible with every base change. Its Theorem 3.4, with the universal coefficient calculation of Lemma 3.3, proves smoothness and identifies its tangent functor with the invariant tangent functor of the chart. The fixed schemes agree on chart overlaps because they represent the same universal fixed-point condition. They glue to the closed smooth scheme $X^T \subset X$. Thus X^T is projective and complete. This uses the earlier full infinitesimal proof directly and includes nonreduced test schemes.

At a fixed point $x = gB$, put $B' = gBg^{-1}$. The equality of its schematic stabilizer with B' shows that $T \subset B'$. The quotient map $H \rightarrow X$ is smooth: on the fppf torsor trivializations it is projection from the smooth group B , and smoothness descends. Its differential is therefore surjective, with kernel the tangent space of the fibre B' . It gives the exact T -representation sequence

$$0 \longrightarrow \mathfrak{b}' \longrightarrow \mathfrak{h} \longrightarrow T_x X \longrightarrow 0.$$

The exact weight projections of RG01 give

$$T_x(X^T) = (T_x X)^T = \mathfrak{h}^T / (\mathfrak{b}')^T = \mathrm{Lie} C / \mathrm{Lie} C_{B'}(T).$$

The last identities are the fixed-subgroup tangent calculation of that same AG-GS Theorem 3.4, applied to conjugation on H and B' . The differential of the orbit map $C \rightarrow X^T$ is consequently surjective at the identity. Lemma 1.1 already proves that C is smooth and connected. S02 Lemma 3.A now makes this orbit map smooth near the identity, and translation makes it smooth everywhere. Its image is open by S08 Lemma G.4.F. S08 Theorem G.4.C supplies the full schematic orbit, its fppf stabilizer torsor and its immersion. Flatness of the smooth orbit map descends through the fppf map from C to its homogeneous orbit, so that immersion is flat. Locally it is a quotient $A \rightarrow A/I$. Tensoring its ideal sequence with the flat A/I gives $I/I^2 = 0$; at a point of the orbit the ideal is finite and lies in the maximal ideal, so the determinant Nakayama proof of S01 Lemma 1.2 gives $I = 0$ locally. Thus the orbit immersion is an open immersion, and the full scheme orbit is precisely this open subscheme.

These disjoint open orbits cover X^T . Indeed they contain every k -point, and a nonempty closed complement would have a k -point by the Nullstellensatz. Quasi-compactness gives a finite subcover, hence finitely many orbits. Each is also closed, since its complement is the union of the other open orbits. Thus every orbit is smooth and complete. Since $C = T \times U_C$ and T fixes the entire fixed scheme, each is also the full schematic U_C -orbit of its chosen point.

Such a U_C -orbit is affine, with the following explicit earlier proof route. Let $K \subset U_C$ be its schematic stabilizer. The RG01 detecting-line construction gives a representation and a line kv whose schematic stabilizer in U_C is K . The subgroup K is unitriangular, so RG01 Lemma 2.A makes its line character trivial, including for nonreduced K . Its vector stabilizer is therefore exactly K . S08 Theorem G.4.C identifies U_C/K , retaining the entire K , with the vector orbit in affine space. Lemma 1.2 makes that orbit closed there. The original orbit and this vector orbit represent the same fppf quotient by K , so they are isomorphic. This proves affineness without replacing the stabilizer by its rational points.

An orbit just obtained is integral: it is smooth, and its underlying space is the surjective image of the irreducible smooth connected U_C . As a closed projective subscheme, its algebra of global

functions is finite-dimensional over k , by S01 Theorem 8.4 and its closed-immersion cohomology calculation. This algebra is a domain. Multiplication by any nonzero element is injective and hence surjective on that finite-dimensional vector space; thus it is a field, and algebraic closedness makes it k . The orbit is affine, so it is $\text{Spec } k$. We conclude that X^T is a finite disjoint union of such reduced points. The action of C on each of these point orbits is trivial as a scheme morphism. In particular, at every gB fixed by T , its full schematic stabilizer contains C ; equivalently $C \subset B'$ for every Borel B' containing T . $\square$

The normalizer acts transitively on X^T . If gB is fixed, conjugate $g^{-1}Tg$ to T within B to obtain a representative in $N_H(T)$. Its stabilizer is $N_B(T) = C_B(T) = C$, since conjugation in the solvable decomposition induces the identity on $B/R_u(B)$. Thus X^T is the set $N_H(T)/C_H(T)$, with a simply transitive action of that quotient. A cocharacter separating the finitely many weights in an equivariant projective embedding has exactly the same fixed points as T .

Lemma 1.D (Borel intersections). For a torus $A \subset B$, the full subgroup $K = C_H(A) \cap B = C_B(A)$ is a Borel subgroup of $C = C_H(A)$. Its full quotient C/K is a closed projective smooth orbit in $(H/B)^A$.

Proof. Lemma 1.1 makes C smooth and connected, and Lemma 1.B applied to the solvable B makes K smooth and connected. The equivariant projective embedding and smooth quotient $X = H/B$ are those of RG01 Lemma 2.1 and S08 Theorem G.4.C. Choose a weight basis for A in the embedding representation. The projective weight-coordinate charts are A -stable affine opens. AG-GS-05 Proposition 3.2 and Theorem 3.4 construct their closed smooth fixed schemes with invariant tangent spaces; they agree on overlaps by their universal fixed-point condition. They therefore give a closed smooth projective scheme $X^A \subset X$, with its full fixed-point condition on arbitrary test schemes.

For any $x = gB \in X^A(k)$, put $B' = gBg^{-1}$. Its full stabilizer contains A . The map $H \rightarrow X$ is smooth: it is a flat finitely presented B -torsor whose geometric fibres are translates of the smooth B , so the geometric-fibre criterion applies. Its tangent sequence at x is exact. Exact weight projections therefore give

$$T_x(X^A) = (\mathfrak{h}/\mathfrak{b}')^A = \text{Lie } C / \text{Lie } C_{B'}(A).$$

These centralizer tangent identities are the fixed-subgroup calculation of AG-GS-05 Theorem 3.4. The differential of $C \rightarrow X^A$ is surjective at the identity. S02 Lemma 3.A makes the orbit map smooth near the identity, and translation makes it smooth throughout C . Its image is open by S08 Lemma G.4.F. S08 Theorem G.4.C gives its full orbit immersion and its fppf stabilizer torsor. That immersion is flat, since its composite with the faithfully flat torsor is smooth and flatness descends. In a local closed-immersion presentation $R \rightarrow R/I$, flatness gives $I/I^2 = 0$. At a point of the orbit the finite ideal I lies in the maximal ideal, so Nakayama gives $I = 0$ locally. Hence the full orbit is an open subscheme of X^A .

The disjoint open C -orbits cover all rational points of X^A . Their complement, if nonempty, would be a closed finite-type subscheme with a rational point, so they cover X^A . Quasi-compactness gives finitely many such orbits; each is also closed, because its complement is the union of the others. In particular the orbit O of eB is closed and projective, smooth and integral, and its full stabilizer is exactly K . S08 Theorem G.4.C identifies O with the full quotient C/K , without passing to pointwise stabilizers.

The smooth connected solvable K is contained in a Borel D of C : choose a containing smooth connected solvable subgroup of largest dimension, and use the strict dimension drop for proper closed integral subgroups, as in RG01 Lemma 2.1. Its action on the complete nonempty O has a fixed point by RG01 Lemma 2.M. At eK this says $D \subset eKc^{-1}$ schematically. Thus $\dim D \leq \dim K$. Since $K \subset D$ and both are smooth connected integral groups, a proper inclusion would give a strict dimension increase. They are equal. Consequently K is a Borel of C . Finally RG01 Lemma 2.1 gives conjugacy of Borels in C , so all its Borels are conjugates of this intersection. $\square$

Theorem 1.E (Borel normalizers). Every Borel subgroup of a smooth connected affine group

H over an algebraically closed field satisfies

$$N_H(B) = B$$

as group schemes.

Proof. We first construct the full normalizer. In the conjugation representation on $k[H]$, AG-GS-01 Lemma 5.1 gives a finite-dimensional stable space W containing a finite generating list for the ideal I_B . Put $J = I_B \cap W$. Its stabilizer in H is a closed finite-type subgroup: in a basis of W extending one of J , preservation of J and of J by the inverse matrix is a finite list of equations in the representation matrix and its inverse. This stabilizer represents $N_H(B)$. Indeed, J generates I_B , so stabilizing J preserves that ideal in both directions. Conversely a normalizing section preserves I_B and W , hence their intersection. These statements remain true over every k -algebra R : exact scalar extension identifies $J_R = I_{B,R} \cap W_R$, and the chosen ideal generators still generate. Thus this construction includes arbitrary nonreduced test schemes. Write N for the resulting subgroup. Choose $T \subset B$ maximal in H , as in the preceding fixed-coset argument. The torus-fixed-coset proof gives $C_H(T) \subset B$, and hence $\mathfrak{h}^T \subset \mathfrak{b}$. We compute the tangent space of N directly. An infinitesimal element with tangent vector $X \in \mathfrak{h}$ normalizes B precisely when, for its universal section b , infinitesimal conjugation stays in B . Translating the resulting tangent at b to the identity gives

$$X - \text{Ad}(b)X \in \mathfrak{b} \otimes_k k[B].$$

To check this criterion schematically, substitute the infinitesimal conjugation into every generator of I_B : its constant coefficient is zero on B , and its first-order coefficient is the pairing with this translated tangent vector. Vanishing of all such coefficients is exactly membership in the tangent space of B ; translation identifies that space with $\mathfrak{b}$ on all test algebras. The inverse infinitesimal conjugation gives the negative coefficients, so the criterion gives preservation in both directions. Consequently

$$\text{Lie } N/\mathfrak{b} = (\mathfrak{h}/\mathfrak{b})^B \subset (\mathfrak{h}/\mathfrak{b})^T = 0,$$

where the last equality uses exact weight projections and $\mathfrak{h}^T \subset \mathfrak{b}$. Thus the tangent dimension of N at the identity is $\dim B$. Its local dimension is at least $\dim B$, since it contains the smooth B , and at most its tangent dimension, by the local embedding-dimension bound. Equality makes this local ring regular; the perfect-field Jacobian criterion makes N smooth at the identity. Translation proves smoothness at every rational point, and hence everywhere. It will therefore suffice to show $N(k) = B(k)$, since equations vanishing at all points of the reduced N vanish by the strong Nullstellensatz.

We also need the following consequence of projectivity: every rational element of H lies in a Borel. Choose $t \in T(k)$ avoiding the equations $\chi(t) = 1$ for every nonzero adjoint weight χ . Such a point exists, because these are finitely many proper closed subsets of the integral torus; the weak Nullstellensatz supplies a rational point of their open complement. For $T = 1$ the same calculation has no nonzero weights. The differential of

$$H \times C_H(T) \longrightarrow H, \quad (g, c) \longmapsto gcg^{-1},$$

at $(1, t)$, with tangent spaces translated to the identity, is $(1 - \text{Ad } t)X + Y$. It is surjective: the first summand covers every nonzero weight and $\text{Lie } C_H(T) = \mathfrak{h}^T$ covers weight zero. S02 Lemma 3.A and S08 Lemma G.4.F give a nonempty open image, dense in the integral H . The image lies among conjugates of B , since $C_H(T) \subset B$.

The incidence of pairs (h, x) with $hx = x$ is closed in $H \times H/B$, by the separated diagonal. Its projection to H has closed image by the projectivity of RG01 Lemma 2.1 and S01 Lemma 7.A and Corollary 7.B. This image contains the preceding dense open, so is all of H . At any rational h , its nonempty finite-type fibre has a rational fixed coset; lifting that coset through the full B -torsor gives $h \in gBg^{-1}$ for some $g \in H(k)$. In particular, if B is normal in H , then $B(k) = H(k)$ and reducedness gives $B = H$ schematically.

We now induct on $\dim H$; the zero-dimensional smooth connected group is trivial. Let $n \in N(k)$. RG01 Lemma 2.N conjugates nTn^{-1} back to T within B . Multiplying n by that element of B preserves both its membership in $N(k)$ and the question whether it belongs to $B(k)$. We may assume that n normalizes T . If $T = 1$, RG01 Lemma 2.D makes H unipotent, hence solvable; its Borel is then H itself. Otherwise consider the torus homomorphism

$$f : T \longrightarrow T, \quad t \longmapsto ntn^{-1}t^{-1}.$$

If f is not surjective, its character matrix has rank smaller than $\dim T$. The integer basis calculation of S02 Lemma 3.M expresses its kernel as a diagonalizable group with a free character summand of positive rank. The corresponding split factor supplies a nontrivial torus $A \subset \ker f$. Thus $n \in C_H(A)$, and it normalizes $K = C_H(A) \cap B$, a Borel of this smooth connected centralizer by Lemma 1.D. If the centralizer is proper in H , its dimension is strictly smaller; induction gives $n \in K \subset B$.

If instead $C_H(A) = H$, the torus A is central. The quotient construction in Lemma 1.C uses only centrality through its construction of the smooth connected affine $Q = H/A$ and its full smooth A -torsor q . S08 Lemma G.4.D gives $\dim Q = \dim H - \dim A < \dim H$. The closed subgroup $B \subset H$ descends along q to a closed subgroup $\bar{B} \subset Q$: right multiplication by $A \subset B$ preserves its defining ideal, so the ideal effectivity argument of S04 Lemmas A1.1 and A1.2 gives

$$q^{-1}(\bar{B}) = B, \quad B \longrightarrow \bar{B} \text{ an fppf } A\text{-torsor}.$$

Its coordinate ring injects into the reduced $k[B]$, so $\bar{B}$ is reduced and connected. RG01 Lemma 2.L makes it smooth; derived commutator identities descend through the torsor, so it is solvable. The central A acts trivially on H/B , since $ahB = haB = hB$. Morphism descent gives an action of Q there with full stabilizer $\bar{B}$, and the same local representatives identify its full quotient with the projective H/B . Any containing Borel of Q has a fixed point on this quotient by RG01 Lemma 2.M, and the stabilizer inclusion and strict dimension argument of Lemma 1.D force it to equal $\bar{B}$. Thus $\bar{B}$ is a Borel. Induction in Q gives $q(n) \in \bar{B}$, so the exact inverse-image identity gives $n \in B$.

It remains to handle surjective f . The RG01 detecting-line construction gives an H -representation and a line kv with full stabilizer N ; write $\chi : N \rightarrow \mathbf{G}_m$ for its character. This character kills commutators, so $\chi(f(t)) = 1$ on every test algebra. Surjectivity of the torus homomorphism makes its character pullback injective, hence $\chi|_T = 1$. RG01 Lemma 2.A also makes χ trivial on the unipotent radical of B . The split-solvable decomposition therefore makes B fix v as a vector. The map $H \rightarrow V$, $h \mapsto hv$, descends along its full B -torsor to $H/B \rightarrow V$. The global-function algebra of this integral projective quotient is finite-dimensional by S01 Theorem 8.4; being a domain over algebraically closed k , it is k . All coordinate functions of the descended map are constant, with value v at eB . Thus H fixes v as a scheme morphism, and the detecting-line stabilizer gives $H = N$. Now B is normal in H , so the incidence argument above gives $H = B$. This proves $n \in B$ in the final case and completes the induction. The smooth normalizer and the strong Nullstellensatz turn equality of rational points into the required full scheme equality. $\square$

We may consequently identify H/B with the variety of Borel subgroups. The preceding simply transitive action on its torus-fixed points is also the action on the Borels containing T .

Lemma 1.3. Let H be smooth connected affine over the algebraically closed field k , and let $T \subset H$ be a maximal torus. If $\mathcal{B}^T$ is the collection of Borels containing T , put

$$J = \bigcap_{B' \in \mathcal{B}^T} R_u(B'), \quad I = (J_{\text{red}})^0.$$

Then $I \subset R_u(H)$, as a schematic inclusion.

Proof. Write $X = H/B$, where B contains T . The full normalizer theorem identifies its rational points with the Borel subgroups, and the torus-fixed-coset proof makes X^T a finite disjoint union of reduced points. Thus $\mathcal{B}^T$ is finite and nonempty, and J is the full finite intersection of closed

subgroup schemes. RG01 Lemma 2.L makes I smooth connected. It is unipotent, since it embeds in the unitriangular $R_u(B)$. For each $n \in N_H(T)(k)$, conjugation permutes the Borels in $\mathcal{B}^T$ and their unipotent radicals, which are characteristic by Lemma 1.A. It preserves the full J and hence I . This invariance under actual rational conjugating elements is sufficient for the chart transport below.

RG01 Lemma 2.1 embeds X equivariantly as a closed subscheme of $\mathbf{P}_{\text{lines}}(V) = \mathbf{P}(V^\vee)$. Replace V by the span of its point lines. This span is H -stable on all test algebras: the matrix entries into its quotient vanish on $H(k)$, and hence on the reduced H by the strong Nullstellensatz. The embedding factors through this span by the same argument on reduced X . Thus X spans V .

Choose a cocharacter λ separating the distinct T -characters in V . Here is an explicit choice when $T = \mathbf{G}_m^r$, $r > 0$. Write each difference of two characters as $(a_1, \dots, a_r)$. The nonzero integer polynomial $\sum_i a_i M^{i-1}$ has finitely many integer roots by S02 Lemma 4.A. Choose one positive integer M outside these finitely many root sets and put $\lambda(u) = (u, u^M, \dots, u^{M^{r-1}})$. For $r = 0$ use the unique cocharacter. On weight-coordinate projective charts, the ideals of the T -fixed and λ -fixed schemes are generated by the same coordinates: a difference character is nonzero precisely when its pairing with λ is nonzero. The universal weight-ideal calculation of AG-GS-05 Proposition 3.2 therefore gives $X^\lambda = X^T$, including arbitrary test algebras.

Let V_0 be the weight space of lowest λ -weight among those in V . The open $U \subset X$ where projection to V_0 is nonzero is nonempty, since X spans V . For a point in this open, multiply its weight coordinates by u^{w-w_0} . All exponents are nonnegative, so this defines the limit at $u = 0$, in $\mathbf{P}_{\text{lines}}(V_0)$. The limit is in X : on each projective chart its equations vanish in the Laurent polynomial ring and hence in the polynomial ring. Thus projection defines a morphism

$$U \longrightarrow X \cap \mathbf{P}_{\text{lines}}(V_0) \subset X^T.$$

The integral U has irreducible image in this finite reduced scheme, so that image is one point $x = [v]$. Every nonzero projected point line is consequently kv ; projection of the point lines spanning V spans V_0 , so $V_0 = kv$. Choose the coordinate covector ℓ with $\ell(v) = 1$ and zero on the other weight spaces. The chart

$$X_x = X \cap D(\ell)$$

is affine, contains x , is T -stable, and λ contracts every one of its rational points to x .

The group $N_H(T)(k)$ is transitive on $X^T(k)$. Indeed a fixed coset gB has $g^{-1}Tg \subset B$; this is a maximal torus of B . RG01 Lemma 2.N supplies $b \in B(k)$ with $b^{-1}g^{-1}Tgb = T$, so $gb \in N_H(T)(k)$ represents that same coset. Apply this to two fixed cosets to obtain an element carrying one to the other. Transport X_x to an affine T -stable chart X_y containing each $y \in X^T(k)$. These charts cover X . For a rational point z , the reduced closure of its T -orbit is projective and T -stable by S08 Theorem G.4.2. RG01 Lemma 2.M gives a fixed point y in this closure. If $z \notin X_y$, its entire orbit and closure would be contained in the closed T -stable complement, contradicting $y \in X_y$. Thus all rational points lie in these finitely many charts; the Nullstellensatz rules out a nonempty closed complement.

We prove their invariance under I . Every $H(k)$ -orbit in $\mathbf{P}_{\text{lines}}(V^\vee)$ meets the affine chart of covectors nonzero on v . Otherwise a nonzero covector would vanish on every translate of v ; these translates span V , since H is transitive on X . On this dual chart λ^{-1} contracts to $[\ell]$, by the same coordinate calculation. S08 Theorem G.4.2 gives a closed H -orbit in this projective space. It meets the dual chart, so its closedness includes $[\ell]$. It is therefore exactly the orbit $H[\ell]$. S08 Theorem G.4.C gives this closed projective orbit with its full stabilizer P and its fppf torsor $H \rightarrow H[\ell]$.

The action of B on this nonempty projective orbit has a fixed point by RG01 Lemma 2.M. Its full stabilizer supplies an H -Borel $B_1 \subset P$. Both B_1 and the torus T , which fixes $[\ell]$, lie in the smooth connected $(P_{\text{red}})^0$, by RG01 Lemma 2.L. A maximal torus T_1 of H in B_1 , and T , are maximal there too. RG01 Theorem 2.2 conjugates T_1 to T by an element of this reduced identity component. It carries B_1 to a Borel $B_2 \subset P$ containing T . By its definition $I \subset R_u(B_2) \subset P$ schematically.

It therefore preserves the covector line $k\ell$, and hence the chart $D(\ell)$, on every test algebra. This proves invariance of X_x . The previously proved invariance of I under each transporting normalizer element gives invariance of every X_y . Neither smoothness nor connectedness of P has been assumed. For $z \in X(k)$, the reduced projective closure of Iz is I -stable, by S08 Theorem G.4.2. The smooth connected solvable I has a fixed point z_0 there, by RG01 Lemma 2.M. Choose an invariant affine chart X_y containing z_0 . If the orbit met its invariant closed complement, that complement would contain the whole point orbit and its closure, contradicting $z_0 \in X_y$. Hence the orbit lies in X_y . Lemma 1.2 makes its full orbit closed in this chart, so it contains z_0 ; transitivity makes it a point. S08 Theorem G.4.C gives its reduced scheme structure, so the full I -action fixes z . This holds for every rational z ; the reducedness of $I \times X$ and the strong Nullstellensatz give equality of the action with the projection on the whole scheme.

For the final normality argument, form the schematic intersection

$$K = \bigcap_{g \in H(k)} gBg^{-1}.$$

It is a closed finite-type subgroup: the sum of these conjugate ideals in the Noetherian $k[H]$ is generated by finitely many elements, each from a finite subsum. The resulting finite intersection represents the same condition on every test algebra. The stabilizer of gB is the entire gBg^{-1} , so the preceding scheme action gives $I \subset K$. Put $K_0 = (K_{\text{red}})^0$. It is smooth connected, contains I , and is solvable because it is contained in B . Every rational conjugation by H preserves K_0 ; the reducedness of $H \times K_0$ and the strong Nullstellensatz make K_0 normal in H schematically.

RG01 Theorem 2.K gives $K_0 = U \rtimes D$, with $U = R_u(K_0)$ smooth connected unipotent and D a torus. Every character of I is trivial by RG01 Lemma 2.A, so its map to the torus quotient D is trivial and $I \subset U$. The uniqueness of the unipotent radical makes U invariant under every rational conjugation by H ; reducedness of $H \times U$ and the same equation argument make it normal schematically. Lemma 1.A then gives $U \subset R_u(H)$, proving $I \subset R_u(H)$. $\square$

This is the geometric mechanism in Chevalley's description of the unipotent radical. It gives exactly the part needed for centralizers without presupposing roots.

2. Centralizers and weight zero

Theorem 2.1. If G is connected reductive over an algebraically closed field and T is maximal, then $C_G(T) = T$. More generally, the centralizer of any torus in G is connected reductive.

Proof. For a maximal T , Lemmas 1.1 and 1.C give the full decomposition

$$C = C_G(T) = T \times U_C,$$

where U_C is smooth connected unipotent. The torus-fixed-coset proof gives $C \subset B'$ for every Borel B' containing T . Its split-solvable quotient $B'/R_u(B')$ is a torus by RG01 Theorem 2.K. The homomorphism from U_C to this quotient is trivial: every character pulled back to a unitriangular group is trivial by RG01 Lemma 2.A, and the torus characters generate its coordinate algebra. Thus $U_C \subset R_u(B')$ on all test algebras. It lies in their full intersection J , and its reduced connected identity-containing source factors through $I = (J_{\text{red}})^0$, by RG01 Lemma 2.L. Lemma 1.3 gives $U_C \subset R_u(G)$. Reductivity makes this radical trivial, so $C_G(T) = T$ as group schemes.

Now let $A \subset G$ be any torus. Choose a maximal torus $T \supset A$: among containing tori choose one of largest dimension, bounded by $\dim G$; every proper closed inclusion of integral tori increases dimension, so it is maximal. Put $C = C_G(A)$, smooth connected by Lemma 1.1, and put $R = R_u(C)$, whose existence and normality are Lemma 1.A. For every Borel B' of G containing T , Lemma 1.D makes $K' = C \cap B'$ a Borel of C , with projective full quotient C/K' . The action of the smooth connected solvable R on this quotient has a fixed point by RG01 Lemma 2.M. Lift that point

through its fppf stabilizer torsor to $c \in C(k)$; its full stabilizer gives $R \subset cK'c^{-1}$. Since R is normal in C , this is equivalent to $R \subset K' \subset B'$. Its map to $B'/R_u(B')$ is again trivial by the character argument, so $R \subset R_u(B')$. Thus R factors through the same I attached to G, T , and Lemma 1.3 gives

$$R \subset R_u(G) = 1.$$

The smooth connected affine C therefore has trivial unipotent radical, which is precisely reductivity over this field. This proves both assertions with their full scheme structures. $\square$

The Borel-intersection step here is the complete projective fixed-orbit argument of Lemma 1.D; no additional closure or rigidity argument is needed.

Theorem 2.2. If $G \rightarrow S$ is reductive and T is a maximal S -torus, then $C_G(T) = T$. For any subtorus A , the group $C_G(A)$ is reductive over S . These centralizers commute with arbitrary base change.

Proof. A torus splits étale locally by its definition. On a split affine chart, conjugation by A grades the affine coordinate algebra of G . AG-GS-05 Proposition 3.2 represents the fixed group by the ideal generated by all nonzero-weight elements, proves finite presentation, and proves compatibility with every base change, including nonreduced bases. Its Theorem 3.4 proves smoothness of this fixed group when G is smooth. The fixed-point functor is the centralizer functor on every test scheme. Thus the constructions agree canonically on overlaps; the ideal and algebra effectivity in S04 Lemmas A1.1 and A1.2 gives the global affine closed subgroup $C_G(A)$. Finite presentation and flatness descend by S04 Lemmas A1.3 and A1.3a; its geometric fibres are smooth by the split calculation, so the geometric-fibre criterion of AG-FSE-05 Theorem 3.1 proves global smoothness. The same canonical fixed condition and the local base-change calculation prove arbitrary base-change compatibility globally.

Each geometric fibre is the connected reductive centralizer in Theorem 2.1. Consequently this smooth affine group of finite presentation is reductive by the definition of a reductive group scheme. In particular this proves the assertion for arbitrary A before treating maximal T .

For maximal T , its closed inclusion $i : T \rightarrow C_G(T)$ is an isomorphism on every geometric fibre by Theorem 2.1 and the just-proved base-change compatibility. Both groups are smooth of finite presentation over S . Their relative tangent map is an isomorphism at every point of T , since it becomes one over a geometric residue field; the exact tangent and differential scalar-extension calculation identifies that map with the one on the fibre. S02 Lemma 3.A therefore makes i étale everywhere. Its finite presentation follows also from the affine-pair presentation calculation of AG-MO-02 Theorem 3.1. Its étale smoothness makes it flat by AG-CA-18 Theorem 6.1.

Here is the scheme equality over the original base, without a fibrewise-isomorphism shortcut. On an affine chart of $C_G(T)$, the closed inclusion has algebra $D \rightarrow D/I$, with I finitely generated by finite presentation. Flatness of D/I makes the tensor of $I \hookrightarrow D$ injective; that tensor map is zero, so

$$I \otimes_D D/I = I/I^2 = 0.$$

Every target point is in the image of i : extend its residue field to an algebraic closure and use the fibre isomorphism. Thus, at every prime $\mathfrak{p}$ of D , the finite ideal $I_{\mathfrak{p}}$ is contained in the maximal ideal and satisfies $I_{\mathfrak{p}} = I_{\mathfrak{p}}^2 \subset \mathfrak{p}D_{\mathfrak{p}}I_{\mathfrak{p}}$. The determinant Nakayama proof of S01 Lemma 1.2 gives $I_{\mathfrak{p}} = 0$. Localization detects the zero module, so $I = 0$. All these affine charts therefore give $T = C_G(T)$, with the original base and every nilpotent retained. $\square$

Theorem 2.3. A reductive group scheme admits maximal tori étale locally. Any two maximal tori are conjugate étale locally.

Proof. Fix $s \in S$. AG-RG-01 Theorem 4.2 gives a geometrically maximal torus $M \subset G_s$ over $\kappa(s)$ itself. AG-RG-01 Lemma 3.1 extends that specified closed embedding to a torus $T \subset G_U$ on a pointed étale neighbourhood $(U, u) \rightarrow (S, s)$, with $\kappa(u) = \kappa(s)$ and fibre embedding equal to $M \subset G_s$. No separate residue extension or assertion about an unrepresented embedding functor is

needed.

Theorem 2.2, applied to this subtorus before it is known to be maximal on U , makes $C = C_{G_U}(T)$ a smooth affine reductive group of finite presentation and identifies every geometric fibre with the centralizer of the fibre torus. Write $r = \dim M$. The torus rank is locally constant, so restrict to an open neighbourhood of u on which it is r . The relative Lie module

$$\mathfrak{c} = e_C^* \mathcal{T}_{C/U}$$

is finite locally free: the standard smooth-coordinate calculation in AG-RG-S02 Lemma 3.1 makes the relative differential bundle locally free, and dualizing and pulling back by the identity gives this module. Its scalar extension is the Lie algebra of the fibre centralizer, by the differential base-change calculation of AG-CA-15 Theorem 2.1. At a geometric point above u , Theorem 2.1 gives $C_{G_{\bar{u}}}(T_{\bar{u}}) = T_{\bar{u}}$, so $\mathfrak{c}$ has rank r there. A local free basis of $\mathfrak{c}$ consequently gives an open neighbourhood of u on which its rank is r . Restrict to that neighbourhood.

For every geometric point v of this smaller U , the smooth connected group C_v has dimension r , since its dimension equals its tangent dimension at the identity. The closed subgroup T_v is a torus of the same dimension. A smooth connected finite-type group over an algebraically closed field is integral by the component calculation in AG-RG-01 Lemma 2.L. A proper closed subset of an integral finite-type scheme has smaller dimension, by AG-CA-09 Theorem 4.2. Thus T_v has the whole underlying space of C_v . Its defining ideal is radical because T_v is reduced, and its radical is zero because C_v is reduced and the underlying space is the whole space. Therefore $T_v = C_v$ as schemes. Any torus containing T_v centralizes it and hence lies in $C_v = T_v$; it equals T_v . Thus every fibre torus is maximal, which makes T a maximal U -torus. These open restrictions and étale neighbourhoods exist around every s , proving local existence over the original, possibly nonreduced base.

Now let $T_1, T_2 \subset G$ be maximal S -tori. On each geometric fibre they are conjugate by AG-RG-01 Theorem 2.2. The full equality transporter

$$P = \text{Transp}_G(T_1, T_2)$$

is represented by a scheme of finite presentation and is smooth by AG-RG-01 Lemma 3.1: its Laurent-coefficient equations test equality of the subgroup schemes, including on algebras with nilpotents. Each geometric fibre has a conjugating point, so $P \rightarrow S$ is surjective.

Apply AG-FSE, *Smooth morphisms*, Theorem 5.2 to this smooth surjection. Its proof first chooses a point of the smooth fibre with finite separable residue extension, realizes that extension by a pointed étale neighbourhood, and then pulls an étale coordinate chart back along a section of affine space. The resulting étale neighbourhood $V \rightarrow S$ has an S -map $V \rightarrow P$. The represented transporter condition says exactly that its image $g \in G(V)$ satisfies $gT_{1,V}g^{-1} = T_{2,V}$ as subgroup schemes. This proves étale-local conjugacy. $\square$

The statement is local around every point, and the conjugating element belongs to the group over the chosen neighbourhood. It does not assert a global section of the transporter.

If T is split, write

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_{\alpha}.$$

Here Φ is the finite collection of nonzero weights on a neighbourhood where their ranks are constant. Theorem 2.2 says that the zero-weight summand is exactly $\mathfrak{t}$. In the next lesson we prove that the nonzero summands have rank one and construct their root groups. The arguments about regularity below need the weight decomposition and weight zero; they do not need that rank-one conclusion.

3. Three uses of the word regular

Let G be connected reductive over an algebraically closed field, with rank $r = \dim T$. A group element is **dimension-regular** if its centralizer has the minimum dimension r . A **regular semisimple** element is a semisimple element g such that $C_G(g)^0$ is a maximal torus. A **strongly regular semisimple** element additionally has $C_G(g)$ itself equal to that torus. This last condition is stronger.

There is also the convention used in SGA 3, Exposé XIII: a group element is regular if the generalized eigenspace for eigenvalue 1 in its adjoint action has the minimum dimension, the nilpotent rank. For reductive groups this minimum is r , and this condition selects exactly the regular semisimple elements. We verify this identification below. Thus dimension-regular unipotent elements are not regular in that convention.

For $t \in T$, define

$$T_{\text{rs}} = \{t : \alpha(t) \neq 1 \text{ for every } \alpha \in \Phi\}.$$

The inequalities mean invertibility of the indicated functions over a general base. Since every α is a nontrivial character, its equation $\alpha = 1$ is a proper closed subset of a torus over an algebraically closed field. A finite union of such subsets cannot cover the irreducible torus. Hence T_{rs} is open and dense, and is nonempty in every geometric fibre in the relative case.

For a semisimple t , its centralizer is smooth: the closure of the subgroup generated by t is of multiplicative type, and invariants for such a group are exact. Its Lie algebra is the fixed space of $\text{Ad}(t)$. Consequently

$$\text{Lie}(C_G(t)) = \mathfrak{t} \oplus \bigoplus_{\alpha(t)=1} \mathfrak{g}_\alpha.$$

The identity centralizer equals T precisely on T_{rs} . Indeed when no moving weight is fixed, the smooth identity centralizer has the same tangent dimension as T . Its closed torus inclusion has an isomorphic differential, so S02 Lemma 3.A and the étale-monomorphism proof of S04 make it open. It is also closed; connectedness makes it the whole identity component. If a moving weight is fixed, its nonzero summand gives strictly larger tangent dimension, and smoothness makes the identity centralizer strictly larger than T .

Theorem 3.1. The regular semisimple locus G_{rs} is a dense open subset. Its elements lie in a unique maximal torus.

Proof. First construct an open set of conjugates of torus elements. Consider the conjugation map

$$q : G \times T_{\text{rs}} \longrightarrow G, \quad (g, t) \longmapsto gtg^{-1}.$$

After translations identify tangent spaces, its differential at $(1, t)$ is

$$(X, Y) \longmapsto (1 - \text{Ad}(t))X + Y.$$

It is surjective: the second term covers $\mathfrak{t}$, while the first is invertible on every nonzero weight summand. AG-RG-S02, Lemma 3.A, therefore makes q smooth, so its image is nonempty and open. We justify density precisely. The affine finite-type G is Noetherian by the Hilbert basis proof and has finitely many irreducible components. If two met, their nonempty closed intersection would have a rational closed point by the Nullstellensatz. Smoothness makes its local ring regular by the field Jacobian proof; AG-CA-14 Theorem 1.1 makes that local ring a domain, contradicting two component primes there. Thus the components are disjoint and each is open and closed.

Connectedness leaves one component. Hence G is irreducible and this nonempty open image is dense.

Here is a useful consequence of that density. Choose a Borel B containing T . In $G \times G/B$ the incidence scheme

$$I = \{(g, z) : gz = z\}$$

is closed, since the flag variety is separated. Its projection to G is proper: AG-RG-01, Lemma 2.1 and AG-RG-S01, Lemma 7.A and Corollary 7.B make G/B proper, and Corollary 7.B preserves properness under this base change and restriction to the closed incidence. Its image contains the dense open just constructed, and is consequently all of G . Over our algebraically closed field, a nonempty fibre has a rational point. Thus **every element of G belongs to a Borel subgroup**.

Let b be a semisimple element of $B = U \rtimes T$. In a faithful representation diagonalize b and take the reduced closure D of its integral powers. It is a closed subgroup of a diagonal torus, hence diagonalizable by the explicit Hopf-quotient proof in AG-GS-05, Theorem 4.2. Since it is reduced, its finite character torsion has order invertible in k , by the equation calculation of AG-GS-05, Theorem 4.4.

The intersection $D \cap U$ is trivial by AG-RG-01, Lemma 2.A. Thus $D \rightarrow T$ is a monomorphism and a closed immersion by a direct character calculation. Write $D = D(M)$ and $T = D(\mathbf{Z}^r)$. The kernel has character group $M/\text{im}(\mathbf{Z}^r \rightarrow M)$, by AG-GS-05 Theorem 4.1. A diagonalizable group is trivial only when its character group is zero, by its monomial basis. Thus the character map is surjective, and so is the group-algebra map defining $D \rightarrow T$. Let A denote that image. In $U \rtimes A$, the subgroup D is the graph of a regular crossed homomorphism $A \rightarrow U$. AG-RG-01, Lemma 2.B supplies a T -stable filtration of U , hence an A -stable one, and AG-RG-01, Lemma 2.C makes this cocycle principal. Conjugation by its correcting $u \in U(k)$ places D and b in T .

The incidence proof just given puts every group element in a Borel. Its multiplicative Jordan factors belong to that same closed group, by the Jordan proof in AG-RG-01, Section 4. Applying the preceding paragraph to the semisimple factor proves that every semisimple group element belongs to a maximal torus. Hence the conjugation map $G \times T_{\text{rs}} \rightarrow G$, whose differential was computed above, has image exactly the regular semisimple locus.

Any maximal torus containing g belongs to $C_G(g)^0$, because a torus is connected. That identity centralizer is already a maximal torus. This proves uniqueness. $\square$

Minimum centralizer dimension is indeed r . On the dense open just constructed it is r . Here is the complete local dimension bound needed for the converse. The universal centralizer $H \rightarrow G$ is an affine scheme of finite presentation: in the finite coordinate generators of G , impose equality of the two products with the universal element. Its identity is a section. Suppose at a rational point g_0 its fibre has dimension $d < r$. The full finite Noether-normalization proof in AG-CA-09 Theorem 4.2 gives d functions on that fibre making its coordinate algebra finite over a polynomial algebra. Lift these functions to the universal centralizer algebra and subtract their restrictions to its identity section; subtracting constants on the fibre preserves finiteness. They define $f : H \rightarrow \mathbf{A}_G^d$ with the identity sent to zero. Its fibre at g_0 is finite, so f is locally quasi-finite there. S07 Theorem 3.4 proves that its locally quasi-finite locus is open. Pulling that open back along the identity section gives an open neighbourhood V of g_0 in G .

For every rational $g \in V$, the fibre morphism is quasi-finite near its identity. The full affine Zariski Main factorization of S04 Theorem D7.1 writes that neighbourhood as an open in a finite scheme over affine d -space. The integral dimension calculation in AG-CA-09 therefore bounds its local dimension at the rational identity by d . For any finite-type group over this algebraically closed field, that local dimension is its whole dimension: choose a rational closed point on each irreducible component by the Nullstellensatz and translate it to the identity. Translation carries the entire component to one through the identity, without changing its dimension. The exact closed-point

component/local-dimension calculation of AG-CA-09 Corollary 4.4 then puts each component dimension below the local identity dimension; the reverse inequality is immediate. No smoothness of the fibre group is used. Thus $\dim H_g \leq d$ throughout $V(k)$. The open V meets the dense regular semisimple open and has a rational point in their intersection, again by the Nullstellensatz. At that point its fibre dimension is r , a contradiction. This proves the minimum bound without importing an unwritten semicontinuity theorem.

For the SGA convention, write the adjoint characteristic polynomial as

$$P_g(x) = \det(x - \text{Ad}(g)) = (x - 1)^r Q_g(x).$$

This divisibility holds for all g , since it holds on the dense open locus already proved, and the coefficients are regular functions. The SGA regularity condition is $Q_g(1) \neq 0$. Jordan decomposition $g = g_s g_u$ preserves the eigenvalues of the adjoint action, so this condition first implies that $g_s \in G_{\text{rs}}$.

Choose a Borel containing g , by the incidence argument. Its Jordan factors belong to this Borel. Conjugate within it to arrange $g_s = t \in T_{\text{rs}}$. Its commuting unipotent factor g_u belongs to U , because a unipotent element in the torus quotient is one.

Let D be the reduced diagonalizable closure of the powers of $\mathfrak{t}$. Then $C_U(t) = U^D$: regular conjugation identities holding on all powers hold on their schematic closure, and the reverse containment is immediate. Lemma 1.B makes this group smooth connected. Its Lie algebra is the $\mathfrak{t}$ -fixed part of $\text{Lie } U$, by AG-GS-05, Theorem 3.4. Since $U \cap T = 1$, $\text{Lie } U \cap \text{Lie } T = 0$. Regular semisimplicity gives $(\text{Lie } G)^{\mathfrak{t}} = \text{Lie } T$; consequently $\text{Lie}(U^D) = 0$. A smooth connected zero-dimensional algebraic group over k is the identity. Thus $g_u = 1$.

Conversely a regular semisimple element has exactly r eigenvalue-one directions by the weight calculation. This proves the claimed equivalence in all characteristics. $\square$

The minimum generalized zero-eigenspace dimension for $\text{ad}(X)$ similarly defines SGA regularity in a Lie algebra. It is distinct from minimum kernel dimension. Its openness has a complete polynomial proof: write $\det(x - \text{ad}(X)) = \sum_j c_j(X) x^j$ on the affine space $\mathfrak{g}$, and let d be the least index with coefficient polynomial not identically zero. Earlier coefficients vanish identically, and the open $c_d \neq 0$ is nonempty by the polynomial-avoidance proof of S06 Section 1. The generalized-eigenspace decomposition there, followed by the block determinant calculation, identifies the zero-eigenvalue multiplicity with the dimension of the generalized zero space. Thus its minimum is exactly d , attained precisely on that open. A semisimple $X \in \mathfrak{t}$ has Lie centralizer $\mathfrak{t}$ precisely when every occurring nonzero weight differential $d\alpha(X)$ is nonzero, by the weight decomposition. Each nonzero differential is a nonzero linear polynomial; their finite product is nonzero in the polynomial domain, so avoidance supplies such a vector whenever no differential vanishes identically. In characteristic zero every nonzero integer character has a nonzero differential: in a torus basis at least one of its integer exponents is nonzero and remains nonzero in the field. In positive characteristic that exponent can vanish, and group regularity still works.

Lemma 4.P (the projective matrix group used below). Over every base on which two is invertible, PGL_2 is the smooth affine reductive group of invertible two-by-two matrices modulo locally invertible scalar multiples. Its diagonal torus is split and geometrically maximal. The scalar quotient and all centralizer computations below are sheaf statements over arbitrary test algebras.

Proof. We construct the quotient first, over $\mathbf{Z}$. Let $B = \mathbf{Z}[a, b, c, d]$, $f = ad - bc$, with each variable of degree one, and let $A = (B[f^{-1}])_0$ be its degree-zero subring. It is generated by the ten degree-two monomials divided by f : a monomial of degree $2m$ factors into m pairs. For each variable $x_i \in \{a, b, c, d\}$ put $g_i = x_i^2/f$. The four opens $D(g_i)$ cover $\text{Spec } A$. Indeed, a prime containing all g_i would contain ad/f and bc/f , whose squares are $g_a g_d$ and $g_b g_c$; their difference is one, a contradiction. On $D(g_i)$ one has

$$x_j/x_i = (x_i x_j/f)/g_i, \quad f/x_i^2 = g_i^{-1}.$$

Conversely these ratios express every generator of A and g_i^{-1} . Thus this localized ring is exactly the three-variable polynomial chart, with $x_i = 1$, localized at its normalized determinant. The equalities are identities inside $B[f^{-1}, x_i^{-1}]$, and remain the displayed mutually inverse presentations after every base change. Gluing these four charts by the ratio transitions of S01 Section 7 gives exactly the determinant-nonzero open in projective three-space. This proves its affineness, finite presentation and smoothness by explicit affine charts, without importing a general Proj affineness assertion.

A point of this projective open is locally a tuple of four entries, at least one a unit, whose determinant is a unit, and different local tuples differ by a unit scalar. Thus the map $\mathrm{GL}_2 \rightarrow \mathrm{Spec} A$ is a Zariski locally trivial $\mathbf{G}_m$ -torsor: normalize that chosen entry to one, and retain its scalar as the other coordinate. In particular it is fppf surjective on sheaves. Matrix multiplication and the cofactor inverse are compatible with these scalar transitions. They therefore glue to a group law and inverse on the projective open, with scalar kernel, giving precisely PGL_2 . No assertion that every projective class has a global matrix lift is used. On a geometric field, its space is the surjective image of the irreducible GL_2 of RG01 Lemma 1.M, and is consequently irreducible and connected.

The diagonal locus $b = c = 0$ is a closed torus with unit ratio a/d , hence is $\mathbf{G}_m$ on every test algebra. The torsor charts compute the tangent algebra as $\mathfrak{gl}_2/kI$, since multiplying a lift $I + \epsilon X$ by $1 + \epsilon q$ changes X by qI . Conjugation by the ratio torus has nonzero full weights $1, -1$ on E_{12}, E_{21} and a one-dimensional zero-weight diagonal quotient. If N is a smooth connected normal unipotent subgroup over an algebraically closed field, its intersection with this torus is diagonalizable and trivial, by GS05 Theorem 4.2 and RG01 Lemma 2.A. Thus its tangent space misses the diagonal quotient. The full weight decomposition forces any nonzero tangent space to contain E_{12} or E_{21} . Normality gives the mixed dual-number Lie-ideal identity proved in RG01 Lemma 1.M. The bracket of these two vectors is $E_{11} - E_{22}$ modulo kI , which is nonzero when two is invertible; it is diagonal, a contradiction. Hence $\mathrm{Lie} N = 0$, and the complete smooth zero-dimensional calculation of S07 Lemma 5.3A makes connected N the identity. This proves reductivity in the stated characteristics. A containing torus has tangent space in the same one-dimensional zero weight, so has rank one. Its closed inclusion of the displayed torus is an isomorphism by the free-character argument of RG01 Lemma 1.M. This proves geometric maximality, and therefore the stated assertions over every base with two invertible. $\square$

4. Matrix centralizers and the missing component

For a diagonal matrix $d = \mathrm{diag}(a_1, \dots, a_n)$ over a field,

$$(Ad - dA)_{ij} = (a_j - a_i)a_{ij}.$$

These equations hold on every algebra over the field: a nonzero difference of field eigenvalues stays a unit. With distinct eigenvalues they force every off-diagonal entry to vanish, and invertibility makes every remaining diagonal entry a unit. The centralizer scheme is therefore precisely the diagonal torus, which is maximal by RG01 Lemma 1.M; the element is strongly regular semisimple. With multiplicities $m_1, \dots, m_a$, the same equations leave exactly the square diagonal blocks, each invertible, so the entire centralizer scheme is $\prod_i \mathrm{GL}_{m_i}$. Its dimension is $\sum_i m_i^2$: each factor is an open in affine m_i^2 -space by the earlier matrix proof. This exceeds n when some multiplicity exceeds one.

In SL_2 , a noncentral diagonal element has centralizer the diagonal torus. For

$$u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

commutation says $c = 0$ and $d = a$ for a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and the determinant condition says $a^2 = 1$. Scheme-theoretically

$$C_{\mathrm{SL}_2}(u) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a^2 = 1 \right\} \simeq \mu_2 \times \mathbf{G}_a,$$

via $(a, x) \mapsto a \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$. Its dimension is one; u is dimension-regular and is not semisimple. In characteristic two this centralizer is non-smooth. This is why tangent dimension alone cannot define dimension-regularity in every characteristic.

For PGL_2 over a field of characteristic different from two, take the class t of $\mathrm{diag}(-1, 1)$. Its root character has value -1 , so t is regular semisimple. On any test algebra R , lift a projective matrix class fppf locally to an invertible matrix A . It centralizes the class precisely when $AtA^{-1} = ct$ for a unit scalar c . Taking determinants gives $c^2 = 1$. Since two is invertible, $e = (1 + c)/2$ is an idempotent, and $R = eR \times (1 - e)R$; on those factors $c = 1$ and $c = -1$ respectively. For $c = 1$, the entry equations and invertibility of two make A diagonal; for $c = -1$ they make it anti-diagonal. The remaining two entries are units by the determinant condition. Conversely all such matrices centralize the class. Taking the scalar quotient and descending these unique locally constant factors proves the entire group-scheme identity, not merely equality on field points. The standard swap matrix has square one in the projective quotient and inverts the diagonal torus, so

$$C_{\mathrm{PGL}_2}(t) = T \rtimes (\mathbb{Z}/2\mathbb{Z}).$$

The identity component is the unique torus containing t , while the full centralizer has a second component. Regular semisimple must not silently be strengthened to strongly regular semisimple. In characteristic two, $\mathfrak{t} \subset \mathfrak{sl}_2$ consists of scalar matrices. Every $\mathrm{ad}(X)$ for $X \in \mathfrak{t}$ is zero. Nevertheless a diagonal group element with $t^2 \neq 1$ acts on the upper root space by $t^2 \neq 1$. Differentiation has lost a nontrivial character.

5. Regular sections in a family

Let $G \rightarrow S$ be reductive. Define G_{rs} fibrewise using geometric regular semisimplicity. This is an open subscheme, compatible with every base change. To prove this without making assumptions about S , choose a maximal torus étale locally by Theorem 2.3 and split it étale locally. The same differential calculation makes

$$G \times_S T_{\mathrm{rs}} \longrightarrow G$$

smooth by [AG-RG-S02, Lemma 3.A](#). Its image is open and, on every geometric fibre, precisely the required locus. These opens agree on overlaps and descend. Their fibres are dense and nonempty.

Theorem 5.1. A geometrically regular semisimple section $s : S \rightarrow G$ of a reductive group lies in a unique maximal S -torus. This construction commutes with arbitrary base change.

Proof. The existence and conjugacy of maximal tori étale locally have already been proved in this lesson, Theorem 2.3. On such a neighbourhood choose and split a maximal torus T . Write the adjoint weight decomposition

$$\mathrm{Lie} G = \mathrm{Lie} T \oplus \bigoplus_{\chi \neq 0} M_\chi.$$

The zero-weight equality is Theorem 2.2. Here the nonzero characters are just Lie weights; their root-group construction is unnecessary. Put $T_{\text{rs}} = \{t : 1 - \chi(t) \text{ is invertible for every occurring } \chi \neq 0\}$. The differential of

$$q : G \times T_{\text{rs}} \longrightarrow G, \quad (g, t) \longmapsto gtg^{-1}$$

at $(1, t)$, after translations, is $(Y, Z) \mapsto (1 - \text{Ad } t)Y + Z$. It is surjective on the displayed weight decomposition. [AG-RG-S02, Lemma 3.A](#), makes q smooth. Its image on each geometric fibre is exactly the regular semisimple locus proved in Section 3. Thus its pullback over s is smooth and surjective. A smooth surjection has sections étale locally; such a section gives $s = gtg^{-1}$ with $t \in T_{\text{rs}}$, and the torus gTg^{-1} contains s .

We prove uniqueness over all test schemes, including nilpotent ones. It suffices to prove that, when $s \in T_{\text{rs}}(S)$, the closed subgroup T is also open in the scheme centralizer $C = C_G(s)$. This is a local assertion on S , so write $S = \text{Spec } A$. First assume that A is Noetherian. Set $B = A[G]$, let I be its augmentation ideal, let K be the ideal of T , and let $J \subset K$ be the equalizer ideal defining C .

After shrinking S , choose a homogeneous basis of I/I^2 . Such a choice is possible because the torus weight summands are finite locally free, and exactness of weight projections lifts their basis elements to homogeneous elements $q_i \in I$. Smoothness of G/A identifies its formal completion along the identity with

$$\widehat{B}_I \simeq A[[q_1, \dots, q_N]].$$

For completeness, the coordinate assertion follows by taking the associated graded along the smooth identity section: $\text{gr}_I B = \text{Sym}_A(I/I^2)$. The chosen lifts give an isomorphism at each successive graded piece, hence by induction modulo each power of I , and then on inverse limits. Locally this associated-graded assertion is the regular-parameter calculation for a section of a smooth morphism: the chosen-coordinate conclusion of [AG-RG-S02, Lemma 3.A](#), makes these parameters an étale chart after a local shrink. Subtracting their values on the section sends it to zero, and its ideal is generated by those parameters.

The cotangent map to T is an isomorphism on weight zero and is zero on every nonzero weight. Indeed conjugation on T is trivial and the zero-weight Lie space of G is exactly $\text{Lie } T$. Consequently the completed ideal $\widehat{K}$ is generated by the q_i of nonzero weight. One can check this directly: these elements vanish on T ; the remaining weight-zero elements are formal coordinates on the smooth torus, so the quotient by the displayed generators is exactly its formal coordinate ring.

If q_i has weight $\chi_i \neq 0$, conjugation by s gives the exact equality $s \cdot q_i = \chi_i(s)q_i$. Thus $(\chi_i(s) - 1)q_i \in J$, and the coefficient is a unit. Every moving parameter belongs to $\widehat{J}$. Since $J \subset K$, we have $\widehat{J} = \widehat{K}$. In particular,

$$K \subset J + I^n \quad \text{for every } n \geq 1.$$

Here is the ideal-separation step. At a prime $\mathfrak{p} \supset I$, work in the Noetherian local ring $R = B_{\mathfrak{p}}/J_{\mathfrak{p}}$, with ideal L the image of I . It is contained in the maximal ideal. The preceding inclusions put the image of K inside $\bigcap_n L^n R = 0$. To see the last equality without importing a separatedness theorem, set $N = \bigcap_n L^n R$. The Rees ring $R[Lt]$ is Noetherian, since L is finitely generated and this ring is a quotient of a polynomial ring. The submodule $\bigoplus_n (L^n R \cap N)t^n$ of the finite Rees module $\bigoplus_n L^n R, t^n$ has finitely many homogeneous generators. If their degrees are at most c , then

$$L^n R \cap N = L^{n-c}(L^c R \cap N) \quad (n \geq c).$$

Both intersections on the left and at degree c equal N , so taking $n = c + 1$ gives $N = LN$. The ideal N is finitely generated. Nakayama's lemma gives $N = 0$: in generators, the equality writes

the identity minus a matrix with entries in the maximal ideal times their column as zero; its determinant is a unit and annihilates every generator. Thus $K_{\mathfrak{p}} = J_{\mathfrak{p}}$.

The finite B -module K/J consequently vanishes at every point of the identity section. Its support is closed. The complementary open in C is an open W , contains the entire identity section, and is contained scheme-theoretically in T . Translation now supplies openness along all of T : the action map $T \times C \rightarrow C$ is smooth, being a projection after the isomorphism $(t, c) \mapsto (t, tc)$. Its image on $T \times W$ is therefore open. This image is contained in T and contains T , because W contains the identity section. Hence T is open in C , as claimed.

For a general ring A , descend this finite collection of data to a finitely generated $\mathbf{Z}$ -algebra A_0 : the finitely presented smooth group G , the closed torus embedding, its conjugation action, the section s , the finite Lie weight decomposition, the equality of its zero part with $\mathrm{Lie} T$, and the inverses of the finitely many $1 - \chi(s)$. AG-MO-03, Theorem 5.1, Lemma 3.1 and Theorems 4.1–4.2 descend these presentations, maps and finite-module isomorphisms. The finite splitting-witness proof in AG-RG-S02, Lemma 3.1 and Proposition 3.2 retains smoothness at a sufficiently large stage. Enlarging that stage makes the finite identities valid there. No spreading of reductivity is needed: the proof just given uses smoothness and these explicit weight conditions. Thus it proves that T_0 is open and closed in $C_{G_0}(s_0)$. Equalizers commute with base change, and openness and closedness persist under it. We obtain the same conclusion over A .

Let T' be any maximal torus containing s . It maps to C . The inverse image of the open and closed T is open and closed in T' and contains the identity section. Each geometric fibre of a torus is connected, so this inverse image has every point of every geometric fibre and is all of T' . Thus $T' \subset T$. The inclusion of two smooth maximal tori is an isomorphism on geometric fibres. Its differential is therefore an isomorphism. AG-RG-S02, Lemma 3.A, makes the inclusion an étale monomorphism and hence an open immersion. Fibrewise surjectivity makes it an isomorphism. This proves uniqueness on every base change.

The local tori obtained above consequently agree on overlaps as subgroup schemes, and their unique descent identifications satisfy the cocycle condition. Affine faithfully flat descent produces the desired global maximal torus. The same uniqueness shows that pulling it back gives the torus associated with the pulled-back section. $\square$

The full centralizer can have additional components supported on special loci. Work over $A = k[a, a^{-1}, (a - 1)^{-1}]$, with two invertible, and take the projective section $\mathrm{diag}(a, 1)$. On any test algebra a lift centralizes it up to a scalar c , and determinants again give $c^2 = 1$. The same idempotent decomposition separates $c = 1$ and $c = -1$. In the first piece, the equations and the unit $a - 1$ force diagonal matrices. In the second, the two diagonal entries vanish because $2a$ and 2 are units; the off-diagonal entries are units, and their equations force $a + 1 = 0$. Conversely these conditions suffice. Thus the centralizer is the disjoint union of its diagonal torus over all of $\mathrm{Spec} A$ and an anti-diagonal torus over $\mathrm{Spec} A/(a + 1)$. The latter is not flat: tensoring the injection of the principal ideal $(a + 1) \hookrightarrow A$ with its nonzero coordinate algebra gives the zero map on a nonzero free rank-one module, violating AG-CA-07 Theorem 2.1. The diagonal torus remains open and closed and is the unique maximal torus containing the section.

Theorem 5.2. Let G/S be reductive. Let X be a section of $\mathrm{Lie} G$ such that every geometric fibre is conjugate into the Lie algebra of a maximal torus and every occurring nonzero Lie-weight differential there is nonzero on X . There is a unique maximal S -torus whose Lie algebra contains X , and the construction commutes with every base change.

Proof. Existence is the smooth conjugation map with the same differential calculation as Theorem 5.1: after splitting a local maximal torus, the differential of $(g, X) \mapsto \mathrm{Ad}(g)X$ is $(Y, Z) \mapsto [Y, X] + Z$. Its moving weights have invertible coefficients, and its zero part is the torus Lie algebra by Theorem 2.2. AG-RG-S02, Lemma 3.A, therefore makes it smooth; its fibrewise image contains the section, so a smooth surjective pullback supplies étale local lifts.

For uniqueness work on one such chart, with $X \in \mathrm{Lie} T$. Let C be its scheme stabilizer in G .

It is affine finitely presented: in a finite local frame of $\mathrm{Lie} G$, the finitely many coordinates of $\mathrm{Ad}(g)X - X$ are defining equations. Their formation commutes with all base changes. Let A be the local affine base, first Noetherian, let $B = A[G]$, let I be its identity ideal, let K define T , and let $J \subset K$ define C .

Choose homogeneous identity parameters of the conjugation action of T . The completed-coordinate proof in Theorem 5.1 gives

$$\widehat{B}_I = A[[z_1, \dots, z_r, m_1, \dots, m_d]], \quad \widehat{K} = (m_1, \dots, m_d),$$

where the z_i have weight zero and the m_j have nonzero weights. The linearization of the equations defining J is $Y \mapsto [Y, X]$. It is zero on the z -part and is invertible on the m -part by the hypotheses. Select d coordinate equations f_i whose d by d matrix on these moving directions is invertible.

Each f_i vanishes on T , so its completed expansion lies in (m) . Factor it as

$$f_i = \sum_j a_{ij}(z, m) m_j.$$

The constant matrix $(a_{ij}(0, 0))$ is the invertible moving differential. Its determinant has unit constant term, hence is a unit in $A[[z, m]]$. The adjugate matrix therefore expresses every m_j as a completed linear combination of the f_i . It follows that

$$\widehat{J} = \widehat{K}.$$

This is the complete implicit-system argument: it does not infer ideal equality merely from equality of tangent spaces.

Consequently $K \subset J + I^n$ for every n . At any prime containing I , pass to the Noetherian local quotient $B_{\mathfrak{p}}/J_{\mathfrak{p}}$. The Rees-module Artin–Rees and Nakayama argument written in Theorem 5.1 makes the intersection of powers of the image of I zero. Thus $K_{\mathfrak{p}} = J_{\mathfrak{p}}$. The finite module K/J has closed support avoiding the identity section, so its complementary open in C lies in T and contains that section. Translation by the smooth T makes T open along all of itself. It is already closed.

For arbitrary A , descend G , T , X , the finite weight decomposition, the zero-weight isomorphism, and the inverses of the moving differentials to a finitely generated $\mathbf{Z}$ -algebra. The affine limit and presentation proofs are AG-MO-03, Theorem 5.1, Lemma 3.1, and Theorems 4.1–4.2. The finite module presentations, their maps and inverse maps involve finite lists of coefficients and identities, so the same generator-and-relation argument retains them at a common stage. Smoothness of the stage is supplied by the full finite splitting-witness proof in AG-RG-S02, Lemma 3.1 and Proposition 3.2. The proof just given uses only smoothness and those finite identities, so applies there without requiring a reductive stage. Its open-and-closed conclusion pulls back to A .

Any torus T' with $X \in \mathrm{Lie} T'$ centralizes X as a section of its Lie algebra: conjugation on the Lie algebra of a commutative group is trivial. Hence it maps to C . The inverse image of the open-and-closed T in T' contains its identity section and is all of each connected geometric torus fibre, so is all of T' . Thus $T' \subset T$. If T' is maximal, its inclusion is a fibrewise isomorphism of smooth tori; the differential is an isomorphism, so [AG-RG-S02, Lemma 3.A](#), makes it an étale monomorphism and hence an open immersion. Surjectivity on every fibre makes it an isomorphism. These statements persist on every test scheme. The local tori consequently agree as subgroup schemes on overlaps; the affine descent proof in AG-RG-S04 Section A glues them, and uniqueness proves arbitrary base-change compatibility. $\square$

In characteristic zero the fibre condition holds for every semisimple Lie element with centralizer dimension equal to the rank. We prove each step over an algebraically closed geometric field. RG01 Lemma 4.K gives the entire polynomial Jordan decomposition and invariant-derivation construction: the formal exponential of a matrix Lie vector lies in $G(k[[t]])$, and the distinct-exponent independence proof gives its exact Laurent relations. For a matrix-semisimple X , write

its diagonal eigenvalues λ_i . The full closed exponential image is

$$D = D_k(\mathbf{Z}^n/N), \quad N = \{(a_i) : \sum_i a_i \lambda_i = 0\}.$$

Indeed substitution sends a Laurent monomial to $e^{(\sum a_i \lambda_i)t}$; RG01 Lemma 4.K makes a polynomial vanish precisely when the coefficient sum in each equal-exponent class is zero. The full coset-basis ideal proof of RG01 Lemma 4.S then gives exactly the displayed group algebra and closed subgroup of G , including every scheme equation. Its character group embeds into the additive field as the group generated by the finitely many λ_i . Choose a rational basis of their finite-dimensional $\mathbf{Q}$ -span, using the finite basis algorithm, and clear the finitely many coordinate denominators. This identifies that group with a subgroup of a finite $\mathbf{Z}^r$. The explicit integer subgroup induction in RG01 Lemma 4.S makes it finite free. Hence D really is a torus. Its tangent point $z_i = 1 + \epsilon \lambda_i$ satisfies every relation, so $X \in \text{Lie } D$.

Choose a torus containing D of largest possible dimension; dimensions are bounded by $\dim G$, so this nonempty finite set of integer dimensions has a largest member. A strictly larger containing torus would have larger dimension: a closed equal-rank inclusion corresponds to a surjection of free character lattices of equal rank, whose free kernel is zero by the integer subgroup proof. Thus the chosen torus is maximal and its Lie algebra contains X . The full group stabilizer $C_G(X)$ is smooth by the characteristic-zero Cartier theorem proved in S08 Theorem G.3.4, and its tangent space is $\ker \text{ad}(X)$ by the dual-number conjugation formula. Its dimension is therefore this kernel dimension. On the chosen torus decomposition, dimension equal to the rank says exactly that every moving weight differential is nonzero. This is the fibre condition of Theorem 5.2.

If semisimplicity is instead specified by diagonalizability of $\text{ad}(X)$, it gives the same conclusion. RG01 Lemma 4.K makes its nilpotent part $\text{ad}(X_n)$, so this is zero. S08 Theorem G.3.4 makes $C_G(X_n)$ smooth with Lie algebra the whole $\mathfrak{g}$, hence with dimension $\dim G$. The proper-closed-subset dimension proof of AG-CA-09 and irreducibility proved in Theorem 3.1 force $C_G(X_n) = G$. In particular X_n commutes with the maximal torus. Theorem 2.2 puts it in $\text{Lie } T$, whose matrices are diagonal in a torus weight basis. A diagonal nilpotent matrix is zero, so $X_n = 0$. General SGA-regular Lie sections in small characteristic need not meet the fibre condition; the characteristic-two example above explains that restriction.

For a general smooth connected affine group over a field, a **Cartan subgroup** is $C_G(T)$ for a geometrically maximal torus. The complete field centralizer proof of Lemma 1.1 gives smoothness and geometric connectedness. Over an algebraic closure T is central and still maximal in this centralizer, so the entire central-maximal-torus proof of Lemma 1.C gives $C_G(T) = T \times U_C$ with U_C smooth connected unipotent. Here nilpotence is explicit. Embed U_C in upper unitriangular m -by- m matrices by RG01 Lemma 2.A. Let I_h be the matrices whose entries vanish unless column minus row is at least h . Matrix multiplication gives $I_h I_j \subset I_{h+j}$, and $I_m = 0$. The inverse of $1 + X$ is the finite geometric series, since $X^m = 0$. Modulo I_{h+j} the products of an I_h and an I_j entry vanish in either order, so $[1 + I_h, 1 + I_j] \subset 1 + I_{h+j}$ on every test algebra. Intersecting these subgroups with U_C makes its lower central series vanish by step m . The central product with T is therefore nilpotent as well. These nested commutator identities descend through the faithful field extension, so the original Cartan group is nilpotent. Conjugacy of maximal tori, proved in RG01 Theorem 2.2, conjugates their entire centralizers. In a reductive group Theorem 2.1 makes that centralizer the torus itself. Thus Cartan subgroups are precisely the maximal tori in the reductive case. We make no additional general-base Cartan representability assertion here.

6. Exercises

Exercise 6.1 (first steps). Compute the centralizer of a diagonal matrix with distinct eigenvalues in GL_n . If its eigenvalues have multiplicities $m_1, \dots, m_a$, compute the dimension of its centralizer.

Exercise 6.2 (centralizers). In characteristic different from two, prove $C_{\text{SL}_2}(u) = \mu_2 \times U$ for the

upper unipotent element displayed above. Determine its component group. Explain what changes in characteristic two.

Exercise 6.3 (universal equations). Over an arbitrary base scheme, prove that the centralizer of the diagonal torus in GL_n is that torus. Why is checking only the base field's rational points insufficient?

Exercise 6.4 (geometry). Show that the regular semisimple locus of GL_n is exactly the nonvanishing locus of the discriminant of its characteristic polynomial, and is dense over every algebraically closed field.

Solutions

Solution 6.1

The equations $(a_j - a_i)a_{ij} = 0$ allow matrix entries only between equal-eigenvalue subspaces. The centralizer is $\prod_i \mathrm{GL}_{m_i}$ and has dimension $\sum_i m_i^2$. For distinct eigenvalues all $m_i = 1$, giving the diagonal torus and dimension n .

Solution 6.2

Multiplying out gives $c = 0$, $a = d$ and $a^2 = 1$. The map $(a, x) \mapsto \begin{pmatrix} a & ax \\ 0 & a \end{pmatrix}$ is an isomorphism of group schemes. In characteristic different from two, μ_2 is étale with two points, so the component group is the constant group of order two. In characteristic two μ_2 is connected and nonreduced, and the centralizer has one geometric component and a two-dimensional tangent space despite having dimension one.

Solution 6.3

Use the universal diagonal matrix with independent Laurent coordinates z_i . The equation $a_{ij}(z_i - z_j) = 0$ forces $a_{ij} = 0$ for $i \neq j$ by independence of Laurent monomials over every ring. The diagonal entries are units since the matrix is invertible. Over a finite field the rational points of a torus may not separate all its characters, and over nonreduced algebras rational-point equations may miss infinitesimal conditions; the universal computation avoids both problems.

Solution 6.4

If the roots are a_i , the discriminant is $\prod_{i < j} (a_i - a_j)^2$, a symmetric polynomial in them. The complete lexicographic elementary-symmetric reduction of S06 Section 1 proves that it is a polynomial in the characteristic coefficients, including in positive characteristic. It is nonzero precisely when all eigenvalues are distinct. A matrix with this property is diagonalizable, and Exercise 6.1 gives its torus centralizer. Conversely a semisimple matrix with repeated eigenvalues has larger centralizer, so is not regular semisimple. The polynomial does not vanish identically: choose distinct nonzero elements in the infinite algebraically closed field and take their diagonal matrix. Its nonvanishing locus in the irreducible variety GL_n is therefore open dense. No assertion that this locus has a rational point over every finite field is needed.

The [course prerequisite guide](#) records the exact supporting statements, their full proof routes, and the hypotheses needed in their applications.

References

- [Milne's freely accessible *Algebraic Groups*, version 2.00 \(2015\)](#). This exact free author edition provides comparison material; its citations do not replace the programme proofs.
- Brian Conrad, [Reductive group schemes](#), 2014, §§1.1–1.2, 2.2 and 3.2. These references receive mathematical credit; the centralizer proof above is written out independently.
- A. Grothendieck and the SGA 3 contributors, [Schémas en groupes](#), Exposé XIII, Theorem 3.1 and Corollary 3.2, for the relative regular-section theorem and its convention; Exposé XIV for the applications to maximal tori.