

Tori, maximal tori and their conjugacy

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Codex, Ultra setting

October 2026

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Course sources: <https://github.com/KokunoYumeto/open-math-courses/tree/main/docs/courses/AG-RG>.

Contents

1	Tori, maximal tori and their conjugacy	1
	Diagonal actions and their descent	1
1.	What maximality means	2
2.	Moving tori over an algebraically closed field	4
3.	Deforming an embedding rather than an eigenvalue	18
4.	A torus over the ground field	27
5.	The finite symmetry left after centralizing	47
6.	Exercises	47
	Solutions	48
	Exact supporting prerequisites and proof order	49
	References	49

Lesson 1

Tori, maximal tori and their conjugacy

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A diagonal matrix is easy to understand because its eigenspaces separate the action into one-dimensional pieces. A torus is the group-scheme version of this diagonal action. Over a field that is not algebraically closed, or over a scheme, the eigenspaces may appear only after changing the base. We therefore keep track of which conclusions hold on the original base and which hold locally for the étale topology.

We assume affine group schemes, smooth morphisms, faithfully flat descent, and the character description of groups of multiplicative type. Basic references are Conrad's *Reductive group schemes*, SGA 3, and Milne's *Algebraic Groups*. The first two treat arbitrary bases; the third provides complementary field arguments.

Diagonal actions and their descent

For a scheme S , a **torus** is a group scheme T that is étale locally isomorphic to $\mathbf{G}_{m,S}^r$, with r locally constant. Its character sheaf is

$$X^*(T) = \underline{\mathrm{Hom}}_{S\text{-groups}}(T, \mathbf{G}_m).$$

It is a locally constant sheaf of finite free abelian groups on the étale site. Its dual is the cocharacter sheaf $X_*(T)$. The pairing is fixed by

$$\chi \circ \lambda(t) = t^{\langle \chi, \lambda \rangle}.$$

For $T = \mathbf{G}_m^r$, the characters $e_i(t_1, \dots, t_r) = t_i$ and coordinate cocharacters form dual bases. A representation of a split torus on a finite locally free module M is a grading

$$M = \bigoplus_{\chi \in \mathbb{Z}^r} M_\chi,$$

with finitely many nonzero summands locally on S . Indeed, write its comodule map as $m \mapsto \sum_\chi m_\chi \otimes z^\chi$. Coassociativity implies that each m_χ has weight χ , and the counit gives $m = \sum m_\chi$. Linear independence of the Laurent monomials makes the sum direct. Each summand is a direct summand of M , hence finite locally free.

Lemma D.1 (maps of split tori on every base). Let X, Y be finite free abelian groups and $D_S(X) = \mathrm{Spec}_S \mathcal{O}_S[X]$. Homomorphisms $D_S(X) \rightarrow D_S(Y)$ correspond to locally constant

functions on S with values in $\mathrm{Hom}_{\mathbf{Z}}(Y, X)$. Automorphisms of $D_S(X)$ correspond to locally constant functions with values in $\mathrm{Aut}_{\mathbf{Z}}(X)$. These assertions commute with every base change, including nonreduced and non-quasi-compact schemes.

Proof. On an affine chart $\mathrm{Spec} R$, write the image of a group-algebra basis element under a Hopf map as $a = \sum_x c_x e^x$, a finite sum. Comparison of coefficients in $\Delta(a) = a \otimes a$ and the counit identity gives $c_x^2 = c_x$, $c_x c_z = 0$ for $x \neq z$, and $\sum_x c_x = 1$. For an idempotent c , the open $D(c)$ is also the closed set $V(1 - c)$: a prime contains either c or $1 - c$ because their product is zero, and cannot contain both. The finitely many $D(c_x)$ therefore partition the chart into open-and-closed parts. On the part $D(c_x)$, that coefficient is one and every other coefficient is zero, so $a = e^x$ there. Apply this to a finite basis of Y and refine their finite partitions together. On each resulting part, multiplicativity and inverses extend the chosen images to one homomorphism $Y \rightarrow X$. Linear independence of the exponent basis makes it unique. These locally constant exponent maps agree on chart overlaps for the same reason. Conversely a locally constant homomorphism defines the Hopf map $e^y \mapsto e^{f(y)}$ on each part where its value is fixed. The maps glue by S04 Lemma A1.5, giving the required group morphism. For an automorphism, apply the same calculation to its inverse; the two integer maps are inverse, and conversely inverse integer maps give inverse Hopf maps. Every coefficient identity persists under an arbitrary ring map. On a non-quasi-compact scheme the argument is chartwise and the image of the locally constant function need not be finite. This proves all the assertions. $\square$

Example. Multiplication on $\mathbb{C}$ gives an embedding

$$\mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbf{G}_m \longrightarrow \mathrm{GL}_{2,\mathbb{R}}, \quad a + bi \longmapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

The determinant is $a^2 + b^2$, so the displayed functor is represented by $\mathrm{Spec} \mathbb{R}[a, b, (a^2 + b^2)^{-1}]$, with matrix multiplication giving its group law and a closed inclusion into GL_2 by the two linear entry equations. After base change to $\mathbb{C}$, put $u = a + bi$ and $v = a - bi$. The inverse formulas $a = (u + v)/2$, $b = (u - v)/(2i)$ identify this coordinate algebra with $\mathbb{C}[u^{\pm 1}, v^{\pm 1}]$, since $a^2 + b^2 = uv$. This proves the torus assertion on every test algebra. Complex conjugation interchanges u, v , and hence the two character generators, by Lemma D.1. A split real torus has trivial conjugation action on its character lattice: its coordinate Laurent monomials are already real. A real isomorphism would intertwine these actions after complex base change, whereas the transposition action is nontrivial. Thus this torus is not split.

The anti-equivalence between tori and locally constant free character sheaves follows by descent from the split case: a change of basis in $\mathbb{Z}^r$ gives the corresponding monomial automorphism of $\mathbf{G}_m^r$, and these constructions carry inverse descent data to inverse descent data. The alternative definition using an fpqc splitting cover gives the same tori; we prove this after Lemma 3.1. One cannot replace the character sheaf by a single lattice with trivial action on an arbitrary base.

1. What maximality means

Lemma 1.M (the matrix groups used in the examples). For $n \geq 1$, GL_n and SL_n over every base are smooth affine reductive group schemes. Their diagonal tori are split and geometrically maximal. All assertions follow from the matrix functors themselves, in every residue characteristic.

Proof. The determinant identities, inverses and polynomial matrix calculations have their complete finite proofs in S06 Section 1. The determinant open in affine n^2 -space is GL_n , hence smooth affine of finite presentation over $\mathbf{Z}$. Its geometric coordinate ring is a polynomial domain localized at its nonzero determinant, so it is geometrically irreducible by the prime-spectrum proof of AG-CA-08. The equation $\det M = 1$ defines SL_n . The cofactors of such a matrix are the entries of its inverse transpose, and therefore generate the unit ideal. On the open where a chosen cofactor is a unit,

solve for its corresponding entry in the determinant equation, which is linear in that entry and has this cofactor as coefficient. This is an open chart in affine $(n^2 - 1)$ -space; the equation/inverse maps are explicit and commute with any base change. Thus SL_n is smooth affine of finite presentation over every base as well.

We prove its geometric connectedness on an algebraically closed field. In an invertible matrix, choose a nonzero pivot. Determinant-one row and column swaps are supplied in a two-coordinate block by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Move the pivot into the first position, and use elementary additions to clear its column and row. Induct on the remaining block; its pivots are nonzero because it remains invertible. This leaves a determinant-one diagonal. Write $x(u) = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ and $y(v) = \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix}$. Direct two-by-two multiplication over any ring gives

$$y(-d^{-1})x(d-1)y(1)x(-(d-1)d^{-1}) = \mathrm{diag}(d, d^{-1}) \quad (d \text{ a unit}).$$

For a diagonal $(c_1, \dots, c_n)$ with product one, multiply these adjacent-coordinate pairs with $d_i = c_1 \cdots c_i$, $1 \leq i < n$; the successive diagonal entries telescope to exactly $c_1, \dots, c_n$. Thus every determinant-one field matrix is an elementary word. Each fixed-length word is the image of an affine space of its additive parameters, which is connected and has identity at all parameters zero. Every such matrix consequently lies in the identity component. A nonempty complement would have a rational closed point by the complete Nullstellensatz, a contradiction. This proves connectedness; for $n = 1$ the determinant-one group is the identity directly.

The diagonal tori have Laurent coordinate algebras, with characters $\mathbf{Z}^n$ for GL_n and $\mathbf{Z}^n/\mathbf{Z}(1, \dots, 1)$ for SL_n , by GS05 Theorem 2.2. Conjugation on the tangent matrices has full characters $e_i - e_j$ on E_{ij} , and zero-weight diagonal part, trace zero in the determinant-one case. These are characters of the torus scheme; they remain distinct in characteristic two. Let N be a smooth connected normal unipotent subgroup of either geometric group. Its intersection with the torus is diagonalizable by GS05 Theorem 4.2 and trivial by the entire unitriangular exclusion of Lemma 2.A below. A tangent point in both closed subgroups satisfies the sum of their defining ideals, so $\mathrm{Lie} N$ has zero intersection with the diagonal Lie space. Its full character grading is supplied by GS05 Lemma 2.1. If nonzero, it therefore contains some E_{ij} . Normality makes it a Lie ideal: conjugate dual-number tangent points over $k[\epsilon, \eta]/(\epsilon^2, \eta^2)$; the mixed coefficient of their commutator is $[X, Y]$, and the normal-subgroup equations force it to be in $\mathrm{Lie} N$. Taking $X = E_{ij}$ and $Y = E_{ji}$ gives $E_{ii} - E_{jj}$ in that Lie algebra, a nonzero diagonal vector even in characteristic two. This contradicts the intersection calculation. Hence $\mathrm{Lie} N = 0$. The complete smooth zero-dimensional field calculation of S07 Lemma 5.3A gives an étale group with finitely many geometric points; connectedness and its identity make it the identity scheme. Thus both matrix groups have reductive geometric fibres.

A torus containing the diagonal torus is commutative, so its tangent matrices have only the displayed zero weight. Its rank is at most the diagonal rank and at least that rank by inclusion. The closed inclusion of equal-rank tori gives, by GS05 Theorem 4.2, a surjection of finite free character groups of equal rank. Its kernel is a free subgroup of rank zero by the full integer subgroup proof of Lemma 4.S below, and hence is zero. Thus the containing torus is the diagonal torus. These computations prove geometric maximality and all reductive assertions over any base by the stated geometric-fibre definition. $\square$

A closed subtorus $T \subset G$ is **maximal** if $T_{\bar{s}}$ is a maximal torus of $G_{\bar{s}}$ at every geometric point of S . We always use this fibrewise definition. A **reductive group scheme** is a smooth affine group scheme of finite presentation whose geometric fibres are connected reductive algebraic groups. A connected smooth affine group over an algebraically closed field is reductive when it has no nontrivial smooth connected normal unipotent subgroup.

This definition of maximality is preserved by every base change. Here is the field-extension detail for the affine finite-presentation groups used here. Let $k \subset K$ be algebraically closed fields and suppose

a larger torus Q_K contains a maximal T_K . Choose split coordinates on $T = \mathbf{G}_m^t$ and $Q_K = \mathbf{G}_m^r$, with $r > t$. Lemma D.1 describes the inclusion of these tori by one fixed surjection $\mathbb{Z}^r \rightarrow \mathbb{Z}^t$. Write $k[G] = k[z_1, \dots, z_m]/(f_1, \dots, f_a)$. The closed group inclusion $Q_K \rightarrow G_K$ sends the z_i to finitely many Laurent polynomials; surjectivity of its coordinate map expresses all r coordinates and their inverses as polynomials in these images. Collect the finitely many coefficients in these expressions in a finitely generated k -subalgebra $D \subset K$. Enlarge D to include the coefficients in the finite identities expressing the $f_j = 0$, the coproduct/counit/inverse group-map identities on the z_i , and equality of the composite $T_D \rightarrow Q_D \rightarrow G_D$ with the given inclusion. These identities hold coefficient by coefficient because $D \hookrightarrow K$ is injective. The inverse-coordinate expressions consequently make $Q_D \rightarrow G_D$ a closed group inclusion, and it contains T_D through the same surjective integer map. The nonzero finite-type algebra D has a k -rational closed point by the complete Nullstellensatz. Specializing at that point gives a closed r -dimensional torus containing T in G , contradicting maximality. This proves persistence without assuming that an arbitrary specialization preserves a closed immersion.

Example 1.1. Diagonal matrices. In $\mathrm{GL}_{n,S}$, the diagonal torus $D = \mathbf{G}_m^n$ is maximal. For a matrix $A = (a_{ij})$ over an S -algebra R , commuting with the universal diagonal matrix says

$$a_{ij}(z_i - z_j) = 0 \quad \text{in } R[z_1^{\pm 1}, \dots, z_n^{\pm 1}].$$

Comparison of distinct Laurent monomials forces $a_{ij} = 0$ for $i \neq j$. Thus $C_{\mathrm{GL}_n}(D) = D$ as group schemes over every base. Any torus containing D centralizes it and is therefore equal to it.

Example 1.2. Two real tori. In $\mathrm{SL}_{2,\mathbb{R}}$, compare

$$D = \{\mathrm{diag}(t, t^{-1})\}, \quad K = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a^2 + b^2 = 1 \right\}.$$

Over $\mathbb{C}$, the second group has characters $a + bi$ and its inverse, so both groups become the same type of one-dimensional torus. An eigenbasis gives a conjugation in $\mathrm{GL}_2(\mathbb{C})$; rescaling one eigenvector makes its determinant one, giving a conjugation in $\mathrm{SL}_2(\mathbb{C})$. They are maximal because their complex fibres are maximal. They are not conjugate over $\mathbb{R}$: the character lattice of D has trivial conjugation action, while that of K has the sign action. A real conjugation would be an isomorphism of real tori and would preserve these actions.

2. Moving tori over an algebraically closed field

We first establish the classical tools used in the conjugacy argument. The proofs work in every characteristic. Orbit representability and the closed-orbit lemma are the exact supporting results of the supporting lesson [Group schemes over a field](#); finite free-action quotients are provided by [Quotients and torsors](#).

A line detecting a subgroup. Every closed subgroup H of an affine algebraic group K is the stabilizer of a line in a finite-dimensional representation. Let $A = k[K]$ and let I define H . For the right regular action write the coproducts of a finite list f_α as

$$\Delta(f_\alpha) = \sum_j u_{\alpha j} \otimes v_j$$

with the second factors v_j linearly independent. Put $W = \mathrm{span}_k\{u_{\alpha j}\}$. The counit puts every f_α in W . Coassociativity and linear functionals selecting the v_j give $\Delta(u_{\alpha j}) \in W \otimes A$, so W is a finite-dimensional right regular subrepresentation. This is also the complete local-finiteness calculation in AG-GS-01, Lemma 5.1.

The Hilbert basis proof makes I finitely generated. Choose W containing these generators and put $V = I \cap W$. For every k -algebra R , exact tensoring of vector spaces gives $V_R = I_R \cap W_R$. Right translation by an element of $H(R)$ preserves I_R and W_R , hence V_R . Conversely, an element stabilizing V_R preserves its generated ideal I_R . Evaluating the translated functions at the identity shows $f(g) = 0$ for every $f \in I$, so that element belongs to $H(R)$. Stabilizing V_R is equivalent to stabilizing its wedge line in $\bigwedge^{\dim V} W_R$: choose a basis of W starting with a basis of V . Preservation of that line makes the wedge coefficient in the first block a unit; wedging each transformed vector with its generator makes every component outside V_R zero. Applying the inverse transformation gives equality with V_R . For $\dim V = 0$ the wedge line is k and the generated ideal is zero, so both stabilizers are all of K . This proves the equality of subgroup schemes, including nonreduced test rings.

Finally take W containing algebra generators of A . Write its coaction as $\Delta(w_j) = \sum_i w_i \otimes a_{ij}$. Applying the counit in the first factor gives $w_j = \sum_i \epsilon(w_i) a_{ij}$. The matrix coefficients therefore generate A as an algebra. The antipode gives the inverse matrix, so the corresponding map to $\mathrm{GL}(W)$ is defined, and its coordinate-ring map is surjective. It is a closed immersion, proving linearity. $\square$

Lemma 2.A (all representations of a unitriangular subgroup). Over any field k , let N be a closed subgroup scheme of the upper unitriangular group over k . Every finite-dimensional representation of N has a filtration whose nonzero successive quotients are one-dimensional trivial representations. Consequently N has no nontrivial character, and a diagonalizable subgroup scheme of N is trivial.

Proof. For the defining representation, the coordinate flag has trivial quotients. Tensor products inherit this property by ordering tensor basis vectors lexicographically. Direct sums inherit it by concatenating flags. A subrepresentation inherits the intersections with a flag; a quotient inherits its images, deleting zero quotients. These statements use ordinary vector-space intersections and remain statements about comodules, hence about every test algebra.

The coordinate algebra $k[N]$ is generated by the upper matrix entries. Its right regular coaction sends a row by

$$\Delta(x_{ij}) = \sum_l x_{il} \otimes x_{lj}.$$

The span of a row is therefore a quotient of the defining representation. Multiplication is a comodule map. The finite-dimensional span of polynomials of degree at most d in the entries is consequently a quotient of a finite direct sum of their tensor powers of orders at most d , and has the same trivial-quotient filtration.

Let W be an arbitrary finite-dimensional comodule. Its coaction embeds it in $W_{\mathrm{triv}} \otimes k[N]$: the counit is a left inverse to the underlying linear map, and coassociativity makes it a comodule map when N acts only on the second factor of its target. Finitely many matrix coefficients of this map occur, and all belong to one of the preceding finite-degree spaces. Thus W embeds into a finite direct sum of such spaces and inherits the required filtration.

An invariant line in W has trivial action, by intersection with that filtration. Applied to the group-like function of a character in $k[N]$, this proves character triviality as a coordinate-algebra identity, including nonreduced N . If a diagonalizable subgroup D embeds in N , each representation of D splits into weight spaces by AG-GS-05, Theorem 3.1. Its faithful defining representation has only trivial constituents by the filtration just proved, so it is trivial. Faithfulness gives $D = 1$. $\square$

A subgroup of multiplicative type cannot have a faithful unitriangular representation: its representations split into character spaces after the splitting cover, whereas the same filtration has only trivial quotients. Its action is therefore the identity. Consequently a subgroup of multiplicative type and a unitriangular subgroup of a linear group have trivial scheme-theoretic intersection. This conclusion includes finite nonreduced groups of multiplicative type.

Lemma 2.H (the connected reduced derived subgroup). For a smooth connected affine

B/k , with k algebraically closed, the reduced closure of the subgroup generated by commutators is a smooth connected normal subgroup D . If B is solvable of derived length $l > 0$, then D has derived length at most $l - 1$.

Proof. The regular-local-domain and component argument in Lemma 2.G makes B integral. Let C_r be the image of the product of r commutators, a morphism $B^{2r} \rightarrow B$, and let J_r be its reduced closure. The product is integral by S08 Lemma G.4.1, so J_r is integral. Inserting identity commutators makes the J_r nested. Their dimensions are bounded by $\dim B$; choose R attaining their largest dimension. For every $r \geq R$ the inclusion $J_R \subset J_r$ is equality, because a proper closed subset of an integral finite-type variety has smaller dimension, by the normalization dimension proof. Put $D = J_R$.

The maps $B^{2R} \rightarrow D$ and their products are schematically dominant: the reduced image closure gives injective coordinate maps, and tensoring injections over a field preserves injectivity. Multiplication of two products of R commutators gives a product of $2R$ commutators, whose image is still in D . Thus multiplication factors through D . Inversion reverses the factors and exchanges the two inputs of each commutator, so also factors. Conjugation preserves these commutator products; the same schematic-dominance argument, now after tensoring with $k[B]$, proves that D is normal as a group scheme. Its identity is the empty product. It is integral, hence connected, and the generic field-rank, Jacobian and translation calculation in Lemma 2.G proves it smooth.

Products of commutators of $B(k)$ are dense in D : a function vanishing on their image pulls back to zero on the dense rational points of B^{2R} and is therefore zero on D . If B has derived length l , its abstract commutator subgroup satisfies the derived word of length $l - 1$. The same word vanishes on products of these dense sets. Tensoring their injective coordinate maps detects this identity on D , including all test rings. Thus the derived length drops as asserted. $\square$

Triangularization. Every finite-dimensional representation of a smooth connected solvable group B has an invariant full flag.

Proof. Induct on the derived length, with the trivial group as base case. Lemma 2.H gives a smooth connected normal derived subgroup D of smaller length. Its representation has an invariant flag by induction, hence a nonzero common character eigenspace. The nonzero common character spaces form a finite, linearly independent collection. Here is the independence calculation: distinct group-like functions are linearly independent, since applying a linear functional to one factor of a minimal comultiplication relation and subtracting a scalar multiple eliminates one term while retaining another. Distinct group-like functions cannot be scalar multiples, because each has counit one. Applying this independence to the coaction of a relation among character vectors makes the character spaces a direct sum; there are at most $\dim V$ nonzero ones.

Normality makes $B(k)$ permute these spaces. The conditions for a fixed permutation are closed matrix conditions on B . Their finitely many closed loci cover every rational point, hence all of the reduced B by the Nullstellensatz. The integral B lies in one of them. That permutation is the identity because the locus contains the identity matrix and the distinct character spaces are disjoint. Its matrix equations vanish identically, so each space is stable under the group scheme B on every test ring.

Choose one, V_χ , of dimension d . The determinant of every commutator of B on this invariant space is one. On D the determinant is χ^d ; density of commutator products therefore makes $\chi^d = 1$ as a scheme identity. Thus χ maps D into the finite scheme μ_d . Reducedness kills its nilpotents, and connectedness makes its image in the finite reduced set a single point. That point is one, so D acts trivially on V_χ . The matrices of B on this space consequently commute, since their commutators factor through D .

Commuting matrices over an algebraically closed field have a common eigenvector. If all are scalar, any line works. Otherwise an eigenspace of a nonscalar matrix is a nonzero proper space preserved by every commuting matrix, and induction on its dimension gives the eigenvector. This line is $B(k)$ -stable; its matrix equations vanish on the reduced B , so it is a group-scheme subrepresentation.

Repeat on its quotient, using induction on representation dimension, and lift the resulting flag. This proves the full invariant flag in every characteristic. $\square$

Lemma 2.M (the Borel fixed-point argument). A smooth connected solvable affine group B over algebraically closed k acting on a nonempty complete variety has a fixed k -point, fixed as a scheme morphism.

Proof. S08 Theorem G.4.2 supplies a closed reduced orbit O . It is complete because it is closed in the given complete variety. S08 Theorem G.4.C proves directly that its orbit map is an fppf torsor under its full schematic stabilizer H , and that O represents B/H . The detecting-line construction gives a finite-dimensional B -representation V and a line kv with schematic stabilizer H , including its possible nilpotents. Here $\mathbf{P}_{\text{lines}}(V) = \mathbf{P}(V^\vee)$ denotes lines in V , or equivalently rank-one quotients of its dual. The map $b \mapsto b[kv]$ is constant on H -cosets over every test scheme. The torsor identity and S04 Lemma A1.5 therefore descend it to an equivariant morphism $O \rightarrow \mathbf{P}_{\text{lines}}(V)$.

Its image is closed: the graph in $O \times \mathbf{P}_{\text{lines}}(V)$ is closed by separatedness, and projection is proper by base change of the proper $O \rightarrow \text{Spec } k$. Give that image its reduced structure Z . Every k -point of Z lifts to a k -point of O , and then to one of B : both fibres are nonempty finite-type schemes, so the weak Nullstellensatz supplies the lifts. Thus $Z(k)$ is the single $B(k)$ -orbit of $[kv]$. The action preserves Z schematically by reduced-product and rational-point detection, as in S08 Theorem G.4.2.

Triangularize V by the proof above, with B -stable full flag $V_0 = 0 \subset V_1 \subset \cdots \subset V_n = V$. Choose the least d for which $Z \cap \mathbf{P}_{\text{lines}}(V_d)$ is nonempty. Its intersection with $\mathbf{P}_{\text{lines}}(V_{d-1})$ is empty, so $Y = Z \cap \mathbf{P}_{\text{lines}}(V_d)$ is a closed projective subscheme entirely contained in the affine chart $\mathbf{P}_{\text{lines}}(V_d) \setminus \mathbf{P}_{\text{lines}}(V_{d-1})$. It is closed in that chart, hence affine as well. S01 Theorem 8.4 and its closed-immersion cohomology calculation make $H^0(Y, \mathcal{O}_Y)$ finite-dimensional over k . Since Y is affine, its coordinate algebra is that finite-dimensional algebra, so Y has finitely many points. Indeed each prime quotient is a finite-dimensional domain, hence a field by injective multiplication; the Noetherian zero-dimensional spectrum is a finite union of its irreducible components, each a single point. Each residue field is k by algebraic closedness.

The finite subscheme Y is B -stable. Choose $y \in Y(k)$. Its orbit map from connected B has connected image in this finite discrete space and contains y , so its image is the point y . Since B is reduced, that map kills nilpotent functions and factors through the reduced point $\text{Spec } k$. Hence y is fixed on every test scheme. Choose $b \in B(k)$ with $b[kv] = y$, using the lifts above. The full schematic stabilizer of y is bHb^{-1} . Fixedness gives $B \subset bHb^{-1}$, and the reverse inclusion is automatic. Thus $H = B$, and the quotient $O = B/H$ is a point by its proved fppf quotient description. Its point in the original variety is the required fixed point. $\square$

Solvable structure. We prove the splitting $B = U \rtimes T$ first. The conjugacy of maximal tori is then the separate consequence of Lemmas 2.B and 2.C below.

Lemma 2.J (the diagonal image and its filtration). In a faithful upper triangular representation of a smooth connected affine B/k , the diagonal map is an fppf quotient $B \rightarrow D$ onto a torus, with unitriangular kernel U . The upper-entry filtration induces a finite B -normal filtration $U = U_0 \supset \cdots \supset U_m = 1$. Each fppf quotient U_i/U_{i+1} is a closed subgroup scheme of $\mathbf{G}_a$, with linear D -action, and $[U, U_i] \subset U_{i+1}$.

Proof. The coordinate algebra of B is a domain by the argument in Lemma 2.G. The diagonal image algebra is generated by the diagonal characters and their inverses. Distinct group-like elements are independent by the preceding calculation, so this algebra is $k[M]$, where M is the finitely generated group of their distinct images. It has no torsion: an element of order $n > 1$ would give two nonzero elements $e^a - 1$ and $1 + e^a + \cdots + e^{(n-1)a}$ with product zero. Thus M is free by S02 Lemma 3.M, and the image D is a torus. Its schematically dominant image map is faithfully flat and of finite presentation by AG-GS-04, Lemma 11.1c. The kernel is B intersected with the unitriangular group, and its kernel identity identifies D with the fppf quotient.

Order the upper entries by increasing distance $j - i$, refining equal distances arbitrarily, and let F_i be the unitriangular subgroup whose first i entries vanish. The multiplication formula for an entry of distance r has cross terms of smaller distances, all zero on F_i at the stage where that entry is removed. Hence the next entry gives $F_i \rightarrow \mathbf{G}_a$ with kernel F_{i+1} . Conjugation by a unitriangular matrix changes it only by smaller-distance terms, so $[\text{UT}, F_i] \subset F_{i+1}$. Diagonal conjugation scales it by the character d_a/d_b for that entry (a, b) . These are polynomial identities, using the finite inverse series for a strictly upper triangular matrix, over every test ring. Thus each $U_i = U \cap F_i$ is closed and normal in B and has the stated commutator property.

The quotients $Q_i = B/U_i$ exist as smooth connected affine groups by Lemma 2.G, with $Q_0 = D$ and $Q_m = B$. The induced maps $Q_{i+1} \rightarrow Q_i$ descend by S04 Lemma A1.5. They are surjective and flat: $B \rightarrow Q_i$ is faithfully flat, and flatness descends through the faithfully flat $B \rightarrow Q_{i+1}$ by Lemma A1.3. Finite presentation follows from their finite-type affine coordinate algebras over the field and the Hilbert basis proof. Thus they are fppf. Their kernels K_i have the fppf cover $U_i \rightarrow K_i$, the base change of $B \rightarrow Q_{i+1}$, with kernel U_{i+1} . The next-entry map descends through this cover to $K_i \rightarrow \mathbf{G}_a$. Its kernel is trivial, since zero entry means membership in U_{i+1} after a local lift. It is a monomorphism and hence a closed immersion by Lemma 2.I. Therefore $K_i = U_i/U_{i+1}$ is the required closed additive subgroup. The universal conjugation formula on its entry gives the linear D -action; its descent is unique by Lemma A1.5. $\square$

Lemma 2.E (additive subgroup quotients). A proper closed subgroup scheme $N \subset \mathbf{G}_a$ over k is finite and is the kernel of a monic additive polynomial P . Its fppf quotient is $\mathbf{G}_a$ via P , and this map is finite faithfully flat. A linear torus action preserving N induces a linear action on this quotient.

Proof. Its nonzero ideal in $k[X]$ has one monic generator P of positive degree d , by the polynomial-division calculation in S02, Section 3. The identity belongs to N , so $P(0) = 0$. The subgroup identity makes $P(X + Y)$ vanish modulo $(P(X), P(Y))$. The polynomial $P(X + Y) - P(X) - P(Y)$ has degree less than d in each variable. Its vanishing in that quotient, whose basis is $X^i Y^j$ for $i, j < d$, makes it zero. Thus P is additive and $N = \ker P$ as schemes.

The ring $k[X]$ is free over $k[P(X)]$ on $1, X, \dots, X^{d-1}$. The monic relation expressing X^d in terms of $P(X)$ and lower powers spans all higher powers by induction. Independence follows because the leading degrees of nonzero terms $a_i(P(X))X^i$, for $0 \leq i < d$, are distinct modulo d and cannot cancel. Hence $P : \mathbf{G}_a \rightarrow \mathbf{G}_a$ is finite faithfully flat. For two points over the same quotient coordinate, additivity says $P(x - y) = 0$; subtraction and addition give inverse scheme maps between $\mathbf{G}_a \times N$ and $\mathbf{G}_a \times_{P, \mathbf{G}_a, P} \mathbf{G}_a$. Local lifts through this faithfully flat map therefore identify its target with the fppf quotient.

A preserving torus action descends through this quotient: its maps and identities descend uniquely by S04 Lemma A1.5. At every k -point of the torus it is an automorphism of $k[X]$ fixing zero. Degrees multiply under composition of nonconstant polynomials, so an automorphism and its inverse both have degree one. In the universal action the coefficients of higher powers vanish at every k -point of the reduced torus, and are zero by the Nullstellensatz. The constant term is zero, the slope is a unit by the inverse action, and the action identity makes the slope a character. This proves linearity on every test scheme. $\square$

An extension of an affine torus D by a vector line has a regular scheme section. Here is a proof for its fppf vector-line torsor over $\text{Spec } A = D$. Choose a faithfully flat affine cover trivializing the torsor, with coordinate ring B . On the double overlap the two line coordinates differ by translation by $c \in B \otimes_A B$. The torsor relation makes c satisfy the additive cocycle identity on the triple overlap.

Consequently the matrices $\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$ give descent data on B^2 . The affine module-descent proof in

AG-GS-04, Section “Faithfully flat affine descent”, or AG-RG-S04, Section A, descends them to an A -module E and an exact sequence $0 \rightarrow A \rightarrow E \rightarrow A \rightarrow 0$: exactness is checked after the faithfully flat cover. Choose a preimage of $1 \in A$ under the surjection $E \rightarrow A$. Its local coordinate vectors are

(a,1), and the displayed transition matrices give exactly the torsor translation law. They therefore define a descended section of that torsor. This proof supplies the fppf assertion directly and needs no unproved comparison of fppf and Zariski cohomology. The multiplication defect is a regular degree-two cocycle. Exactness of torus invariants makes the regular group cohomology in positive degrees vanish, so a correction makes the section a homomorphism. The same vanishing in degree one makes two sections conjugate by the vector line. The regular cochain contraction is proved in AG-GS-05, Proposition 7.14, and the homomorphism-lifting and conjugacy calculation is AG-GS-05, Theorem 7.15.

Lemma 2.F (finite additive extensions of a torus). Let D be a torus over k , let a character $\lambda : D \rightarrow \mathbf{G}_m$ preserve a finite subgroup $N \subset \mathbf{G}_a$ by scalar multiplication, and let

$$1 \longrightarrow N \longrightarrow E \longrightarrow D \longrightarrow 1$$

be an extension of affine algebraic groups, with $E \rightarrow D$ an fppf N -torsor and conjugation on N given by λ . This extension has a group section.

Proof. Form $\mathbf{G}_a \rtimes E$ with multiplication

$$(a, e)(b, f) = (a + \lambda(\bar{e})b, ef),$$

where $\bar{e}$ is the image in D . Its subgroup $K = \{(-n, n) : n \in N\}$ is normal. Conjugation by the vector line fixes it, and conjugation by e sends $(-n, n)$ to $(-\lambda(\bar{e})n, ene^{-1})$, which belongs to K by the specified conjugation action. This verifies normality as a scheme identity.

Construct its quotient P by affine descent, without assuming a finite-action quotient theorem. Over the faithfully flat affine cover $E \rightarrow D$ it is an affine line. If two lifts of d differ by $e \mapsto en$, change its coordinate by $a \mapsto a - \lambda(d)n$. These translations and their inverses satisfy the triple cocycle because the differences of three lifts add in N . S04 Lemma A1.2 descends this affine line to $P \rightarrow D$. It represents the sheaf of classes of (a, e) under

$$(a, e) \sim (a - \lambda(\bar{e})n, en).$$

The preceding normality calculation makes multiplication and inversion well defined on these classes. S04 Lemma A1.5 descends them and their identities. Thus P is a group extension of D by the vector line, with action λ . The map $j : E \rightarrow P$, $e \mapsto [(0, e)]$, is a closed immersion: after the cover $E \rightarrow D$ it is the inclusion of a coset of N in the affine line; the scalar $\lambda(d)$ is a unit. Closed immersions descend by S04 Corollary A3.2. In particular its kernel is zero and its vector-line intersection is exactly N .

The preceding vector-line extension proof, equivalently the complete regular-cochain calculation in AG-GS-05, Proposition 7.14 and Theorem 7.15, splits P . Identify it with $\mathbf{G}_a \rtimes D$ using that section. Lemma 2.E identifies $\mathbf{G}_a/N$ with a vector line; write P_N for its additive quotient polynomial and ρ for the induced character. In each fibre $j(E)$ is a coset of N . Its quotient coordinate therefore descends through the fppf cover $E \rightarrow D$ to a regular map $c : D \rightarrow \mathbf{G}_a/N$. The subgroup multiplication gives the scheme identity

$$c(de) = c(d) + \rho(d)c(e).$$

The degree-one coefficient-of-identity contraction displayed below gives $c(d) = v - \rho(d)v$ for $v \in (\mathbf{G}_a/N)(k)$. The finite surjective map P_N has a k -point a over v , by the Nullstellensatz. Conjugate $j(E)$ by $(-a, 1)$ on the left and $(a, 1)$ on the right. Its quotient coordinate becomes

$$c(d) - P_N(a) + \rho(d)P_N(a) = 0.$$

Consequently this conjugate is $N \rtimes D$ as a scheme, not only on its k -points: equality with the inverse image of the zero quotient coordinate is checked after the N -torsor cover, where both are

the same coset subscheme. Its standard section of D transports back through j to a group section in E . $\square$

These two lemmas supply the finite additive-kernel step in the splitting argument. The case $N = \mathbf{G}_a$ is the vector-line case already proved above.

Lemma 2.G (affine normal unitriangular quotients). Let B be smooth, connected and affine over the algebraically closed field k , and let $N \subset B$ be a closed normal subgroup admitting a closed unitriangular embedding. The fppf quotient B/N is a smooth connected affine algebraic group, and its quotient map is faithfully flat and of finite presentation. The subgroup N may be nonreduced or have positive dimension.

Proof. Choose a detecting line kv for N by the complete construction above. Its character on N is trivial by Lemma 2.A, so N fixes v over every test algebra. Let E be the span of the vectors gv for $g \in B(k)$. Normality of N makes each of these vectors N -fixed: for every test ring, conjugating an N -point by this fixed g gives another N -point. The space E is preserved by $B(k)$. It is preserved by the group scheme B too: the matrix coefficients for its map to the quotient vector space vanish on every k -point of the reduced B , hence vanish by the Nullstellensatz. This gives a representation $B \rightarrow \mathrm{GL}(E)$ in which N acts trivially. Conversely an element in its scheme-theoretic kernel fixes v and its detecting line, so belongs to N . Its kernel is therefore exactly N on all test rings.

The coordinate ring $A = k[B]$ is an integral domain. Indeed smooth local rings are regular by AG-CA-18, Theorem 2.1, and regular local rings are domains by the complete regular-parameter proof. A finite-type scheme has finitely many irreducible components; two meeting components would give two minimal primes in the local ring at an intersection point. Thus these components are disjoint and open and closed. Connectedness leaves one, and smoothness also gives reducedness.

Let $C \subset A$ be the image of $k[\mathrm{GL}(E)]$ under the representation. It is a finitely generated domain. Matrix comultiplication, inversion and the counit preserve C , and their Hopf identities restrict, since tensoring inclusions over the field is injective. Consequently $Q = \mathrm{Spec} C$ is an affine algebraic group and $q : B \rightarrow Q$ is schematically dominant with kernel N . The complete monic-division and translation proof in AG-GS-04, Lemma 11.1c, applies to this integral target and proves that q is faithfully flat and of finite presentation. It assumes no smoothness of the kernel and no pre-existing quotient.

For clarity, smoothness of Q has its own earlier proof inputs. Let $r = \mathrm{trdeg}_k \mathrm{Frac} C$. Noether normalization makes this field finite over $k(t_1, \dots, t_r)$, and S02 Lemma 3.F makes its differential dimension r . At the generic point of Q the local ring is that field, so the dimension and differential numbers in AG-CA-18, Theorem 2.1 are equal. Thus Q is smooth on a nonempty open, by that theorem and Corollary 2.2. Choose a k -point in the open. Its translates show smoothness at every k -point of Q ; a nonempty closed complement would have a k -point by the Nullstellensatz. Hence Q is smooth everywhere. It is connected because C is a domain.

The kernel identity gives inverse scheme maps between $B \times N$ and $B \times_Q B$, by $(b, n) \mapsto (b, bn)$ and $(b, b') \mapsto (b, b^{-1}b')$. Every section of Q lifts fppf locally through the faithfully flat finite-presentation map q ; any two lifts differ by a unique section of N . These statements identify Q with the fppf quotient sheaf, proving the assertion. $\square$

Lemma 2.I (monomorphisms into the additive line). Every monomorphism $X \rightarrow \mathbf{G}_a$ of affine algebraic groups over an algebraically closed field is a closed immersion, including nonreduced X .

Proof. Write its coordinate map as $k[T] \rightarrow A$. If it is injective, AG-GS-04 Lemma 11.1c makes the morphism faithfully flat. Monomorphy identifies $A \otimes_{k[T]} A$ with A by multiplication, so the two maps from A to that tensor product agree. The faithfully flat equalizer in S04 Lemma A1.1 then gives $k[T] = A$, and the morphism is an isomorphism.

Otherwise its nonzero kernel cuts out a finite subscheme of the line, by polynomial division. Injectivity on rational points makes $X(k)$ finite. Every positive-dimensional reduced irreducible

component of X would have infinitely many rational points: Noether normalization gives a finite surjection to a positive-dimensional affine space, and lying over and the Nullstellensatz give a rational lift of every rational base point. The algebraically closed field is infinite, since for a finite field the polynomial $\prod_{a \in k} (T - a) + 1$ would have no root. Thus all components of X have dimension zero. Normalization makes their coordinate domains finite over k , so A_{red} , which injects into the product of these finitely many domains, is finite-dimensional. The Noetherian nilradical of A is nilpotent: finitely many nilpotent generators give a uniform vanishing power by expanding products. A monic polynomial for the image of each algebra generator in A_{red} , raised to that power, annihilates the generator in A . The finite-integral calculation of S04 Lemma P0.2 therefore makes A finite over k .

In particular A is a finite $k[T]$ -module. At every prime of $k[T]$ its finite fibre algebra F satisfies $F \otimes_{\kappa} F = F$, by monomorphy. Its dimension satisfies $r^2 = r$, so the fibre is either zero or the scalar field. The finite module $A/(k[T] \cdot 1)$ consequently has zero fibre at every prime. Local Nakayama and the localization zero test make it zero. The coordinate map is surjective, proving a closed immersion. $\square$

Theorem 2.K (split solvable structure). Every smooth connected solvable affine group B over an algebraically closed field is $U \rtimes T$, where T is a torus and U is smooth, connected and unitriangular. Moreover U is its largest smooth connected normal unipotent subgroup.

Proof. The detecting-line construction supplies a faithful representation, and the triangularization proof supplies an invariant flag. Lemma 2.J then gives an fppf extension of a torus D by a unitriangular kernel U and a finite normal central filtration with closed additive quotients and linear D -actions. We prove splitting by induction on the length of such a supplied filtration for any smooth connected affine extension with these properties. Length zero is already D .

Let N be its last nontrivial term. It is central in U , normal in B , and a closed subgroup of the additive line. Lemma 2.G constructs the smooth connected affine quotient B/N , with its fppf map and kernel identity. Its induced map to D is fppf: surjectivity is inherited from $B \rightarrow D$, flatness descends through $B \rightarrow B/N$ by S04 Lemma A1.3, and finite presentation follows from the affine finite-type algebras over the field. The kernel U/N is affine. It is unitriangular: a faithful representation of it exists by the linearity proof; pulled back through the fppf cover $U \rightarrow U/N$, it has the trivial-quotient flag of Lemma 2.A. Its flag and trivial constituent identities are detected by the injective faithfully flat coordinate map, so that faithful representation is unitriangular already on U/N .

The other terms descend as closed subgroups along $B \rightarrow B/N$, because their defining ideals are invariant under right N -translation; apply S04 Lemma A1.2. Their quotient maps and conjugation identities descend by Lemma A1.5. Cancelling the common subgroup N preserves their fppf additive quotients: local lifts identify $(U_i/N)/(U_{i+1}/N)$ with U_i/U_{i+1} , with the same linear action and centrality. This is a shorter filtration. The induction hypothesis gives a torus section $D \rightarrow B/N$. Its inverse image in B is an affine fppf extension of D by N , whose conjugation is the specified linear action, since N is central in U . If $N = \mathbf{G}_a$, the vector-line section and regular-cochain calculation above splits it. If N is proper, Lemmas 2.E and 2.F split it, including finite nonreduced N . The resulting section of D lies in B .

For this section s , the maps

$$U \times D \longrightarrow B, \quad (u, d) \longmapsto us(d), \quad b \longmapsto (bs(q(b))^{-1}, q(b))$$

are inverse scheme maps and give the semidirect product law. The section is a closed immersion, since its coordinate map is surjective using $q \circ s = 1$. Its image T is the torus D .

Here smoothness of U also has a proof. Write $D = \text{Spec } k[t_1^{\pm 1}, \dots, t_r^{\pm 1}]$. At any rational point $(u, 1)$ of $U \times D \simeq B$, the local ring is regular. The classes $t_i - 1$ in its cotangent space are independent: evaluation at u gives a retraction to the torus local ring and its cotangent space. Complete them to a minimal parameter list. The regular-parameter theorem makes their quotient, the local ring

of U at u , regular. The rational-point criterion of AG-CA-18, Proposition 3.1, gives smoothness there. Its open smooth locus has no nonempty closed complement without a rational point, by the Nullstellensatz, so U is smooth everywhere. A disconnection of U would disconnect $U \times D \simeq B$, since D is nonempty, so U is connected. It was already a closed unitriangular subgroup by Lemma 2.J.

Finally every unipotent subgroup has trivial characters by Lemma 2.A, so its map to the torus B/U is trivial on every test ring. In particular each smooth connected normal unipotent subgroup is contained in U . This proves the largest-subgroup assertion and finishes the theorem. $\square$

Lemma 2.L (reduced identity components). For a finite-type affine group K over algebraically closed k , its reduction is a smooth subgroup. Its component containing the identity, with its reduced structure, is a smooth connected closed subgroup K_{red}^0 . Every morphism from a connected reduced k -scheme to K whose image contains the identity factors through this subgroup. Each component of K_{red} is a translate of it by a k -point.

Proof. The nilradical of $k[K]$ is a Hopf ideal. Indeed, the product of two reduced finite-type k -schemes is reduced by S08 Lemma G.4.1. Pullback of a nilpotent function under multiplication is therefore zero on $K_{\text{red}} \times K_{\text{red}}$; inversion and the identity kill it too. These factorizations give the subgroup structure on the reduction.

Here is its smoothness proof, including the generic input. A Noetherian scheme has finitely many irreducible components: a minimal closed subset failing this assertion would split into two proper closed subsets and give a contradiction. Remove the other components from one component, giving a nonempty integral open. On an affine part of that open, Noether normalization and its dimension formula make the generic field finite over $k(t_1, \dots, t_d)$, with d the component dimension. S02 Lemma 3.F makes its differential dimension d . The field criterion in AG-CA-18 Theorem 2.1 and its open smooth locus in Corollary 2.2 therefore give a nonempty open on which the reduction is smooth over k . The weak Nullstellensatz supplies a k -point a there. For any $g \in K_{\text{red}}(k)$, translation by ga^{-1} takes that smooth neighbourhood to a smooth neighbourhood of g . The nonsmooth closed complement consequently has no closed point, and is empty by the same Nullstellensatz. Thus the reduction is smooth everywhere.

Its local rings are regular domains by the same smooth criterion and the regular-parameter proof. Distinct irreducible components cannot meet: at a common point their distinct minimal primes would remain distinct in the local ring, contradicting its being a domain. The finitely many components are therefore disjoint, open and closed, and each is integral. The identity component is one of these. Its product with itself is integral by S08 Lemma G.4.1, so multiplication has connected image containing the identity and stays in that component. Inversion does likewise. This proves the subgroup assertion.

A map from a reduced scheme kills all nilpotent functions, so factors through K_{red} . Its connected image is contained in one of the disjoint open and closed components; if it contains the identity it is contained in the identity component. This gives the asserted factorization as schemes, since that component is open and closed. Finally every component contains a k -point a by the Nullstellensatz. Translation by a^{-1} sends it to the identity component, proving the translate assertion. $\square$

Lemma 2.B (the smooth central filtration). Let B be a smooth connected solvable affine group with the splitting $B = U \rtimes T$ just constructed, where U is smooth connected and unitriangular. There is a B -stable filtration

$$U = U_0 \supset U_1 \supset \dots \supset U_a = 1$$

by smooth connected normal subgroups such that every successive quotient is $\mathbf{G}_a$, each T -action on it is multiplication by a character, and

$$[U, U_i] \subset U_{i+1}.$$

Proof. Use the splitting $B = U \rtimes T$ just proved. In the faithful triangular representation choose a T -weight basis compatible with the full B -stable flag: exactness of the T -weight projections lets

one choose a weight basis in each flag quotient and lift it inside that flag term. In this basis T is diagonal and B remains upper triangular. In its unitriangular group, order the entries by increasing distance $j - i$ from the diagonal and refine each distance arbitrarily. Let F_i be the subgroup on which the first i entries vanish. The multiplication formula for an entry of distance r has cross terms of smaller positive distances. On F_i , all distances smaller than the distance currently removed have vanished. The next entry therefore gives a homomorphism $F_i \rightarrow \mathbf{G}_a$ with kernel F_{i+1} . Conjugation by a unitriangular matrix changes an entry only by terms involving smaller distances; hence each F_i is normal and $[UT, F_i] \subset F_{i+1}$. Diagonal conjugation scales the next entry by a character. Thus the entire upper triangular group preserves the filtration.

Put $V_i = (U \cap F_i)_{\text{red}}^0$. Lemma 2.L makes these smooth connected closed subgroups. The connected reduced subgroup V_{i+1} maps into $U \cap F_i$ and contains the identity, so the factorization in that lemma gives $V_{i+1} \subset V_i$. They are B -stable: conjugation preserves $U \cap F_i$, and the morphism from the smooth connected $B \times V_i$ has image in its reduced identity component by the same factorization. Similarly, the commutator morphism $U \times V_i \rightarrow U \cap F_{i+1}$ has smooth connected reduced source and contains the identity in its image, so factors through V_{i+1} . This proves the schematic central-filtration assertion. We have $V_0 = U$ and the final $V_i = 1$.

Lemma 2.G, applied to the smooth connected affine group V_i and its normal unitriangular subgroup V_{i+1} , constructs the smooth connected affine quotient $Q = V_i/V_{i+1}$ and its faithfully flat finite-presentation torsor map q . The next entry $e : V_i \rightarrow \mathbf{G}_a$ vanishes on V_{i+1} and is constant on its cosets over every test algebra. The torsor identity and the faithfully flat equalizer in S04 Lemma A1.1 therefore descend it to a group map $\bar{e} : Q \rightarrow \mathbf{G}_a$.

Let $N_i = \ker e = V_i \cap F_{i+1}$. Its reduced identity component equals V_{i+1} . One inclusion follows by mapping it into $U \cap F_{i+1}$ and using Lemma 2.L; the other follows from the already proved $V_{i+1} \subset V_i$ and its inclusion in F_{i+1} . The same lemma makes every component of $(N_i)_{\text{red}}$ a translate of V_{i+1} by a k -point. Put $L = \ker \bar{e}$. Base change of q gives the faithfully flat surjection $N_i \rightarrow L$. Each of the finitely many components of $(N_i)_{\text{red}}$ maps to one k -point of L , since q is constant on the corresponding coset. Consequently the underlying space of L is a finite set of closed k -points.

This implies that L is a finite scheme, retaining its nilpotents. Explicitly, let $A = k[L]$. The prime-intersection calculation in AG-CA-01 Theorem 1.2 embeds A_{red} into the finite product of the residue fields of those points, each equal to k . Thus A_{red} is finite-dimensional. The Noetherian nilradical J has finitely many nilpotent generators, so $J^m = 0$ for some m by expanding monomials. Each J^r/J^{r+1} is a finite module over A/J , hence finite-dimensional over k . The filtration of A by these powers makes A finite-dimensional too.

Here is the dimension bound without a general torsor-dimension theorem. Lemma 2.G makes $k[Q]$ a domain. The coordinate map $k[T] \rightarrow k[Q]$ of $\bar{e}$ therefore has either zero kernel or a prime kernel $(T - a)$, since k is algebraically closed and polynomial division gives the prime ideals of $k[T]$. In the latter case the identity makes $a = 0$, so $Q = L$ is finite. In the zero-kernel case the complete Hopf-subalgebra proof in AG-GS-04 Lemma 11.1c makes $\bar{e}$ faithfully flat. Its generic fibre over $k(T)$ becomes, after algebraic closure of that field, a translate of L after choosing a point: faithful flatness gives a nonempty finite-type fibre, the weak Nullstellensatz supplies the point, and multiplication by its inverse identifies the fibre with the kernel on every test algebra. That fibre's algebra is therefore finite-dimensional after the field extension, and already finite-dimensional before it: linearly independent vectors remain independent under field extension. The domain $k[Q] \otimes_{k[T]} k(T)$ is thus a field finite over $k(T)$, and is the fraction field of $k[Q]$. Noether normalization and its dimension formula give $\dim Q = 1$. In the finite case its coordinate domain is a finite field extension of algebraically closed k , so $Q = 1$ and $V_i = V_{i+1}$ by the kernel identity of q .

Otherwise the quotient is a one-dimensional smooth connected unipotent group. A faithful representation of the affine quotient, pulled back to V_i , has a trivial-quotient filtration by Lemma 2.A. The matrix coefficients expressing preservation of the subspaces and identity on their quotients vanish in $k[V_i]$, and injectivity of $k[Q] \rightarrow k[V_i]$ makes them vanish already in $k[Q]$. Thus its faithful

image is unitriangular. The complete curve, genus and automorphism-family proof in earlier S03 Theorem 4.2 now identifies Q with $\mathbf{G}_a$ as a group scheme.

The T -action is a regular action by automorphisms of this affine line fixing zero. On each closed point of T the automorphism is a linear polynomial. In the universal action its coefficients of higher powers therefore vanish on all closed points of the reduced torus, and are zero. Its slope is a unit and the action identity makes that slope a character of T . Removing the repeated V_i gives the required filtration. This argument retains finite and infinitesimal kernels of the original entry maps; it does not mistake those maps for isomorphisms. $\square$

Here are the degree-one and degree-two calculations explicitly. For $D = D_k(M)$, let $\int : k[M] \rightarrow k$ extract the coefficient of the identity character. It is translation invariant on every scalar extension because $\Delta(e^m) = e^m \otimes e^m$. If V is a vector representation and c is a regular cocycle

$$c(ab) = c(a) + a c(b),$$

put $v = \int_b c(b)$. Integration of this equality in b gives

$$v = c(a) + av, \quad c(a) = v - av.$$

This is an equality of regular maps, also when D is finite nonreduced. It divides by no group order. For a regular two-cocycle c , the identity

$$ac(b, d) - c(ab, d) + c(a, bd) - c(a, b) = 0$$

and integration in d give $c(a, b) = ah(b) - h(ab) + h(a)$, with $h(a) = \int_d c(a, d)$. These are the two calculations used in the splitting argument.

Lemma 2.C (unipotent cocycles on all test schemes). Work over an algebraically closed field k . Let a diagonalizable D act on a smooth connected affine group U of finite type, with a given finite D -stable filtration by smooth connected normal subgroups, as specified in Lemma 2.B. Its successive fppf quotient sheaves are assumed to be represented vector lines with linear D -actions. Every regular crossed homomorphism $D \rightarrow U$ is principal. More generally, over any k -algebra R it is principal after an fppf extension of R .

Proof. Project the cocycle to U/U_1 , a vector line. The displayed degree-one identity, or the complete split regular-cochain contraction in AG-GS Proposition 7.14, makes its projection principal with parameter v ; the coefficient-of-the-identity integral works over every R , including one with nilpotents. Because the quotient in the hypothesis is the represented fppf quotient sheaf, v lifts to $u \in U$ fppf locally. Its fibre is a torsor under U_1 on all test schemes: two lifts differ uniquely by an element of that kernel, and translation identifies a locally chosen lift with every other lift. Over k this nonempty fibre is of finite type, so the earlier Nullstellensatz supplies a k -point and no extension is required. Replace the cocycle by

$$c'(d) = u^{-1}c(d)d(u).$$

Its projection to U/U_1 is now the identity, because the additive quotient cocycle is $v - d(v)$ and the projection of u is v . It takes values in the full kernel U_1 . The cocycle identity survives this correction on the universal pair of D_R -points:

$$c'(ab) = u^{-1}c(a)a(c(b))ab(u) = c'(a)a(c'(b)).$$

Repeat through the finite filtration. Over k the same finite-type fibre argument supplies every correcting lift without extending the field. Over R compose the finitely many fppf refinements. If the corrected cocycle is $c'(d) = w d(w)^{-1}$, then the original one is

$$c(d) = uw d(uw)^{-1}.$$

Thus the correcting parameters multiply in their actual order, and induction makes the original cocycle principal. Every equation is a regular comodule identity over the universal D_R , so detects nonreduced test rings. Applying the affine assertion on an affine cover of an arbitrary test scheme gives the stated fppf local result there as well. $\square$

Lemma 2.N (tori in a split solvable group). In the proved decomposition $B = U \rtimes T$, every torus of B is conjugate by an element of $U(k)$ to a subtorus of T . Every maximal torus is consequently conjugate to T . If B is commutative, the decomposition is $U \times T$.

Proof. Let $M \subset B$ be a torus and put $A = M \cap U$ with its full scheme structure. A torus over algebraically closed k is split. The algebra $k[A]$ is a quotient of the group algebra $k[X^*(M)]$, so is generated by group-like elements. The group of their distinct images is some abelian group Γ . The independence calculation in S02 Lemma 3.N makes its distinct group-like elements linearly independent. Hence $k[A] = k[\Gamma]$ and A is diagonalizable, including any possible finite or infinitesimal part. It embeds in the unitriangular U , so Lemma 2.A gives $A = 1$ on all test schemes.

Projection $q : M \rightarrow T$ has trivial schematic kernel. Let $\Lambda \subset X^*(M)$ be the image of its character map $X^*(T) \rightarrow X^*(M)$, calculated by Lemma D.1. The coordinate algebra of its kernel is the quotient of $k[X^*(M)]$ by $e^\lambda - 1$ for $\lambda \in \Lambda$. The map taking a monomial to its coset identifies this quotient with $k[X^*(M)/\Lambda]$: cosets supply its basis, and the displayed relations identify precisely the monomials in one coset. Triviality of the kernel therefore gives $X^*(M)/\Lambda = 0$, because a nontrivial group algebra has more than the identity basis element. Thus the coordinate map of q is surjective, so q is a closed immersion onto a subtorus $T' \subset T$.

Identify M with T' through q . Its inclusion in $U \rtimes T'$ is a regular section $d \mapsto c(d)d$. The multiplication law gives $c(de) = c(d)d(c(e))$ on every test algebra. Lemma 2.B supplies the smooth central filtration of U , with linear actions after restriction from T to T' . Lemma 2.C therefore gives $c(d) = u d(u)^{-1}$ for some $u \in U(k)$. Conjugation by u^{-1} sends M to the standard T' . If M is maximal, then $T' = T$ since otherwise this conjugate is properly contained in the torus T . This proves conjugacy. In a commutative B conjugation of U by T is trivial, so its semidirect product law is the direct product law. $\square$

Lemma 2.1. If G is smooth connected affine over an algebraically closed field k , its maximal smooth connected solvable subgroups are conjugate, and G/B is complete for each such subgroup B .

Proof. We first construct the flag scheme and its equivariant projective embedding from earlier proved results. Let V be a finite-dimensional G -representation of dimension n . We use the quotient convention for Grassmannians and projective spaces. In the product

$$P = \prod_{i=1}^{n-1} \text{Gr}_{n-i}(V)$$

the universal quotient $V \otimes \mathcal{O}_P \twoheadrightarrow Q_i$ has locally free kernel F_i of rank i . These universal bundles, their arbitrary-base-change property and the Grassmannian charts are proved in AG-HP, *Grassmannians*, Theorem 1.1. Impose the vanishing of each composite

$$F_i \longrightarrow V \otimes \mathcal{O}_P \longrightarrow Q_{i+1}, \quad 1 \leq i < n-1.$$

In local bundle frames this means that all entries of a finite matrix vanish. Changing either frame multiplies by invertible matrices, so the generated ideals agree on overlaps. They define a closed subscheme $\text{Fl}(V) \subset P$.

For every test scheme S , these equations say exactly that $F_i \subset F_{i+1}$. The resulting surjection $Q_i \twoheadrightarrow Q_{i+1}$ splits locally, since its target is locally free. Its kernel F_{i+1}/F_i is consequently locally free of rank one. With $F_0 = 0$ and $F_n = V \otimes \mathcal{O}_S$, a point of this incidence scheme is therefore a full flag of locally direct-summand subbundles. Conversely such a flag gives the indicated locally free quotients and satisfies every equation. Thus the construction represents full flags over arbitrary

test schemes, including nonreduced schemes. For $n = 0$ or $n = 1$, the empty product and the flag scheme are the point $\text{Spec } k$.

AG-HP, *Grassmannians*, Theorem 2.1 proves the closed Plücker embedding

$$\text{Gr}_{n-i}(V) \hookrightarrow \mathbf{P}\left(\bigwedge^{n-i} V\right)$$

by its complete minor-chart calculation. Products of these closed immersions are closed immersions: on affine charts a quotient map stays surjective after tensoring, and a composite of surjective maps is surjective. Factor the product map into base changes of its individual closed immersions. Iterating the Segre embedding proved in AG-MO, *Very ample invertible sheaves, Segre and Veronese embeddings*, Theorem 3.1, gives a closed immersion

$$\text{Fl}(V) \hookrightarrow \mathbf{P}(W), \quad W = \bigotimes_{i=1}^{n-1} \bigwedge^{n-i} V.$$

For an empty product put $W = k$. The incidence equations, exterior quotients and tensor product of quotient lines are all natural under automorphisms of V . Hence the flag scheme is G -stable and this embedding is equivariant for the linear representation on W . In particular its restriction to any torus is a linear projective action. This proves both projectivity and the precise equivariance required later in the rank-one lesson. S01 Lemma 7.A and Corollary 7.B prove that this projective flag scheme is proper.

Choose a smooth connected solvable subgroup B of largest dimension. Such a subgroup exists: the trivial group is one candidate and every candidate has integer dimension at most $\dim G$. The detecting-line construction above provides a representation and a line whose schematic stabilizer is B . Add a faithful representation as a direct summand and retain that line in the first summand, obtaining a faithful V . Apply the triangularization proved above to the quotient by this line and lift the resulting flag. We obtain a B -stable full flag x beginning with the detecting line. Its schematic stabilizer G_x is exactly B : preserving the flag implies preserving that line, whereas every B -point over every test algebra preserves the full flag.

For any full flag $y \in \text{Fl}(V)(k)$, its stabilizer G_y lies in an upper triangular group, since V is faithful. Lemma 2.L proves that its reduced identity subgroup $(G_y)_{\text{red}}^0$ is smooth, and that every component of its reduction is a translate of that subgroup. It is connected; the polynomial upper-entry commutator identities of Lemma 2.J make the upper triangular group solvable, and hence make this subgroup solvable. Its dimension is therefore at most $\dim B$. Reduction does not change dimension, and translations identify the finitely many components of an algebraic group; consequently

$$\dim G_y = \dim (G_y)_{\text{red}}^0 \leq \dim B.$$

The complete generic-fibre and field-extension calculation in S08 Lemma G.4.D gives

$$\dim(G \cdot y) = \dim G - \dim G_y \geq \dim G - \dim B = \dim(G \cdot x).$$

Thus $G \cdot x$ has minimum dimension among all these orbits; this calculation includes nonreduced stabilizers.

We apply the reduced orbit argument to $\text{Fl}(V)_{\text{red}}$. The action restricts to this reduction: S08 Lemma G.4.1 makes $G \times \text{Fl}(V)_{\text{red}}$ reduced, so nilpotent defining functions pull back to zero. The reduced orbit $O = G \cdot x$ is locally closed. Its closure is irreducible, because smooth connected G is irreducible, as proved in S08, Section G.4. The boundary argument of S08 Theorem G.4.2 shows that any nonempty boundary is a closed invariant subset of strictly smaller dimension. Such a boundary has a k -point, by the earlier Nullstellensatz. Its orbit lies in the boundary and has smaller dimension than O , contradicting the preceding inequality. Therefore O has closed underlying image in $\text{Fl}(V)$.

We also retain the full scheme structure. S08 Theorem G.4.C constructs the full quotient directly by the monic-division flatness and torsor calculation and gives the orbit immersion

$$G/B \longrightarrow \mathrm{Fl}(V)$$

and the fppf B -torsor $G \rightarrow G/B$. The quotient is reduced: on an affine chart its functions inject under faithfully flat pullback into functions on a reduced open subscheme of the smooth G , so it has no nonzero nilpotents. An immersion from a reduced scheme with closed underlying image identifies that scheme with the reduced closed subscheme on its image. Indeed, on an open neighborhood where the immersion is closed its defining ideal is radical; it agrees there with the radical ideal defining the reduced closed image, and outside the image the ideal is the unit ideal. These descriptions glue. Thus this orbit immersion is a closed immersion. Composing with the embedding above realizes G/B as a closed subvariety of $\mathbf{P}(W)$, with its linear G -action. It is projective and hence complete.

Now let B' be any maximal smooth connected solvable subgroup. Lemma 2.M applies to its action on the nonempty complete variety G/B , giving a B' -fixed k -point. The fppf torsor fibre above this point is nonempty and of finite type over the algebraically closed field, so the Nullstellensatz supplies a lift $g \in G(k)$. The schematic stabilizer of gB is gBg^{-1} . Fixedness consequently gives $g^{-1}B'g \subset B$ as subgroup schemes. Maximality of B' forces equality, because gBg^{-1} is smooth, connected and solvable. Conjugation identifies G/B' with G/B , transferring projectivity and completeness. This proves the assertion. For additional reading, compare the freely accessible [Milne author draft, v2.00](#), Theorem 4.19; the complete argument used here is the one just given. $\square$

Lemma 2.D (the no-torus criterion). A smooth connected affine group H over an algebraically closed field is unipotent if it has no nontrivial torus. Consequently any such H which is not unipotent has a positive-dimensional maximal torus.

Proof. Choose a Borel B . The solvable structure just proved gives $B = U \rtimes T$; the hypothesis makes $T=1$, so B is unitriangular. Detect B as the stabilizer of a line kv in a finite-dimensional H -representation. Its line character is trivial by Lemma 2.A, so B fixes v as a vector, and its schematic vector stabilizer is still B . The map $h \mapsto hv$ descends through the full fppf quotient $H \rightarrow H/B$ to a morphism $H/B \rightarrow V$. Lemma 2.1 realizes H/B as a closed subvariety of the projective flag variety. It is integral: its underlying space is the surjective image of the irreducible smooth connected H , and its charts are reduced because their functions inject under the faithfully flat pullback into functions on reduced open subschemes of H . AG-RG-S01, Theorem 8.4 and its closed-immersion cohomology calculation make $H^0(H/B, \mathcal{O})$ a finite-dimensional k -domain. A finite-dimensional domain is a field, and the algebraically closed k makes that field k . Thus every coordinate of the morphism to V is constant. Its value is v at the identity coset, so H fixes v . The vector-stabilizer identity forces $H = B = U$, proving the assertion. For the final sentence, the contrapositive gives a nontrivial torus. Choose a torus containing it with largest dimension, bounded by the dimension of H . A proper inclusion of tori strictly increases dimension, by the domain dimension theorem for their reduced integral schemes. Thus this torus is maximal and has positive dimension. $\square$

Theorem 2.2. Any two maximal tori in a smooth connected affine group over an algebraically closed field are conjugate.

Proof. A torus is smooth, connected and solvable. Among the smooth connected solvable closed subgroups containing it, choose one of largest dimension; the torus itself is a candidate. A proper inclusion of such groups strictly increases dimension: their underlying reduced schemes are integral, and a proper closed subset of an integral finite-type scheme has smaller dimension by the earlier domain dimension theorem. Thus the chosen subgroup is maximal. For maximal tori T_1, T_2 , choose B_1, B_2 containing them. Lemma 2.1 gives $gB_2g^{-1} = B_1$. Both T_1 and gT_2g^{-1} are maximal tori of B_1 . Lemma 2.N conjugates them within this split solvable group, giving $u \in B_1(k)$ with $ugT_2g^{-1}u^{-1} = T_1$. $\square$

No conclusion here asserts conjugacy under $G(k)$ when k is not algebraically closed. Example 1.2 is already a counterexample.

3. Deforming an embedding rather than an eigenvalue

We need a mechanism that transports a torus from one fibre to nearby fibres. It uses the entire torus action, which retains information even when differentiating its characters loses information in positive characteristic.

For tori $T_1, T_2 \subset G$, define

$$\text{Transp}_G(T_1, T_2)(R) = \{g \in G(R) : g(T_1)_R g^{-1} = (T_2)_R\}.$$

The normalizer $N_G(T)$ is the transporter from T to itself. The centralizer $C_G(T)$ asks that the induced automorphism of T be the identity. Both conditions concern subgroup schemes after arbitrary base change, rather than just the rational points of a fibre.

Lemma 3.1. Torus deformation. Let G be smooth affine of finite presentation over S .

1. The normalizer and centralizer of a subtorus are closed subgroup schemes of finite presentation. Transporters between tori of the same rank are schemes of finite presentation.
2. These normalizers, centralizers, and transporters are smooth over S .
3. A torus embedded in a fibre G_s extends, after an étale neighbourhood with unchanged residue field, to a torus embedded in G .

Proof. We explain the equations, the deformation obstruction, and then the passage from formal to étale neighbourhoods.

After splitting a torus, its finite subgroups $T[n]$ are schematically dense, universally on the base. A Laurent polynomial has finite support; take n larger than all differences between exponent coordinates. Its distinct monomials remain distinct modulo n , so a nonzero Laurent polynomial remains nonzero on $T[n]$. This works over any ring. The closed conditions on every $T[n]$ therefore detect both inclusion and equality of maps on T .

Here is a finite presentation of the conditions directly over the original base. Work on an affine cover $\text{Spec } R$ splitting the specified tori $T_i = D_R(X_i)$, and put $C = \mathcal{O}(G)$. The ideal J_i of each closed torus inclusion is finitely generated: the source and target are finitely presented over R , so the morphism is finitely presented by the affine-pair presentation calculation in AG-MO-02, Theorem 3.1. For the universal conjugation $G \times T_1 \rightarrow G$, the pullback of each chosen generator of J_2 is a finite Laurent sum in $C[X_1]$. Set each of its finitely many coefficients in C equal to zero. The quotient by these coefficients represents $gT_1g^{-1} \subset T_2$ on every R -algebra, because the exponent basis of its group algebra stays free after every base change. Impose also the coefficient conditions from $g^{-1}T_2g \subset T_1$. The two inclusions are precisely equality of the subgroup schemes, so together they represent the equality transporter as a closed subscheme of finite presentation. Taking $T_1 = T_2$ gives the normalizer. For the centralizer, choose finitely many algebra generators f_j of C and compare their pullbacks along $(g, t) \mapsto gtg^{-1}$ and $(g, t) \mapsto t$. Each difference again has finitely many Laurent coefficients; their vanishing is exactly equality of the two maps on every test algebra. These equations represent the centralizer as a closed subscheme of finite presentation. On overlaps the schemes represent the same conditions. S04 Lemmas A1.2–A1.3 descend their compatible ideal modules, their multiplication stability and their finite generation, giving the closed subgroups and transporter inside the original G . The coefficient tests commute with arbitrary base change, including to rings with nilpotents. Thus no finite generation or base-change assertion about an infinite intersection is needed.

Now let $R \rightarrow R/I$ have $I^2 = 0$. Two liftings of a torus homomorphism $f_0 : T_0 \rightarrow G_0$ differ by a regular crossed homomorphism with values in

$$M = \mathrm{Lie}(G_0) \otimes_{R/I} I,$$

with the action through $\mathrm{Ad} \circ f_0$. More explicitly, lifting a scheme map first gives a multiplication defect $c(t, u)$, and associativity is exactly the degree-two cocycle equation. Correcting a lift by a function $b(t)$ changes c by its coboundary. Once the defect vanishes, the difference between two homomorphisms is a degree-one cocycle. Conjugating by an element infinitesimally equal to one changes that cocycle by a degree-one coboundary.

For a split torus, let $\int$ take the coefficient of the trivial character in its group algebra. On an exponent basis, $(\int \otimes 1)\Delta(f) = (\int f)1$, so this integral is invariant under translation after every scalar extension. For a regular cochain c of degree $j \geq 1$, put $(hc)(g_1, \dots, g_{j-1}) = \int_a a^{-1}c(a, g_1, \dots, g_{j-1})$. In $dhc + hdc$, the internal multiplication faces and the last faces cancel in pairs. The first face of hdc gives c ; its next face, with the substitution $b = ag_1$, cancels the first face of dhc by translation invariance. Hence $dh + hd = 1$ in positive degrees, as an identity of regular sections, over any ring. A closed degree-two defect is therefore a coboundary and can be removed. A difference cocycle b between homomorphism lifts is dw ; conjugating the first lift by the kernel element corresponding to $-w$ changes it by $(-w) - g(-w) = dw(g)$ and gives the second lift. The square-zero kernel and its additive law are proved in AG-GS-03, Propositions 1.4 and 1.6. The full cochain calculation and homomorphism correction are AG-GS-05, Proposition 7.14 and Theorem 7.15.

We now check smoothness on a cover splitting both specified tori, so write $T_i = D_R(X_i)$. Let g_0 be an equality-transporter point over R/I , with $I^2 = 0$. Smooth lifting over an affine test scheme, including the gluing of local lifts, is proved in AG-FSE, *Infinitesimal lifting and invariance under thickenings*, Theorem 4.1; it gives a lift $g \in G(R)$. Conjugation by g_0 induces an isomorphism $\psi_0 : T_{1,R/I} \rightarrow T_{2,R/I}$. Every prime of R contains I , so $\mathrm{Spec}(R/I) \rightarrow \mathrm{Spec} R$ is a homeomorphism. Lemma D.1 identifies ψ_0 with a locally constant invertible integer matrix on this space; the same matrix defines its unique lift $\psi : T_1 \rightarrow T_2$. Thus $\mathrm{Int}(g) \circ i_1$ and $i_2 \circ \psi$, where $i_i : T_i \hookrightarrow G$, are two homomorphism lifts of exactly the same map with the same source T_1 . The preceding degree-one correction gives $u \in G(R)$ reducing to one such that $\mathrm{Int}(u) \circ \mathrm{Int}(g) \circ i_1 = i_2 \circ \psi$. Consequently ug is an equality-transporter point lifting g_0 .

Taking $T_1 = T_2$ proves the normalizer lifting assertion. If g_0 centralizes T , its induced map ψ_0 is the identity, and the unique lift supplied by Lemma D.1 is $\psi = 1_T$. The corrected element ug therefore centralizes the entire torus. This proves formal smoothness of all three representing schemes on the splitting cover. Their finite presentation was proved by the coefficient equations above, so the same infinitesimal criterion gives smoothness there. S04 Lemma A1.3a descends smoothness to the original base.

Finally, in the dual-number kernel, conjugation by a torus element acts by its adjoint action on $\mathrm{Lie}(G)$, by the derivation calculation of AG-GS-03, Proposition 1.4. The centralizer equations require this action to fix the tangent vector on every test algebra. They therefore give, as a sheaf on the base,

$$\mathrm{Lie}(C_G(T)) = \mathrm{Lie}(G)^T.$$

Work on an affine base $\mathrm{Spec} R$ at the chosen prime p . Descend the finitely presented smooth G to a finitely generated $\mathbf{Z}$ -subalgebra R_0 using AG-MO-03, Theorems 4.1–4.2, and retain smoothness by the finite conormal splitting witnesses of AG-RG-S02, Lemma 3.1 and Proposition 3.2. Include the specified fibre torus in the same finite data before taking a local model. Its Hopf presentation, finite separable splitting algebra, splitting maps and inverses, descent maps, embedded coordinate pullback and finitely many surjectivity witnesses use only finitely many coefficients in $\kappa(p) = \mathrm{Frac}(R/p)$. Express those coefficients as fractions of elements of R modulo p and include all their numerators

and denominators in R_0 . With $p_0 = p \cap R_0$, the residue field $\kappa(p_0)$ embeds in $\kappa(p)$. Enlarge R_0 until it contains these finite presentations, expressions and relation witnesses. Their identities then hold over $\kappa(p_0)$, because this field embeds in $\kappa(p)$; their chosen nonzero determinants remain nonzero. The resulting source is a torus split by the retained finite étale algebra and is embedded in the fibre of the smooth stage. Now take the Noetherian local ring $(R_0)_{p_0}$, essentially of finite type over $\mathbf{Z}$, and its completion. The argument below lifts that exact stage fibre torus. After the final pointed étale approximation and pullback to R , its residue data recover the originally specified torus and embedding, with unchanged residue field at p .

Over its complete local ring A , lift the fibre torus as follows. AG-RG-S05, Lemma 3.A constructs, from a finite Galois residue splitting field L/κ , a finite free étale faithfully flat local algebra E/A with lifted Galois automorphisms and $E \otimes_A E \simeq \prod_{\Gamma} E$. Descend $E[M]$ with its fixed semilinear character-lattice action. That lemma proves that this is an actual finitely presented torus T_A , and that every reduction $(T_A)_{A/\mathfrak{m}^n}$ is the descent of the same character data on $(E/\mathfrak{m}^n E)[M]$. Thus the source is already defined over A . Apply AG-GS-05, Theorem 7.15 to this fixed source and the smooth target to lift the specified fibre homomorphism successively modulo $\mathfrak{m}^n$, retaining each previous lifting. Its existence is supplied by the regular cochain contraction of AG-GS-05, Proposition 7.14. We now prove effectivity of the compatible homomorphisms.

Let A be Noetherian and complete for an ideal J , and first let the source be $D_A(M)$. Its completed coordinate module consists of families $(a_m)_{m \in M}$ tending J -adically to zero, while its actual coordinate module consists of families of finite support. Compatible homomorphisms modulo J^n give a map from $A[G]$ to the former. For $f \in A[G]$, write its coproduct as a finite sum $\sum_{i=1}^r f_i \otimes g_i$. Compatibility with the coproduct says that the coefficient matrix of its completed image has entries

$$\delta_{m,n} a_m = \sum_{i=1}^r b_{i,m} c_{i,n}, \quad (3.1)$$

where the right side consists of the coefficient families of the images of the f_i and g_i . Regard these columns as elements of A^M . Every $a_n e_n$ belongs to the finite module spanned by the r families $(b_{i,m})_m$. The submodule generated by all $a_n e_n$ is therefore finitely generated, since A is Noetherian. It is the direct sum of the ideals $A a_n$ in their distinct coordinate positions. Finite generation forces all but finitely many of these ideals to be zero. Thus the image of f has finite support. This holds for every f , so the completed map takes values in $A[M]$ itself. Its algebra and Hopf identities hold because $A[M]$ and $A[M \times M]$ inject into their completions. This proves existence and uniqueness of the actual homomorphism. For the torus source just constructed, use the particular finite free cover E/A of AG-RG-S05, Lemma 3.A. Both E and $E \otimes_A E \simeq \prod_{\Gamma} E$ are complete for the extended ideal, by their finite free bases; they are Noetherian because they are finite over the Noetherian ring A . Apply the split argument over E . Its two pullbacks to $E \otimes_A E$ have the same reductions at every order, since the original formal maps respected the character descent data in (3.5) of that lemma. Uniqueness of the split effectivity argument makes those pullbacks equal as actual Hopf maps. AG-RG-S04, Lemmas A1.1–A1.2 therefore descend the actual homomorphism to A , and faithful flatness verifies all of its Hopf identities and the prescribed reductions. This proves the effectivity statement needed here.

Lemma 3.E (a torus lift remains a closed embedding). Let $(A, \mathfrak{m}, \kappa)$ be any local ring, let T/A be a torus, and let G/A be an affine group scheme of finite presentation. If a homomorphism $f : T \rightarrow G$ is a closed immersion on the closed fibre, then f is a closed immersion. No Noetherian, completeness or smoothness hypothesis on the local base or target is needed.

Proof. For every positive integer n , the group scheme $T[n]$ is finite locally free. On a splitting cover its coordinate algebra is $B[M/nM]$, with the finite exponent basis of M/nM , where M is the finite free character lattice. Its rank is $n^{\text{rank } M}$. The finite locally free module and finite affine algebra descend by S04 Lemmas A1.2–A1.3; thus this is the full torsion scheme, including when n is not invertible. Put $K_n = \ker f \cap T[n]$, with its scheme-theoretic structure. It is a closed subscheme

of this finite affine scheme, so its coordinate algebra C_n is a finite A -module. Its augmentation $C_n \rightarrow A$ has the unit map as a section. The augmentation ideal J_n is consequently an A -direct summand of C_n and is finite: project a finite generating list of C_n to it. The closed-fibre immersion gives $(K_n)_\kappa = 1$ as schemes. Tensoring the direct sum $C_n = A \oplus J_n$ gives $J_n/\mathfrak{m}J_n = 0$. Nakayama, S01 Lemma 1.2, makes $J_n = 0$, so $K_n = 1$ over A , for every n .

Now take any algebraically closed residue overfield at any point of $\text{Spec } A$. The full geometric kernel is a closed subgroup of its split torus, hence is $D(Q)$ for a finitely generated quotient Q of M , by Lemma 4.S. Its intersection with $T[n]$ is its own n -torsion, namely $D(Q/nQ)$, by the same group-algebra kernel calculation; it is trivial by the preceding paragraph. If $Q \neq 0$, it has a maximal proper subgroup. Indeed, in a chain of proper subgroups the union remains proper: if it contained all of a finite generating list, one chain member would contain that list and would be all of Q . Zorn therefore supplies a maximal proper subgroup L . The nonzero simple abelian group Q/L is cyclic, since any nonzero element generates a nonzero subgroup; it cannot be infinite cyclic, which has the proper nonzero subgroup $2\mathbb{Z}$. A nontrivial finite cyclic simple group has prime order, say ℓ , since a composite order supplies a proper nonzero cyclic subgroup. Therefore $Q/\ell Q$ surjects onto $Q/L \neq 0$. Its group algebra has more than one exponent-basis vector, so $D(Q/\ell Q) \neq 1$, a contradiction. Thus $Q = 0$: every full geometric kernel is trivial. This uses the subgroup's finite character quotient, not ambient torsion density.

Let $K = \ker f = \text{Spec } C$, and let $J = \ker(C \rightarrow A)$ be its augmentation ideal. The finite-type affine algebra C , being a quotient of the torus algebra, has generators $c_1, \dots, c_r$ over A . Then $J = (c_i - \epsilon(c_i))_i$: subtracting the evaluation of a polynomial from that polynomial, one variable at a time, expresses the difference in these generators. Thus J is finite as a C -module. The exact sequence $0 \rightarrow J \rightarrow C \rightarrow A \rightarrow 0$ is split as A -modules. Every geometric fibre of K is the identity, and faithful field extension detects the kernel and cokernel of the map from the residue field to its coordinate algebra. Hence every residue fibre is already the identity and $J \otimes_A \kappa(p) = 0$ for every prime $p \subset A$. At a prime $q \subset C$, put $p = q \cap A$. Localizing this equality gives $J_q/pJ_q = 0$; the ideal pC_q is contained in the maximal ideal of C_q . Nakayama applied to the finite C_q -module J_q gives $J_q = 0$. A module zero at all primes is zero: if an element were nonzero, its annihilator would be proper, contained in a maximal ideal by S01 Lemma 1.1, and the element would survive that localization by the fraction-zero criterion. Therefore $J = 0$, and $K = 1$ as a group scheme. For any test algebra, two T -points with the same image differ by a kernel point; they are equal. Thus f is a monomorphism on all test schemes.

It remains to prove that this monomorphism is a closed immersion over the whole base. Work on an affine faithfully flat splitting cover, write $T_B = D_B(M)$, and put $C_G = \mathcal{O}(G_B)$. Right translation by T_B , through f , grades this algebra as $C_G = \bigoplus_{m \in M} (C_G)_m$, by the complete weight decomposition of AG-GS, *Diagonalizable groups*, Theorem 3.1. Pullback maps $(C_G)_m$ into Bz^m . Its coefficients form an ideal $I_m \subset B$. The weight projections commute with every base change, since they are the coefficient projections in a direct exponent-basis decomposition. On any residue field the full kernel is trivial. Lemma 4.S, applied after splitting its torus, proves that this field homomorphism is a closed immersion; it uses the faithful linearity proof of Section 2, the generating weight lattice and the inverse determinant, with no smoothness or reducedness assumption on the target. Its coordinate pullback is therefore surjective on each residue fibre. Projecting to weight m shows that I_m is contained in no prime of B . A proper ideal has a maximal ideal containing it, by S01 Lemma 1.1. Consequently $I_m = B$ for every m . Each z^m has a preimage, so $C_G \rightarrow B[M]$ is surjective. Finally faithful flatness reflects zero cokernels, by CA08 Theorem 1.1, and descends this surjectivity to the original affine map. It follows that f is a closed immersion. $\square$

Apply Lemma 3.E to the actual homomorphism over the complete local ring constructed above. Its closed-fibre map is the specified torus embedding, so its coordinate pullback is surjective. This supplies the actual closed embedding, and therefore the finite surjectivity witnesses, for the approximation step that follows.

To make the approximation step precise, use **Artin approximation for polynomial equations**

in the following form. If $(R, \mathfrak{m})$ is a Noetherian local G-ring, a solution $\hat{a} \in \hat{R}^n$ of finitely many polynomial equations over R can, for every $N \geq 1$, be approximated modulo $\mathfrak{m}^N \hat{R}$ by a solution on an étale neighbourhood with the same residue field. Equivalently the approximating solution belongs to the henselization R^h and descends to a pointed étale neighbourhood. Comparison with the formal solution uses the canonical map from the chosen local étale algebra to $R^h \rightarrow \hat{R}$; no embedding of an entire, possibly disconnected, étale algebra is required.

A **G-ring** is a Noetherian ring whose local completion maps are regular, meaning flat with geometrically regular fibres. In particular the map $R \rightarrow \hat{R}$ above is regular.

Theorem 1.1 of the preceding [Artin approximation lesson](#) proves this statement by its written desingularization sequence D.1–D.38, followed by the standard smooth residue-point lift in Lemma 1.3. The finite-order congruence is imposed in that same finite polynomial system. Its G-ring proof for every local ring essentially of finite type over $\mathbb{Z}$ is F.132–F.135, including the completed-local faithful-flatness argument F.99A. These proofs retain arbitrary characteristic, imperfect residue fields, nilpotent elements and nonregular local G-rings.

Here is the finite system to which the theorem is applied. Choose finite presentations for the completed torus and a finite étale algebra splitting it. Since the completed base is local, that algebra is finite free; choose a basis and its multiplication table. Include the finitely many coefficients of the splitting isomorphism to a Laurent polynomial algebra, its inverse, the descent maps on the double overlap, and the homomorphism into G . Each chosen algebra expression is a finite polynomial or Laurent polynomial. The algebra, Hopf, inverse-map and descent identities are tested on finite generating sets; equality in a finitely presented algebra is recorded using finitely many witnesses in its relation ideal. Invertibility conditions are encoded by adjoining inverse variables. In particular, invertibility of the trace-pairing determinant keeps the chosen finite free splitting algebra étale and faithfully flat.

The closed embedding must be kept in the same system. Its coordinate pullback is surjective by the preceding argument. Choose preimages of a finite set of generators of the completed torus algebra and include the equations stating that their images are those generators. After approximation, these equations still give a surjective pullback. The splitting maps and their inverses still identify the source with a split torus on the finite étale cover, and the preserved descent identities give a torus on the approximating base. Thus approximation retains both the torus structure and the closed embedding, rather than approximating only an arbitrary scheme map.

Apply the theorem at order one to this entire coefficient tuple. The solution then agrees with the specified fibre torus, its embedding and splitting data on the residue field; the equations themselves hold exactly on the étale neighbourhood. Spread its finite presentation and all chosen data from the local model to an open part of the Noetherian model, then pull back to the original base. This gives the required embedded torus after an étale neighbourhood, without a Noetherian assumption on S . $\square$

The positive-characteristic proof behind approximation

The full Néron–Popescu argument in [Artin approximation \(Reductive group schemes\)](#), Section 4, includes inseparable residue fields. The following calculation gives the positive-characteristic exponent and approximation bounds used in that construction. The Stacks Project authors' [smoothing chapter](#) supplies further reading.

Let k be a field of characteristic $p > 0$, and write $k \rightarrow A \rightarrow \Lambda$, where A is of finite type over k and Λ is Noetherian and geometrically regular. Let $H = H_{A/k}$ be the radical ideal defining the non-smooth locus, and choose a prime $\mathfrak{q}$ minimal over $\sqrt{H\Lambda}$. Put $\Gamma = \Lambda_{\mathfrak{q}}$ and $d = \dim \Gamma$. The construction chooses $N \geq 1$ with $\mathfrak{q}^N \Gamma \subset H\Gamma$, then improves an auxiliary presentation and chooses an exponent $c \geq 1$. Its lifting step requires $e = 8c$. At the [order-selection step](#), take

$$n = \max\{N + dc, d(e - 1) + 1\}, \quad e = 8c.$$

This replaces the printed $n = N + dc$. The earlier choices of N , the auxiliary algebras and c do not depend on n . Define the finite set of coefficient images in $\Gamma/\mathfrak{q}^n\Gamma$ with this order, then apply the [Artinian approximation lemma](#), D.35 with its global-selection proof D.34A, which permits every positive n . All subsequent maps, Artinian subalgebras and parameters are chosen for this same quotient.

That lemma supplies a local polynomial ring $P = k[y_1, \dots, y_m]_{\mathfrak{p}}$ and a flat map $P \rightarrow \Gamma$ for which $\mathfrak{p}\Gamma = \mathfrak{q}\Gamma$. The chosen parameters $\pi_1, \dots, \pi_d$ generate $\mathfrak{p}P$ and their images generate $\mathfrak{q}\Gamma$. Put $J = (\pi_1^e, \dots, \pi_d^e) \subset P$. A degree- n monomial in d parameters contains some π_i^e : otherwise every exponent is at most $e - 1$ and the total degree is at most $d(e - 1) < n$. Therefore

$$\mathfrak{p}^n P \subset J, \quad \mathfrak{q}^n \Gamma \subset J\Gamma.$$

This proves the [inclusions used to descend the smooth factorization](#) from the quotient of order n to the quotient by J . The unit rescalings of the parameters preserve these inclusions. In the auxiliary polynomial ring containing the t_i , the selected target point sends t_i to a unit δ_i . Hence the ideals generated by $(\pi_i t_i)^e$ and by π_i^e agree after localization at that point. Reducing a smooth algebra modulo J is the required base change. The other lifting conditions involve e , strict standardness and the annihilator equalities; these are unchanged by the larger order.

The coefficient calculation also survives that choice. At the [denominator-clearing step](#), write $H = (a_1, \dots, a_t)$ and $\pi_i^N = \sum_j a_j d_j$ in the Artinian intermediate algebra D' . Lift d_j to $b_j \in \Gamma$. The error in this equality lies in $\mathfrak{q}^n \Gamma \subset \mathfrak{q}^{n-N} H\Gamma$, so it is $\sum_j a_j b'_j$ with $b'_j \in \mathfrak{q}^{n-N} \Gamma$. Multiplication by π_i^N kills each b'_j modulo $\mathfrak{q}^n \Gamma$. Choose a denominator unit $s \in \Lambda \setminus \mathfrak{q}$; replace it by a suitable power and enlarge D' so that its residue also belongs to D' , as explained below. Clearing denominators then gives coefficients $\mu_{ij} \in \Lambda$ whose residues satisfy

$$\mu_{ij} \bmod \mathfrak{q}^n \Gamma = s^{2N} \pi_i^N d_j \in D'.$$

The printed expression at line 2967 omits π_i^N . This factor belongs to D' through the polynomial coordinate map, so the needed coefficient membership holds with the corrected expression. The subsequent unit and annihilator adjustments preserve the exact defining relations.

One further detail concerns [enlarging the Artinian intermediate algebra](#). Let $\bar{\Gamma}$ be its Artinian target with maximal ideal $\mathfrak{n}$ satisfying $\mathfrak{n}^n = 0$, and let $\bar{\lambda}$ be a unit whose residue α belongs to the residue field of D . Choose a unit $x \in D$ lifting α^{-1} and put $u = x\bar{\lambda} - 1 \in \mathfrak{n}$. For a power $Q = p^r \geq n$, Frobenius gives

$$(x\bar{\lambda})^Q = (1 + u)^Q = 1 + u^Q = 1, \quad \bar{\lambda}^Q = x^{-Q} \in D.$$

Thus the exponent must be a sufficiently large power of p ; divisibility by p alone does not give this identity. If α is algebraic over the residue field F of D , first take a p -power Q_0 making α^{Q_0} separable over F . Lift that finite separable residue extension to the finite étale Artinian algebra D_1 and its map to $\bar{\Gamma}$. The same calculation with another sufficiently large p -power Q_1 gives $\bar{\lambda}^{Q_0 Q_1} \in D_1$. The transcendental case uses the localized polynomial extension in the linked proof. These steps require no perfection assumption on k or on its residue extensions.

If $d = 0$, the localized regular target Γ is a field and the parameter lists are empty. Skip the sequence-lifting lemma, which is stated for a positive-length sequence, and use the Artinian smooth factorization and the height-zero delocalization step directly. The separable and inseparable cases therefore both cover dimension zero as well. These details are included in the preceding written

desingularization proof; its Theorem D.38 and Theorem 1.1 supply the arbitrary-characteristic approximation input used above.

Why fpqc tori split étale locally. Suppose T/S is fpqc locally $\mathbf{G}_m^r$. Affine descent makes it affine, smooth and of finite presentation, with geometric fibres tori; its relative dimension is locally constant. A fibre torus splits over a finite separable extension, by the separable-splitting proof in the earlier character lesson, with its full semilinear Hopf descent supplied in [Supporting group-scheme proofs](#), G.5.1–G.5.3. After making that residue extension, apply Lemma 3.1(3) with target $G = T$ and source the entire fibre torus. We obtain an embedded étale torus $T' \subset T$ near the point. Shrink so that both groups have the same relative dimension. On every geometric fibre the closed subtorus T' now has the dimension of the connected torus T , so is the entire fibre. The inclusion between these smooth groups has invertible differential. [AG-RG-S02, Lemma 3.A](#), makes it an étale monomorphism, hence an open immersion. Fibrewise surjectivity makes it an isomorphism. Finally split T' by its étale character cover. This proves the asserted equivalence of definitions, using the deformation lemma for a general smooth affine target, rather than presupposing the splitting assertion.

Subgroups obtained by taking a limit

Over a ring R , let $\lambda : \mathbf{G}_m \rightarrow H$ be a cocharacter of a smooth affine group. Define

$$\begin{aligned} P_H(\lambda)(A) &= \{h : \lambda(t)h\lambda(t)^{-1} \text{ extends to } t = 0\}, \\ U_H(\lambda)(A) &= \{h : \text{that extension has value } 1\}, \\ L_H(\lambda) &= C_H(\lambda(\mathbf{G}_m)). \end{aligned}$$

The extension is a morphism from $\mathbf{A}_A^1$, not a numerical limit. All definitions commute with base change.

Lemma 3.3. These are smooth closed subgroups of finite presentation. The limit map identifies

$$P_H(\lambda) = L_H(\lambda) \ltimes U_H(\lambda),$$

and multiplication gives an open immersion

$$U_H(-\lambda) \times P_H(\lambda) \longrightarrow H.$$

If $\mathfrak{h} = \bigoplus \mathfrak{h}_n$ is the cocharacter-weight decomposition, their Lie algebras are respectively $\bigoplus_{n \geq 0} \mathfrak{h}_n$, $\bigoplus_{n > 0} \mathfrak{h}_n$ and $\mathfrak{h}_0$. The fibres of $U_H(\lambda)$ are connected unipotent.

Proof. Since H is smooth and affine over R , the earlier S04 Lemma A1.8 makes its coordinate algebra finitely presented over R . Give $R[H]$ its conjugation grading, with $f(t.h) = t^n f(h)$ for a homogeneous function of degree n . Existence of the limit is equivalent to vanishing of all negative-degree functions at h , so these functions generate the ideal of $P_H(\lambda)$. Killing all nonzero-degree functions gives $L_H(\lambda)$; adding to the ideal of $P_H(\lambda)$ every degree-zero $f - f(1)$ gives $U_H(\lambda)$. These ideals are finitely generated over any R . Indeed replace a finite algebra generating list of $R[H]$ by all the finitely many homogeneous components appearing in it. A negative-weight monomial contains a negative-weight generator, so those generators generate the first ideal. The nonzero-weight generators generate the second. Modulo the first ideal, a weight-zero monomial involves only weight-zero generators: a product of remaining positive-weight generators cannot have weight zero. Thus the first ideal together with $a - a(1)$ for the finitely many weight-zero generators a generates the third ideal. Adding these finite lists to a finite presentation gives finite presentation of all three subgroups. The coefficient projectors for the grading commute with every scalar extension, so these equations also prove arbitrary base-change compatibility.

Evaluation at zero is a homomorphism $P \rightarrow L$ and is the identity on L . The mutually inverse maps $(l, u) \mapsto lu$ and $p \mapsto (\lim t.p, (\lim t.p)^{-1}p)$ prove the semidirect product assertion.

Here is a direct smoothness check over every base ring. The fixed subgroup $L = H^{\mathbf{G}_m}$ is smooth by AG-GS-05, Theorem 3.4. To prove smoothness of P , test a square-zero surjection $C \rightarrow \bar{C} = C/J$. A point of $P(\bar{C})$ gives the equivariant extension $F_0 : \mathbf{A}_{\bar{C}}^1 \rightarrow H_{\bar{C}}$ of its conjugation orbit. Smoothness of the affine H lifts it to a scheme map $F : \mathbf{A}_C^1 \rightarrow H_C$. On these maps let the universal $a \in \mathbf{G}_m$ act by

$$(aF)(t) = \lambda(a)F(a^{-1}t)\lambda(a)^{-1}.$$

Since F_0 is fixed, the ratio $(aF)F^{-1}$ is a regular one-cocycle in the vector module

$$V = \text{Lie}(H_{\bar{C}}) \otimes_{\bar{C}} J[t], \quad (av)(t) = \text{Ad}(\lambda(a))v(a^{-1}t).$$

This square-zero kernel identification is made on the universal free algebras $C[t, a, a^{-1}]$ and $C[t, a, a^{-1}, b, b^{-1}]$, so the displayed tensor ideal is exact there. The ratio identity $c(ab) = c(a) + ac(b)$ follows from composition of this action; multiplication in the square-zero kernel is addition. AG-GS-05, Proposition 7.14 supplies $w \in V$ with $c(a) = w - aw$. Left multiplication of F by its kernel section $w(t)$ therefore gives a fixed map, since $aw + c(a) = w$. A fixed map is equivariant and, on $\mathbf{G}_{m,C}$, is the conjugation orbit of its value at 1. Its extension to $\mathbf{A}_C^1$ thus makes that value a lift in $P(C)$.

For U , start with $F_0(0) = 1$. Correct any scheme lift first by the inverse of its value at 0, a square-zero kernel element, so that $F(0) = 1$. Its equivariance defect now belongs to the stable submodule $\text{Lie}(H_{\bar{C}}) \otimes_{\bar{C}} J[t]t$. Apply the same degree-one calculation to this submodule. The correcting $w(t)$ vanishes at zero, so the resulting equivariant lift still has $F(0) = 1$ and gives a lift in $U(C)$. We have proved formal smoothness of P and U . Their already proved finite presentation and the infinitesimal criterion of AG-FSE, Infinitesimal lifting and invariance under thickenings, Theorem 4.1, give smoothness everywhere. Finally an identity tangent vector $X = \sum_n X_n$ has orbit $\sum_n t^n X_n$. It extends exactly when $X_n = 0$ for $n < 0$, and has value zero at the origin exactly when also $X_0 = 0$. Fixedness means only weight zero remains. These are the three asserted Lie algebras.

The intersection $U_H(-\lambda) \cap P_H(\lambda)$ is trivial: both positive and negative degrees must vanish, and the limit is the identity. The multiplication map is a monomorphism. Its differential is an isomorphism at the identity; at (u, p) , translations reduce it to the decomposition of $\mathfrak{h}$ by $\mathfrak{u}_- \oplus \mathfrak{p}$, followed by $\text{Ad}(p^{-1})$, which induces an automorphism modulo $\mathfrak{p}$. AG-RG-S02, Lemma 3.A, therefore makes it étale everywhere; the complete flat-monomorphism proof in the earlier S04 Lemma A1.9 makes it an open immersion.

Every point of U is connected to the identity by its extended orbit map $\mathbf{A}^1 \rightarrow U$, so the geometric fibres are connected. Over a field, embed $H \rtimes \mathbf{G}_m$ faithfully in a general linear group and choose a weight basis for the acting torus. The condition that the limit be the identity makes U strictly triangular relative to that weight ordering. It is therefore unipotent. $\square$

Lemma 3.D (contracted one-dimensional groups). Let U be a smooth affine group of finite presentation over a scheme S . Suppose an action of $\mathbf{G}_m$ by group automorphisms extends to $\mathbf{A}^1 \times U \rightarrow U$, with zero sending every point to the identity. Suppose $\text{Lie}(U/S)$ is an invertible module on which the action has a locally constant positive integer weight d . There is a unique $\mathbf{G}_m$ -equivariant isomorphism

$$\text{Exp}_U : \mathbf{V}(\text{Lie}(U/S)) \longrightarrow U$$

whose differential is the identity. It is an isomorphism of groups and commutes with every base change. Any further group action commuting with this $\mathbf{G}_m$ -action also commutes with Exp_U , using its differential action on the source. The integer d need not be invertible on S .

Proof. Work on an affine open $S = \text{Spec } R$ where $d > 0$ is constant and the Lie line has a frame v . Put $A = R[U]$. We use the point-pullback convention for weights: a homogeneous function a

of weight j satisfies $a(t.u) = t^j a(u)$ over the universal Laurent polynomial algebra. The complete weight decomposition in AG-GS, *Diagonalizable groups and groups of multiplicative type*, Theorem 3.1 gives $A = \bigoplus_{j \in \mathbf{Z}} A_j$. The extension to $\mathbf{A}^1$ forces $A_j = 0$ for $j < 0$: the coefficient at^j must belong to $A[t] \subset A[t, t^{-1}]$. Evaluation at zero is the identity section, so a weight-zero element equals its identity value. The unit $R \rightarrow A$ and the identity evaluation are inverse on that summand. Thus

$$A_0 = R, \quad I = \ker(A \rightarrow R) = \bigoplus_{j>0} A_j.$$

The relative tangent-functor calculation in AG-GS, *Lie algebras and smoothness*, Proposition 1.4 identifies $I/I^2 = e^* \Omega_{U/S}$ with the dual of the Lie line. Explicitly, an identity derivation kills R and I^2 , and is precisely an R -map $I/I^2 \rightarrow R$. Smoothness makes this cotangent module finite projective, so its double dual is itself. In our point-pullback convention its sole weight is d : pulling a cotangent functional back along multiplication by t^d multiplies that functional by t^d . Choose a homogeneous lift $f \in A_d$ of the functional taking v to one. This lift exists because the surjection $I \rightarrow I/I^2$ is surjective in each weight, by the same weight-decomposition theorem.

Every homogeneous element of A is a polynomial in f . Here is the induction, including possible nilpotents in R . For $a \in A_j$ with $j > 0$, its class in I/I^2 is zero if $j \neq d$, and is $r\bar{f}$ for some $r \in R$ if $j = d$. Subtract rf in the latter case. The remainder lies in $(I^2)_j$. Write it as a finite sum of products of homogeneous elements of positive degrees adding to j , by decomposing both factors of each product. Each factor has degree strictly smaller than j , and induction expresses it as a polynomial in f . The degree-zero case is $A_0 = R$. This proves $A = R[f]$. In particular all A_j with $0 < j < d$ vanish. Hence $(I^2)_d = 0$, and the chosen lift f is unique. The map

$$q : U \longrightarrow \mathbf{A}_R^1, \quad q^*(x) = f,$$

is a closed immersion, is equivariant for weight d on the target, and has invertible differential at the identity.

We prove that it is an isomorphism over the original ring. The sheaf $\Omega_{U/R}$ is an invertible sheaf. One can see its rank everywhere directly: on every geometric fibre, translation by a point identifies its tangent space with the one-dimensional tangent space at the identity. Smoothness makes $\Omega_{U/R}$ finite locally free, as proved in Proposition 1.4 and S02 Lemma 3.A, so these fibre ranks give rank one at every point. The noninvertibility locus of the differential $q^* \Omega_{\mathbf{A}^1/R} \rightarrow \Omega_{U/R}$ is closed: in local frames it is the zero locus of its single coefficient. It is $\mathbf{G}_m$ -stable, because q is equivariant and multiplication by an invertible t^d is an automorphism on both tangent lines. This locus misses the identity section.

If it contained a geometric point u with algebraically closed residue field K , its preimage under the extended orbit $\mathbf{A}_K^1 \rightarrow U$, $t \mapsto t.u$, would be a closed subset containing $\mathbf{G}_{m,K}$. That open subset is dense, since $K[t]$ is a domain. The preimage would contain zero, whose image is the identity, a contradiction. Thus the differential is invertible everywhere. S02 Lemma 3.A makes q étale everywhere; its proof includes the finite-presentation calculation for a map between smooth schemes of finite presentation. The earlier S04 Lemma A1.9 makes this monomorphism an open immersion. Its image is also closed, because q was already a closed immersion. On every geometric fibre it is an open and closed subset of the connected $\mathbf{A}_K^1$ containing zero, hence is the whole fibre. Every target point belongs to a geometric fibre, so the open image is the whole target. Consequently q is an isomorphism of schemes, including over nonreduced bases.

Transport the group law of U to this affine line. It is a polynomial $m(x, y) \in R[x, y]$, and equivariance gives, in $R[t, t^{-1}, x, y]$,

$$m(t^d x, t^d y) = t^d m(x, y).$$

The Laurent monomials are an R -basis. A coefficient of $x^i y^j$ can be nonzero only if $d(i + j) = d$, hence only if $i + j = 1$. This is equality of integer exponents; it uses no division by d in R . Thus $m = ax + by$, and the two identity laws give $a = b = 1$. The inverse of q , using the frame v on the source Lie line, is the required group isomorphism with identity differential.

A weight- d equivariant morphism between two such affine lines is also linear: comparing the coefficient of x^i gives $di = d$, so only $i = 1$ occurs. Its differential determines it, even when the coefficients have nilpotents. Composing two candidate isomorphisms proves uniqueness. Therefore the local constructions agree after changes of frame and glue to $\mathbf{V}(\text{Lie}(U/S))$. Relative differentials and the finite-projective dual commute with arbitrary base change by Proposition 1.4, and the uniquely characterized isomorphism after any base change is the base change of this one.

Finally let another group scheme act on U by automorphisms commuting with $\mathbf{G}_m$. On its universal test algebra, applying that automorphism after Exp_U and applying its differential before Exp_U give equivariant isomorphisms with the same differential. Compose with the inverse of that differential on the Lie line; uniqueness makes the two maps equal. This is an equality of scheme morphisms over every test algebra, proving the final assertion. $\square$

4. A torus over the ground field

The finite-field and infinite-field arguments have quite different mechanisms. The first adjusts descent by Lang's theorem. The second finds a semisimple infinitesimal direction and uses dimension induction. The infinitesimal argument is essential over imperfect fields.

Lemma 4.L (Lang's equation). Let $G/\mathbf{F}_q$ be a smooth connected algebraic group, of finite type, and let $F : G \rightarrow G$ be its q -power Frobenius. For every algebraically closed extension $K/\mathbf{F}_q$, the map $h \mapsto h^{-1}F(h)$ is surjective on $G(K)$. Affineness is not required.

Proof. S08 Lemma G.0.3 makes G_K connected. Frobenius kills every square-zero infinitesimal point at the identity. Here is the full coordinate check, including for a nonaffine group. Choose an affine $\mathbf{F}_q$ -open neighbourhood U of the identity. A point δ reducing to the identity across a square-zero ideal $I \subset R'$ factors through U_K , since a nilpotent closed immersion leaves the underlying topological space unchanged. A coordinate $a \in \mathbf{F}_q[U]$ vanishing at the identity has value in I . Frobenius pulls it back to a^q , whose value is zero. These coordinates, together with constants, show $F(\delta) = 1$. The restriction of Frobenius to this open is legitimate: over $\mathbf{F}_q$ absolute Frobenius has the identity underlying topological map.

For $x \in G(K)$, consider the twisted orbit map

$$\theta_x : G_K \longrightarrow G_K, \quad g \longmapsto gx F(g)^{-1}.$$

It is formally étale. Given a square-zero extension $R' \twoheadrightarrow R$, a point $g \in G(R)$, and a prescribed lift $y \in G(R')$ of $\theta_x(g)$, smoothness of G_K gives a lift g_0 , locally on $\text{Spec } R'$ if necessary. Put

$$\delta = g_0^{-1} y F(g_0) x^{-1}.$$

It reduces to the identity, so $F(\delta) = 1$, and direct multiplication gives $\theta_x(g_0 \delta) = y$. Any other lift is $g_0 \epsilon$ with ϵ reducing to the identity; its equation and $F(\epsilon) = 1$ force $\epsilon = \delta$. This uniqueness glues the local lifts. Thus existence and uniqueness hold on every square-zero test, including all nilpotent directions.

The map is of finite presentation. On affine source and target charts their algebras are finite type over the Noetherian field K , by S01 Lemmas 1.0 and 1.3; the relative presentation kernel is finitely generated. The finite affine covers supplied by finite type give the global assertion. Formal étaleness implies formal smoothness, so the definition and lifting criterion of AG-CA, *Formally smooth, unramified and étale maps*, Section 3, make θ_x smooth. Its flatness is the complete arbitrary-base

proof of AG-CA, *Smooth algebras over a field and the Jacobian criterion*, Theorem 6.1. S08 Lemma G.4.F consequently makes its image O_x open.

These open images are either disjoint or equal. If $O_x \cap O_z$ is nonempty, it has a K -rational point by the weak Nullstellensatz on an affine chart. Each of its nonempty finite-type orbit-map fibres has a K -rational point by the same argument. Hence x and z belong to the same twisted point orbit. The identity $g \cdot (a \cdot x) = (ga) \cdot x$ then identifies the two entire orbit-map images. The union of the O_x , for $x \in G(K)$, covers G_K : a nonempty closed complement would have a rational point, which belongs to its own orbit image. Connectedness makes this disjoint open partition consist of one member. In particular $O_1 = G_K$. Every rational target point has a rational lift in its finite-type fibre. Thus $g \mapsto gF(g)^{-1}$ is surjective on $G(K)$. Replacing g by h^{-1} gives the stated equation. $\square$

Lemma 4.F (a fixed basis for an invertible semilinear operation). Let K be algebraically closed of characteristic $p > 0$, and let V be finite-dimensional with an invertible Frobenius-semilinear operation Φ . Then V has a basis fixed by Φ .

Proof. The assertion is empty in dimension zero. Otherwise, in a chosen basis write $\Phi(v) = Av^{(p)}$, with $A \in \mathrm{GL}_d(K)$. The group $\mathrm{GL}_d/\mathbf{F}_p$ is smooth and geometrically connected: its coordinate ring is the polynomial ring in matrix entries localized at the determinant, a domain over every field extension; entries lift across a square-zero surjection and the determinant remains a unit because an element invertible modulo a nilpotent ideal is invertible, using the finite geometric-sum inverse. This proves its finite-presentation and formal smoothness directly. Apply Lemma 4.L to choose h with $h^{-1}F(h) = A$, and put $C = h^{-1}$. Then

$$C(C^{(p)})^{-1} = A, \quad AC^{(p)} = C.$$

The columns of the invertible C are a basis and are fixed by Φ , as required. $\square$

Lemma 4.H (PBW, Frobenius kernels and height-one groups). Ordinary PBW holds for every finite-dimensional Lie algebra over any field. In characteristic $p > 0$, the restricted enveloping algebra $u(L)$ of every finite-dimensional restricted Lie algebra has the ordered basis $x_1^{r_1} \cdots x_d^{r_d}$, $0 \leq r_i < p$, and dimension p^d . It is a cocommutative Hopf algebra with primitive generators. Its finite dual defines a height-one group H_L , and $L \mapsto H_L$ is inverse to the Lie functor on finite height-one group schemes over the field, with their full scheme structures. In particular, for a smooth affine algebraic group G ,

$$k[G[F]] \simeq k[t_1, \dots, t_n]/(t_1^p, \dots, t_n^p), \quad \mathrm{Dist}(G[F]) = u(\mathrm{Lie} G),$$

where $n = \dim_k \mathrm{Lie} G$ and the first isomorphism concerns algebras, with its Hopf maps supplied by G . Restricted subalgebras $L \subset \mathrm{Lie} G$ correspond to closed subgroup schemes of $G[F]$. Their centralizers are closed and have Lie algebra $\{Y : [Y, L] = 0\}$. All these constructions commute with field extension. If L is commutative and its p -operation is geometrically invertible, then H_L is of multiplicative type and is geometrically $\mu_p^{\dim L}$. No perfect-field hypothesis is imposed.

Proof. We first prove ordinary PBW, including the confluence step. In the tensor algebra on an ordered basis, replace $x_j x_i$, $j > i$, by $x_i x_j + [x_j, x_i]$. On words, order first by length and then by inversion count. A swapped word has one fewer inversion, and every bracket term has shorter length, so all reductions terminate. Disjoint reductions commute as linear substitutions. The only overlapping pair occurs in $x_k x_j x_i$, $k > j > i$. Reducing its left or right pair and then the remaining inverted pairs gives expressions whose difference is

$$[[x_j, x_i], x_k] + [x_j, [x_k, x_i]] + [[x_k, x_j], x_i] = 0$$

by Jacobi. Here a commutator between two linear combinations of basis vectors reduces to their given Lie bracket by the defining two-letter rules. These terms have smaller length than the original triple, even inside a larger word.

For completeness, well-founded induction now proves uniqueness of the final linear combination of ordered words. Compare two first reductions. In the disjoint case apply the other substitution

to obtain the same expression; all resulting words have smaller order. In the overlap case their difference has the zero normal form just computed, using the induction for shorter words. Subsequent reductions are unique by the same induction. Extend this normal-form operation linearly. Every defining relation in any word context has normal form zero, so its entire two-sided ideal does as well. Reduction changes a word only by that ideal, while an ordered word is its own normal form. Hence the ordered monomials both span the quotient and are independent. This proves ordinary PBW in every characteristic.

We specify the restricted convention and verify the identity needed for all vectors, rather than only a basis. In characteristic p , a restricted operation satisfies $(aX)^{[p]} = a^p X^{[p]}$, $\text{ad}(X^{[p]}) = (\text{ad } X)^p$, and

$$(X + Y)^{[p]} = X^{[p]} + Y^{[p]} + \sum_{i=1}^{p-1} s_i(X, Y),$$

where $is_i(X, Y)$ is the coefficient of t^{i-1} in $(\text{ad}(tX + Y))^{p-1}(X)$. These are defining axioms; the integers $1, \dots, p-1$ are invertible in the field. In any associative algebra the same s_i , using its commutator bracket, satisfy

$$(X + Y)^p = X^p + Y^p + \sum_{i=1}^{p-1} s_i(X, Y).$$

Indeed set $Z = tX + Y$. Differentiate the product Z^p formally in the central variable t , obtaining $\sum_{j=0}^{p-1} Z^j X Z^{p-1-j}$. Expanding the iterated commutator gives this same sum, since $\binom{p-1}{j} = (-1)^j$ in characteristic p . To see the coefficient identity, multiply $\sum_{j=0}^{p-1} (-t)^j$ by $1 + t$ and compare with $(1 + t)^p = 1 + t^p$; its intermediate binomial coefficients vanish because their denominators are prime to p . The coefficient of t^{i-1} is consequently i times the coefficient of t^i in Z^p . The pure terms have zero derivative, and this determines every mixed coefficient. Evaluating at $t = 1$ proves the identity.

In the ordinary enveloping algebra put $z_i = x_i^p - x_i^{[p]}$. The binomial expansion of an iterated commutator gives $[x_i^p, y] = (\text{ad } x_i)^p(y) = [x_i^{[p]}, y]$, so z_i is central. The products $z_1^{a_1} \cdots z_d^{a_d} x_1^{r_1} \cdots x_d^{r_d}$, with $0 \leq r_i < p$, have distinct leading PBW monomials, whose exponents are $pa_i + r_i$. They are independent. They span by replacing each basis exponent $\geq p$ by a z_i term plus $x_i^{[p]}$, which has lower degree, and inducting on degree. Thus the ordinary algebra is free over the polynomial algebra in its central z_i , on those truncated monomials. Quotienting by the z_i retains precisely that basis. The preceding associative identity and the restricted axioms make the defect $X^p - X^{[p]}$ additive and Frobenius-semilinear in this quotient. It vanishes on each basis vector, hence on every X . The quotient is therefore exactly $u(L)$, not a larger algebra imposing too few relations. The coproduct $\Delta X = X \otimes 1 + 1 \otimes X$, counit zero and antipode $SX = -X$ preserve the bracket relations. Each z_i is primitive, because its two tensor terms commute before taking the p -th power. Its ideal is consequently a Hopf ideal. The quotient is cocommutative and has the asserted basis and dimension.

We now prove its relation with the first Frobenius kernel from coordinates. Put $A = k[G]$ and $\mathfrak{m} = \ker \epsilon$. Relative Frobenius pulls a twisted augmentation element back to its p -th power, so

$$C = k[G[F]] = A/(a^p : a \in \mathfrak{m}).$$

The ideal $\mathfrak{m}$ is finitely generated by S01 Lemmas 1.0 and 1.3. Its images in C generate a nilpotent ideal: if there are s generators, every word of length $s(p-1) + 1$ contains one with exponent p . The only underlying point of $\text{Spec } C$ is the identity. Iteratively replacing coefficients by their identity values shows that C is generated by those nilpotent images over k , so it is finite-dimensional.

Choose $t_1, \dots, t_n \in \mathfrak{m}$ dual to a basis of the tangent space. The complete smooth-coordinate proof of S02 Lemma 3.A makes their map $f : U \rightarrow \mathbb{A}_k^n$ étale on an open neighbourhood U of the identity. An étale map has unique lifts across any nilpotent ideal: apply its proved square-zero uniqueness and existence successively to I^j/I^{j+1} , whose square is zero. The lifted points glue by uniqueness.

For any k -algebra R and any list $b_i \in R$ with $b_i^p = 0$, the ideal $I = (b_i)$ is nilpotent by the same word bound. The identity over R/I has a unique lift $g \in U(R)$ with coordinates b_i . The twisted étale coordinate map $f^{(p)}$ sends $F_{G/k}(g)$ to zero; it reduces to the twisted identity. Uniqueness for that étale map makes $F_{G/k}(g)$ the identity, so $g \in G[F](R)$.

Conversely any $G[F](R)$ -point factors through U and has p -nilpotent coordinates. Its finitely many augmentation-generator values generate a nilpotent ideal J , and it reduces to the identity modulo J . The point constructed from its coordinates also reduces to the identity modulo J . Étale uniqueness, with their common coordinate map, makes the two points equal. This gives mutually inverse natural maps on every algebra R , including nonreduced ones. Hence $C \simeq k[t_1, \dots, t_n]/(t_i^p)$, and $\dim_k C = p^n$.

We describe invariant derivatives and their pairing, without a distribution-algebra classification assumption. A tangent vector is an identity derivation $v : A \rightarrow k$, so $v(ab) = v(a)\epsilon(b) + \epsilon(a)v(b)$. Its invariant derivation is $D_v = (\text{id} \otimes v)\Delta$. The Hopf identities prove the product rule and invariance, and evaluation at the identity recovers v . Invariance conversely determines the field from that value, by translation. Commutators of these derivations are invariant. Their p -th powers are invariant derivations as well: iteration of the product rule gives the binomial formula, whose intermediate coefficients vanish in characteristic p . Embed G faithfully in $\text{GL}(V)$ by Section 2. On its matrix-coordinate entries, $D_X(z) = zX$, and therefore $D_X^p(z) = zX^p$. This proves that $\text{Lie } G$ is closed under matrix p -th powers and that its restricted operation is $X^{[p]} = X^p$. The same calculation gives its bracket as the matrix commutator and the restricted axioms just verified.

Each D_X preserves the Frobenius-kernel ideal, since $D_X(a^p) = 0$. The finite dual $C^\vee$ is a Hopf algebra; its convolution is $(vw)(a) = (v \otimes w)(\Delta a)$. Directly from the coaction formula, this is $\epsilon D_v D_w(a)$ for tangent vectors, with the same formula for longer products. These primitive elements satisfy the bracket and restricted relations, giving a Hopf map $u(\text{Lie } G) \rightarrow C^\vee$. Choose the tangent basis dual to the t_i . Then $D_{x_i} t_j = \delta_{ij}$ modulo the augmentation ideal. On its associated graded algebra D_{x_i} lowers degree by one and acts as $\partial/\partial t_i$. Consequently a distribution monomial of total degree r vanishes on a coordinate monomial of degree greater than r , and its pairing at degree r is

$$\langle x_1^{a_1} \cdots x_n^{a_n}, t_1^{b_1} \cdots t_n^{b_n} \rangle = \begin{cases} \prod_i a_i! & a_i = b_i \text{ for all } i, \\ 0 & \text{otherwise,} \end{cases} \quad \sum_i a_i = \sum_i b_i = r.$$

All factorials are nonzero when $a_i < p$. In any relation, evaluate on monomials of the largest participating degree to kill all its coefficients of that degree, then descend on degree. Thus the p^n truncated distribution monomials are independent. They form a basis of $C^\vee$, proving the Hopf isomorphism with $u(\text{Lie } G)$.

For a restricted subalgebra $L \subset \text{Lie } G$, extend an ordered basis of L to one of $\text{Lie } G$. The PBW basis makes $u(L) \rightarrow u(\text{Lie } G)$ injective. Finite duality gives a surjective map of commutative Hopf coordinate algebras $C \rightarrow u(L)^\vee$, hence a full closed subgroup of $G[F]$. Its order is $p^{\dim L}$.

We prove that every subgroup is obtained this way, including the needed order assertion. Let $H \subset G[F]$ be closed and put $d = \dim_k \mathfrak{m}_H / \mathfrak{m}_H^2$. Lifts $a_1, \dots, a_d$ of a cotangent basis generate $k[H]$: first $\mathfrak{m}_H = (a_i) + \mathfrak{m}_H^2$, and iteration to its nilpotence exponent gives ideal generation; expanding coefficient identity values then gives algebra generation. Their p -th powers are zero. Thus

$$P = k[T_1, \dots, T_d]/(T_i^p) \twoheadrightarrow k[H]$$

has kernel $J \subset (T_i)^2$. It has no nonzero kernel. Otherwise choose the smallest augmentation degree $r \geq 2$ of a nonzero kernel element, and let f_r be its first nonzero homogeneous component. Lift the coproduct of each a_i to $T_i \otimes 1 + 1 \otimes T_i + u_i$, with u_i in the tensor product of the two augmentation ideals. This defines an algebra map out of P : every augmentation element in $P \otimes P$ has p -th power zero. The kernel element maps into $J \otimes P + P \otimes J$: this is the tensor quotient kernel, as choosing a vector-space complement to J in P shows. Its terms of bidegree $(s, r-s)$, $0 < s < r$, must vanish, since every element of J has augmentation degree at least r . The first homogeneous component is

therefore primitive for the standard additive coproduct on P . Its bidegree $(1, r - 1)$ coefficients are the partial derivatives of f_r . A nonzero positive-degree polynomial whose individual exponents are all below p has a nonzero partial derivative: differentiate in an index with positive exponent in any nonzero monomial; that exponent is a unit, and distinct monomials give distinct derivatives in that index. This contradiction proves $J = 0$ and $|H| = p^d$.

Its tangent derivations preserve its defining Hopf ideal I , because $\Delta I \subset I \otimes A + A \otimes I$ and the identity derivations kill I . Thus $\text{Lie } H$ is restricted in $\text{Lie } G$. The preceding distributions of its tangent vectors lie in $k[H]^\vee \subset C^\vee$. Their generated algebra $u(\text{Lie } H)$ has dimension p^d , exactly that of $k[H]^\vee$, so it equals that dual. Conversely this argument applied to the subgroup built from $u(L)$ gives its Lie dimension $\dim L$; the original L consists of primitive distributions in that Lie algebra, so it is the whole Lie algebra. This proves the subgroup correspondence.

For an abstract restricted L , left multiplication on $u(L)$ gives a faithful restricted embedding $L \rightarrow \text{End}_k u(L)$: it preserves the bracket and p -operation, and evaluating at 1 detects a nonzero basis-linear element by PBW. Apply the subgroup construction to this subalgebra of $\text{Lie GL}(u(L))$. Its coordinate algebra is exactly $u(L)^\vee$, so it is height one and has Lie algebra L . Every finite height-one group embeds in a general linear group by Section 2; Frobenius naturality puts it in that group's first kernel, and the preceding converse identifies its distributions with $u(\text{Lie } H)$. A restricted linear map extends uniquely to a Hopf map of enveloping algebras because those algebras are generated by their primitive Lie elements. Finite duality gives the corresponding group homomorphism. Conversely a group homomorphism gives this same map on primitive distributions, which generate the whole algebra. This proves the equivalence, including uniqueness of every morphism.

The schematic centralizer of a finite $H_L \subset G$ is closed: expand the conjugation identities for a finite generating list of $k[G]$ in a basis of the finite algebra $k[H_L]$. Their finitely many coefficient equations in $k[G]$ express exactly that each test point fixes every H_L -point. For a tangent Y , conjugation by $1 + \epsilon Y$ acts on the distribution generator X as $X + \epsilon[Y, X]$, by matrix multiplication. It fixes the full subgroup exactly when it fixes all generators of $u(L)$, equivalently $[Y, L] = 0$. The finite-dual basis formulas remain valid over the dual-number algebra, so this tests the entire subgroup scheme. PBW bases, coordinate equations, tangent derivations and finite duals all tensor exactly with field extensions; this proves the asserted compatibility.

Finally suppose L is commutative with geometrically invertible p -operation. Over an algebraic closure Lemma 4.F gives a fixed basis, so $u(L)$ is the tensor product of the primitive Hopf algebras $\bar{k}[x]/(x^p - x)$. Evaluation at the p distinct elements of $\mathbf{F}_p$, using the Chinese remainder proof, identifies each with functions on the additive group $\mathbf{F}_p$; its coproduct is pullback of addition. The dual has basis e_a with $e_a e_b = e_{a+b}$ and $\Delta(e_a) = e_a \otimes e_a$. It is $\bar{k}[\mathbf{Z}/p\mathbf{Z}]$, the coordinate algebra of μ_p . Thus H_L is geometrically $\mu_p^{\dim L}$. S08 Lemma G.0.T gives its finite-separable splitting and multiplicative type over the original field. $\square$

Lemma 4.J (multiplicative Jordan components). Let G be an affine algebraic group over a field k , and choose a faithful finite-dimensional linear representation. Over an algebraic closure, every $g \in G(\bar{k})$ has unique commuting factors $g = g_s g_u$, where g_s is semisimple and g_u is unipotent. Both factors belong to $G(\bar{k})$. Every homomorphism of affine algebraic groups preserves the factors. If k is perfect and $g \in G(k)$, both factors belong to $G(k)$. These conclusions do not require G to be smooth or connected.

Proof. Work first over $K = \bar{k}$, and let the representation space have dimension n . Dimension zero gives the trivial group, so assume $n > 0$. Cayley–Hamilton, S06 Section 1, and the polynomial Bézout/Chinese remainder calculation, CA03 Lemma 4.1, decompose the space as the sum of the generalized eigenspaces of g . On the summand for a nonzero eigenvalue λ , put $g_s = \lambda I$ and $g_u = \lambda^{-1}g$. Thus g_s is diagonalizable, $g_u - I$ is nilpotent, and the factors commute and multiply to g . This also proves uniqueness: if commuting semisimple S and unipotent U multiply to g , every eigenspace of S is preserved by U , and on it g has only the corresponding eigenvalue of S .

Therefore S acts by exactly λI on the generalized λ -space of g , and $U = S^{-1}g$.

We use the following elementary sequence calculation, and give its proof. Over any field, the sequences α^m , for distinct nonzero α and integers $m \geq 0$, are linearly independent. In characteristic zero the same is true of exponential polynomial sequences $\alpha^m P_\alpha(m)$: if their finite sum vanishes, each polynomial P_α is zero. Let E shift a sequence one place. On $\alpha^m P(m)$, the operator $E - \beta$ gives $\alpha^m(\alpha P(m+1) - \beta P(m))$. When $\beta = \alpha$, it lowers a positive polynomial degree by one, with leading coefficient multiplied by $\alpha \deg P$, and kills a constant. When $\beta \neq \alpha$, it preserves the degree and multiplies the leading coefficient by $\alpha - \beta$. To isolate one nonzero term, apply the product of $(E - \beta)^{\deg P_\beta + 1}$ over the other terms. It kills them and leaves $\alpha^m Q(m)$ with Q nonzero. Over a characteristic-zero field, a nonzero polynomial cannot vanish at all nonnegative integers, by the root bound in S02 Lemma 4.A. This is a contradiction. For constant polynomials the same operator proof works in every characteristic, since it leaves a nonzero constant and uses no integer denominators.

Let $\lambda_1, \dots, \lambda_r$ be the distinct eigenvalues, and write $D = (\mathbf{G}_m)^r$, acting by $z_i I$ on the corresponding generalized eigenspace. For $a \in \mathbf{Z}^r$, put $\lambda^a = \prod_i \lambda_i^{a_i}$, and let $N = \{a : \lambda^a = 1\}$. The quotient $K[\mathbf{Z}^r] \rightarrow K[\mathbf{Z}^r/N]$ defines a closed diagonalizable subgroup $D_\lambda \subset D$, as in Lemma 4.S. A finite Laurent sum $\sum_a c_a z^a$ belongs to its defining ideal exactly when the sum of its coefficients in each coset of N is zero. This follows directly from the coset basis of the group algebra. By the constant-sequence calculation, these are exactly the Laurent sums vanishing at every $(\lambda_1^m, \dots, \lambda_r^m)$.

Suppose the characteristic is $p > 0$, and choose $p^q \geq n$. Then $g_u^{p^q} = I$, by the binomial theorem for $I + (g_u - I)$. Hence $g^{p^q m} = g_s^{p^q m} \in G(K)$. A defining regular function f of $G \subset \mathrm{GL}_n$, restricted to D , is a finite Laurent sum. Its evaluations on these points vanish. The map $\alpha \mapsto \alpha^{p^q}$ is injective on K : $(\alpha - \beta)^{p^q} = 0$ implies $\alpha = \beta$. The constant-sequence calculation therefore makes the coefficient sum in every λ^a -class zero. Thus f restricts to zero on the full group scheme D_λ , and $D_\lambda \subset G_K$. In particular $g_s \in G(K)$, and $g_u = g_s^{-1}g \in G(K)$. This argument makes no isomorphism assertion about the Frobenius power map on a torus.

Now suppose the characteristic is zero. Put $T = g_u - I$, and define the nilpotent matrix $L = \log(I + T) = \sum_{j=1}^{n-1} (-1)^{j+1} T^j / j$. For a nilpotent matrix, $\exp(tL) = \sum_{j=0}^{n-1} t^j L^j / j!$ is a polynomial matrix. These finite logarithm and exponential formulas are inverse: in formal rational-coefficient power series, differentiating $\log(1+x)$ gives $1/(1+x)$, so $\exp(\log(1+x))$ has derivative equal to itself divided by $1+x$ and constant term one. Comparing coefficients gives $1+x$. Substituting the nilpotent T proves $\exp L = g_u$. Multiplying the exponential series and using the binomial coefficients also gives $\exp(tL) \exp(sL) = \exp((t+s)L)$, hence $g_u^m = \exp(mL)$.

The matrix L preserves the generalized eigenspaces and commutes with D . On each summand a basis adapted to the kernels of its successive powers makes it strictly upper triangular: extend bases through that kernel filtration, and note that L lowers the filtration. Consequently $\exp(tL)$ is upper triangular with diagonal one, and has determinant one. The substitution $(z, t) \mapsto D(z) \exp(tL)$ therefore pulls every regular function on GL_n , including the inverse determinant, back to a finite sum $F(z, t) = \sum_a z^a P_a(t)$ in $K[z_1^{\pm 1}, \dots, z_r^{\pm 1}, t]$.

For a defining function f of G , evaluation on g^m gives $\sum_a (\lambda^a)^m P_a(m) = 0$ for all $m \geq 0$. Group by the distinct values $\alpha = \lambda^a$. The exponential-polynomial calculation shows $\sum_{\lambda^a = \alpha} P_a(t) = 0$ as a polynomial for each α . Setting $t = 0$ shows f vanishes on the full D_λ . Setting every $z_i = 1$ shows $f(\exp(tL)) = 0$ as a polynomial. Thus $D_\lambda \subset G_K$, and the whole polynomial one-parameter homomorphism $t \mapsto \exp(tL)$ factors through G_K . In particular both Jordan factors belong to $G(K)$.

For functoriality, embed the target of a homomorphism $\phi : G \rightarrow H$ faithfully in a general linear group by Section 2. The restriction to D_λ has the complete weight decomposition of AG-GS, *Diagonalizable groups*, Theorem 3.1. Therefore $\phi(g_s)$ is semisimple. In characteristic p , $\phi(g_u)^{p^q} = I$, so $(\phi(g_u) - I)^{p^q} = 0$, and its image is unipotent. In characteristic zero, write the polynomial matrix of the homomorphism $\mathbf{G}_a \rightarrow H \rightarrow \mathrm{GL}$, obtained above, as $B(t)$. It satisfies $B(0) = I$

and $B(t + s) = B(t)B(s)$. Differentiate in s at zero to obtain $B'(t) = B(t)Y$, with $Y = B'(0)$. Coefficient comparison gives $B_j = Y^j/j!$. Since B is polynomial of some degree d , the coefficient of t^d in this differential equation gives $Y^{d+1} = 0$. Thus $B(1)$ is unipotent. In either characteristic the two images commute, their product is $\phi(g)$, and uniqueness makes them its Jordan factors. Applying this to two faithful representations and their direct sum also proves independence of the chosen representation.

Finally let k be perfect and $g \in G(k)$. The geometric factors have algebraic matrix coefficients. Perfectness makes their finite field of definition separable. Adjoin all roots of the finitely many separable minimal polynomials to obtain a finite normal separable extension K_0/k containing those coefficients. Its k -embeddings all map it to itself; the separable embedding count in S02 Lemma 4.A and the finite automorphism-field calculation in S08 Lemma G.5.2 therefore give $K_0^{\text{Gal}(K_0/k)} = k$. Every such automorphism fixes g and, by uniqueness, fixes its two geometric factors. Their coefficients are consequently in k . Their defining equations already vanish after faithful field extension, so they belong to $G(k)$. $\square$

Lemma 4.K (additive Jordan components and their functoriality). Let G be a smooth affine algebraic group over a field k , with a faithful finite-dimensional linear representation. Over an algebraic closure every $X \in \text{Lie } G$ has unique commuting matrix components $X = X_s + X_n$, with X_s semisimple and X_n nilpotent. Both components belong to $\text{Lie } G$. For every homomorphism from G to an affine algebraic group, its differential carries them to the additive Jordan components of its image, in any faithful representation of the target. Over a perfect field a rational X has rational components. In particular these assertions apply to the adjoint representation.

Proof. Work first over $K = \bar{k}$. Cayley–Hamilton and the same polynomial primary projectors used in Lemma 4.J decompose the representation space into the generalized eigenspaces of X . On its λ -summand define $X_s = \lambda I$ and $X_n = X - \lambda I$. These components commute, and X_n is nilpotent. They are unique: if a commuting semisimple matrix S and nilpotent matrix N sum to X , the eigenspaces of S are preserved by N , and on the eigenvalue- λ space the sum has only eigenvalue λ . Its generalized eigenspace is exactly that space, so S is the specified scalar on every generalized eigenspace of X . Dimension zero gives two zero matrices and is included.

In characteristic $p > 0$, choose p^q at least the matrix dimension. Since X_s and X_n commute, the characteristic- p binomial formula gives $X^{p^q} = X_s^{p^q}$. Let $V \subset K$ be the $\mathbf{F}_p$ -span of the eigenvalues λ^{p^q} . The assignment $\lambda^{p^q} \mapsto \lambda$ respects every $\mathbf{F}_p$ -linear relation: raising a proposed relation among the λ 's to the p^q -power gives the corresponding relation in V , and Frobenius on the field is injective. It therefore defines an $\mathbf{F}_p$ -linear map $h : V \rightarrow K$.

If $V = 0$, all eigenvalues are zero, so $X_s = 0$ and $X_n = X$. Otherwise choose an $\mathbf{F}_p$ -basis $v_1, \dots, v_d$ of V . The square matrix $(v_i^{p^j})_{1 \leq i \leq d, 0 \leq j < d}$ is invertible over K . Indeed, a vector in its kernel gives a polynomial $P(T) = \sum_{j < d} a_j T^{p^j}$ vanishing on that basis. This polynomial is additive and $\mathbf{F}_p$ -linear, by the characteristic- p binomial formula, so it vanishes on all p^d elements of V . If it were nonzero its degree would be at most $p^{d-1} < p^d$, contradicting the polynomial root bound in S02 Lemma 4.A. Thus the kernel is zero; the finite-dimensional linear algebra and adjugate calculation in S01 Lemmas 1.0 and 1.2 give invertibility. Solve $\sum_{j < d} a_j v_i^{p^j} = h(v_i)$. The resulting linearized polynomial agrees with h on all of V , and hence

$$X_s = P(X^{p^q}) = \sum_{j < d} a_j X^{p^{q+j}}.$$

Lemma 4.H proves closure of $\text{Lie } G$ under all these restricted matrix powers. It follows that X_s , and then $X_n = X - X_s$, belong to $\text{Lie } G$.

We give the full characteristic-zero argument using only formal power series, with no analytic exponential. For a matrix X , the series $E(t) = \sum_{j \geq 0} t^j X^j / j!$ is an invertible matrix over $K[[t]]$, since its constant term is I ; coefficient multiplication gives inverse $E(-t)$. It satisfies $E'(t) = E(t)X$. Let $A = K[\text{GL}_n]$, and let I_G be the Hopf ideal defining G . A tangent vector X defines the invariant

derivation D_X with $D_X(z) = zX$ on the matrix of coordinate functions. The identity derivation kills I_G , and the Hopf-ideal inclusion $\Delta I_G \subset I_G \otimes A + A \otimes I_G$ shows $D_X(I_G) \subset I_G$. This is the invariant-derivation calculation already proved in Lemma 4.H, and it uses no smoothness at this step.

For every $f \in A$, the chain rule for the substitution $z \mapsto E(t)$ gives

$$\frac{d}{dt}f(E(t)) = (D_X f)(E(t)), \quad f(E(t)) = \sum_{j \geq 0} \frac{t^j}{j!} \epsilon(D_X^j f).$$

To check the chain rule on all of A , check it first on the matrix entries, using $E' = EX$, and extend by sums and products. For the inverted determinant, differentiate the identity $bb^{-1} = 1$ to get $(b^{-1})' = -b^{-2}b'$; this gives exactly the localized derivation formula. Repeated differentiation and evaluation at $t = 0$ then give the coefficients in the second identity; characteristic zero makes every factorial invertible. If $f \in I_G$, every $D_X^j f$ still belongs to I_G , and its identity value is zero. Therefore $E(t)$ belongs to $G(K[[t]])$.

Write $\lambda_1, \dots, \lambda_r$ for the distinct eigenvalues of X , including zero if it occurs, and let $D = (\mathbf{G}_m)^r$ act by scalar z_i on the corresponding generalized eigenspace. Put

$$N = \{a \in \mathbf{Z}^r \mid \sum_i a_i \lambda_i = 0\}, \quad D_0 = D_K(\mathbf{Z}^r/N) \subset D.$$

Lemma 4.S gives its full defining ideal and coset basis. The matrices X_n and D commute. Since X_n is nilpotent, $\exp(tX_n)$ is polynomial, is upper triangular with diagonal one in a kernel-filtration basis, and has determinant one, by the calculation in Lemma 4.J. Consequently substitution $(z, t) \mapsto D(z) \exp(tX_n)$ sends any regular f on GL_n to a finite Laurent polynomial $F(z, t) = \sum_a z^a P_a(t)$. The formal identity $E(t) = \exp(tX_s) \exp(tX_n)$, obtained by multiplying commuting series and the binomial formula, gives for $f \in I_G$

$$\sum_a e^{(\sum_i a_i \lambda_i)t} P_a(t) = 0 \quad \text{in } K[[t]].$$

The formal exponential polynomials with distinct exponents are independent. Here is the exact proof. On $e^{\alpha t} P(t)$, the operator $d/dt - \beta$ gives $e^{\alpha t} (P' + (\alpha - \beta)P)$. For $\beta = \alpha$ it lowers a positive degree by one and kills a constant; for $\beta \neq \alpha$ it preserves the degree and multiplies the leading coefficient by $\alpha - \beta$. Apply the product of $(d/dt - \beta)^{\deg P_\beta + 1}$ over all exponents but one. If that term were nonzero, the resulting identity would be $e^{\alpha t} Q(t) = 0$, with Q a nonzero polynomial. Since the formal exponential has constant term one, it is a unit, so this is impossible. This proves independence and makes every grouped polynomial $\sum_{\sum a_i \lambda_i = \alpha} P_a(t)$ zero.

Setting $t = 0$ in these coefficient-sum identities shows that f vanishes on the full D_0 , since its cosets are exactly the equal-exponent classes. Setting all $z_i = 1$ shows $f(\exp(tX_n)) = 0$ as a polynomial. Thus $D_0 \subset G_K$, and $\exp(tX_n) : \mathbf{G}_a \rightarrow G_K$ is a polynomial group homomorphism. The element $z_i = 1 + \varepsilon \lambda_i$ in the dual numbers satisfies every relation $z^a = 1$, $a \in N$: products and inverses give $z^a = 1 + \varepsilon \sum_i a_i \lambda_i$. It is therefore a D_0 -tangent vector whose matrix is X_s . Differentiating the polynomial one-parameter subgroup gives X_n . Both components consequently belong to $\mathrm{Lie } G$.

We verify differential functoriality, including the positive-characteristic restricted operation. For a homomorphism $\phi : G \rightarrow H$, the invariant derivations satisfy $D_X \phi^* = \phi^* D_{d\phi(X)}$, by the coproduct identity for ϕ^* and evaluation of the tangent vector. Taking p -th powers of this identity and evaluating at the identity proves $d\phi(X^{[p]}) = d\phi(X)^{[p]}$. In a closed matrix group the same invariant-derivation calculation as in Lemma 4.H identifies this operation with the matrix p -th power; no smoothness of H is required. Differentials preserve brackets too: expand the commutator of $I + \varepsilon X$ and $I + \eta Y$ over $K[\varepsilon, \eta]/(\varepsilon^2, \eta^2)$; its mixed coefficient is $[X, Y]$, and a group homomorphism preserves this commutator. Thus the images of our commuting components commute.

In characteristic p , the image of X_n is nilpotent, since $X_n^{p^q} = 0$ for large q and the differential preserves these powers. Let L be the span of $X_s, X_s^p, X_s^{p^2}, \dots$ in $\mathrm{Lie } G$. It is a finite-dimensional

commutative restricted subalgebra. In a diagonalization its p -operation is injective, since a diagonal matrix whose p -th power is zero is zero. Over the perfect field K , an injective semilinear endomorphism of a finite-dimensional space is bijective: its images of a basis are independent by taking p -th roots of any linear relation, and hence form a basis. Lemma 4.H supplies a multiplicative-type height-one subgroup with Lie algebra L . Restrict ϕ to this subgroup. The full diagonalizable weight decomposition of AG-GS, *Diagonalizable groups*, Theorem 3.1 shows that its differential acts diagonally on every target representation. In particular $d\phi(X_s)$ is semisimple. This includes $L = 0$.

In characteristic zero, restrict ϕ to the full diagonalizable subgroup D_0 just constructed. Its weight decomposition again makes $d\phi(X_s)$ semisimple. The polynomial homomorphism $\phi(\exp(tX_n))$, in a faithful target representation, has the form $B(t)$ with $B(0) = I$ and $B(t+s) = B(t)B(s)$. The coefficient calculation in Lemma 4.J gives $B(t) = \sum_j t^j Y^j / j!$, with $Y = B'(0) = d\phi(X_n)$ nilpotent, since $B(t)$ is polynomial. In either characteristic the images are a commuting semisimple and nilpotent pair with sum $d\phi(X)$. The uniqueness proved at the start identifies them as its Jordan components. Applying this to two representations and their direct sum gives independence of faithful representation.

For perfect k and rational X , the finite-normal-separable-extension argument at the end of Lemma 4.J applies verbatim to the algebraic entries of these unique matrix components: all field automorphisms fixing X fix both components, so their coefficients lie in k . The linear equations defining $\text{Lie } G$ descend through faithful field extension, hence both are rational Lie vectors. Finally conjugation on $\text{Lie } G$ is an algebraic linear representation: differentiate the regular conjugation morphism at the identity in its second variable, or use the matrix formula $Y \mapsto gYg^{-1}$ in a faithful embedding. Therefore the functoriality just proved applies to the adjoint representation and gives $\text{ad}(X_s)$ and $\text{ad}(X_n)$ as the semisimple and nilpotent components of $\text{ad}(X)$. $\square$

Lemma 4.S (field character subgroups, quotients and closed embeddings). Let k be any field and M an abelian group. Every closed subgroup of $D_k(M)$ is uniquely $D_k(M/N)$, with its natural embedding, for a subgroup $N \subset M$. If M is finitely generated, its fppf quotient by that subgroup is $D_k(N)$; the quotient map is faithfully flat of finite presentation with that full schematic kernel. In particular the quotient of a split torus by any closed subgroup is a split torus. A homomorphism from a split torus to an affine algebraic k -group with trivial schematic kernel is a closed immersion. No characteristic or reducedness hypothesis is imposed on the closed subgroup or the target.

Proof. First, distinct group-like elements in a coalgebra over a field are linearly independent. In a shortest nonzero relation $\sum_{i=1}^s a_i g_i = 0$, all coefficients are nonzero. A one-term relation is impossible because $\epsilon(g_i) = 1$. The first $s-1$ elements are independent by minimality. Apply comultiplication and replace $a_s g_s$ in the first factor by $-\sum_{i<s} a_i g_i$. This gives

$$\sum_{i<s} a_i g_i \otimes (g_i - g_s) = 0.$$

Coefficient functionals for those independent first factors give $g_i = g_s$, a contradiction.

For a closed subgroup $H \subset D_k(M)$, the coordinate algebra is a Hopf quotient $k[M] \twoheadrightarrow C$. The images g_m of the basis elements e^m are nonzero unit group-like elements, with $g_m g_{m'} = g_{m+m'}$. Their equality relation is $g_m = g_{m'}$ precisely when $m - m' \in N$, where $N = \{m : g_m = 1\}$ is a subgroup. Their distinct images span C , by surjectivity, and are independent by the preceding paragraph. Thus $C = k[M/N]$, with the group-algebra Hopf operations and the specified quotient map. This proves the assertion, uniqueness and its full scheme structure, including finite nonreduced subgroups.

For any subgroup $N \subset M$, restriction of characters defines $D_k(M) \rightarrow D_k(N)$ through $k[N] \hookrightarrow k[M]$. Its kernel has coordinate algebra

$$k[M]/(e^n - 1 : n \in N) = k[M/N].$$

Indeed each difference of basis elements in one coset is $e^{m'}(e^{m-m'} - 1)$; every linear combination with zero coefficient sum in each coset is a sum of these differences. The resulting coset basis proves the equality of algebras. Choose a representative r_c of each coset. The decomposition

$$k[M] = \bigoplus_{c \in M/N} k[N]e^{r_c}$$

makes it a free $k[N]$ -module on a nonempty basis. Exact tensor with this free module preserves injections, and its direct-sum formula detects a nonzero module. The map is therefore faithfully flat, also after every base change.

We give the finite-generation argument needed for finite presentation and for the torus assertion. Every subgroup $L \subset \mathbf{Z}^r$ is finite free. Induct on r . The image of its first-coordinate projection is either zero or $d\mathbf{Z}$, where $d > 0$ is its least positive element; division with remainder proves that description. In the zero case apply induction in $\mathbf{Z}^{r-1}$. Otherwise choose $z \in L$ with first coordinate d . Subtracting the appropriate multiple of z puts every element in the projection kernel, and the intersection of $\mathbf{Z}z$ with that kernel is zero. Induction gives the finite free direct sum. For a subgroup of any finitely generated abelian M , lift it to a subgroup of a finite free group surjecting onto M ; the images of the just-constructed finite basis generate it. Thus N is finitely generated when M is. The group algebras $k[N]$ and $k[M]$ are generated by the finitely many basis characters of group generators and their inverses. The Hilbert basis proof of S01 Lemma 1.3 makes $k[N]$ Noetherian, and a finite polynomial presentation of $k[M]$ over it has finitely generated kernel. The quotient map is consequently of finite presentation.

It is an fppf cover and its full torsor relation is

$$D_k(M) \times D_k(M/N) \simeq D_k(M) \times_{D_k(N)} D_k(M), \quad (g, h) \mapsto (g, gh),$$

with inverse $(g, g') \mapsto (g, g^{-1}g')$. These maps are inverse on every test scheme by the kernel calculation. Local lifts along this cover and that relation identify the fppf quotient with $D_k(N)$. For $M = \mathbf{Z}^r$, the proved finite freeness of N makes it a split torus, as asserted.

Finally let $T = D_k(L)$, with L finite free, map to an affine algebraic group K with trivial schematic kernel. The full linearity construction of Section 2 embeds K in $\mathrm{GL}(V)$. The weight decomposition of AG-GS, *Diagonalizable groups*, Theorem 3.1 gives a basis of V in which the T -action is diagonal with characters $\lambda_1, \dots, \lambda_a \in L$. Its full kernel is $D_k(L/\Lambda)$, where Λ is the subgroup generated by these characters, by the preceding kernel calculation. Triviality forces $\Lambda = L$: a nonzero quotient group has at least two independent basis elements in its group algebra, so its diagonalizable scheme is nontrivial even when its geometric point set has only one point. The matrix coefficients supply e^{λ_i} , and the inverse determinant supplies $e^{-\sum_i \lambda_i}$. For each i ,

$$e^{-\lambda_i} = e^{-\sum_j \lambda_j} \prod_{j \neq i} e^{\lambda_j}.$$

They generate all of $k[L]$. Hence $k[\mathrm{GL}(V)] \rightarrow k[T]$ is surjective and $T \rightarrow \mathrm{GL}(V)$ is a closed immersion. It factors through the closed K , so $k[K] \rightarrow k[T]$ is also surjective. This proves the final assertion on full schemes. $\square$

Lemma 4.1. If a smooth connected affine k -group has a nontrivial k -torus, and geometrically maximal tori exist for groups of smaller dimension, then it has a geometrically maximal k -torus.

Proof. Let $M \subset G$ be a nontrivial k -torus and put $C = C_G(M)^0$. Lemma 3.1 makes the centralizer smooth. The identity-component and scalar-extension calculation of AG-RG-S08 Lemma G.0.3 makes C a smooth geometrically connected affine group containing M ; the torus M is central in C . We construct the affine quotient before applying induction.

The detecting-line construction in Section 2 works over k : its finite coproduct, ideal-generation and exterior-power calculations use no algebraic closedness. It gives a finite-dimensional C -representation

V and a line kv whose full schematic stabilizer is M . Let $W \subset V$ be the subrepresentation generated by v . Explicitly, write the coaction of v as $\sum_i w_i \otimes a_i$ with linearly independent $a_i \in k[C]$; the counit puts v in the span of the w_i , and coassociativity puts the coaction of each w_i in that span tensored with $k[C]$. Thus their span is the required W . The action of M on the line is a character $\chi : M \rightarrow \mathbf{G}_m$. Centrality says, on the universal $C \times M$ point, that $m(cv) = c(mv) = \chi(m)cv$. Comparing the independent coaction coefficients proves that M acts by this same scalar on all of W , on every test algebra.

Let $\rho : C \rightarrow \mathrm{GL}(\mathrm{End}_k W)$ act by conjugation through the representation on W . Its kernel is exactly M . Indeed a matrix which commutes with every matrix unit is scalar over any k -algebra: the diagonal units kill its off-diagonal entries, and the other matrix units make all diagonal entries equal. An invertible scalar matrix has a unit scalar. Such a matrix fixes the line kv , so its preimage in C belongs to the schematic stabilizer M . Conversely the scalar action just proved makes every M -point lie in the kernel. This avoids any assumption that a projective linear image is already the desired quotient.

Let H be the schematic image of ρ . It is an affine finite-type group: its coordinate algebra is the finitely generated Hopf subalgebra

$$B = \mathrm{im}(k[\mathrm{GL}(\mathrm{End} W)] \longrightarrow k[C]),$$

and the kernel of the coordinate map is a Hopf ideal. To check the full Hopf structure, write the ambient coordinate algebra as P and its kernel as J . The identity $\ker(P \otimes_k P \rightarrow B \otimes_k B) = J \otimes_k P + P \otimes_k J$ follows by taking vector-space complements to J . The injection $B \otimes_k B \hookrightarrow k[C] \otimes_k k[C]$ and the group homomorphism identity make comultiplication preserve that kernel; the identity and inverse maps give the counit and antipode identities. Thus all Hopf operations descend. For every field extension L/k , the injection $B \hookrightarrow k[C]$ remains injective after tensoring. Since $C_{\bar{k}}$ is smooth connected and therefore integral by Lemma 2.L, $H_{\bar{k}}$ is integral as well. The reduced-target image proof in AG-GS, *Quotients and torsors*, Lemma 11.1c applies over $\bar{k}$ and makes $C_{\bar{k}} \rightarrow H_{\bar{k}}$ faithfully flat of finite presentation. Its full kernel is $M_{\bar{k}}$. The integral algebraic group $H_{\bar{k}}$ is smooth by the reduction argument of Lemma 2.L. Faithfully flat descent, S04 Lemmas A1.3 and A1.3a, consequently makes $C \rightarrow H$ faithfully flat of finite presentation and H smooth over k . Its geometric fibre is connected, because it is the surjective image of $C_{\bar{k}}$.

For every test scheme, two C -points have the same image in H exactly when their quotient lies in M . Multiplication and quotient give inverse maps

$$C \times M \xrightarrow{\sim} C \times_H C, \quad (c, m) \longmapsto (c, cm), \quad (c, d) \longmapsto (c, c^{-1}d).$$

The faithfully flat map $C \rightarrow H$ therefore is an M -torsor and identifies H with the fppf quotient C/M . Applying the full orbit-dimension calculation of S08 Lemma G.4.D over $\bar{k}$ gives

$$\dim H = \dim C - \dim M < \dim G.$$

Dimension is unchanged by field extension by that same lemma. The induction hypothesis now supplies a geometrically maximal k -torus $Q \subset H$.

Put $E = C \times_H Q$, an affine closed subgroup of C . Its map to Q is the base change of the M -torsor. Every geometric fibre is a nonempty finite-type M -torsor over an algebraically closed field. The weak Nullstellensatz gives a rational point; translating by that point identifies the full fibre scheme with the split torus M over that field. The map is flat of finite presentation; the fibre criterion of AG-FSE, *Smooth morphisms*, Theorem 3.1 makes $E \rightarrow Q$ smooth. Hence E is smooth over k . It is geometrically connected. To check this over $\bar{k}$, its smooth components are disjoint open and closed by Lemma 2.L. Their images in $Q_{\bar{k}}$ are open by S08 Lemma G.4.F. A connected fibre cannot meet two such components, so those images are disjoint; surjectivity makes them cover the connected $Q_{\bar{k}}$. There is only one component.

The quotient Q is commutative and the kernel M is central, so the commutator of E takes values in M , and the next commutator is trivial. Thus $E_{\bar{k}}$ is a smooth connected solvable group. Theorem 2.K writes it as its smooth connected unitriangular radical semidirect a torus. Every character of that radical is trivial by Lemma 2.A, so its map to $Q_{\bar{k}}$ is trivial and it lies in $M_{\bar{k}}$. Lemma 2.A also makes its intersection with this diagonalizable group trivial, with full scheme structure. The radical is therefore the identity, and $E_{\bar{k}}$ is a torus. The complete field splitting proof in AG-RG-S08, Lemma G.0.T gives a finite separable extension over which E is the split torus with its free geometric character group. Thus E is a k -torus.

Finally suppose a $\bar{k}$ -torus $A \subset G_{\bar{k}}$ contains $E_{\bar{k}}$. It contains $M_{\bar{k}}$, hence centralizes it; its connected image lies in the identity component $C_{\bar{k}}$, with full scheme factorization from S08 Lemma G.0.1. Lemma 4.S represents the fppf quotient $A/M_{\bar{k}}$ as a split torus: its character group is the kernel of the surjection $X^*(A) \rightarrow X^*(M_{\bar{k}})$, a finite free subgroup of $X^*(A)$. The full torsor relation for $C \rightarrow H$ makes the map $A \rightarrow H_{\bar{k}}$ constant on the M -fibres, so S04 Lemma A1.5 descends it to this quotient.

Its induced map $A/M_{\bar{k}} \rightarrow H_{\bar{k}}$ has trivial schematic kernel. On any test scheme, lift a kernel point fppf-locally to A . Its image in H is the identity, so that lift lies in the full kernel M of $C \rightarrow H$; its quotient point is therefore the identity. Faithful descent of this equality proves triviality before the cover, including nilpotent test schemes. The closed-embedding assertion of Lemma 4.S now makes its image a closed torus in $H_{\bar{k}}$. It contains $Q_{\bar{k}}$, because $E \subset A$ and $E \rightarrow Q$ is fppf-surjective. Geometric maximality of Q makes that image $Q_{\bar{k}}$. Consequently the map of A factors through $C_{\bar{k}} \times_{H_{\bar{k}}} Q_{\bar{k}} = E_{\bar{k}}$. Since the opposite inclusion was assumed, $A = E_{\bar{k}}$ as closed subgroup schemes. Thus E is geometrically maximal in G , proving the lemma. $\square$

Lemma 4.Q (the central affine quotients used in the induction). Let G be a smooth geometrically connected affine group over a field k , and let $D \subset G$ be any closed central subgroup. Its fppf quotient is represented by a smooth geometrically connected affine k -group H . The map $G \rightarrow H$ is faithfully flat of finite presentation with full kernel D , and is a D -torsor. If D is finite, this map is finite locally free of rank $\dim_k k[D]$.

Proof. Smoothness makes G locally of finite type, and S08 Lemma G.0.3 makes its connected identity component quasi-compact; thus G is of finite type. The complete detecting-line construction of Section 2 gives a finite-dimensional representation V and a nonzero vector v whose line has full schematic stabilizer D . Let W be the cyclic subrepresentation generated by v . Concretely write its coaction as $\sum_i w_i \otimes a_i$ with independent $a_i \in k[G]$. The counit puts v in their span; coassociativity and coefficient functionals make that span a subrepresentation. The character χ of D on kv acts by the same scalar on W : on the universal $G \times D$ point, centrality gives $d(gv) = g(dv) = \chi(d)gv$, and independence of the a_i compares every coefficient. This is an equality on every test algebra, including nilpotent ones.

Let $\rho : G \rightarrow \mathrm{GL}(\mathrm{End}_k W)$ be conjugation through the representation on W . Its kernel is D . A matrix commuting with all matrix units is scalar over any algebra: the diagonal units force every off-diagonal entry to vanish and the other units make all diagonal entries equal. Its invertibility makes that scalar a unit. Such an element fixes kv in V , hence belongs to D ; conversely the scalar action of D just proved makes it lie in the kernel.

Put $B = \mathrm{im}(k[\mathrm{GL}(\mathrm{End} W)] \rightarrow k[G])$ and $H = \mathrm{Spec} B$. This is a finite-type Hopf algebra and defines an affine group. To check the Hopf ideal explicitly, the injection $B \hookrightarrow k[G]$ remains injective on tensor squares over the field. Write $P = k[\mathrm{GL}(\mathrm{End} W)]$ and J for the kernel. Exact field tensor gives $\ker(P \otimes P \rightarrow B \otimes B) = J \otimes P + P \otimes J$. Comultiplication of $u \in J$ maps to zero in $k[G] \otimes k[G]$, hence in its injected $B \otimes B$; therefore $\Delta(u) \in J \otimes P + P \otimes J$. Compatibility of the counit and antipode gives $\epsilon(J) = 0$ and $S(J) \subset J$. The group operations therefore descend to B . Exact field tensor also makes $B_{\bar{k}}$ the image algebra after extension. Since $G_{\bar{k}}$ is smooth connected, its algebra is a domain by Lemma 2.L; hence $B_{\bar{k}}$ is a domain.

The complete integral-target image argument of AG-GS, *Quotients and torsors*, Lemma 11.1c

applies to $G_{\bar{k}} \rightarrow H_{\bar{k}}$, whose coordinate map is injective. It is faithfully flat of finite presentation. S04 Lemma A1.3 descends flatness, and Lemma A1.3a descends finite presentation along $B \rightarrow B \otimes_k \bar{k}$. Faithfulness also descends: if tensoring a B -module with $k[G]$ gives zero, extending to $\bar{k}$ and using the faithful map upstairs makes its extended module zero, and faithful field tensor then makes the original module zero. Thus $G \rightarrow H$ is faithfully flat of finite presentation. The integral group $H_{\bar{k}}$ is smooth by Lemma 2.L and connected as the surjective image of $G_{\bar{k}}$. The full field smoothness-descent proof of S08 Lemma G.2.1 makes H smooth over k ; the same geometric image argument proves geometric connectedness.

The kernel calculation gives the full torsor relation

$$G \times D \xrightarrow{\sim} G \times_H G, \quad (g, d) \mapsto (g, gd), \quad (g, h) \mapsto (g, g^{-1}h).$$

Both are scheme morphisms and are inverse on every test scheme, because equality of the two images is exactly membership of their quotient in D . The fppf cover $G \rightarrow H$, with this relation, identifies its sheaf quotient by D with H : every H -point lifts fppf-locally, and two lifts differ uniquely by D .

If D is finite, pull back the quotient map along its own faithfully flat cover $G \rightarrow H$. Its coordinate module becomes $k[G] \otimes_k k[D]$, free of rank $\dim_k k[D]$ over $k[G]$. The finite generation, finite presentation and flatness descent in S04 Lemma A1.3 makes the original B -module $k[G]$ finite locally free. The rank can be read after the same cover, and is that constant dimension. Thus the quotient map is finite locally free of precisely that rank. $\square$

Lemma 4.A (a commutative finite extension of multiplicative-type groups). A finite commutative k -group E in an fppf exact sequence

$$1 \longrightarrow N \longrightarrow E \longrightarrow Q \longrightarrow 1$$

with N and Q of multiplicative type is of multiplicative type.

Proof. Work first over an algebraic closure. Both end groups are diagonalizable, and their invariant functors are exact by the complete weight decomposition in AG-GS, *Diagonalizable groups*, Theorem 3.1. For a finite-dimensional E -representation W , the space W^N is E -stable, since N is normal. Its action factors through Q on all test schemes. Indeed the kernel identity gives the full torsor relation $E \times N \simeq E \times_Q E$, with inverse $(a, b) \mapsto (a, a^{-1}b)$. The fppf-surjective quotient becomes N times its base after an fppf lift. The finite Q has a finite discrete spectrum by Artinian decomposition. Choose finitely many affine pieces of such lifts covering those points and combine them into one affine faithfully flat cover. The trivial projection is flat; flat descent from AG-CA, *Faithful flatness and the local criterion for flatness*, Corollary 3.2 makes the quotient flat, and its surjectivity makes it faithfully flat. Each matrix coefficient of the action on W^N has equal pullbacks along that relation, because N acts trivially. The faithfully flat equalizer proved in AG-RG-S04, Lemma A1.0 descends these coefficients to Q ; its algebra and group identities descend by the same faithful comparison. Hence

$$W^E = (W^N)^Q,$$

and E -invariants are exact.

For a subrepresentation $V \subset W$, the map

$$\mathrm{Hom}(W/V, W) \longrightarrow \mathrm{Hom}(W/V, W/V)$$

is a surjective equivariant linear map: lift the images of a vector-space basis to prove surjectivity, and use the representation $(g \cdot f)(v) = gf(g^{-1}v)$ on each Hom space for equivariance. Exactness of invariants lifts its invariant identity to an equivariant section. A nonzero finite-dimensional representation has a simple subrepresentation by choosing a nonzero subrepresentation of smallest dimension. The section splits it off, and induction on dimension decomposes every finite-dimensional representation into simples.

Put $C = k[E]$ and $H = C^\vee$. Choose a basis c_j of C with dual basis h_j of H . A coaction λ defines $hw = (1 \otimes h)\lambda(w)$, and an H -action defines $\lambda(w) = \sum_j (h_j w) \otimes c_j$. These formulas are inverse by the dual-basis identity. Pairing both tensor factors with that basis identifies coassociativity with associativity of the action, and the counit identity with its unit identity. Thus C -comodules and H -modules are exactly the same representations. Since E is commutative, its comultiplication on C is cocommutative, so the finite-dimensional algebra H is commutative. Its regular module is semisimple by the preceding paragraph. Decompose it into simple ideals $H = \bigoplus_i L_i$, and project one to $e_i \in L_i$. For $x \in L_i$, the products xe_j lie in $L_i \cap L_j$, hence vanish for $j \neq i$, while $x = \sum_j xe_j = xe_i$. Thus e_i is the identity of L_i , the e_i are orthogonal idempotents and each commutative L_i has no proper nonzero ideal. Every nonzero element is invertible, so each L_i is a field. It is a finite extension of the algebraically closed base, and therefore equals that base. Hence $H = k^M$ for a finite set M .

The Hopf maps of H give a group structure on this finite set: a map between finite products of the field sends a standard idempotent to a sum of standard idempotents, and hence specifies a map between their indexing sets. The Hopf identities give associativity, identity and inverse. Commutativity of C makes the dual comultiplication cocommutative, so M is abelian. Dualizing its idempotent basis gives the group-algebra Hopf algebra $C = k[M]$: basis products encode the multiplication of M , and each basis element has comultiplication $e^m \otimes e^m$ and counit one. Thus E is diagonalizable over the algebraic closure. The full finite-separable splitting proof of AG-RG-S08, Lemma G.0.T now proves the assertion over the original k , including imperfect fields. $\square$

Lemma 4.B (composite kernels stay central). Suppose G is smooth geometrically connected, N is a finite connected central multiplicative-type subgroup, and Q is a finite connected central multiplicative-type subgroup of G/N . Its inverse image E in G is finite connected central and of multiplicative type.

Proof. The kernel identity gives $E \times N \simeq E \times_Q E$, and fppf-local lifts identify $E \rightarrow Q$ with the projection from N times the base. The finite Q has a finite discrete spectrum by Artinian decomposition, so finitely many affine pieces of these lifts give a single affine faithfully flat cover. Effective affine descent in S04 Lemma A1.2, with comparison morphisms from Lemma A1.5, represents this torsor by an affine scheme. Its coordinate module becomes free of rank $\dim_k k[N]$ after that cover. S04 Lemma A1.3 descends finite local freeness, and the local lifts give surjectivity. Thus $E \rightarrow Q$ is finite faithfully flat and E is finite over k . After algebraic closure, Lemma G.0.2 of S08 makes the finite connected N and Q connected; each has one underlying point and has its identity as a rational point. The single geometric fibre of $E \rightarrow Q$ is a nonempty finite-type N -torsor. A rational point supplied by the weak Nullstellensatz identifies its full scheme with N . Thus E is geometrically connected, hence connected.

Since Q is commutative and N is central, the commutators in E take values in N . Replacing either lift by its product with an element of N leaves that commutator unchanged. The fppf comparison proof in S04 Lemma A1.5 therefore descends the commutator to $b : Q \times Q \rightarrow N$. The identities $[aa', c] = [a, c][a', c]$ and $[a, cc'] = [a, c][a, c']$, when the commutators are central, make b a homomorphism in each variable. They follow by inserting $a'^{-1}a^{-1}$ and the other inverse products into the commutator formula and commuting only the central commutators.

Over the algebraic closure write $Q = D(M_Q)$, $N = D(M_N)$, and put $R = k[Q]$. As a family of homomorphisms in the second variable, b is a Hopf map $R[M_N] \rightarrow R[M_Q]$. The coefficient calculation in S08 Lemma G.0.T makes the image of each character a sum of basis characters with orthogonal idempotent coefficients summing to one. A connected ring spectrum has only idempotents zero and one: $D(c)$ and $D(1 - c)$ are disjoint clopen subsets covering it, and a nonzero idempotent is not nilpotent, so has a nonempty principal open by S04 Lemma P0.1. Consequently the image is one basis character, independent of the first variable even along nilpotent test directions. Evaluation at that variable's identity gives the trivial homomorphism. Thus every character maps to one and b is trivial as a scheme map. Faithful scalar extension proves that E is commutative over the original field. Lemma 4.A makes E of multiplicative type.

Conjugation preserves E , as the preimage of the normal Q . Over the algebraic closure write $E = D(M_E)$. On any nonempty affine open of G , conjugation is a family of group homomorphisms of this diagonalizable group. That open is irreducible by S08 Lemma G.0.3, hence connected. The same coefficient argument makes every character image a fixed basis character throughout the open, including its nilpotents. Any two such opens meet and their families agree there, so their basis characters agree; for example evaluate at a rational point of the nonempty intersection. The open containing the identity gives the identity homomorphism. Thus conjugation is the identity family on all of G . Faithful field descent proves centrality schematically over k .

In the affine case used in Theorem 4.2, the full central-quotient proof of Lemma 4.Q applies to the now-central E . It represents G/E as a smooth geometrically connected affine group and makes $G \rightarrow G/E$ finite locally free of rank $\dim_k k[E]$. This additional affineness assertion uses the affine hypothesis; the central-kernel statement above retains its stated generality. $\square$

Lemma 4.C (descending the torus in a central affine group). Let Z be an affine commutative algebraic k -group. The torus factor of $(Z_k^0)_{\text{red}}$ is defined over k . In particular, a nontrivial such torus descends even if k is imperfect.

Proof. Let n range over the positive integers invertible in k , and let $I_n \subset A = k[Z]$ define $Z[n]$. Its coordinate algebra is finite and is a product of finite separable field extensions. Here is the full calculation. The differential of multiplication at the identity is addition: its restrictions to either tangent summand are the identity, so its linear map is their sum. Iterating gives $d[n] = n$, whose kernel is zero. Thus $Z[n]$ has zero tangent space at the identity and, by translation, at every rational point over an algebraic closure. On each of its Noetherian local rings at such a point, $\mathfrak{m}/\mathfrak{m}^2 = 0$; Nakayama gives $\mathfrak{m} = 0$. Every prime lies under a maximal ideal, and the local field there has no smaller prime, so every prime is maximal. The zero-dimensional Noetherian and Artinian decomposition proofs of AG-CA, *Noetherian and Artinian rings*, Theorems 3.3 and 4.2 make the geometric coordinate algebra a finite product of copies of $\bar{k}$, using the weak Nullstellensatz for their residue fields. The finite-algebra argument in S08 Lemma G.0.T now applies: independence of vectors proves finite dimensionality before extension; injectivity proves reducedness; and an inseparable element in a field factor would give a nonzero nilpotent after extension. Thus all its finite field factors are separable. In particular every $Z[n]$ is geometrically reduced.

Define the ideal and closed subscheme directly over the original field by

$$I = \bigcap_{\substack{n \in \mathbf{Z}_{>0} \\ n \cdot 1_k \neq 0}} I_n, \quad Y = \text{Spec}(A/I).$$

Here the indexing condition means that the integer n is invertible in k . This intersection commutes with every field extension L/k . Write an element of $A \otimes_k L$ as $\sum_i f_i \otimes a_i$, with the finitely many a_i linearly independent over k . Its image in $(A/I_n) \otimes_k L$ is zero precisely when each f_i lies in I_n , by applying linear functionals selecting these coefficients. Membership in every extended ideal is therefore precisely membership in $I \otimes_k L$. Consequently $(A/I) \otimes_k L$ embeds in the product of the reduced algebras $(A/I_n) \otimes_k L$. It is reduced. Thus Y is geometrically reduced, and the closure construction commutes with all field extensions without a perfectness assumption.

Over $\bar{k}$, put $H = \bigcup_n Z[n](\bar{k})$. These finite schemes are reduced, so their rational-point evaluations detect their functions; the preceding intersection equality says exactly that $Y_{\bar{k}}$ is the reduced schematic closure of H . The set H is a subgroup, since Z is commutative and the product of elements killed by n and m is killed by nm . It is schematically dense in $Y_{\bar{k}}$, and $H \times H$ is schematically dense in its product. For the latter assertion write a tensor as $\sum_i b_i \otimes c_i$ with the c_i independent. If its values on $H \times H$ vanish, evaluation at each first-coordinate point gives an element vanishing on all second-coordinate points, hence zero. Independence makes each b_i vanish on H , hence zero. Multiplication, inverse and identity therefore restrict to $Y_{\bar{k}}$: the equations of Y pull back to zero on those dense sets. The same equations hold before extension by faithful field tensor. Thus Y is a k -subgroup of Z , with its full scheme structure.

Over $\bar{k}$, put $R = (Z_{\bar{k}}^0)_{\text{red}}$. Reduction is a subgroup here: products of reduced finite-type schemes are reduced by S08 Lemma G.4.1, so the group operations factor through it. This connected commutative group is smooth by Lemma 2.L. The complete solvable structure theorem 2.K gives $R = T_Z \times U_Z$, since commutativity makes the semidirect action trivial. The unitriangular factor has no nontrivial torsion of order invertible in the field. In characteristic p , a matrix $1 + N$, with N nilpotent, has p -power order by $(1 + N)^{p^r} = 1 + N^{p^r}$. In characteristic zero, if $(1 + N)^n = 1$, its binomial expansion is $N(nI + NB) = 0$ for a polynomial B in N ; the second factor is invertible by a finite geometric sum, so $N = 0$. Hence $H \cap R$ consists exactly of the invertible-order torsion points of T_Z .

These points are schematically dense in T_Z . For a nonzero Laurent polynomial choose an invertible integer n larger than every absolute coordinate difference between its finitely many exponent vectors. Those vectors are distinct modulo $n\mathbf{Z}^r$, so their monomials stay independent in the coordinate algebra of $T_Z[n]$. Its restriction is nonzero, and that finite reduced scheme has a rational point where it is nonzero. The identity component of $Z_{\bar{k}}$ is open and closed by S08 Lemma G.0.3, so closure in that component is closure of the intersection with it. Its rational points factor through R . It follows that the intersection of $Y_{\bar{k}}$ with that component has exactly the underlying space of T_Z . Both are reduced closed schemes there, hence equal as schemes; equivalently both defining ideals are the radical ideal of that same closed subset, by S04 Lemma P0.1. This intersection is connected, so $Y_{\bar{k}}^0 = T_Z$.

Finally put $W = Y^0$, with the canonical component structure of S08 Lemma G.0.3. It is an open and closed k -subgroup, and that lemma's full field-base-change assertion gives $W_{\bar{k}} = T_Z$ as schemes. It is affine and of finite type, being closed in the affine finite-type Y . The complete finite-separable splitting proof of S08 Lemma G.0.T makes W a k -torus with that geometric fibre. This is the required descent. $\square$

Lemma 4.O (one order bound for a torus-free reduced centre). Let Z be a connected commutative affine algebraic group over an algebraically closed field K , retaining its possible nilpotents. If its reduction R has no nontrivial torus, there is one integer B bounding the order of every finite multiplicative-type closed subgroup $E \subset Z$.

Proof. Put $A = K[Z]$ and let J be its nilradical. S01 Lemmas 1.0 and 1.3 make A Noetherian, so J is generated by finitely many nilpotent elements u_i . If $u_i^{n_i} = 0$, every monomial of degree $1 + \sum_i (n_i - 1)$ contains one of these zero powers. Consequently choose $N \geq 1$ with $J^N = 0$. Let d generators generate the augmentation maximal ideal $\mathfrak{m}$ of A .

The reduction is a connected commutative subgroup: S08 Lemma G.4.1 makes $R \times R$ reduced, so the multiplication and inverse maps kill the nilradical and factor through R . Its underlying space is the connected space of Z . Lemma 2.L makes this reduced group smooth. The full solvable structure of Theorem 2.K and the hypothesis make R unitriangular. Over K , the subgroup E is diagonalizable by the definition of multiplicative type and S08 Lemma G.0.T. Its full intersection $E \cap R$ is diagonalizable by Lemma 4.S. Lemma 2.A says that a diagonalizable subgroup of a unitriangular group is the trivial scheme, including when that subgroup is finite and nonreduced.

Write $C = K[E]$. The intersection just computed has coordinate ring $C/JC = K$, and the identity map shows that JC is the augmentation maximal ideal $\mathfrak{n}$ of C . It is generated by the images of the d generators of $\mathfrak{m}$, since $A \twoheadrightarrow C$. Moreover $\mathfrak{n}^N = 0$. For $0 \leq j < N$, every class in $\mathfrak{n}^j/\mathfrak{n}^{j+1}$ is a K -linear combination of the at most d^j words of length j in those generators: a coefficient in C can be reduced modulo $\mathfrak{n}$, and its remainder raises the degree. For $j = 0$ the empty word gives the one-dimensional quotient. The finite filtration therefore gives the uniform bound

$$|E| = \dim_K C \leq B := \sum_{j=0}^{N-1} d^j.$$

Take $d^0 = 1$, including when $d = 0$. Both d and N depend only on Z , proving the assertion. $\square$

Lemma 4.G (finite-field Borel and maximal-torus descent). A smooth connected affine group over $\mathbf{F}_q$ has a Borel subgroup and a geometrically maximal torus defined over $\mathbf{F}_q$.

Proof. Over $\overline{\mathbf{F}}_q$, choose a torus T of largest dimension, starting with the trivial torus. Dimensions are bounded by $\dim G$, and a proper inclusion of tori strictly increases dimension by the earlier domain dimension theorem; thus T is maximal. Choose a smooth connected solvable subgroup B of largest dimension containing T , exactly as in the proof of Theorem 2.2; it is a Borel. Borel–torus pairs are conjugate: Lemma 2.1 first conjugates their Borels, and Lemma 2.N conjugates the two maximal tori within the common Borel.

Let $\sigma(a) = a^q$ on the algebraic closure, and let $\sigma(B, T)$ denote the full pair obtained by this coefficient transport. Since G is defined over $\mathbf{F}_q$, it is again such a pair. Choose $g \in G(\overline{\mathbf{F}}_q)$ with $\sigma(B, T) = g(B, T)g^{-1}$. On rational points this coefficient transport is the Frobenius F . Lemma 4.L gives h with $h^{-1}F(h) = g^{-1}$. Consequently $F(h)g = h$, and the full transported pair $(B', T') = h(B, T)h^{-1}$ satisfies $\sigma(B', T') = (B', T')$, as closed subgroup schemes.

We give the finite descent details. The ambient coordinate algebra is finitely presented, and the two ideals have finite generating lists by S01 Lemmas 1.0 and 1.3. Their finitely many coefficients lie in a finite extension $L/\mathbf{F}_q$, since every element of the algebraic closure is algebraic. Include the finitely many coefficients of their Hopf identities and finite algebra presentations as well, so they define closed subgroup schemes over L . If $[L : \mathbf{F}_q] = n$, then $|L| = q^n$. Multiplication by any nonzero $a \in L$ permutes the nonzero elements. Comparing their nonzero products gives $a^{q^n-1} = 1$. Therefore all elements are the q^n distinct roots of $X^{q^n} - X$; its derivative is -1 . All minimal polynomials over $\mathbf{F}_q$ divide this split separable polynomial, so L is normal and separable. Its Frobenius has order exactly n : its n -th power is the identity, whereas for $0 < r < n$, being the identity would give q^n roots to $X^{q^r} - X$, whose degree is q^r . The embedding count and fixed-field proof following S08 Lemma G.5.2 show that its Galois group has order n ; it is consequently generated by this Frobenius.

Both ideals over L are Frobenius-stable. Indeed their scalar extensions to the algebraic closure have that stability, and extension of field scalars is injective and detects equality of subspaces. The whole finite Galois group therefore preserves them. Apply the complete invariant-space proof of S08 Lemma G.5.2 to each ideal and to $L \otimes_{\mathbf{F}_q} \mathbf{F}_q[G]$. It identifies each ideal with the scalar extension of its invariant ideal in $\mathbf{F}_q[G]$. Multiplication and all Hopf maps descend by that lemma; their ideal identities are detected after faithful field extension. This gives closed subgroup schemes B_0, T_0 over $\mathbf{F}_q$ with the prescribed geometric fibres. S08 Lemma G.2.1 descends smoothness, and Lemma G.0.T makes T_0 a torus. Its geometric maximality and the Borel property of B_0 are exactly the properties of those fibres. This proves the lemma. $\square$

Lemma 4.R (rational semisimple directions in positive characteristic). Let k be infinite of characteristic $p > 0$, and let G/k be a smooth affine algebraic group. If $G_{\bar{k}}$ has a nontrivial torus, then $\text{Lie } G(k)$ contains a nonzero matrix-semisimple vector in any faithful linear representation.

Proof. Choose a closed $\mathbf{G}_m$ in that split geometric torus. In a weight basis of the faithful representation its integer weights generate $\mathbf{Z}$, by the full trivial-kernel calculation of Lemma 4.S. Thus at least one weight is not divisible by p . The Lie generator is a nonzero diagonal matrix, so its characteristic polynomial is not a pure power of its variable. At least one coefficient of the characteristic polynomial of the universal matrix in $\text{Lie } G$ is therefore a nonzero polynomial over k .

A nonzero polynomial over an infinite field has a rational point where it is nonzero. Induct on the number of variables: retain a nonzero coefficient polynomial in the last variable, choose values of the preceding variables by induction, and then choose the last value outside the finitely many roots of the resulting nonzero one-variable polynomial. The degree root bound follows by the repeated linear-factor division in S02 Lemma 4.A. Choose the resulting $Y \in \text{Lie } G(k)$. Its characteristic polynomial is not T^n , so Y is not nilpotent.

We give the exact matrix-power argument. Over $\bar{k}$, the characteristic polynomial splits. Cayley–Hamilton, S06 Section 1, and the polynomial Bézout/Chinese remainder proofs decompose the

space into the generalized eigenspaces: the pairwise coprime powers $(T - \lambda)^n$ give polynomial orthogonal projectors summing to one. On a projector's image $Y = \lambda I + N_\lambda$, with $N_\lambda^n = 0$. For $p^r \geq n$, the characteristic- p binomial identity gives

$$Y^{p^r} = \lambda^{p^r} I$$

on that entire summand. Thus Y^{p^r} is semisimple and nonzero, because some eigenvalue is nonzero. The matrix Y^{p^r} is rational over k simply because Y is, and Lemma 4.H puts every iterated p -power in $\text{Lie } G$. This is the required vector; no descent of the semisimple component itself is used. $\square$

Lemma 4.P (a torus in a finite infinitesimal central extension). Suppose P is an affine algebraic k -group and $q : P \rightarrow Q$ is a faithfully flat map of finite presentation to a nontrivial k -torus. Suppose its full kernel E is finite connected central and of multiplicative type. Then P contains a nontrivial k -torus. Smoothness of P and perfection of k are not assumed.

Proof. The full kernel relation gives $P \times E \simeq P \times_Q P$. Pulling q back along its own fppf cover therefore gives the finite free coordinate module $k[P] \otimes_k k[E]$, of rank $\dim_k k[E]$. S04 Lemma A1.3 descends this property, so q is finite locally free. The commutator of P takes values in E , since Q is commutative. Changing either input by a central E -point leaves it unchanged. S04 Lemma A1.5, applied to the full torsor relation in both inputs, descends it to $Q \times Q \rightarrow E$.

Over $\bar{k}$, Lemmas G.0.2 and G.0.T of S08 make E a connected finite diagonalizable scheme with one underlying point, its identity, and reduced algebra $\bar{k}$. The split $Q \times Q$ is reduced. Its map to E kills the nilradical and hence factors through that identity. The commutator is therefore zero after scalar extension, and faithful field extension detects this equality of affine coordinate maps before extension. Thus P is commutative.

Every geometric fibre of the finite surjection q is an E -torsor. The weak Nullstellensatz supplies a rational point on that nonempty finite fibre, and translation identifies its full scheme with E . Its underlying space has one point. This remains true after every base change. A finite map is universally closed by the integral closed-map proof in S04 Lemma P0.4; hence q is a universal homeomorphism. In particular $P_{\bar{k}}$ is connected and its connected reduction R still maps onto the underlying space of $Q_{\bar{k}}$.

The reduction R is a subgroup, because products of reduced affine groups over the algebraically closed field are reduced by S08 Lemma G.4.1. Lemma 2.L makes R smooth, and the full commutative solvable structure of Theorem 2.K writes it as a torus times a smooth connected unitriangular group U . Its map $U \rightarrow Q_{\bar{k}}$ is trivial by Lemma 2.A, so U lies in the finite full kernel $E_{\bar{k}}$. Therefore U is finite. Its coordinate algebra is a finite-dimensional domain by Lemma 2.L; it is a field, and algebraic closedness makes that field $\bar{k}$. Thus $U = 1$ as a scheme. It follows that R is a torus, and it is nontrivial since its underlying image is the nontrivial $Q_{\bar{k}}$. Finally apply Lemma 4.C to the commutative affine group P : it descends precisely this torus factor of its geometric reduced identity group to k . The descended torus is nontrivial and is a closed subgroup of P . $\square$

Theorem 4.2 (Grothendieck). Every smooth connected affine group over a field k has a geometrically maximal torus defined over k .

Proof. Induct on the finite dimension of G , for all fields at once. S08 Lemma G.0.3 shows that a connected field-group is geometrically irreducible and that its identity component commutes with field extension. A smooth connected affine group of dimension zero is trivial: over an algebraic closure, Lemma 2.L makes its coordinate ring a finite-type domain; the dimension/transcendence-degree theorem, AG-CA, *Noether normalization and dimension*, Theorem 4.2, gives transcendence degree zero. Its generators are algebraic over the algebraically closed field, so the ring is a finite field extension and equals that field. Faithful field extension descends the trivial group. Its trivial torus is maximal. For the induction step all smaller-dimensional smooth connected affine groups, over any field, have the asserted torus.

If $G_{\bar{k}}$ has a central maximal torus, it is unique, since all maximal tori are conjugate. Put $Z = Z(G)$, with its scheme-theoretic structure. The torus factor of $(Z_{\bar{k}}^0)_{\text{red}}$ is that maximal torus: it contains

every central torus, and maximality leaves no larger torus. Lemma 4.C descends this factor to a k -torus. Its geometric fibre is the chosen maximal torus. This proves the theorem in the central-maximal-torus case, including the trivial-torus case; the same lemma will later descend a merely nontrivial central torus.

Suppose k is infinite and $G_{\bar{k}}$ has a noncentral torus. We will find a nontrivial k -torus and then use Lemma 4.1, whose smaller-dimension hypothesis is supplied by this induction. Choose a faithful representation by Section 2 and regard $\mathfrak{g} = \text{Lie}(G)$ inside its matrix algebra. Lemma 4.K proves the entire additive Jordan assertions used here: geometric Lie membership, perfect-field rationality and functoriality for all algebraic-group homomorphisms, in particular the adjoint representation.

First assume that $\mathfrak{g}(k)$ contains a semisimple element X with $\text{ad}(X) \neq 0$. In characteristic zero its full stabilizer $C_G(X)$, defined by the matrix equations $\text{Ad}(g)X = X$, is a finite-type closed subgroup and is smooth by S08 Theorem G.3.4. Its Lie algebra is $\ker \text{ad}(X)$: substitute $g = I + \varepsilon Y$ in the conjugation equation to obtain $X + \varepsilon[Y, X] = X$. Thus the stabilizer is proper. Its smoothness makes its defining ideal radical; over $\bar{k}$, the integral coordinate ring of G and the strict-specialization/dimension proofs in AG-CA, *Noether normalization and dimension*, Lemma 4.1 and Theorem 4.2, make its dimension strictly smaller than $\dim G$. Its identity component is smooth geometrically connected by S08 Lemma G.0.3, and contains the tangent vector X , since that component is open at the identity. After algebraic closure it is not unipotent: Lemma 2.A makes the Lie action in a faithful representation strictly upper triangular for a unipotent group, which excludes a nonzero semisimple X . Dimension induction supplies a geometrically maximal k -torus in this smaller-dimensional centralizer component. Lemma 2.D makes its geometric fibre positive-dimensional, so it is nontrivial.

In characteristic $p > 0$, a version of the same argument avoids smoothness questions for a Lie-element stabilizer. Put

$$\mathfrak{h} = \text{span}_k \{X, X^{[p]}, X^{[p^2]}, \dots\}.$$

This is a commutative restricted Lie subalgebra: in a diagonalization of X , its powers and their linear combinations remain diagonal, and the p -operation preserves their span. Over $\bar{k}$ this operation has zero kernel, since a diagonal matrix with zero p -th power is zero. Its image is a vector subspace because every scalar has a p -th root in $\bar{k}$; in finite dimension injectivity therefore gives an invertible semilinear operation. The full subgroup correspondence and multiplicative-type assertion of Lemma 4.H give a closed $H \subset G[F]$ with $H_{\bar{k}} \simeq \mu_p^d$, $d = \dim \mathfrak{h} > 0$.

The centralizer $C_G(H)$ is smooth by the complete fixed-subgroup theorem in AG-GS, *Diagonalizable groups*, Theorem 3.4, whose coefficient and invariant-cohomology proof includes nonreduced multiplicative-type acting groups. Its Lie algebra is the centralizer of $\mathfrak{h}$, by Lemma 4.H, so $\text{ad}(X) \neq 0$ makes it proper. Its identity component contains the connected H , with the full factorization of S08 Lemmas G.0.1 and G.0.3. After algebraic closure that component cannot be unitriangular: Lemma 2.A excludes its nontrivial diagonalizable closed subgroup $H_{\bar{k}}$. Dimension induction therefore supplies a geometrically maximal torus in this smaller-dimensional smooth geometrically connected centralizer, and the no-torus criterion of Lemma 2.D makes it nontrivial.

It remains to find a usable semisimple direction or to remove the central directions obstructing this method. Over $\bar{k}$, split the given noncentral torus into its coordinate copies of $\mathbf{G}_m$. At least one is noncentral, since if each coordinate subgroup centralized G , their product would do so on every test algebra. Its adjoint action on $\mathfrak{g}_{\bar{k}}$ has a nonzero integer weight. Indeed, if all weights were zero, the weight decomposition of AG-GS, *Diagonalizable groups*, Theorem 3.1 and the centralizer Lie formula proved in Lemma 3.1 would make its smooth centralizer have the full Lie algebra and dimension of G . The integral-group dimension argument just used, followed by radicality of the smooth centralizer ideal, would make that entire centralizer equal to G , contradicting noncentrality. In characteristic zero a nonzero integer weight has nonzero differential, so the Lie generator has a nonzero adjoint eigenvalue. Therefore at least one nonconstant coefficient of the characteristic

polynomial of $\text{ad}(Y)$ is a nonzero polynomial on the k -vector space $\mathfrak{g}$. A nonzero polynomial over an infinite field has a rational point where it is nonzero: induct on the number of variables, specialize a nonzero coefficient polynomial in the preceding variables, and then choose the last variable outside the finitely many roots, using S02 Lemma 4.A. Choose such a rational Y . Its adjoint matrix is not nilpotent. The characteristic-zero field is perfect, so Lemma 4.K makes Y_s rational and identifies $\text{ad}(Y_s)$ as the semisimple part of $\text{ad}(Y)$; it is nonzero. This supplies the case already treated.

In characteristic p , the Lie generator of a noncentral torus can be central: the adjoint weights may have zero differential. If a semisimple rational vector has nonzero adjoint action, the preceding case applies. Otherwise all semisimple rational vectors are central in $\mathfrak{g}$. Lemma 4.R nevertheless supplies a nonzero such vector by a rational p -power of a nonnilpotent matrix. That proof establishes rationality directly, including over imperfect fields.

Choose a nonzero semisimple rational X . It is central in $\mathfrak{g}$, and so are its restricted powers, since the matrix commutator formula gives $\text{ad}(X^{[p]}) = (\text{ad } X)^p$. Their span $\mathfrak{h}$ is commutative with geometrically injective, hence invertible, semilinear operation. Lemma 4.H gives its nontrivial height-one multiplicative-type subgroup M . Its full centralizer is smooth by AG-GS, *Diagonalizable groups*, Theorem 3.4, and Lemma 4.H gives its whole Lie algebra as $\mathfrak{g}$. Over $\bar{k}$, its dimension is consequently $\dim G$. The smooth connected $G_{\bar{k}}$ is integral by Lemma 2.L; the earlier domain dimension theorem excludes a proper closed subset of that dimension. The centralizer's defining ideal is radical, since it is smooth, and vanishes on the entire underlying space, so it is zero in the reduced coordinate ring of $G_{\bar{k}}$. Faithful field extension descends this equality. Thus $C_G(M) = G$ as schemes, and M is central. No descent of the span of all semisimple vectors is needed.

Pass to G/M and repeat if all its rational semisimple vectors are central. Lemma 4.B makes every composite kernel E_i finite connected central and of multiplicative type and the quotient smooth geometrically connected affine. The geometric torus needed to continue persists. Indeed its intersection with a finite E_i is a finite diagonalizable subgroup by Lemma 4.S; the quotient torus is represented there and has the same positive rank, since its character kernel has finite index in a nonzero free lattice. Its induced map into $(G/E_i)_{\bar{k}}$ has trivial full kernel by fppf lifting, so Lemma 4.S makes it a closed torus. If a geometric maximal torus of the new group is central, the first case gives a nontrivial k -torus there, because its geometric group still contains the positive-rank torus just constructed; otherwise Lemma 4.R again supplies a nonzero rational semisimple direction. Whenever a quotient torus is found, Lemma 4.P recovers a nontrivial torus in G , and Lemma 4.1 completes the induction. If no quotient torus is found, each next height-one kernel is nontrivial and its order increases as follows.

If the iteration continues indefinitely, its composite kernels E_i are finite connected central and of multiplicative type by Lemma 4.B. Their orders strictly increase: the exact finite locally free rank assertion in that lemma gives $|E_{i+1}| = |E_i| |H_i|$, and the new nontrivial height-one kernel has order $|H_i| = p^{\dim \mathfrak{h}_i} > 1$. After algebraic closure these connected kernels lie in $Z = Z(G_{\bar{k}})^0$, with full factorization through that identity component from S08 Lemmas G.0.1 and G.0.3. If its connected reduction R had no nontrivial torus, the single bound of Lemma 4.O would bound every $|E_i|$, a contradiction. Thus R has a nontrivial torus; Theorem 2.K gives its torus factor, and Lemma 4.C descends that factor to a nontrivial k -torus. Lemma 4.1 finishes this branch of the dimension induction.

If the iteration stops at a nontrivial k -torus $Q \subset G/E_i$, its inverse image P has the full finite connected central multiplicative-type kernel E_i and is its fppf torsor over Q . Lemma 4.P supplies a nontrivial k -torus in $P \subset G$, with the full commutator descent, universal-homeomorphism and geometric-reduction arguments proved there. Lemma 4.1 makes it geometrically maximal in G . This finishes the infinite-field induction.

Finally let $k = \mathbf{F}_q$. Lemma 4.G gives a Borel and a geometrically maximal torus over k , with the Lang equation, full finite-field Galois calculation and descent of the defining Hopf ideals already proved. This closes the induction in every characteristic and over every field. $\square$

The infinitesimal step is needed because a noncentral group torus may have a central Lie algebra. In SL_2 in characteristic two, the Lie algebra of the diagonal torus consists of scalar matrices. The torus itself still acts nontrivially on the upper and lower root groups.

5. The finite symmetry left after centralizing

For a maximal torus in a reductive group scheme, define the fppf quotient

$$W_G(T) = N_G(T)/T.$$

The quotient is a sheaf of groups: the definition of the normalizer makes T normal in it, so multiplying local coset representatives gives a well-defined local coset, compatible with their overlaps. A representative n acts on T by conjugation. Replacing it by nt for $t \in T$ gives the same automorphism, since T is commutative. Thus the action descends through the fppf sheaf quotient, and composition/inverse on characters and cocharacters gives its actions on both sheaves. Representability and finiteness are not assumed in this definition.

The general finite étale Weyl-scheme theorem is taught with its complete proof in [Root data, Weyl chambers and the Bruhat decomposition, Theorem 4.2](#), after the centralizer and reflection arguments it consumes. The matrix examples below have their own direct proofs here.

Example 5.2. Let A normalize the diagonal torus of GL_n over a test algebra R , including its nilpotents. Its induced torus automorphism is, by Lemma D.1, locally a fixed invertible integer map f on the character lattice. On each part where f is fixed, the intertwining equation for the standard representation is

$$(z^{e_i} - z^{f(e_j)})a_{ij} = 0 \quad \text{in } R[z_1^{\pm 1}, \dots, z_n^{\pm 1}].$$

If the two characters differ, their distinct coefficients force $a_{ij} = 0$. If some $f(e_j)$ were not among the e_i , the whole j th column would vanish, contradicting invertibility. The distinct $f(e_j)$ thus permute the n distinct e_i . Only the corresponding one entry per column survives; the determinant is a signed product of these entries and is a unit, so each entry is a unit. Thus A is a permutation matrix times an invertible diagonal matrix on this part. Conversely multiplication by such matrices induces the stated character permutation and normalizes the torus on every test algebra. The local permutation is unique, and therefore glues as a section of the constant finite scheme $\mathfrak{S}_n$. This proves the entire functor identity

$$N_{\mathrm{GL}_n}(D) = D \rtimes (\mathfrak{S}_n)_S, \quad W_{\mathrm{GL}_n}(D) = (\mathfrak{S}_n)_S.$$

The permutation is locally constant; over a disconnected algebra it need not be a single global permutation.

6. Exercises

Exercise 6.1 (first steps). Over an arbitrary scheme, compute the centralizer and normalizer of the diagonal torus of GL_3 . Explain how the answer changes when the test algebra is disconnected.

Exercise 6.2 (descent). Show that the two real tori in Example 1.2 are conjugate over $\mathbb{C}$ but not over $\mathbb{R}$. Show also that the multiplication torus $\mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbf{G}_m$ in $\mathrm{GL}_{2,\mathbb{R}}$ is maximal and nonsplit.

Exercise 6.3 (finite fields). For a smooth connected affine group over $\mathbb{F}_q$, construct a Borel–torus pair over $\mathbb{F}_q$ from one over $\overline{\mathbb{F}}_q$. The only cohomological input permitted is Lang’s theorem.

Exercise 6.4 (a family). Over $\mathbb{Z}$, let D be the diagonal torus in SL_2 . Show that its nontrivial Weyl element is represented by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ even in characteristic two, and explain why the quotient is étale there.

Solutions

Solution 6.1

The universal Laurent-polynomial computation in Example 1.1 makes the centralizer diagonal. A normalizer induces a permutation of the three rank-one character summands in the standard representation. After a decomposition by idempotents, the permutation is fixed on each piece. The normalizer is therefore $D \rtimes (\mathfrak{S}_3)_S$, with quotient the constant finite étale group $\mathfrak{S}_3$. Disconnected test algebras allow different permutations on different open and closed pieces, exactly as sections of a constant étale scheme do.

Solution 6.2

The eigenvectors $(1, -i)$ and $(1, i)$ diagonalize the rotation matrix, with eigenvalues $a + bi$ and $a - bi$. On the norm-one subgroup these are inverse, so the eigenbasis carries it to the diagonal torus of SL_2 . Rescale a column to obtain a determinant-one change of basis. Conjugation on its character lattice is $m \mapsto -m$, whereas the diagonal real torus has trivial action, which rules out a real conjugation. For the multiplication torus, the two eigenvalues vary independently after complex base change; it becomes the full diagonal torus of GL_2 . It is therefore maximal. Its two character generators are exchanged by conjugation, so it is nonsplit over $\mathbb{R}$.

Solution 6.3

Lemma 2.1 conjugates the two Borels, and Lemma 2.N conjugates their maximal tori inside the common Borel. Hence choose g with $F(B, T) = g(B, T)g^{-1}$. Lemma 4.L gives $h^{-1}F(h) = g^{-1}$. Direct substitution then makes both hBh^{-1} and hTh^{-1} Frobenius-stable. For the descent, their finitely generated defining ideals have coefficients in some finite extension $L/\mathbb{F}_q$, because every coefficient in the algebraic closure is algebraic. Frobenius-stability after faithful field extension detects stability of those ideals over L , so the whole cyclic Galois group preserves them. S08 Lemma G.5.2 supplies the full invariant-space proof identifying each ideal with the scalar extension of its invariant ideal; it also descends the Hopf identities. Smoothness descends by S08 Lemma G.2.1, and Lemma G.0.T identifies the descended geometric torus as a torus. Its maximality and the Borel property are those of the unchanged geometric fibres. This is precisely the explicit descent construction of Lemma 4.G. Connectedness of G is a hypothesis of the proved Lang equation.

Solution 6.4

Conjugation by the displayed matrix sends $\mathrm{diag}(t, t^{-1})$ to $\mathrm{diag}(t^{-1}, t)$. A normalizer permutes the two universal weight lines, giving two components, each a coset of D . Its quotient is the constant scheme with two points over $\mathbb{Z}$, not μ_2 . In characteristic two μ_2 is nonreduced, whereas a constant two-point scheme is still the disjoint union of two copies of the field and remains étale. The representative squares to $-I \in D$; it need not itself have order two before taking the quotient.

Exact supporting prerequisites and proof order

The classical triangularization, fixed-point, stabilizer, solvable structure and conjugacy arguments are in Section 2. Section 3 proves formal multiplicative-type homomorphism effectivity and the closed-immersion criterion. Section 4 proves Lang’s theorem and supplies the Jordan and restricted-Lie arguments used in the field existence theorem. Their historical sources remain credited below.

The supporting geometric proofs are [Group schemes over a field](#), for orbits and the closed-orbit lemma; [Quotients and torsors](#), for finite free-action quotients; [Faithfully flat descent](#), for affine schemes, representations and character descent; [Diagonalizable groups and groups of multiplicative type](#), for the field character classification; [Limits of schemes and Noetherian approximation](#), for finite-presentation spreading; the complete [Artin approximation proof and application](#), Sections 1–5, used in Section 3; and *Dualizing sheaves and Serre duality for projective schemes*, for duality of curves. The prerequisite guide records their full statements and proof locations. The tori and reductive-group theorems assigned to this lesson are proved here.

The general cocharacter construction is proved here. The relative maximal-torus application is proved in the second lesson after the centralizer equality, and the finite Weyl-scheme theorem is proved in the fourth lesson after its reflection calculation. Their obligations are retained at those proof locations.

The [course prerequisite guide](#) records the exact supporting statements, their full proof routes, and the hypotheses needed in their applications.

References

- Brian Conrad, *Reductive group schemes*, freely accessible author draft: §§2.1–2.3 and 3.2, Appendix A and §§B.2–B.3.
- M. Demazure and A. Grothendieck, with the SGA 3 contributors, *Schémas en groupes*, re-edited by P. Gille and P. Polo: Exposés III, VII A, IX–X, XII and XIV.
- [Milne’s freely accessible Algebraic Groups, version 2.00 \(2015\)](#). This exact free author edition provides comparison material; its citations do not replace the programme proofs.